Hirota equation and Q-functions

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LPT-ENS (Paris)

[arXiv:1010.4022] V. Kazakov, SL & Z.Tsuboi
[arXiv:1007.1770] V. Kazakov & SL

$\rightsquigarrow$ work in progress N. Gromov, V.Kazakov, SL & D.Volin

LPT-ENS, Mai 25, 2011

Quantum integrability

Key features of Quantum integrable systems :

- a big number of conserved charges
- n-point interactions factorize into 2-points interaction
- Bethe Equations

$\rightsquigarrow$ exact resolution

Questions : Deviations from integrability (finite size effects)
$\rightsquigarrow$ Y-system

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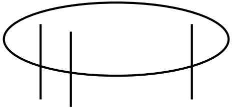
Plan of the talk

Heisenberg Spin Chain



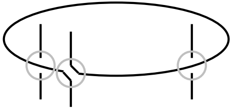
- Hilbert space $\mathcal{H} = \bigotimes_i h_i = \left(\mathbb{C}^2\right)^{\otimes N}$
- $H = \sum_i \vec{S_i} \cdot \vec{S_{i+1}}$

T-operator

$$T(u) = \mathrm{trace} \underbrace{\overbrace{(u\mathbb{I} + 2P)}^{\text{permutation}} \otimes (u\mathbb{I} + 2P) \otimes \cdots (u\mathbb{I} + 2P)}_{N \text{ times}}$$

- $H = \sum_i \vec{S_i} \cdot \vec{S_{i+1}} = \left.\frac{\mathrm{d} \log T}{\mathrm{d} u}\right|_{u \to 0}$
- $[T(u), T(v)] = 0$
- Solved by simultaneous diagonalization of all $T(u)$'s

Heisenberg Spin Chain



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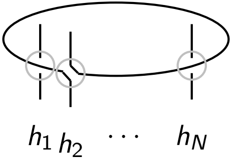

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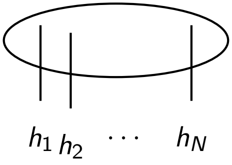
$GL(K|M)$ Spin Chain



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- $u_i = u - \theta_i$
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$$T\quad(u) = \mathrm{trace}\,\Big(\underbrace{(u_1\mathbb{I} + 2P)}_{R_1(u)} \otimes \underbrace{(u_2\mathbb{I} + 2P)}_{R_2(u)} \otimes \cdots \otimes \underbrace{(u_N\mathbb{I} + 2P)}_{R_N(u)} \quad g\Big)$$

- auxiliary space in the irrep $\{\lambda\} = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_a)$
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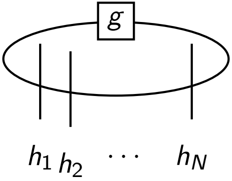
twisted $GL(K|M)$ Spin Chain



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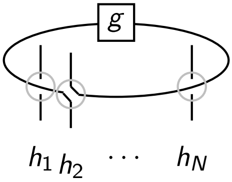

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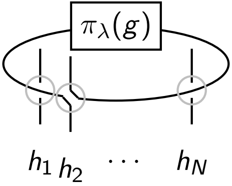

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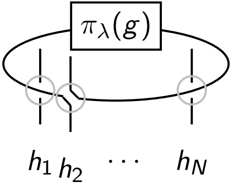

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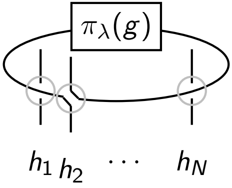

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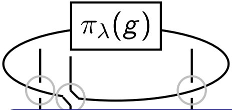

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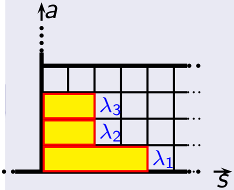
T-lattice



- T-operators for different Young diagrams $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_a)$ where $a \leq K$ $\lambda_i \geqslant \lambda_{i+1}$
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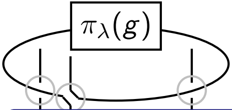

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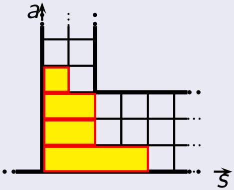

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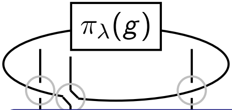

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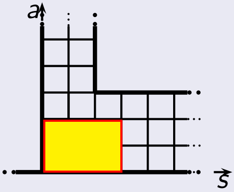

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Characters and T-operators

- The characters $\chi^{\{\lambda\}}(u) = \mathrm{trace}_\lambda \pi_\lambda(g)$ are the $N = 0$ (no spin) case of the T-operator $T^{\{\lambda\}}(u) = \mathrm{trace}_\lambda R_1(u) \otimes R_2(u) \otimes \cdots \otimes R_N(u) \pi_\lambda(g)$
- $GL(K)$ characters obey the Weyl determinant formulae :

$$\chi^{\{\lambda\}} = \frac{\left| x_i^{\lambda_j} \right|_{1 \leqslant i,j \leqslant K}}{\left| x_i^j \right|_{1 \leqslant i,j \leqslant K}} \qquad \chi^{\{\lambda\}} = \left| \chi^{(\lambda_j + i - j)} \right|_{1 \leqslant i,j \leqslant a}$$

where $x_1, \cdots, x_K$ are the eigenvalues of the twist g, $\{\lambda\} = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_a)$, $\chi^{(s)}$ is the character for $\lambda = (s)$

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A Plücker identity

Every matrix satisfies the following identity involving determinants where rows/columns are removed:

$$
\begin{vmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,N-1} & a_{1,N} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,N-1} & a_{2,N} \\
\vdots & & \ddots & & \vdots \\
a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N-1} & a_{N-1,N} \\
a_{N,1} & a_{N,2} & \cdots & a_{N,N-1} & a_{N,N}
\end{vmatrix}
\cdot
\begin{vmatrix}
\square & \square & \cdots & \square & \square \\
\square & a_{2,2} & \cdots & a_{2,N-1} & \square \\
\vdots & & \ddots & & \vdots \\
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\end{vmatrix}
$$

$$
=
\begin{vmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,N-1} & \square \\
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Bilinear relations for rectangular representations

