Hirota equation and its solution through Q-operators / Q-functions

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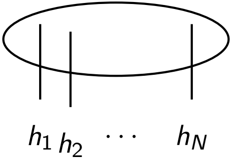
[arXiv:1010.4022] V. Kazakov, SL & Z.Tsuboi

[arXiv:1007.1770] V. Kazakov & SL
[arXiv:1010.2720] N. Gromov, V.Kazakov, SL & Z.Tsuboi

Berlin, February 14, 2011

Outline

Heisenberg Spin Chain



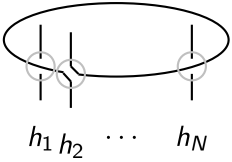
- Hilbert space $\mathcal{H} = \bigotimes_i h_i = \left(\mathbb{C}^2\right)^{\otimes N}$
- $H = \sum_i \vec{S_i} \cdot \vec{S_{i+1}}$

T-operator

$$T(u) = \mathrm{trace} \underbrace{(u\mathbb{I} + 2P) \otimes \overset{\text{permutation}}{(u\mathbb{I} + 2P)} \otimes \cdots (u\mathbb{I} + 2P)}_{N \text{ times}}$$

- $H = \sum_i \vec{S_i} \cdot \vec{S_{i+1}} = \left.\frac{\mathrm{d} \log T}{\mathrm{d} u}\right|_{u \to 0}$
- $[T(u), T(v)] = 0$
- Solved by simultaneous diagonalization of all $T(u)$'s

Heisenberg Spin Chain



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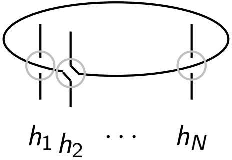

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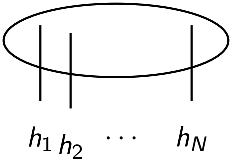
$GL(K|M)$ Spin Chain



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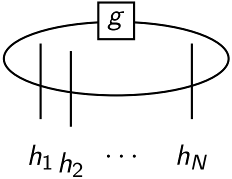
twisted $GL(K|M)$ Spin Chain



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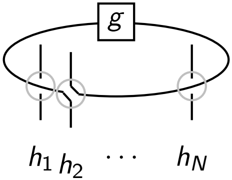

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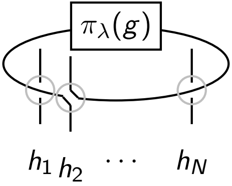

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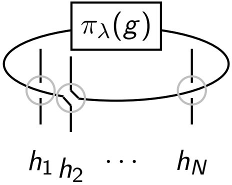

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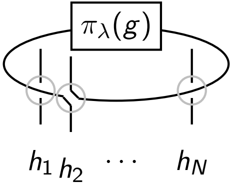

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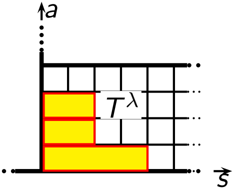
Hirota equations



- T-operators for different Young diagrams $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_a)$ where $a \leq K$ are expressed from T^s corresponding to $\lambda = (s)$ [Bazhanov, Reshetikhin 90] [Cherednik 87] [Tsuboi 97] [Kazakov Vieira 07]

$$T^\lambda(u) = \frac{1}{\prod_{k=1}^{a-1} Q(u-2k)} \det_{1\leq i,j\leq a} \left(T^{\lambda_j+i-j}(u+2-2i) \right).$$

Hirota equation

for rectangular representations, (ie. rectangular Young diagram), the Hirota equation holds:

$$T^{(a,s)}(u+1)T^{(a,s)}(u-1) = T^{(a+1,s)}(u+1)T^{(a-1,s)}(u-1) + T^{(a,s+1)}(u-1)T^{(a,s-1)}(u+1).$$

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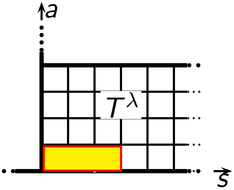

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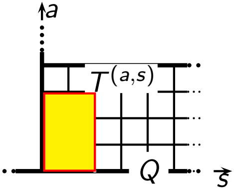

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Expression in terms of co-derivative

