Understanding AdS/CFT Y-system

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[arXiv:1110.0562] N. Gromov, V. Kazakov, SL & D. Volin

[arXiv:1007.1770] V. Kazakov & SL
[arXiv:1010.2720] N. Gromov, V.Kazakov, SL & Z.Tsuboi
[arXiv:1010.4022] V. Kazakov, SL & Z.Tsuboi

Imperial College, October 5, 2011

AdS/CFT spectral problem

AdS/CFT correspondence

Conjectured duality between
4-dimensional ($\mathcal{N} = 4$) Super-Yang-Mills theory and
type IIB string theory on $AdS_5 \times S^5$ background

- weak-strong duality

- Conformal symmetry $\rightsquigarrow$ compute 2-points and 3-points correlation functions.
- The limit of "long operators" is integrable [Beisert Eden Staudacher 07]

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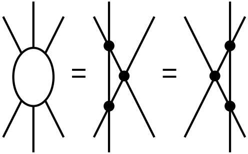

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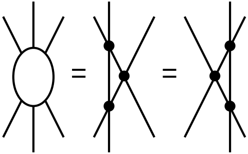
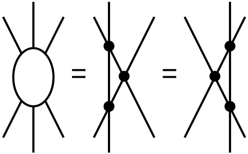


Outline

1. Introduction to AdS/CFT Y-system
   - Thermodynamic Bethe Ansatz
   - Y-system
   - Hirota equation
2. Solving Hirota $\rightsquigarrow$ Q-functions
   - Wronskian solution of Hirota equation
   - Q-Q relations
3. FiNLIE $\rightsquigarrow$ symmetries & analyticity
   - New symmetries
   - a Riemann-Hilbert Problem
   - FiNLIE

"Integrability" and Bethe equations

The limit of "long operators" is integrable



- Bethe equation : $\forall j, e^{iLp_j} = \prod_{k \neq j} S_{j,k}$
- $E = \sum_j E_j$

For relativistic models, $p_j = m_j \sinh \theta_j$, $p_j = m_j \cosh \theta_j$.

For AdS/CFT, $p_j = \frac{1}{i} \log \frac{x_j^{[+a]}}{x_j^{[-a]}}$, $E_j = a + 2i\frac{g}{x_j^{[+a]}} - 2i\frac{g}{x_j^{[-a]}}$.

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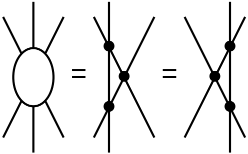

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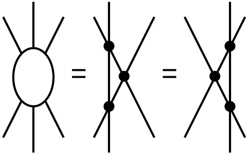

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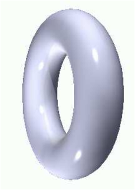
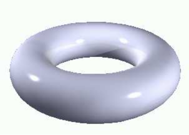
Thermodynamic Bethe Ansatz



short operators
infinite time periodicity $R \to \infty$
Path integral $Z \sim e^{-RE_0(L)}$



Long operators
finite time-periodicity
$\Rightarrow$ finite temperature
S-matrix, Bethe equation,
Bound states

“free Energy” : $f(L) = E_0(L)$

TBA equations

$\rightsquigarrow$ Equations of the form

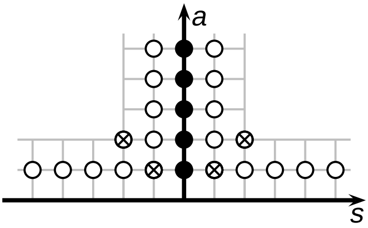
$$Y_{a,s}(u) = \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right)$$
$$+\delta_{s,0}\, L\, \log \frac{x^{[-a]}}{x^{[+a]}} + \langle \text{Source Terms} \rangle$$

[Gromov Kazakov Kozak Vieira 09]

[Bombardelli Fioravanti Tateo 09] [Autyunov Frolov 09]

$$x^{[\pm a]} = x(u \pm a\tfrac{i}{2}) = \frac{1}{2}\frac{u \pm a\frac{i}{2}}{g} + \frac{i}{2}\sqrt{4 - \left(\frac{u \pm a\frac{i}{2}}{g}\right)^2}$$

- $Y_{a,s}(u)$ is a function of $a, s \in \mathbb{Z}$ and u in $\mathbb{R}$



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- Extra assumption : Excited states obey the same equations.
  Each state correspond to a different solution of Y-system, characterized by its zeroes and poles

Resolution of *AdS/CFT* Spectral problem.

