NLIE for $SU(N) \times SU(N)$ Principal Chiral Field via Hirota dynamics

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Outlook

1. Thermodynaamic Bethe Ansatz (TBA) and Y-system
   - Ground state energy
   - Excited states
2. Non linear integral equations (NLIEs) from Hirota equation
   - Spin-chain limit and Asy;ptotic Bethe ansatz (ABA)
   - Finite size solution
   - Numerical results and consistency checks
3. Outlook
   - Principal Chiral Field
   - Other models
   - AdS/CFT Y-system

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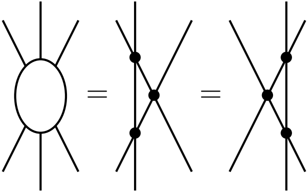
Model definition

The principal chiral field (PCF) is a 1+1 D field theory on the cylinder $0 \leq x < L$, $t \in \mathbb{R}$

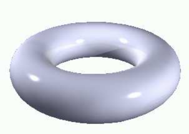
$$\mathcal{S}_{\mathrm{PCF}} = -\frac{1}{2e_0^2} \int dt\, dx \;\; \mathrm{tr}(h^{-1}\partial_\alpha h)^2 \,. \tag{1}$$

Where $h \in SU(N)$

- $SU(N)_L \times SU(N)_R$ symmetry
- Integrable theory with rational $\mathcal{S}$ matrix, identified by Zamolodchikov : $\chi_{CDD}(\theta) \cdot S_0(\theta) \frac{\hat{R}(\theta)}{\theta - i} \otimes S_0(\theta) \frac{\hat{R}(\theta)}{\theta - i}$



Ground state energy : double Wick rotation



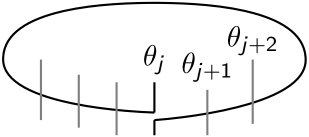
Spatial periodicity L
time-periodicity $R \to \infty$:
Path integral dominated by Ground state $Z \sim e^{-RE_0(L)}$

Spatial periodicity $R \to \infty$
time-periodicity L (finite temperature)

$$\text{free Energy} : f(L) = E_0(L)$$

Solution for large spatial period L

- solutions described by particules having rapidities θ_j:
  $p_j = m_j \sinh(\frac{2\pi}{N}\theta_j)$, $E = \sum_{j=1}^{N} E_j = \sum_{j=1}^{N} m_j \cosh(\frac{2\pi}{N}\theta_j)$
- bound states with mass $m_a = m\frac{\sin\frac{a\pi}{N}}{\sin\frac{\pi}{N}}$
- periodicity condition
  $e^{-imR\sinh(\pi\theta_j)} = -S(\theta_j)\frac{Q^R_{N-1}(\theta_j+i/2)}{Q^R_{N-1}(\theta_j-i/2)}\frac{Q^L_{N-1}(\theta_j+i/2)}{Q^L_{N-1}(\theta_j-i/2)}$

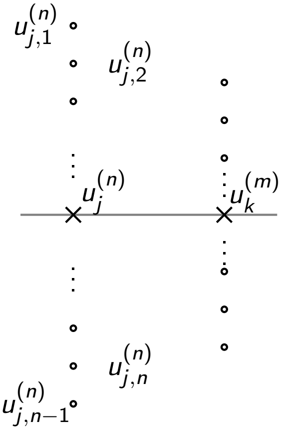


$S(\theta) = \prod_j S_0^2(\theta-\theta_j)\chi_{CDD}(\theta-\theta_j)$

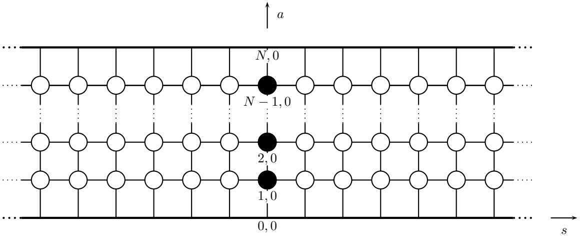
- magnons $1 = \frac{Q^R_{k-1}(u_j^{(k)}-i/2)}{Q^R_{k-1}(u_j^{(k)}+i/2)}\frac{Q^R_k(u_j^{(k)}+i)Q^R_{k+1}(u_j^{(k)}-i/2)}{Q^R_k(u_j^{(k)}-i)Q^R_{k+1}(u_j^{(k)}+i/2)}$
  $(1 \le k \le N-1)$