- $\hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi \cdot e} g)\Big|_{\phi=0}$ $\qquad \phi \in GL(K)$
- If $f(g)$ is an operator on $(\mathbb{C}^K)^{\otimes N}$, then $\hat{D} \otimes f$ is an operator on $(\mathbb{C}^K)^{\otimes N+1}$
- $\hat{D} \otimes \pi_\lambda(g) = \left[\sum_{\alpha,\beta} e_{\beta\alpha} \otimes \pi_\lambda(e_{\alpha\beta})\right] \cdot \mathbb{I} \otimes \pi_\lambda(e_{\alpha\beta})$

  hence

$$\bigotimes_{i=1}^{N} (u_i + 2\mathcal{P})\ \pi_\lambda(g) = \bigotimes_{i=1}^{N} (u_i + 2\hat{D})\ \pi_\lambda(g)$$

and

$$T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N} (u_i + 2\hat{D})\ \chi_\lambda(g)$$

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Skip details ⇝ Bäcklund flow

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Construction of other commuting charges

if for all $g, h \in GL(K|M)$, $f(g) = f(h^{-1} g h)$ and $\tilde{f}(g) = \tilde{f}(h^{-1} g h)$,

then $\left[\!\left[\bigotimes_{i=1}^{N} (u - \theta_i + 2\hat{D}) f(g), \bigotimes_{i=1}^{N} (v - \theta_i + 2\hat{D}) \tilde{f}(g) \right]\!\right] = 0$

Remark : linear combinations of T-operators.

$i=1$ $\qquad i=1$

and

$$T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N} (u_i + 2\hat{D})\ \chi_\lambda(g)$$

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Diagrammatics for coderivative

$\hat{D} \otimes g = \mathcal{P}\left(1 \otimes g\right)$ and Leibniz rule :

$$\hat{D} \otimes (f \cdot \tilde{f}) = [\mathbb{I} \otimes f] \cdot \left[\hat{D} \otimes \tilde{f}\right] + \left[\hat{D} \otimes f\right] \cdot \left[\mathbb{I} \otimes \tilde{f}\right]$$

$\rightsquigarrow$ compute any $\hat{D} \otimes f(g)$

- $w(z) = \det \frac{1}{1-zg} = e^{-\mathrm{trace}\ \log(1-gz)}$

$\hat{D}\ w(z) = \frac{gz}{1-g\ z} w(z) =$  $w(z)$ where  $= \frac{gz}{1-g\ z}$

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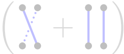
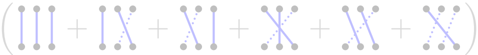
=  $w(z)$ where  $= \frac{1}{1-g\ z}$

$\hat{D} \otimes \hat{D} \otimes \hat{D}\ w(z) =$  $w(z)$

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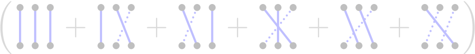
$\hat{D} \otimes \hat{D} \otimes \hat{D}\ w(z) =$  $w(z)$

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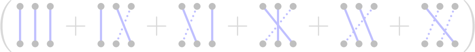
$\hat{D} \otimes \hat{D} \otimes \hat{D}\ w(z) =$  $w(z)$

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$$\hat{D} \otimes (f \cdot \tilde{f}) = [\mathbb{I} \otimes f] \cdot \left[\hat{D} \otimes \tilde{f}\right] + \left[\hat{D} \otimes f\right] \cdot \left[\mathbb{I} \otimes \tilde{f}\right]$$

$\rightsquigarrow$ compute any $\hat{D} \otimes f(g)$

- $w(z) = \det \frac{1}{1-zg} = e^{-\mathrm{trace}\ \log(1-gz)}$

$$\hat{D}\ w(z) = \frac{gz}{1-g\ z} w(z) = \quad w(z)$$

where $\quad = \frac{gz}{1-g\ z}$

$$\hat{D} \otimes \hat{D}\ w(z) = \hat{D} \otimes \left(\frac{gz}{1-g\ z} w(z)\right) =$$

$$= \left(\hat{D} \otimes \frac{gz}{1-g\ z}\right) w(z) + \left[\mathbb{I} \otimes \frac{gz}{1-g\ z}\right] \cdot \left[\hat{D} w(z) \otimes \mathbb{I}\right]$$

$$= \left(\mathcal{P}_{1,2}\left(\frac{1}{1-gz} \otimes \frac{gz}{1-gz}\right) + \frac{gz}{1-g\ z} \otimes \frac{gz}{1-g\ z}\right) w(z)$$

$$= \left(\quad + \quad \right) w(z)$$

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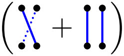
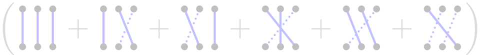
$=$  $w(z)$ where  $= \frac{1}{1-g\ z}$