TBA approach

• infinite set of NLIEs

• complicated kernels (zhukovski cuts)

$\Rightarrow$

Y-system equation

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})}$$

+ Analyticity

[Cavaglia Fioravanti Tateo 09]

$\Leftrightarrow$

Hirota equation

T-system

Gauge

• Finite parameterization

[Gromov Kazakov S.L. Tsuboi 10]

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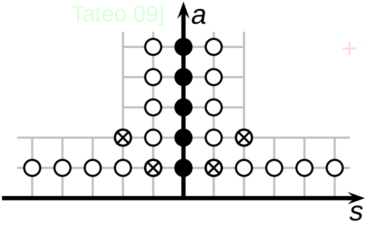

Y-system and Hirota equation

Y-system Equation

The TBA integral equation imply the ‘local’ relation

$$Y^+_{a,s}\,Y^-_{a,s} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$$

[Gromov Kazakov Vieira 09] where $Y^\pm_{a,s} = Y_{a,s}(u \pm \frac{i}{2})$

- change of variable $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$

Hirota equation

$$T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$

Gauge freedom

Y-functions and Hirota equation are invariant under gauge transformations $T_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[-a+s]} g_4^{[-a-s]} T_{a,s}$

$$f^{[\pm k]} \equiv f(u \pm ki/2)$$

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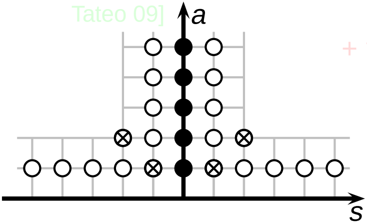

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T-system / Gauge

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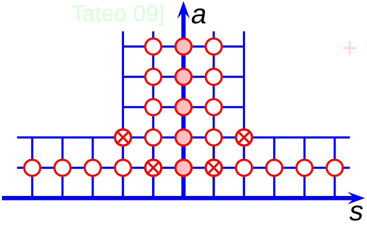
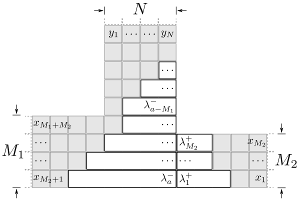


Hirota equation and characters of $PSU(2,2|4)$



Some irreps of $PSU(M_1, M_2|N)$ can be labeled by generalized young diagrams, for which characters are known

[Gromov Kazakov Tsuboi 10]

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- For rectangular young diagrams, $\chi_{a,s}{}^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$

- The Hirota equation $T_{a,s}{}^+ T_{a,s}{}^- = T_{a,s+1}T_{a,s-1} + T_{a+1,s}T_{a-1,s}$ is a generalization of this character identity with an extra parameter u

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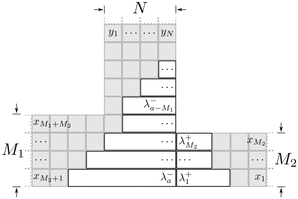

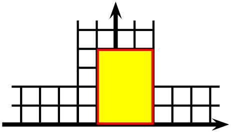

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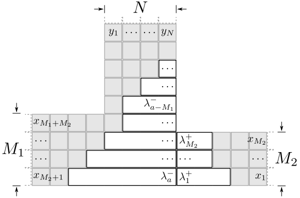

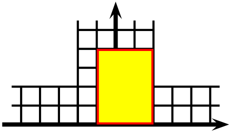

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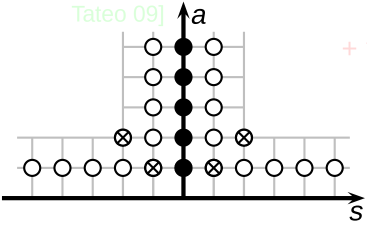

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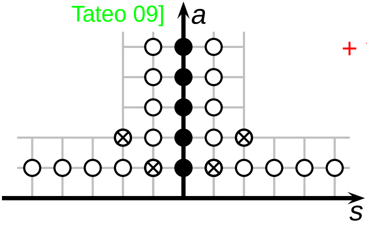

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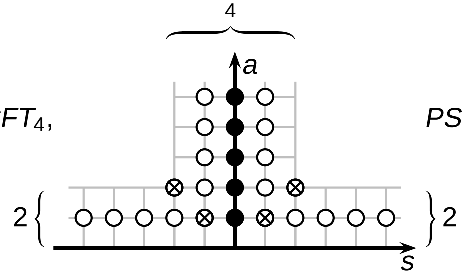
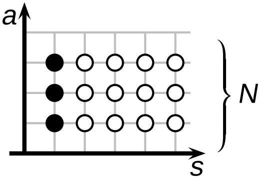
Other Y-systems