Determinant formulae $\xrightarrow{\text{Plücker identities}}$ Bilinear identities

$$\chi^{\{\lambda\}} = \left|\chi^{(\lambda_j+i-j)}\right|_{1\leqslant i,j\leqslant a} \longrightarrow \begin{aligned}&\chi^{(a+1,s)}\chi^{(a-1,s)}\\ &= \chi^{(a,s)}\chi^{(a,s)} - \chi^{(a,s+1)}\chi^{(a,s-1)}\end{aligned}$$

$$\chi_K^{\{\lambda\}} = \frac{\left|x_i^{\lambda_j}\right|_{1\leqslant i,j\leqslant K}}{\left|x_i^{j}\right|_{1\leqslant i,j\leqslant K}} \longrightarrow \begin{cases} \chi_k^{(a,s+1)}\chi_{k-1}^{(a,s)} - \chi_k^{(a,s)}\chi_{k-1}^{(a,s+1)} \\ \quad = x_k\,\chi_k^{(a+1,s)}\chi_{k-1}^{(a-1,s+1)} \\ \\ \dots \end{cases}$$

Where $\chi^{(a,s)}$ is the character, in the irrep labeled by the rectangular young tableau $\lambda = (s^a)$, of the group element $\mathrm{Diag}(x_1, \cdots, x_k) \in GL(k)$

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Bilinear relations for rectangular representations of $GL(K)$

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Hirota equation

For T-operators, this generalizes to

$$T^{(a+1,s)}(u+1)\,T^{(a-1,s)}(u-1) =$$
$$T^{(a,s)}(u+1)\,T^{(a,s)}(u-1) - T^{(a,s+1)}(u-1)\,T^{(a,s-1)}(u+1).$$

and $\quad (Q(u) = T^{\emptyset}(u))$

$$T^{\lambda}(u) = \frac{1}{\prod_{k=1}^{a-1} Q(u-2k)} \det_{1\leq i,j\leq a}\left(T^{(\lambda_j+i-j)}(u+2-2i)\right).$$

[Bazhanov, Reshetikhin 90] [Cherednik 87] [Tsuboi 97] [Kazakov Vieira 07]

Bilinear relations for rectangular representations of $GL(K)$

Determinant formulae $\xrightarrow{\text{Plücker identities}}$ Bilinear identities

$$\chi^{\{\lambda\}} = \left|\chi^{(\lambda_j+i-j)}\right|_{1\leqslant i,j\leqslant a} \longrightarrow \begin{array}{l}\chi^{(a+1,s)}\chi^{(a-1,s)} \\ \quad = \chi^{(a,s)}\chi^{(a,s)} - \chi^{(a,s+1)}\chi^{(a,s-1)}\end{array}$$

Hirota equation

For T-operators, this generalizes to

$$T^{(a+1,s)}(u+1)\,T^{(a-1,s)}(u-1) = T^{(a,s)}(u+1)\,T^{(a,s)}(u-1) - T^{(a,s+1)}(u-1)\,T^{(a,s-1)}(u+1).$$

and $\qquad (Q(u) = T^{\emptyset}(u))$

$$T^{\lambda}(u) = \frac{1}{\prod_{k=1}^{a-1} Q(u-2k)} \det_{1\leq i,j\leq a}\left(T^{(\lambda_j+i-j)}(u+2-2i)\right).$$

[Bazhanov, Reshetikhin 90] [Cherednik 87] [Tsuboi 97] [Kazakov Vieira 07]

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$$\chi_K^{\{\lambda\}} = \frac{\left|x_i^{\lambda_j}\right|_{1\leqslant i,j\leqslant K}}{\left|x_i^j\right|_{1\leqslant i,j\leqslant K}} \quad \longrightarrow \quad \begin{cases} \chi_k^{(a,s+1)}\chi_{k-1}^{(a,s)} - \chi_k^{(a,s)}\chi_{k-1}^{(a,s+1)} \\ \quad = x_k\,\chi_k^{(a+1,s)}\chi_{k-1}^{(a-1,s+1)} \\ \\ \dots \end{cases}$$

Where $\chi_k^{(a,s)}$ is the character, in the irrep labeled by the rectangular young tableau $\lambda = (s^a)$, of the group element $\mathrm{Diag}(x_1, \cdots, x_k) \in GL(k)$

Bäcklund Transformations : linear system

Bäcklund Transformations [Krichever, Lipan, Wiegmann, Zabrodin 96],[Kazakov, Sorin, Zabrodin 07], [Zabrodin 07], [Tsuboi 09]

if $T^{(a,s)}(u)$ is a solution of Hirota equation and

$$T^{(a+1,s)}(u)F^{(a,s)}(u) - T^{(a,s)}(u)F^{(a+1,s)}(u) = \underbrace{x_j}_{\text{eigenvalue of } g} T^{(a+1,s-1)}(u+2)F^{(a,s+1)}(u-2),$$

$$T^{(a,s+1)}(u)F^{(a,s)}(u) - T^{(a,s)}(u)F^{(a,s+1)}(u) = x_j T^{(a+1,s)}(u+2)F^{(a-1,s+1)}(u-2).$$

Then $F^{(a,s)}(u)$ is a solution of Hirota equation.

Moreover, if $T^{(a,s)}(u) = 0, \forall a > K$, one can choose $F^{(a,s)}(u) = 0, \forall a > K-1$.

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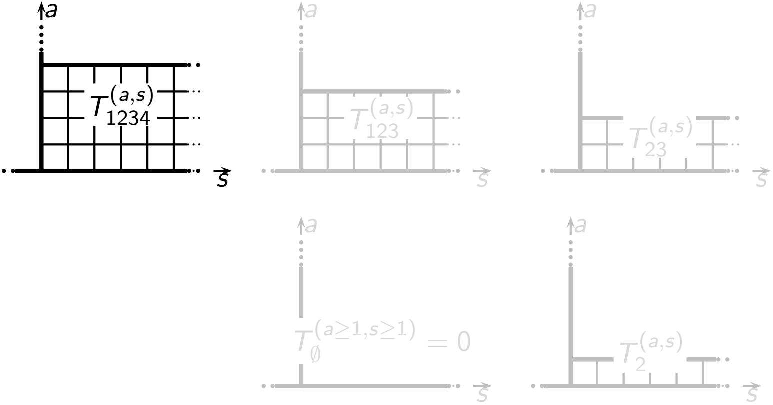
$GL(4)$ Bäcklund flow and lattices' boundaries



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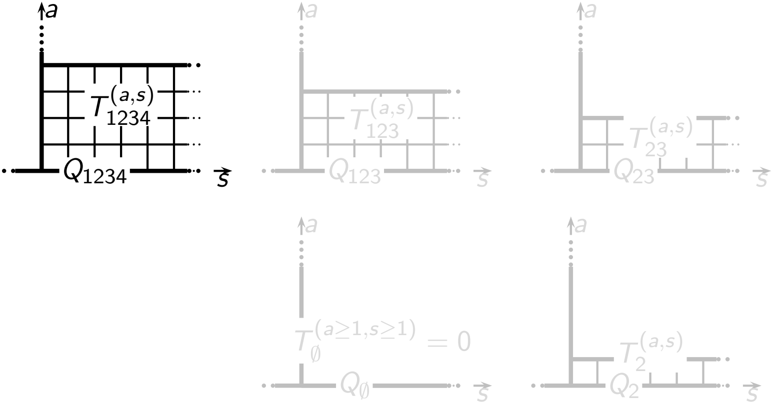