String hypothesis

- due to $\frac{Q^R_{k-1}(u_j^{(k)}-i/2)}{Q^R_{k-1}(u_j^{(k)}+i/2)}\frac{Q^R_k(u_j^{(k)}+i)Q^R_{k+1}(u_j^{(k)}-i/2)}{Q^R_k(u_j^{(k)}-i)Q^R_{k+1}(u_j^{(k)}+i/2)} = 1$, the large number of magnons roots are organized as strings.
- $u_{j,a}^{(n)} = u_j^{(n)} + i\frac{1}{2}(n+1) - ia, \quad a = 1, \ldots, n.$
- Such strings scatter with a shifted product of the original matrix
- the right configuration (described by one density for each type of string) is identified by minimization of the free entropy.



Y-system



$$Y_{a,s}^{+}\,Y_{a,s}^{-}=\frac{1+Y_{a,s+1}}{1+\left(Y_{a+1,s}\right)^{-1}}\,\frac{1+Y_{a,s-1}}{1+\left(Y_{a-1,s}\right)^{-1}},$$

$(a,s)\in\mathbb{Z}\times\{1,2,\cdots,N-1\}$, $\theta\in\mathbb{C}$, $\qquad f^{\pm}=f(\theta\pm i/2)$

- $Y_{a,s}\underset{\theta\gg 1}{\sim} e^{-L\tilde{p}_a(\theta)\delta_{s,0}}\times \mathrm{const}_{a,s}$, $\qquad \tilde{p}_a=\cosh(\frac{2\theta\pi}{N})\frac{\sin(\frac{a\pi}{N})}{\sin(\frac{\pi}{N})}$
- $Y_{0,s}=Y_{N,s}=\infty$
- $E=-\frac{1}{N}\sum_{a=1}^{N-1}\int_{-\infty}^{\infty}p_a(\theta)\log\left(1+Y_{a,0}(\theta)\right)d\theta$

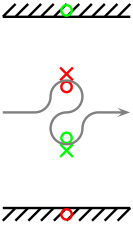
Continuation to excited states

- The Y-system equation is the same as for Vacuum, only the analyticity is different
  - Vacuum : "most analytic state", no pole on the physical strip
  - Excited states characterized by the pole structure
- Once the Y-system is solved, the energy is given by $E = -\frac{1}{N}\sum_{a=1}^{N-1}\int_{\mathcal{C}} p_a(\theta)\log\left(1+Y_{a,0}(\theta)\right)d\theta$ where the right contour of integration has to be identified.

  $\rightsquigarrow$ use the asymptotic limit ($L \to \infty$) as a guide-line.

Claim : the sigma model is completely recast into

- $Y_{a,s}^{+}\, Y_{a,s}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ on a given domain in (a,s)
- Large θ asymptotic
- Contour of integration



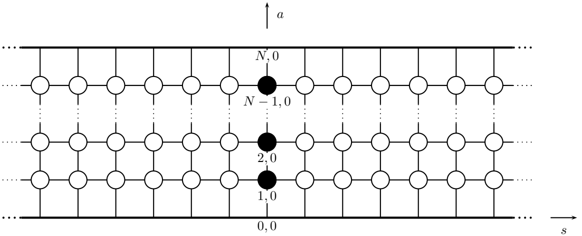
Outline

Hirota equation

If we define $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$, then

$$Y_{a,s}^{+}\,Y_{a,s}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$$

$$\Leftrightarrow T_{a,s}^{+}\,T_{a,s}^{-} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$



$T_{a,s}(\theta)$ is a function of $a \in \{0,1,\cdots N\}$, $s \in \mathbb{Z}$, $\theta \in \mathbb{C}$

$Y_{a,s}$ is invariant w.r.t. the gauge transformation

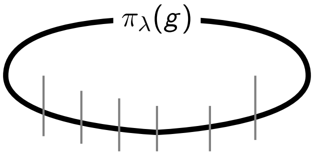
$$T_{a,s} \;\to\; \chi_1^{[a+s]}\chi_2^{[a-s]}\chi_3^{[-a+s]}\chi_4^{[-a-s]}T_{a,s} \qquad f^{[\pm k]} \equiv f(\theta \pm ki/2)$$

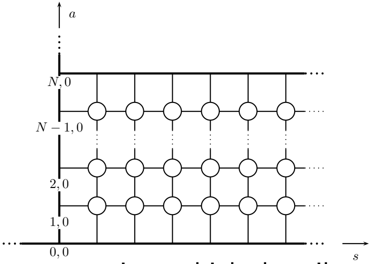
Hirota equation for integrable spin chains