- for AdS_5/CFT_4, $PSU(2,2|4)$



- for $SU(N)$ Gross-Neuveu,



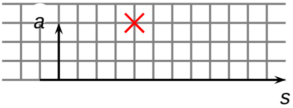
Outline

Q-functions solve Hirota equation

Hirota equation is solved by determinants of Q-functions :
eg. for $SU(4)$,

$$T_{3,s} = \begin{vmatrix} q_1^{[+s+2]} & q_2^{[+s+2]} & q_3^{[+s+2]} & q_4^{[+s+2]} \\ q_1^{[+s]} & q_2^{[+s]} & q_3^{[+s]} & q_4^{[+s]} \\ q_1^{[+s-2]} & q_2^{[+s-2]} & q_3^{[+s-2]} & q_4^{[+s-2]} \\ p_1^{[-s]} & p_2^{[-s]} & p_3^{[-s]} & p_4^{[-s]} \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} 3 \\ \left.\vphantom{a}\right\} 4-3 \end{matrix}$$

- $q_i^{[+k]} = q_i(u + k\frac{i}{2})$



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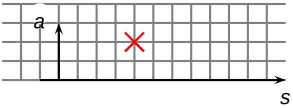

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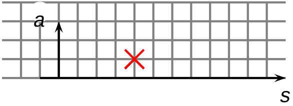

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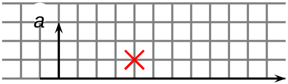
$$T_{1,s} = \begin{vmatrix} q_1^{[+s]} & q_2^{[+s]} & q_3^{[+s]} & q_4^{[+s]} \\ p_1^{[-s+2]} & p_2^{[-s+2]} & p_3^{[-s+2]} & p_4^{[-s+2]} \\ p_1^{[-s]} & p_2^{[-s]} & p_3^{[-s]} & p_4^{[-s]} \\ p_1^{[-s-2]} & p_2^{[-s-2]} & p_3^{[-s-2]} & p_4^{[-s-2]} \end{vmatrix} \quad \begin{matrix} \}\, 1 \\ \}\, 4-1 \end{matrix}$$

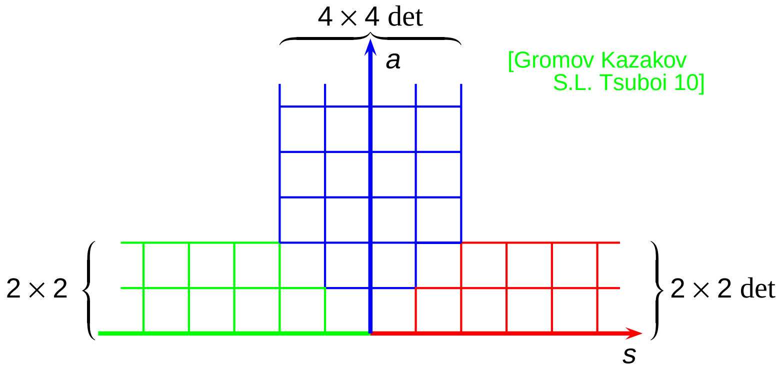
- $q_i^{[+k]} = q_i(u + k\frac{i}{2})$



Finiteness

q-functions are the building blocks of any Hirota solution.
They allow to parameterize the whole Y-system in terms of a finite number of Q-functions.

Wronskian parameterization of AdS/CFT T-functions.



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Skip QQ-relations ⇝ FiNLIE

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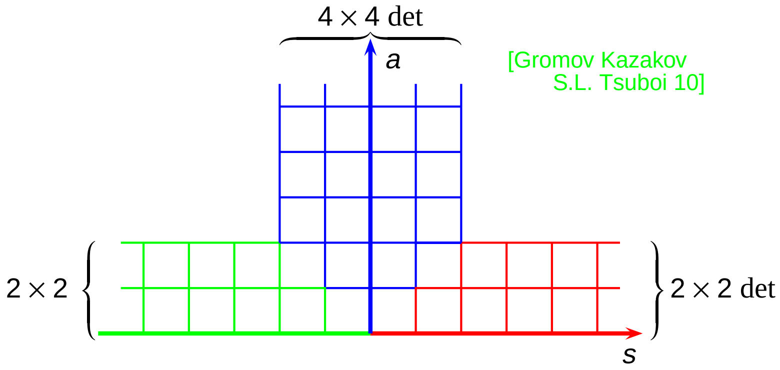