$\hat{D} \otimes \hat{D} \otimes \hat{D}\ w(z) =$  $w(z)$

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Diagrammatics and T-operators

- Generating series $\sum_{s\geq 0} z^s T^s(u) = \sum_{s\geq 0} z^s \bigotimes_{i=1}^{N} (u_i + 2\hat{D})\ \chi_s(g) = \bigotimes_{i=1}^{N} (u_i + 2\hat{D})\ w(z)$
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Outline

Bäcklund Transformations : linear system

Bäcklund Transformations [Krichever, Lipan, Wiegmann, Zabrodin 96],[Kazakov, Sorin, Zabrodin 07], [Zabrodin 07], [Tsuboi 09]

if $T^{(a,s)}(u)$ is a solution of Hirota equation and

$$T^{(a+1,s)}(u)F^{(a,s)}(u) - T^{(a,s)}(u)F^{(a+1,s)}(u) = \underbrace{x_j}_{\text{eigenvalue of } g} T^{(a+1,s-1)}(u+2)F^{(a,s+1)}(u-2),$$

$$T^{(a,s+1)}(u)F^{(a,s)}(u) - T^{(a,s)}(u)F^{(a,s+1)}(u) = x_j T^{(a+1,s)}(u+2)F^{(a-1,s+1)}(u-2).$$

Then $F^{(a,s)}(u)$ is a solution of Hirota equation.

Moreover, if $T^{(a,s)}(u) = 0, \forall a > K$, one can choose $F^{(a,s)}(u) = 0, \forall a > K-1$.

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Bäcklund Transformations : linear system

Hirota equation and its solution through Q-operators / Q-functions

S. Leurent

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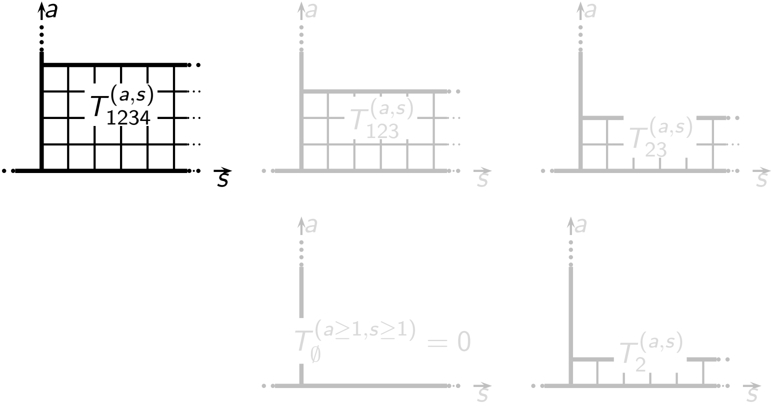
$GL(4)$ Bäcklund flow and lattices' boundaries



$\Rightarrow$ "Q-operators" [Baxter 72], [Pasquier Gaudin 92], [Bazhanov Lykyanov Zamolodchikov 96], [Derkachov, 99], [Bytsko Teschner 06], [Bazhanov Frassek Lukowski Meneghelli Staudacher 10]

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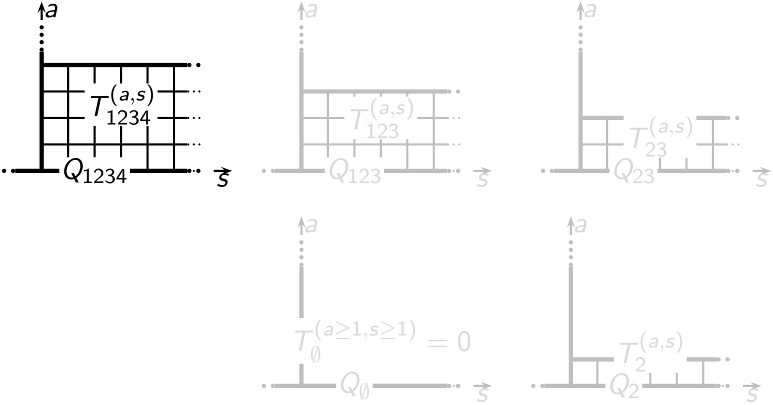