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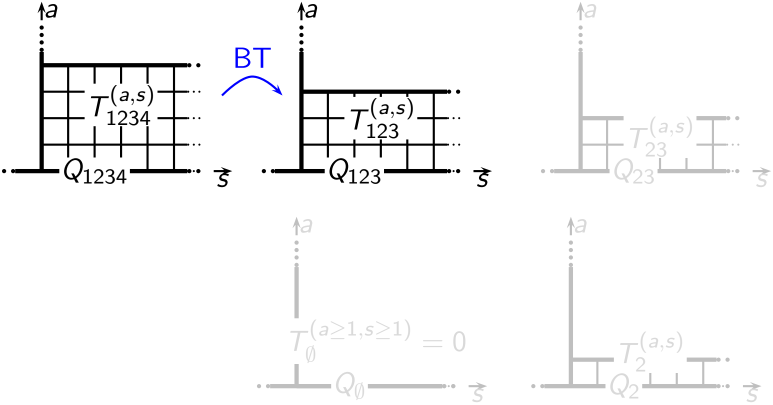

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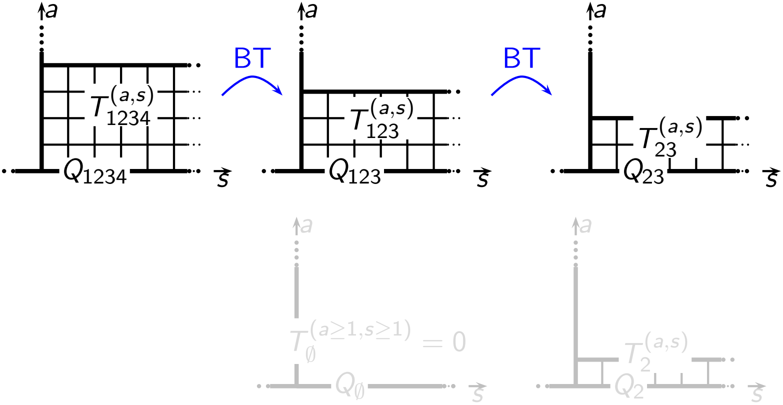

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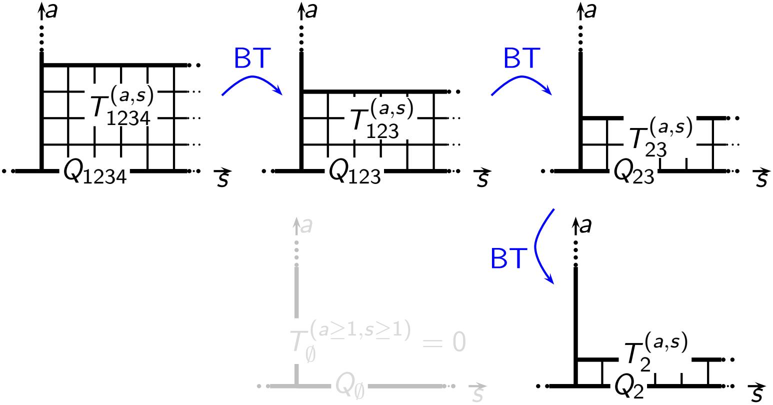

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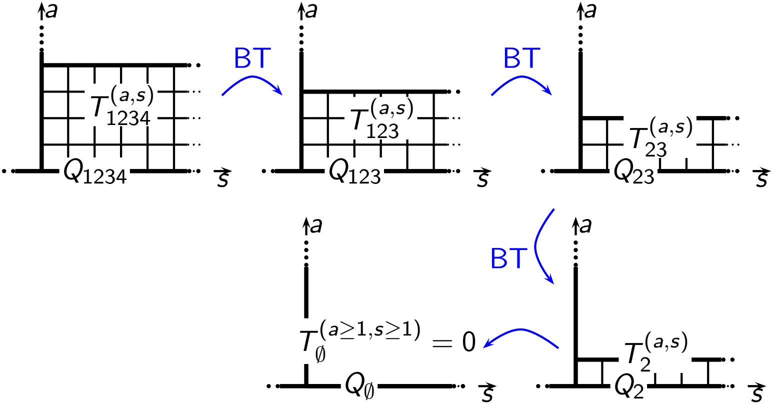

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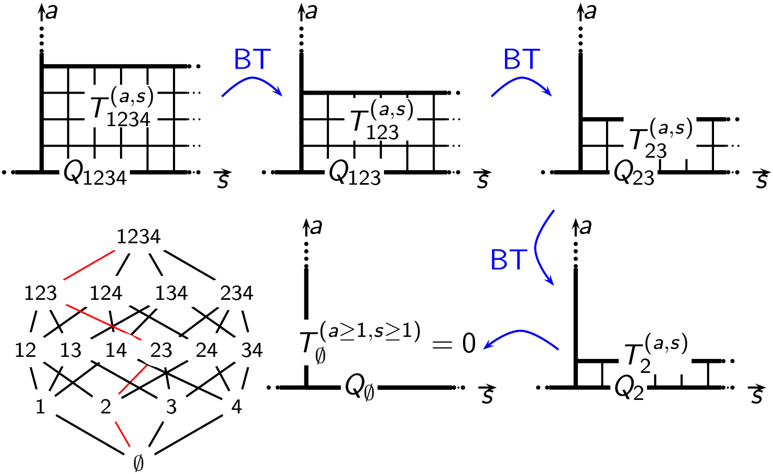

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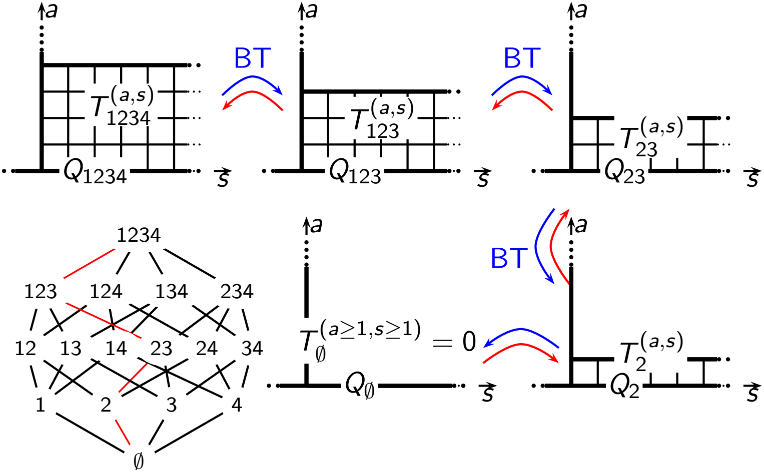