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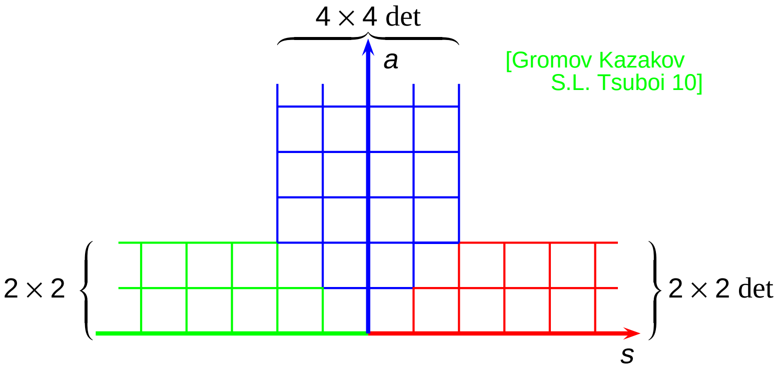

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q-functions for upper band

- right band : $T_{1,s} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix} \in \mathbb{R} \quad \Rightarrow \quad P = -\bar{Q}$
- upper band :

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- one defines this way 2^4 q-functions for the upper band, only 4 of which are independent.

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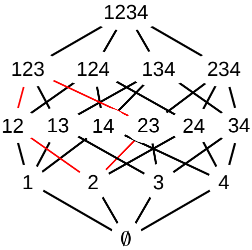
there is one such relation at every facet of the hypercube



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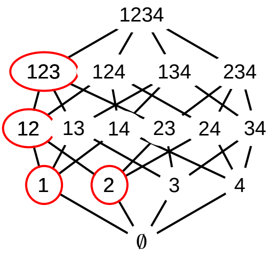

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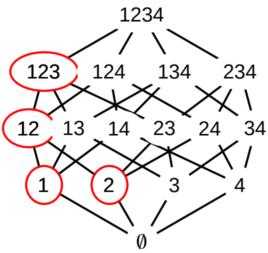

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Outline

Symmetries $\leftsquigarrow$ Classical limit

In the classical limit, $g \to \infty$, and $T_{a,s} \to T_{a,s}(u/g)$.
$\Rightarrow$ shifts by $\pm\frac{i}{2}$ in Hirota equation can be neglected.
$\Rightarrow T_{a,s}(u) = \chi_{a,s}(\Omega(u))$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T\hat{C} \qquad (\text{or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues})$$

[Bena Polchinksi Roiban]

Quantum case

$$T_{1,s} = -\hat{T}_{1,-s}$$

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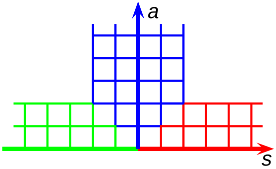

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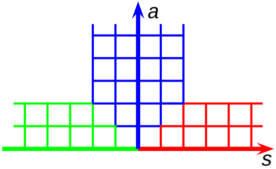

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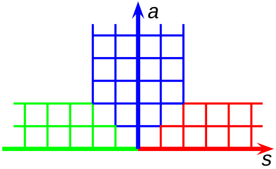

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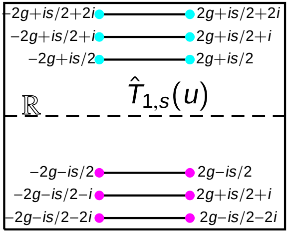

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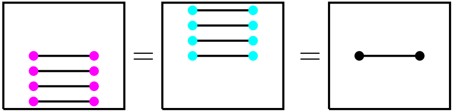

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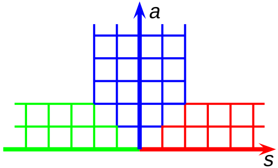

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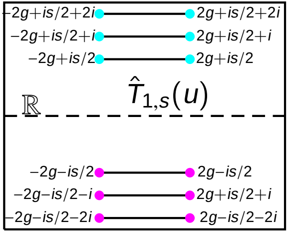

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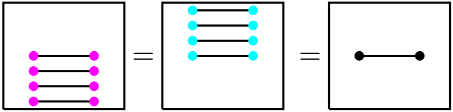

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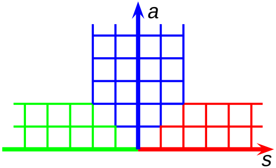

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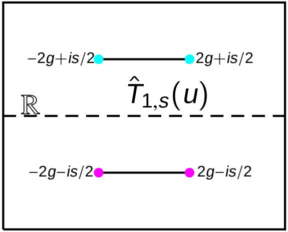

Riemann-Hilbert Problem

General statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

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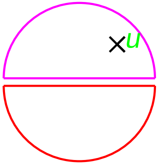

Example : if $Q = -\bar{Q}$ is analytic except on $[-2g, 2g]$ and $Q \xrightarrow[|u|\to\infty]{} -iu$,

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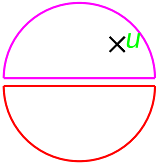

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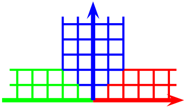
FiNLIE-equations

Appropriate choices of F and G allow to derive non-trivial integral equations from analyticity constraints.