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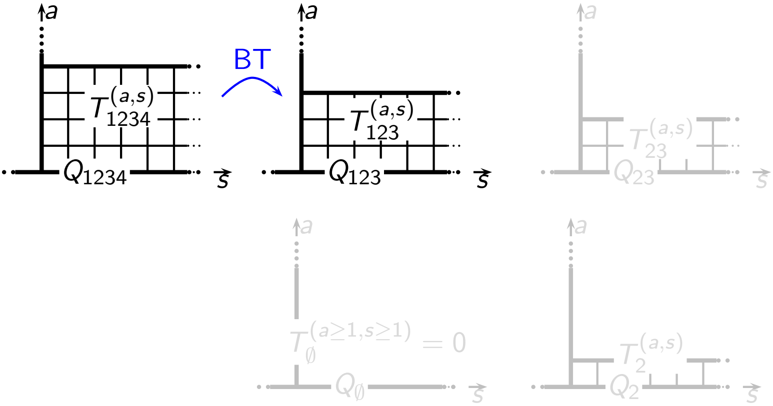

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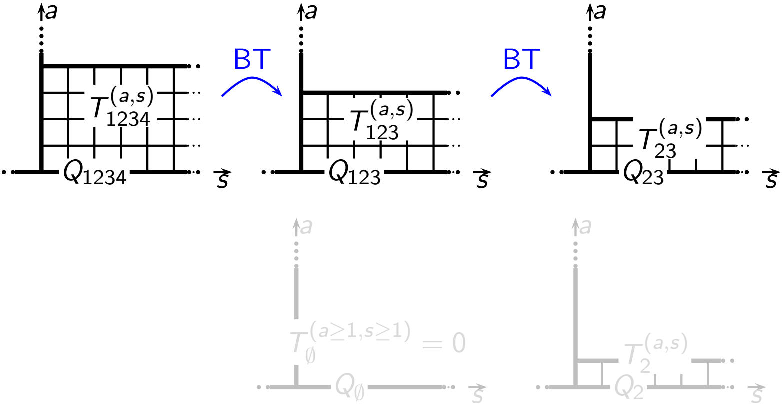

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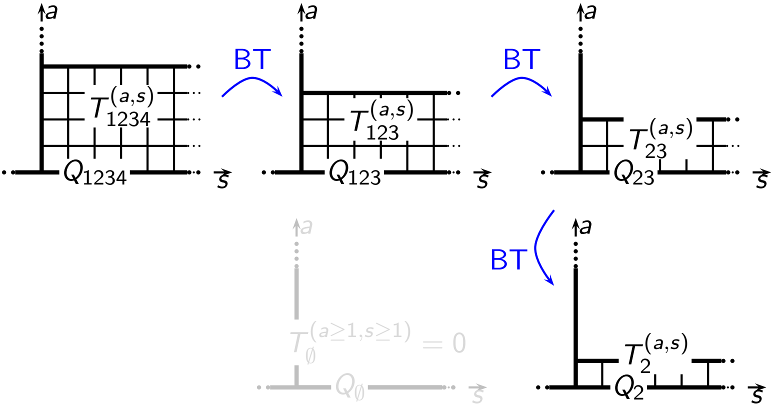

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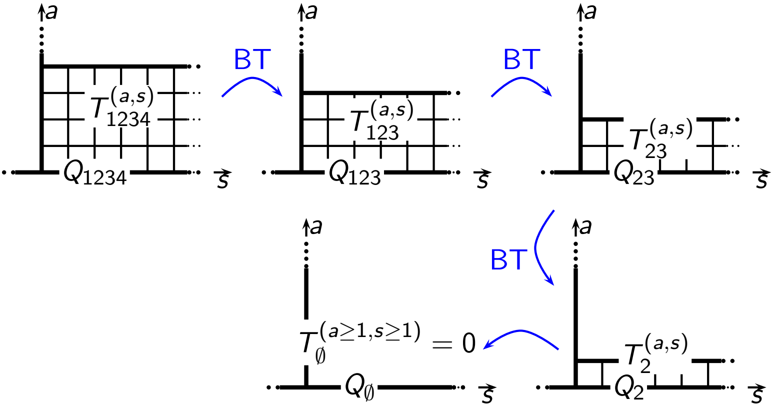