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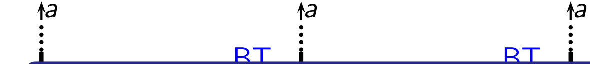

Generalization of the first Weyl Formula

$$T_I^{(a,s)}(u) = \frac{\det\left(x_j^{|I|-1-k+s\Theta} Q_j(u-2k+2s\Theta)\right)_{\substack{j\in I\\ 0\leqslant k\leqslant |I|-1}}}{Q_\emptyset(0)^{|I|-1}\det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\leqslant k\leqslant |I|-1}}}$$

$$\Theta = \begin{cases} 1 \text{ if } k < a \\ 0 \text{ if } k \geq a \end{cases}$$

1

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These Q-operators obey some (bilinear) QQ-relations, which imply Bethe equations.

▸ Skip QQ-relations ⇝ Coderivatives   ▸ Skip QQ-relations and Coderivatives ⇝ PCF

[Derkachov, 99], [Bytsko Teschner 06], [Bazhanov Frassek Lukowski Meneghelli Staudacher 10],[Kazakov Leurent Tsuboi 10]

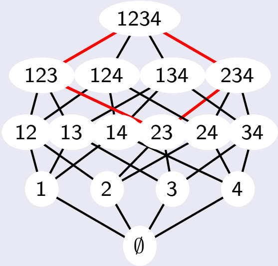
QQ-relations and Bethe Equations

At the level of operators, the QQ-relations

$$(x_i - x_j)\, Q_I(u-2) Q_{I,i,j}(u) = x_i Q_{I,j}(u-2) Q_{I,i}(u) - x_j Q_{I,j}(u) Q_{I,i}(u-2)$$

example : $I = \{23\}, i = 1, j = 4$

$$(x_1 - x_4)\, Q_{23}(u-2) Q_{1234}(u) = x_1 Q_{234}(u-2) Q_{123}(u) - x_4 Q_{234}(u) Q_{123}(u-2)$$



The relation involves Q-operators lying on the same facet of the Hasse diagram

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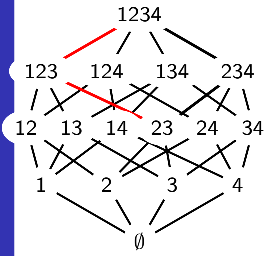
imply

$$Q_{I,i}(u) \mid x_i Q_I(u-2) Q_{I,i,j}(u) Q_{I,i}(u+2) + x_j Q_I(u) Q_{I,i,j}(u+2) Q_{I,i}(u-2).$$



for instance

$$Q_{123}(u) \mid x_1 Q_{23}(u-2) Q_{1234}(u) Q_{123}(u+2) + x_4 Q_{23}(u) Q_{1234}(u+2) Q_{123}(u-2).$$

The relation involves 3 consecutive Q-operators lying on the same nesting path.

QQ-relations and Bethe Equations

At the level of operators, the QQ-relations

$$(x_i - x_j)\, Q_I(u-2) Q_{I,i,j}(u) = x_i Q_{I,j}(u-2) Q_{I,i}(u) - x_j Q_{I,j}(u) Q_{I,i}(u-2)$$

imply

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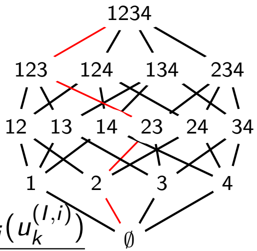
On a given eigen-state,

$$\mathrm{Q}_I(u) = c_I \prod_{k=1}^{K_I} (u - u_k^{(I)}),$$



$$-1 = \frac{x_i}{x_j} \frac{\mathrm{Q}_I(u_k^{(I,i)} - 2) \mathrm{Q}_{I,i}(u_k^{(I,i)} + 2) \mathrm{Q}_{I,i,j}(u_k^{(I,i)})}{\mathrm{Q}_I(u_k^{(I,i)}) \mathrm{Q}_{I,i}(u_k^{(I,i)} - 2) \mathrm{Q}_{I,i,j}(u_k^{(I,i)} + 2)}$$

Expression in terms of co-derivative

- $\hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi \cdot e} g) \Big|_{\phi=0}$ $\qquad \phi \in GL(K)$
- If $f(g)$ is an operator on $\left(\mathbb{C}^K\right)^{\otimes N}$, then $\hat{D} \otimes f$ is an operator on $\left(\mathbb{C}^K\right)^{\otimes N+1}$
- $\hat{D} \otimes \pi_\lambda(g) = \left[\sum_{\alpha,\beta} e_{\beta\alpha} \otimes \pi_\lambda(e_{\alpha\beta})\right] \cdot \mathbb{I} \otimes \pi_\lambda(e_{\alpha\beta})$

  hence

$$\bigotimes_{i=1}^{N} (u_i + 2\mathcal{P})\ \pi_\lambda(g) = \bigotimes_{i=1}^{N} (u_i + 2\hat{D})\ \pi_\lambda(g)$$

and

$$T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N} (u_i + 2\hat{D})\ \chi_\lambda(g)$$

[Kazakov Vieira 07]

$\hat{D}$ is a "spin-chain creation operator"

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[Kazakov Vieira, 0711.2470]

- Q-operators
- Bäcklund flow
- Bethe equations

[Kazakov Leurent Tsuboi, 1010.2720]

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Outline

Model definition

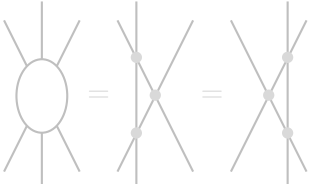
The principal chiral field (PCF) is a 1+1 D field theory on the cylinder $0 \leqslant x < L = \infty$, $t \in \mathbb{R}$

$$\mathcal{S}_{\text{PCF}} = -\frac{1}{2e_0^2}\int dt\, dx \ \operatorname{tr}(h^{-1}\partial_\alpha h)^2 \,.$$

Where $h \in SU(N)$

- $SU(N)_L \times SU(N)_R$ symmetry
- infinite number of conserved charges $\rightsquigarrow$ integrability (factorizability of n-points interactions)

  rational $\mathcal{S}$ matrix $\chi_{CDD}(u) \cdot S_0(u)\frac{\hat{R}(u)}{u-i} \otimes S_0(u)\frac{\hat{R}(u)}{u-i}$



[Zamolodchikov^2 77]
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[Goldsmith Witten 84]
[Wiegmann 84]