These equations can be shown to be equivalent to the TBA-equations.

AdS/CFT FiNLIE

$$
\begin{aligned}
T_{a,+1} &= q_1^{[+a]}\bar{q}_2^{[-a]} + q_2^{[+a]}\bar{q}_1^{[-a]} + q_3^{[+a]}\bar{q}_4^{[-a]} + q_4^{[+a]}\bar{q}_3^{[-a]} \, ,\\
T_{a,0} &= q_{12}^{[+a]}\bar{q}_{12}^{[-a]} + q_{34}^{[+a]}\bar{q}_{34}^{[-a]} - q_{14}^{[+a]}\bar{q}_{14}^{[-a]}\\
&\quad - q_{23}^{[+a]}\bar{q}_{23}^{[-a]} - q_{13}^{[+a]}\bar{q}_{24}^{[-a]} - q_{24}^{[+a]}\bar{q}_{13}^{[-a]} ,\\
q_\emptyset q_{ij} &= q_i^+ q_j^- - q_j^+ q_i^- ,\\
q_{ijk} q_i &= q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^- .
\end{aligned}
$$



$$
Y_{1,1} = -\sqrt{\frac{R^{(+)}}{R^{(-)}}\frac{B^{(-)}}{R^{(+)}}\frac{\mathcal{T}_{1,2}}{T_{2,1}}\left(\frac{T_{1,0}}{Q^+Q^-}\right)^{1+\check{*}\mathcal{Z}}\left(\frac{Q^2}{T_{0,0}}\right)^{*\frac{1}{2}(\mathcal{Z}_1+\mathcal{K}_1)}\left(\frac{T_{1,1}}{\mathcal{T}_{1,1}}\right)^{*\!\!\!\mathcal{K}_1}} \, .
$$

$$
U^2 = \frac{\Lambda^2 T_{00}^-}{\hat{x}^{L-2} Y_{1,1} Y_{2,2} T_{1,0}}\left(\frac{Y_{1,1}Y_{2,2}-1}{\rho/\mathcal{F}^+}\right)^{\hat{*}2\mathcal{Z}}\left(\frac{T_{2,1}\mathcal{T}_{1,1}^-}{\hat{T}_{1,1}^-\mathcal{T}_{1,2}Y_{2,2}}\right)^{*2\Psi}
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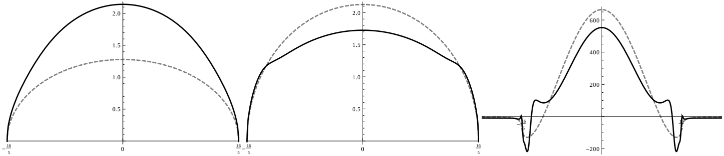
Numeric implementation of FiNLIE

Numerical densities obtained for Konishi state by our FiNLIE algorithm: These three densities (black curves) describe the finite size Konishi state, and are compared to their asymptotic expression dashed gray curve



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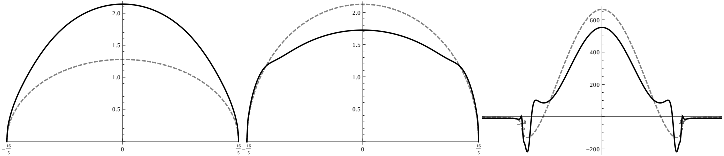

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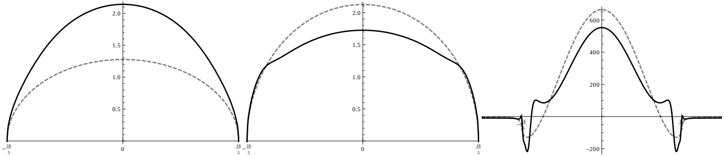

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  - analytic properties
  - new symmetries
  - Finite set of NLIEs
  - $\partial \log T_{0,0} \xrightarrow{u\to\infty} \frac{2E}{u}$
  - Exact Bethe equations arise as absence of poles of T-functions
- to be continued
  - currently restricted to symmetric sl_2 “sector” states
  - $\left\{ \begin{array}{l} \text{numeric efficiency} \\ \text{best FiNLIE formulation} \end{array} \right.$ are to be studied
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  - strong coupling construction of T (? $T = \langle \text{trace } \Omega \rangle$)
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Really

Thank you !

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