For *XXX* Heisenberg spin chain, the transfer matrix $T_\lambda(\theta)$ is a family of commuting operators, defined as

$$T_{\{\lambda\}}(\theta) = \mathrm{tr}_\lambda \left(R_N^\lambda(\theta - \theta_L) \otimes \cdots \otimes R_1^\lambda(\theta - \theta_1)\, \pi_\lambda(g) \right) \qquad (2)$$

Notation : $T_{a,s}(u)$ when λ is a rectular young tableau





In this context, $T_{a,s}$ satisfies the hirota equation, which describes the fusion relations between representations.

Hirota equation for integrable spin chains

- The general solution of Hirota is given by the Bazhanov-Reshetikin determinant

$$T_{a,s} = \frac{\det_{1\leq j,k\leq a} \; T_{1,s+k-j}\left(\theta + (a+1-k-j)i/2\right)}{\mathbf{\Pi}_{a-1}\left(\varphi^{[-s-N/2]}\right)} \qquad (3)$$

where $\mathbf{\Pi}_k[f](\theta) = \prod_{j=-(k-1)/2}^{(k-1)/2} f(\theta + i\, j)$

- For spins chains, there is a generating series for symmetric representations :

$$\sum_{s=0}^{\infty} \frac{T_{1,s}(\theta+(s-1)i/2)}{\varphi(\theta-Ni/4)} e^{is\partial_\theta} =$$
$$\left(1 - X_{(N)}(\theta)\, e^{i\partial_\theta}\right)^{-1} \left(1 - X_{(N-1)}(\theta)\, e^{i\partial_\theta}\right)^{-1} \dots \left(1 - X_{(1)}(\theta)\, e^{i\partial_\theta}\right)^{-1}$$

Hirota equation for integrable spin chains

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- $X_{(k)} = \frac{{Q_{k-1}}^{[N/2-k-1]}}{{Q_{k-1}}^{[N/2-k+1]}} \frac{{Q_k}^{[N/2-k+2]}}{{Q_k}^{[N/2-k]}}$
- T and Q functions are polynomials. In particular $T_{a,0} \sim \varphi^{[a-N/2]}$, where $\varphi = \prod_j(\theta-\theta_j)$
- We will assume that for PCF in the $L \to \infty$ limit, where $Y_{a,0} \sim T_{a,1} T_{a,-1} \ll 1$, there is a gauge where $T_{a,-1} \ll 1$ and where $T_{a,s\geq 0}$ is described by such a spin chain solution of hirota.

Middle node equations

- $1 + Y_{a,s} = \frac{T^+_{a,s} T^-_{a,s}}{T_{a+1,s} T_{a-1,s}}$

$$\frac{Y^+_{a,s} Y^-_{a,s}}{(Y_{a+1,s})^{1-\delta_{a,N-1}} (Y_{a-1,s})^{1-\delta_{a,1}}} = \frac{1+Y_{a,s+1}}{(1+Y_{a+1,s})^{1-\delta_{a,N-1}}} \frac{1+Y_{a,s-1}}{(1+Y_{a-1,s})^{1-\delta_{a,1}}}$$

$$\rightsquigarrow \ Y^{\star\Delta}_{a,0} = \frac{T^{\star\Delta}_{a,1} (T^{(L)})^{\star\Delta}_{a,-1}}{T^{\star\Delta}_{a+1,0} T^{\star\Delta}_{a-1,0}} \times \left(\frac{T^+_{N,0} T^-_{N,0}}{T_{N,1} T^{(L)}_{N,-1}} \right)^{\delta_{a,N-1}} \left(\frac{T^+_{0,0} T^-_{0,0}}{T_{0,1} T^{(L)}_{0,-1}} \right)^{\delta_{a,1}}$$

where $F^{\star\Delta}_a := \frac{F^+_a F^-_a}{(F_{a+1})^{1-\delta_{a,N-1}} (F_{a-1})^{1-\delta_{a,1}}}$

- Inversion of Δ gives

$$Y_{a,0} = e^{-L\tilde{p}_a(\theta)} \frac{T_{a,1} T^{(L)}_{a,-1}}{T_{a+1,0} T_{a-1,0}} \left(\mathbf{\Pi}_{N-a} \left[\frac{T^+_{0,0} T^-_{0,0}}{T_{0,1} T^{(L)}_{N,-1}} \right] \mathbf{\Pi}_a \left[\frac{T^+_{N,0} T^-_{N,0}}{T_{N,1} T^{(L)}_{0,-1}} \right] \right)^{\star K_N}$$