Skip Bethe equations and derivation of Y-system

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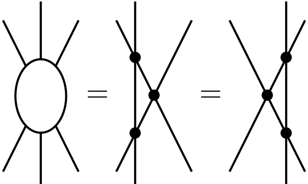

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Skip Bethe equations and derivation of Y-system

Bethe Equations

Solution for $L \to \infty$

- solutions described by particles having rapidities θ_j:

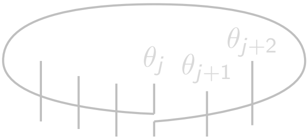
$$p_j = m_j \sinh\left(\frac{2\pi}{N}\theta_j\right) \quad , \quad E = \sum_{j=1}^{N} E_j = \sum_{j=1}^{N} m_j \cosh\left(\frac{2\pi}{N}\theta_j\right)$$

- bound states with mass $m_a = m\frac{\sin\frac{a\pi}{N}}{\sin\frac{\pi}{N}}$
- Their spins carry magnons ($1 \leqslant k \leqslant N-1$)

$$1 = \frac{\mathbb{Q}^R_{k-1}(u_j^{(k)}-i/2)}{\mathbb{Q}^R_{k-1}(u_j^{(k)}+i/2)}\frac{\mathbb{Q}^R_{k}(u_j^{(k)}+i)\mathbb{Q}^R_{k+1}(u_j^{(k)}-i/2)}{\mathbb{Q}^R_{k}(u_j^{(k)}-i)\mathbb{Q}^R_{k+1}(u_j^{(k)}+i/2)}$$

- periodicity condition

$$e^{-imL\sinh(\pi\theta_j)} = -S(\theta_j)\frac{\mathbb{Q}^R_{N-1}(\theta_j+i/2)}{\mathbb{Q}^R_{N-1}(\theta_j-i/2)}\frac{\mathbb{Q}^L_{N-1}(\theta_j+i/2)}{\mathbb{Q}^L_{N-1}(\theta_j-i/2)}$$



$$S(u) = \prod_j S_0^2(u-\theta_j)\chi_{CDD}(u-\theta_j)$$

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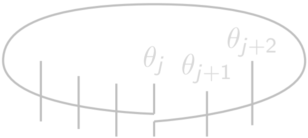
$$1 = \frac{\mathbb{Q}^R_{k-1}(u_j^{(k)}-i/2)}{\mathbb{Q}^R_{k-1}(u_j^{(k)}+i/2)}\frac{\mathbb{Q}^R_{k}(u_j^{(k)}+i)\mathbb{Q}^R_{k+1}(u_j^{(k)}-i/2)}{\mathbb{Q}^R_{k}(u_j^{(k)}-i)\mathbb{Q}^R_{k+1}(u_j^{(k)}+i/2)}$$

- periodicity condition

$$e^{-imL\sinh(\pi\theta_j)} = -S(\theta_j)\frac{\mathbb{Q}^R_{N-1}(\theta_j+i/2)}{\mathbb{Q}^R_{N-1}(\theta_j-i/2)}\frac{\mathbb{Q}^L_{N-1}(\theta_j+i/2)}{\mathbb{Q}^L_{N-1}(\theta_j-i/2)}$$



$$S(u) = \prod_j S_0^2(u-\theta_j)\chi_{CDD}(u-\theta_j)$$

Bethe Equations

Solution for $L \to \infty$

- solutions described by particles having rapidities θ_j:

$$p_j = m_j \sinh\left(\frac{2\pi}{N}\theta_j\right) \quad , \quad E = \sum_{j=1}^N E_j = \sum_{j=1}^N m_j \cosh\left(\frac{2\pi}{N}\theta_j\right)$$

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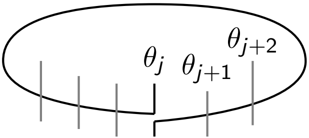

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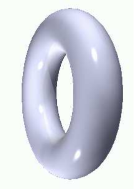
Ground state energy : double Wick rotation



Spatial periodicity L
time-periodicity $R \to \infty$:
Path integral dominated by Ground state $Z \sim e^{-RE_0(L)}$

Spatial periodicity $R \to \infty$
time-periodicity L (finite temperature)

$$\text{free Energy} : f(L) = E_0(L)$$

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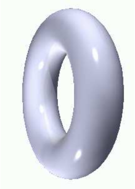

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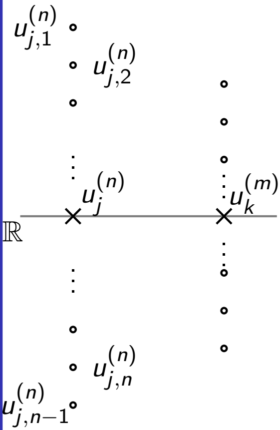
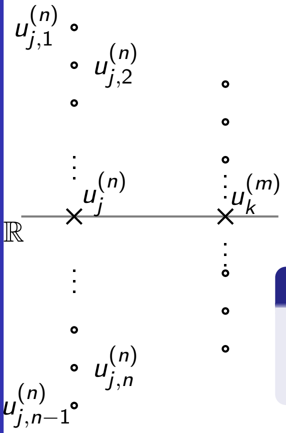
String hypothesis



- The Bethe equations are such that the large number of magnons roots are organized as "strings".
  $u_{j,a}^{(n)} = u_j^{(n)} + i\frac{n+1}{2} - ia, \quad (1 \leqslant a \leqslant n)$
- Such strings scatter with a shifted product of the original $\mathcal{S}$ matrix
- the right configuration (described by one density for each type of string) is identified by minimization of the free entropy.

String hypothesis



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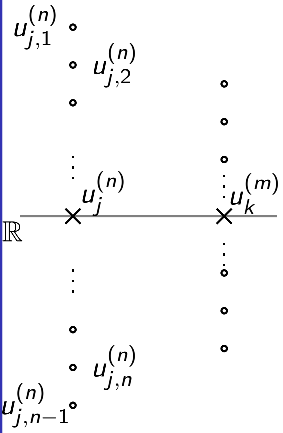
In the Bethe equation

$\prod_k \mathcal{S}(\theta_k - \theta_j)$ becomes
$\prod_m \prod_k \mathcal{S}^{m,n}(u_j^{(n)} - u_k^{(m)})$

String hypothesis



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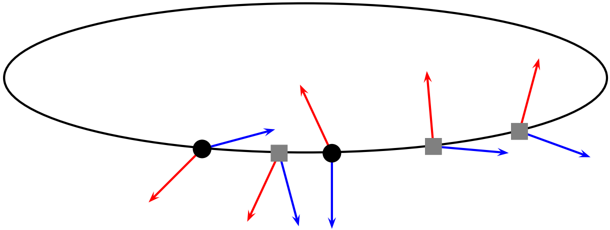
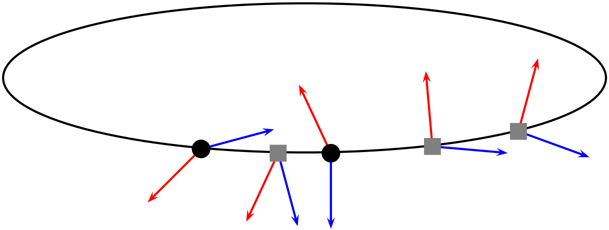
Y-system



- Different types of massive particles $\rightsquigarrow$ a density for each one.
- Massive particles have $SU(N)_R$ and $SU(N)_L$ spins.