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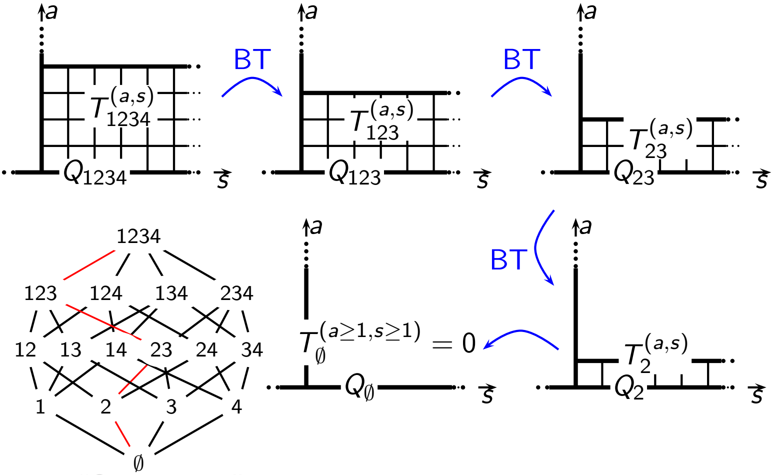

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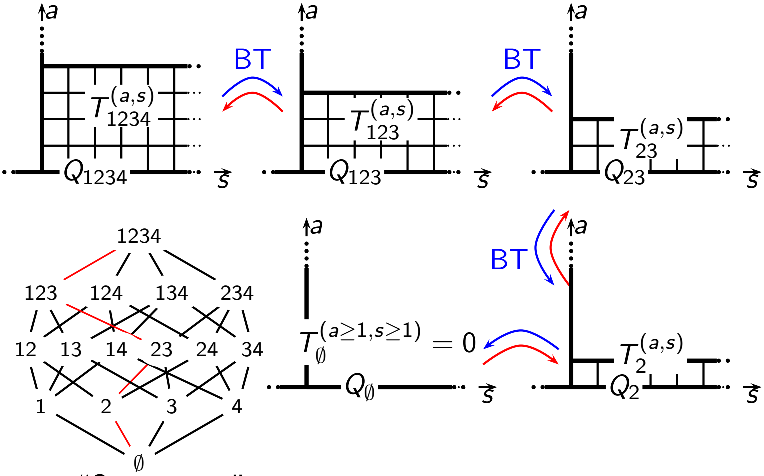

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Explicit Bäcklund flow

Explicit solution of this linear system

$$T_I^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \bar{I}}} B_{\bar{I}} \cdot \left[\bigotimes_{i=1}^{N} (u_i + 2\hat{D} + 2|\bar{I}|)\ \chi_\lambda(g_I) \Pi_{\bar{I}} \right],$$

$$g_{\{j_1, j_2, \cdots, j_k\}} = \mathrm{diag}(x_{j_1}, x_{j_2}, \cdots, x_{j_k}) \qquad \Pi_{\bar{I}} = \prod_{j \in \bar{I}} \det \frac{1}{1 - t_j g}$$

$$B_{\bar{I}} = \prod_{j \in \bar{I}} (1 - x_j\ t_j) \cdot (1 - g\ t_j)^{\otimes N} \qquad Q_I = T_I^{\emptyset}$$

One can algebraically check that

- $[\![T_I^{\{\lambda\}}(u), T_J^{\{\mu\}}(v)]\!] = 0$
- they solve the linear system
- they satisfy TQ and QQ-relations

$\leftsquigarrow$ combinatorics of co-derivative "Master Identity"

Hirota equation and its solution through Q-operators / Q-functions

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Remark

T- and Q-operators are explicitly polynomials of the spectral parameter u

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Bethe Equations ($\rightsquigarrow$ diagonalization of T-operators)

At the level of operators, the QQ-relations

$$(x_i - x_j)\, Q_I(u-2) Q_{I,i,j}(u) = x_i Q_{I,j}(u-2) Q_{I,i}(u) - x_j Q_{I,j}(u) Q_{I,i}(u-2)$$

imply

$$Q_{I,i}(u) \mid x_i Q_I(u-2) Q_{I,i,j}(u) Q_{I,i}(u+2) + x_j Q_I(u) Q_{I,i,j}(u+2) Q_{I,i}(u-2).$$

On a given eigen-state,

$$Q_I(u) = c_I \prod_{k=1}^{K_I} (u - u_k^{(I)}),$$

$$-1 = \frac{x_i}{x_j} \frac{Q_I(u_k^{(I,i)} - 2) Q_{I,i}(u_k^{(I,i)} + 2) Q_{I,i,j}(u_k^{(I,i)})}{Q_I(u_k^{(I,i)}) Q_{I,i}(u_k^{(I,i)} - 2) Q_{I,i,j}(u_k^{(I,i)} + 2)}$$