$\rightsquigarrow$ A density for each type of possible spin wave

Y-system



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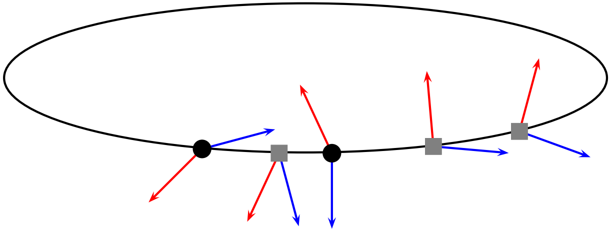

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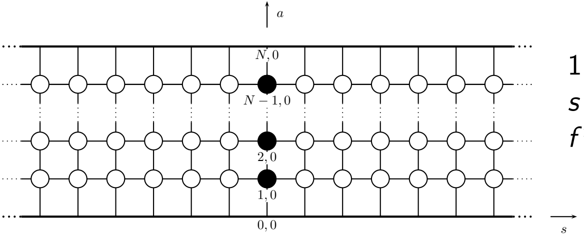

$1 \leqslant a \leqslant N-1;$
$s \in \mathbb{Z},\ u \in \mathbb{C},$
$f^{\pm} = f(u \pm i/2)$

$$Y^+_{a,s}\, Y^-_{a,s} = \frac{1+Y_{a,s+1}}{1+1/Y_{a+1,s}}\,\frac{1+Y_{a,s-1}}{1+1/Y_{a-1,s}},$$

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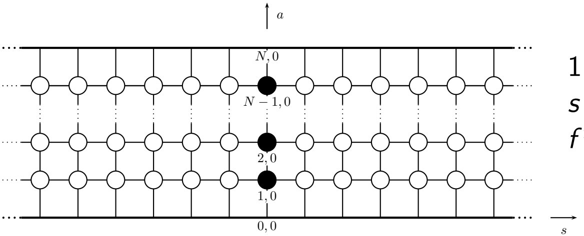

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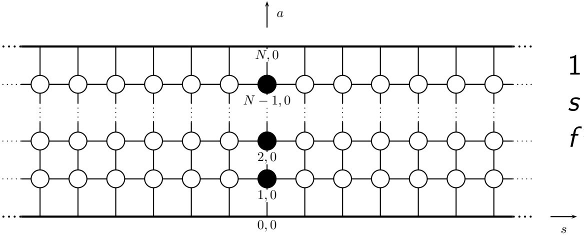
Origin

This comes from integral equations of the form

$$\log Y_{a,s} = \sum_{a',s'} K_{a,s}^{(a',s')} * \log(1+Y_{a',s'})$$

Y-system



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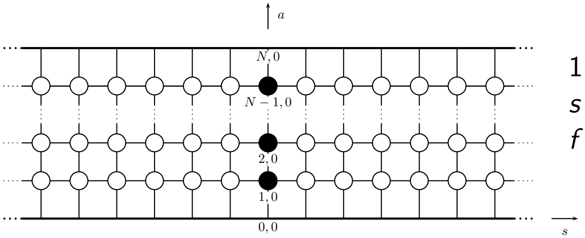

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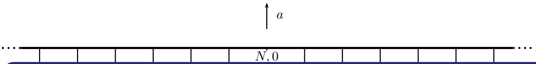

Claim for excited states

for excited states,

- the same Y-system equations hold
- the Y-functions have zeroes/poles which characterize the excitations
- there is a prescription for the integration contour in $E = \sum_a \int_{-\infty}^{\infty} p_a(u) \log\left(1 + Y_{a,0}(u)\right) du$

- $Y_{a,s} \underset{u \gg 1}{\sim} e^{-L\tilde{p}_a(u)\delta_{s,0}} \times \mathrm{const}_{a,s}$, $\quad \tilde{p}_a = \cosh\left(\frac{2u\pi}{N}\right) \frac{\sin\left(\frac{a\pi}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}$
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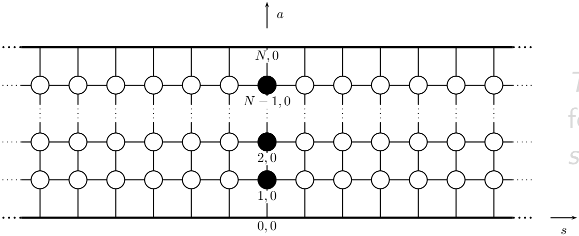
Hirota equation

If $Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$, then

$$Y^{+}_{a,s}\,Y^{-}_{a,s} = \frac{1+Y_{a,s+1}}{1+1/Y_{a+1,s})}\,\frac{1+Y_{a,s-1}}{1+1/Y_{a-1,s}}$$

$$\Leftrightarrow T^{+}_{a,s}\,T^{-}_{a,s} = T_{a+1,s}\,T_{a-1,s} + T_{a,s+1}\,T_{a,s-1}$$



$T_{a,s}(u)$ is nonzero for $0 \leqslant a \leqslant N$, $s \in \mathbb{Z}$

$Y_{a,s}$ is invariant w.r.t. the gauge transformation

$$T_{a,s} \quad \rightarrow \quad \chi_1^{[a+s]}\chi_2^{[a-s]}\chi_3^{[-a+s]}\chi_4^{[-a-s]}\,T_{a,s} \qquad f^{[\pm k]} \equiv f(u \pm ki/2)$$

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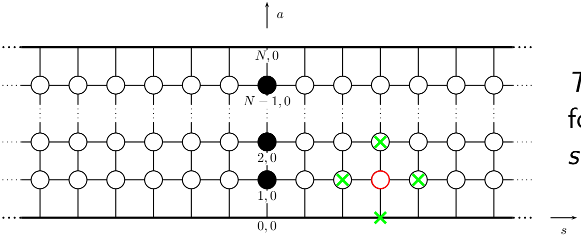

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Y-system = gauge-independent version of Hirota equation