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Wronskian Formulae ($\rightsquigarrow$ diagonalization of T-operators)

from QQ-relations

$$Q_I(u) = \frac{\det\left(x_j^{|I|-1-k}\,Q_j(u-2k)\right)_{\substack{j\in I\\ 0\leq k\leq |I|-1}}}{Q_\emptyset(0)^{|I|-1}\det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\leq k\leq |I|-1}}}.$$

From TQ-relations

$$T_I^s(u) = \frac{\begin{vmatrix} \left(x_k^{|I|-1+s}\,Q_j(u+2s)\right)_{j\in I} \\ \left(x_j^{|I|-1-k}\,Q_j(u-2k)\right)_{\substack{j\in I\\ 1\leq k\leq |I|-1}} \end{vmatrix}}{Q_\emptyset(0)^{|I|-1}\det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\leq k\leq |I|-1}}}.$$

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Wronskian Formulae ($\rightsquigarrow$ diagonalization of T-operators)

from QQ-relations

Wronskian determinant

$$T_I^{(a,s)}(u) = \frac{\det\left(x_j^{|I|-1-k+s\Theta} Q_j(u-2k+2s\Theta)\right)_{\substack{j\in I\\ 0\le k\le |I|-1}}}{Q_\emptyset(0)^{|I|-1}\det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\le k\le |I|-1}}}$$

$$\Theta = \begin{cases} 1 \text{ if } k<a \\ 0 \text{ if } k\ge a \end{cases}$$

$$T_I^s(u) = \frac{\cdots}{Q_\emptyset(0)^{|I|-1}\det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\le k\le |I|-1}}}.$$

Outline

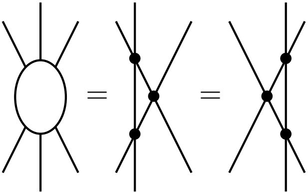
Model definition

The principal chiral field (PCF) is a 1+1 D field theory on the cylinder $0 \leq x < L$, $t \in \mathbb{R}$

$$\mathcal{S}_{\mathrm{PCF}} = -\frac{1}{2e_0^2} \int dt\, dx \;\; \mathrm{tr}(h^{-1}\partial_\alpha h)^2 \,. \qquad (1)$$

Where $h \in SU(N)$

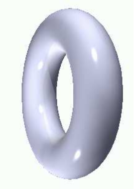
- $SU(N)_L \times SU(N)_R$ symmetry
- Integrable theory with rational $\mathcal{S}$ matrix, of the form : $\chi_{CDD}(u) \cdot S_0(u) \frac{\hat{R}(u)}{u-i} \otimes S_0(u) \frac{\hat{R}(u)}{u-i}$



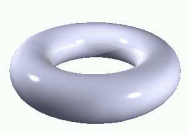
[Zamolodchikov, Zamolodchikov 79]
[Berg, Marowski, Weisz, Kurak 78]
[Wiegmann 84]

Skip details ⇝ Y-system

Ground state energy : double Wick rotation



Spatial periodicity L
time-periodicity $R \to \infty$:
Path integral dominated by Ground state $Z \sim e^{-RE_0(L)}$



Spatial periodicity $R \to \infty$
time-periodicity L (finite temperature)

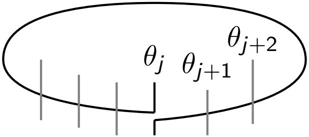
$$\text{free Energy} : f(L) = E_0(L)$$

Solution for large spatial period L

- solutions described by particules having rapidities θ_j:

$$p_j = m_j \sinh\left(\tfrac{2\pi}{N}\theta_j\right) \quad , \quad E = \sum_{j=1}^{N} E_j = \sum_{j=1}^{N} m_j \cosh\left(\tfrac{2\pi}{N}\theta_j\right)$$

- bound states with mass $m_a = m \frac{\sin \frac{a\pi}{N}}{\sin \frac{\pi}{N}}$
- periodicity condition
$$e^{-imR\sinh(\pi\theta_j)} = -S(\theta_j) \frac{Q^R_{N-1}(\theta_j + i/2)}{Q^R_{N-1}(\theta_j - i/2)} \frac{Q^L_{N-1}(\theta_j + i/2)}{Q^L_{N-1}(\theta_j - i/2)}$$