If $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$, then

$$Y_{a,s}^+ Y_{a,s}^- = \frac{1+Y_{a,s+1}}{1+1/Y_{a+1,s})}\frac{1+Y_{a,s-1}}{1+1/Y_{a-1,s}}$$

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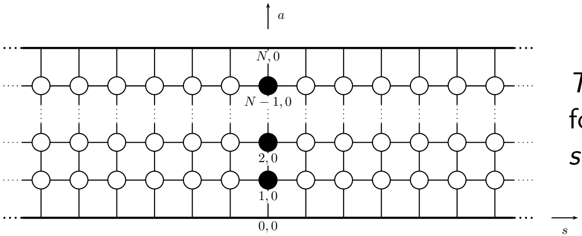

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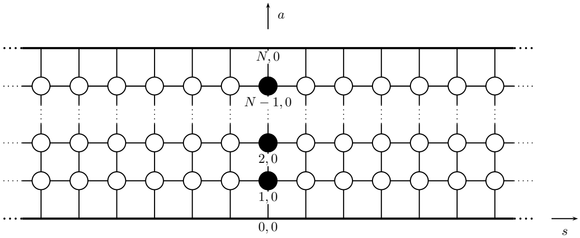
Asymptotic limit

$$\frac{T_{a,1}\,T_{a,-1}}{T_{a+1,0}\,T_{a-1,0}} = Y_{a,0} \underset{u\gg 1}{\sim} e^{-L\tilde{p}_a(u)} \times cste \ll 1 \Rightarrow \text{2 gauges where}$$

$$T_{a,-1} \ll 1\big|_{1\leqslant a\leqslant N-1} \text{ and } \qquad T_{a,s} \sim 1\big|_{s\geqslant 0}$$

$$\text{(resp)} \quad T_{a,1} \ll 1\big|_{1\leqslant a\leqslant N-1} \text{ and } \qquad T_{a,s} \sim 1\big|_{s\leqslant 0}$$



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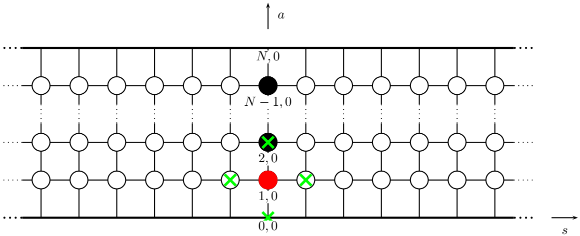

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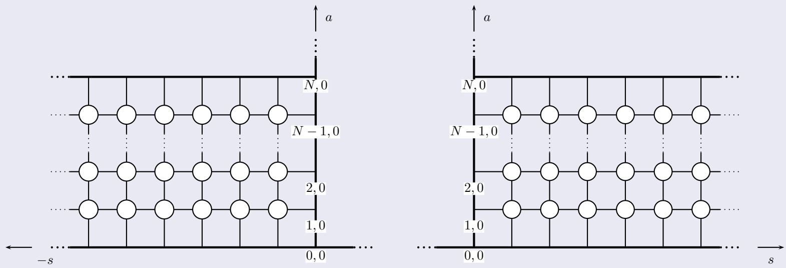
splitting



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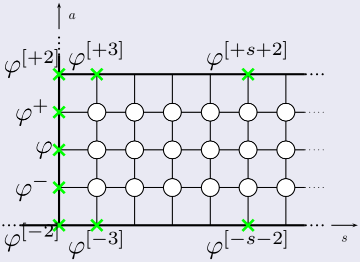
splitting

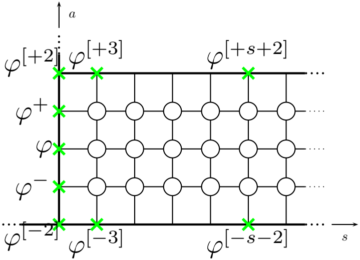
$$\varphi(u) = \prod_j (u-\theta_j)$$

$$\varphi^{[+k]} = \varphi(u + \tfrac{i}{2}k)$$



Existence of "Analyticity Strips"

$$1 + Y_{a,0} = \frac{T^+_{a,0}\, T^-_{a,0}}{T_{a+1,0}\, T_{a-1,0}} \xrightarrow[u \to +\infty]{} \frac{\varphi^+ \varphi^-}{\varphi^+ \varphi^-} = 1$$



$$Y_{a,0} \underset{u \gg 1}{\sim} e^{-L \cosh(2 \frac{\pi}{N} u)}$$

- $\frac{T^+_{a,0}\, T^-_{a,0}}{T_{a+1,0}\, T_{a-1,0}} \xrightarrow[L \to \infty]{} \frac{\varphi^+ \varphi^-}{\varphi^+ \varphi^-} = 1$ only holds if $|\mathrm{Im}(u)| < \frac{N}{4}$

$\Rightarrow$ there exists strips (in $\mathbb{C}$) where $T \xrightarrow[L \to \infty]{}$ polynomial.

- At the boundary of this strip, the limit $L \to \infty$ doesn't commute with analytic continuation (ie the limit $L \to \infty$ isn't analytic)

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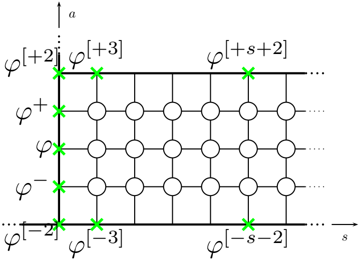

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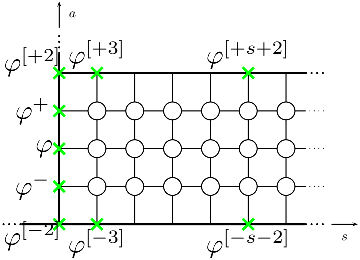

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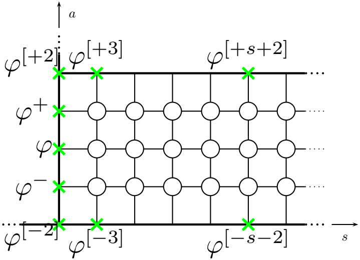

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$\Rightarrow$ there exists strips (in $\mathbb{C}$) where $T \xrightarrow[L\to\infty]{}$ polynomial.

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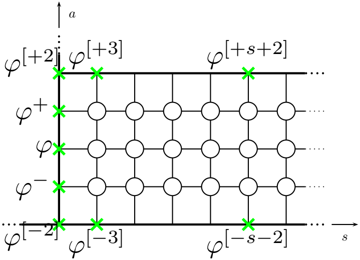

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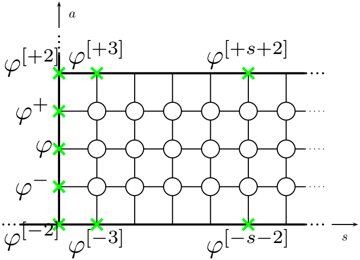

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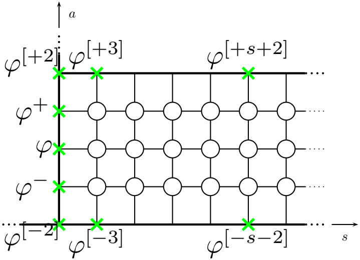

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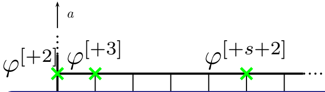

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Claim : existence of analyticity strips

There exists "analyticity strips", in the complex plane, whose size may depend on (a, s), where

- $T_{a,s} \xrightarrow[Re(u) \to \pm\infty]{} \text{polynomial}$
- $T_{a,s} \xrightarrow[L \to \infty]{} \text{polynomial}$
- $Y_{a,s} \xrightarrow[Re(u) \to \pm\infty]{} Y_{a,s}^{(\infty)}$
- $Y_{a,s} \xrightarrow[L \to \infty]{} Y_{a,s}^{(\infty)}$

Where the polynomial and $Y^{(\infty)}$ are extracted from the infinite size Bethe-Equations.

Determination of analyticity strips

Asumption : There exists a gauge where, forall $s \geqslant 0$, $T_{a,s}$ has an analyticity strip which contains the real axis.

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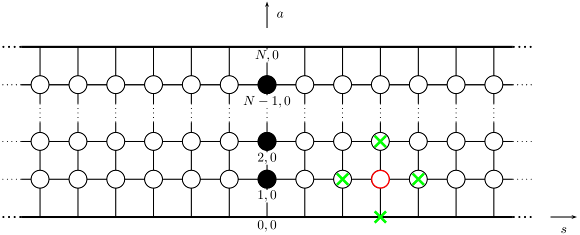

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Conclusion regarding analyticity strips

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Wronskian solution and analyticity strips

The general (up to a gauge) solution of Hirota on this lattice is

$$T_{a,s} = \left| \begin{array}{l} \left(\overline{q_j}^{[s+a+1+\frac{N}{2}-2k]}\right)_{1\leqslant j\leqslant N, 1\leqslant k\leqslant a} \\ \left(q_j^{[-s+a+1+\frac{N}{2}-2k]}\right)_{1\leqslant j\leqslant N, a<k\leqslant N} \end{array} \right|$$

- for spin-chains, q_j and $\overline{q_j}$ would be equal, whereas for the full lattice, they are a priori independent.
- from the analyticity strip, it is very natural to expect that

$$q_i(u) \quad \text{is analytic when} \quad \mathrm{Im(u)} < 1/2 \qquad (1)$$
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Resolvents and densities

$$\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\overline{q_j}(v)-q_j(v)}{u-v}\mathrm{d}v = \begin{cases} \overline{q_j}(u) - P_j(u) & \text{if} \quad \mathrm{Im(u)} > 0 \\ q_j(u) - P_j(u) & \text{if} \quad \mathrm{Im(u)} < 0 \end{cases}$$

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FiNLiE

$N-1$ equations are found for these $N-1$ densities by inverting $Y_{a,0}^{+}\, Y_{a,0}^{-} = \frac{1+Y_{a,1}}{1+1/Y_{a+1,0}} \frac{1+Y_{a,-1}}{1+1/Y_{a-1,0}}$,

Checks of FiNLiEs

Energy of the first excited state of $SU(3)\times SU(3)$ PCF : numerics compared to analytic field-theory prediction

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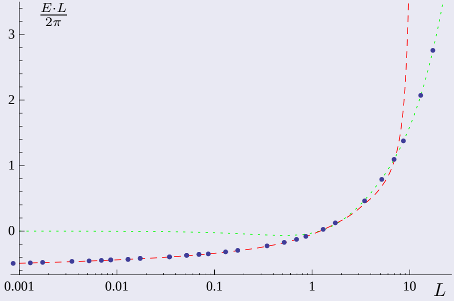

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Outline

Improvements for PCF

- The numerics still deserve to be improved
  $\rightsquigarrow$ better precision, study of $N \geq 4$, states outside the $U(1)$ sector
- The UV limit isn't yet completely understood in terms of these densities
- The $N \to \infty$ limit also requires some investigations...

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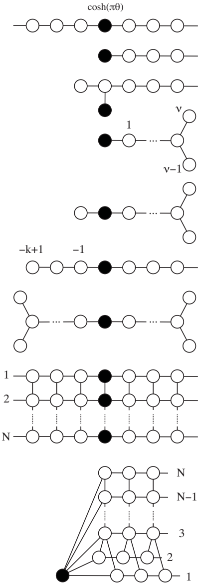
Other models having a “known” Y-system



O(4) σ-model

SU(2) Chiral Gross-Neveu

O(3) σ-model

Sine-Gordon with $\beta^2=8\pi(1-1/\nu)$

Super-sine-Gordon

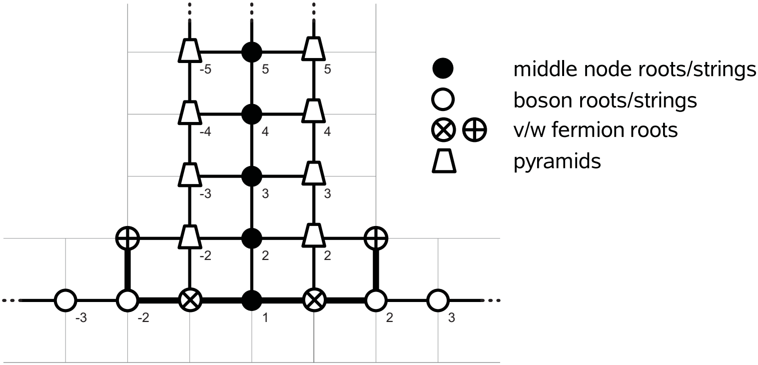
Current-current perturbation of $SU_k(2)$ WZW

Sausage models

SU(N+1) Principal Chiral Field

SO(2N) σ-model

AdS/CFT Y-system



AdS/CFT Y-system

- non relativistic dispersion relation : $\epsilon_a(u) = a + \frac{2ig}{x^{[+a]}} - \frac{2ig}{x^{[+a]}}$ where $\frac{u}{g} = x + \frac{1}{x}$
- the mapping $u \mapsto x$ has zhukowski cuts
- the hirota equation has a wronskian solution, which explains very well the analyticity strips
- infinite number of "middle nodes"
- different reality conditions

Thanks !

Thank you