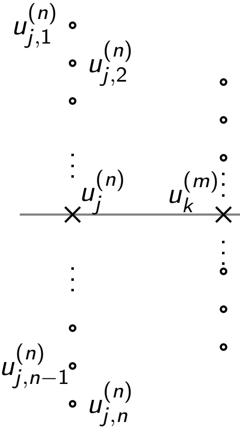


$$S(\theta) = \prod_j S_0^2(\theta - \theta_j)\chi_{CDD}(\theta - \theta_j)$$

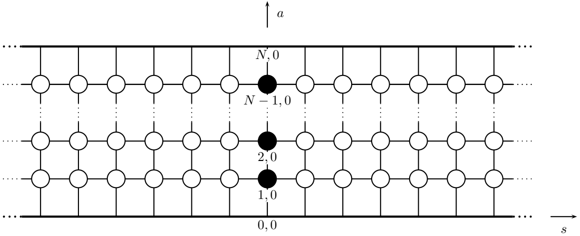
- magnons $1 = \frac{Q^R_{k-1}(u_j^{(k)} - i/2)}{Q^R_{k-1}(u_j^{(k)} + i/2)} \frac{Q^R_k(u_j^{(k)} + i) Q^R_{k+1}(u_j^{(k)} - i/2)}{Q^R_k(u_j^{(k)} - i) Q^R_{k+1}(u_j^{(k)} + i/2)}$
$(1 \le k \le N-1)$

String hypothesis

- The Bethe equations imply that the large number of magnons roots are organized as strings.
- $u_{j,a}^{(n)} = u_j^{(n)} + i\frac{1}{2}(n+1) - ia, \quad a = 1, \ldots, n.$
- Such strings scatter with a shifted product of the original matrix
- the right configuration (described by one density for each type of string) is identified by minimization of the free entropy.



Y-system

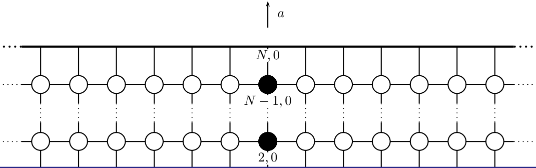


$$Y_{a,s}^{+}\, Y_{a,s}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}},$$

$$(a,s) \in \{1,2,\cdots,N-1\}\times\mathbb{Z},\ u\in\mathbb{C}, \qquad f^{\pm} = f(u\pm i/2)$$

- $Y_{a,s} \underset{u\gg 1}{\sim} e^{-L\tilde{p}_a(u)\delta_{s,0}} \times \mathrm{const}_{a,s}$, $\qquad \tilde{p}_a = \cosh(\frac{2\pi}{N}u)\frac{\sin(\frac{a\pi}{N})}{\sin(\frac{\pi}{N})}$
- $Y_{0,s} = Y_{N,s} = \infty$
- $E = -\frac{1}{N}\sum_{a=1}^{N-1}\int_{-\infty}^{\infty} p_a(u)\log\left(1+Y_{a,0}(u)\right)du$
- A contour prescription is necessary for excited states

Y-system



Hirota relation

If we define $Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$, then

$$Y^+_{a,s}\,Y^-_{a,s} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$$

$$\Leftrightarrow T^+_{a,s}\,T^-_{a,s} = T_{a+1,s}\,T_{a-1,s} + T_{a,s+1}\,T_{a,s-1}$$

- $E = -\frac{1}{N}\sum_{a=1} \int_{-\infty} p_a(u) \log\left(1+Y_{a,0}(u)\right) du$
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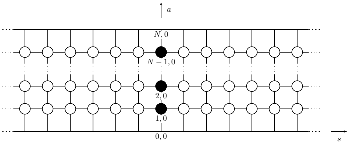
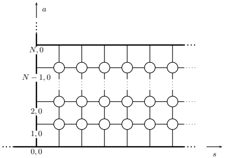
Solution of PCF in terms of q-functions

[1007.1770 ; V. Kazakov & SL] (see also [Gromov Kazakov Vieira 08])





- The corresponding T-system is solved by

$$T_{a,s} = \begin{vmatrix} \left(\overline{q_j}^{[s+a+1+\frac{N}{2}-2k]}\right)_{1\leq j\leq N, 1\leq k\leq a} \\ \left(q_j^{[-s+a+1+\frac{N}{2}-2k]}\right)_{1\leq j\leq N, a<k\leq N} \end{vmatrix}$$

- There exists "analyticity strips", consistent with $q_i(u) = \langle\text{polynomial}\rangle_i + \langle\text{resolvant}\rangle_i$

$\rightsquigarrow$ FiNLIE

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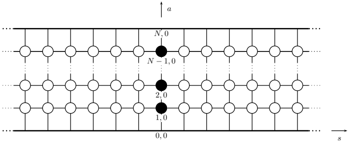

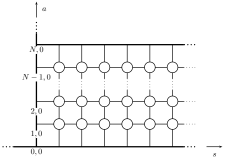

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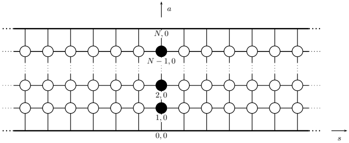

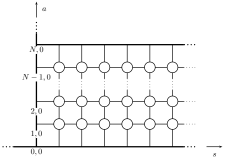

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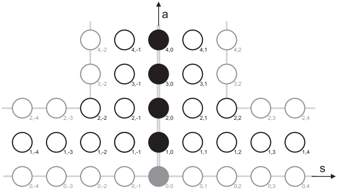
AdS_5 / CFT_4 Y-system



[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- The corresponding T-system is solved by

$$T_{1,s}|_{s\geq 1} = Q_1^{[s]}Q_1^{[-s]} - Q_2^{[s]}Q_2^{[-s]},$$

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$$T_{a,+1}|_{a\geq 1} = (-1)^{a+1}\left(Q_{12\hat{1}}^{[a]}Q_{12\hat{1}}^{[-a]} - Q_{12\hat{2}}^{[a]}Q_{12\hat{2}}^{[-a]} + Q_{12\hat{3}}^{[a]}Q_{12\hat{3}}^{[-a]} - Q_{12\hat{4}}^{[a]}Q_{12\hat{4}}^{[-a]}\right)$$

$$T_{a,0}|_{a\geq 0} =$$

$$Q_{12\hat{1}\hat{2}}^{[a]}Q_{43\hat{4}\hat{3}}^{[-a]} - Q_{12\hat{1}\hat{3}}^{[a]}Q_{43\hat{4}\hat{2}}^{[-a]} + Q_{12\hat{1}\hat{4}}^{[a]}Q_{43\hat{3}\hat{2}}^{[-a]} + Q_{12\hat{2}\hat{3}}^{[a]}Q_{43\hat{4}\hat{1}}^{[-a]} - Q_{12\hat{2}\hat{4}}^{[a]}Q_{43\hat{3}\hat{1}}^{[-a]} + Q_{12\hat{3}\hat{4}}^{[a]}Q_{43\hat{2}\hat{1}}^{[-a]},$$

$\cdots$

[1010.2720 ; N. Gromov, V.Kazakov, SL & Z.Tsuboi]

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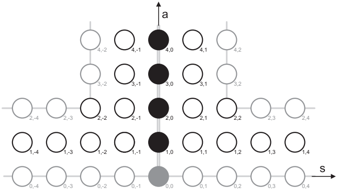

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&T_{a,0}|_{a\geq 0} =\\
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\end{aligned}
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[1010.2720 ; N. Gromov, V.Kazakov, SL & Z.Tsuboi]

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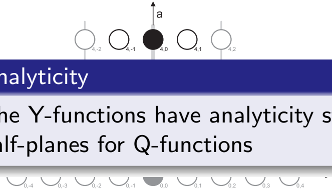

[Gromov Kazakov Kozak Vieira 09]

Analyticity

The Y-functions have analyticity strips, acurately described by analyticity half-planes for Q-functions

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[1010.2720 ; N. Gromov, V.Kazakov, SL & Z.Tsuboi]

Outlook

- The whole integrability of rational spin chains is encoded in combinatorial properties of coderivatives
  - $\rightsquigarrow$ Could the underlying structure be related to these properties ?
- How much can this construction be generalized ?
  - $\rightsquigarrow$ Non-compact representations in auxiliary space
  - $\rightsquigarrow$ Other representations in quantum space
    - The proof of the "Master Identity" suggests that this whole construction can be generalized as soon as the Bazhanov-Reshetikhin formula is known...
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Really

Thank you !

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