- K_N is inverse to $\mathbf{\Pi}_N$: $\forall f \text{ regular}, (\mathbf{\Pi}_N[f])^{*K_N} = f$
- $K_N = \frac{\tan\left(\frac{1}{2}\pi\left(\frac{1}{N} - \frac{2it\theta}{N}\right)\right) + \tan\left(\frac{1}{2}\pi\left(\frac{1}{N} + \frac{2i\theta}{N}\right)\right)}{2N}$

Asymptotic Bethe Ansatz

- crossing relation for $S(\theta)=\prod_j S_0^2(\theta-\theta_j)\chi_{CDD}(\theta-\theta_j)$:

$$\mathbf{\Pi}_N S(\theta)=\left(\frac{\prod_j u-\theta_j-i\frac{N-1}{2}}{\prod_j u-\theta_j+i\frac{N-1}{2}}\right)^2$$

- at $L\to\infty$, $T_{a,0}\sim\varphi^{[a-N/2]}$, and $S(\theta)$ appears eg in $\left(\frac{T^+_{0,0}}{T^-_{0,0}}\right)^{\star K_N}$, giving

$$Y_{a,0}(\theta)\sim e^{-L\tilde{p}_a}\frac{T_{a,1}\,T^{(v)}_{a,-1}}{T_{a+1,0}\,T_{a-1,0}}\frac{\varphi^{[-N/2-a+1]}}{\varphi^{[-N/2+a-1]}}\frac{\varphi^{[-N/2-a+1]}}{\varphi^{[-N/2+a+1]}}\frac{1}{\mathbf{\Pi}_a\left(\left[S^{[-N/2]}\right]^2\chi_{CDD}^{[-N/2]}\right)}$$

- $1+Y_{1,0}(\theta_j+iN/4)=0$ gives the asymptotic Bethe equation :

$$-1=e^{-iL\sinh\frac{2\pi}{N}\theta_j}\frac{1}{\chi_{CDD}(\theta_j)S(\theta_j)^2}\frac{Q^{(R)}_{N-1}(\theta_j-i/2)}{Q^{(R)}_{N-1}(\theta_j+i/2)}\frac{Q^{(L)}_{N-1}(\theta_j-i/2)}{Q^{(L)}_{N-1}(\theta_j+i/2)}$$

General solution of Hirota equation

Any solution of hirota on the lattice $a \in \{0, 1, \cdots, N\}$, $s \in \mathbb{Z}$ is gauge equivalent to a wronskian determinant

$$T_{a,s} = \left| \begin{array}{l} \left(\overline{q_j}^{[s+a+1+\frac{N}{2}-2k]} \right)_{1 \leq j \leq N, 1 \leq k \leq a} \\ \left(q_j^{[-s+a+1+\frac{N}{2}-2k]} \right)_{1 \leq j \leq N, a < k \leq N} \end{array} \right|$$

It involves $2N$ functions $(q_j)_{j=1,\ldots,N}$ and $(\overline{q_j})_{j=1,\ldots,N}$, that we will now express through NLIE.

- On this wronskian form, the solution has two gauge freedoms left, writen as

$$\begin{array}{rcl} q_j(\theta) & \to & g(\theta) \cdot q_j(\theta) \\ \overline{q_j}(\theta) & \to & \overline{g}(\theta) \cdot \overline{q_j}(\theta) \end{array}$$

Analyticity strips

- large $L\cosh(\theta)$: $Y_{a,s} \sim e^{-Lp_a(\theta)\delta_{s,0}}$ $\qquad p_a = \cosh(\frac{2\theta\pi}{N})\frac{\sin(\frac{a\pi}{N})}{\sin(\frac{\pi}{N})}$
- therefore, $T_{a,0} \sim \varphi^{[a-N/2]}$ should hold on the following strips

$$T_{0,0} \xrightarrow[L\cosh(\frac{2\pi\theta}{N})\to\infty]{} \varphi^{[-N/2]} \qquad \text{when } Im(\theta) < \frac{N}{4}$$

$$T_{a,0}|_{0<a<N} \xrightarrow[L\cosh(\frac{2\pi\theta}{N})\to\infty]{} \varphi^{[+a-N/2]} \qquad \text{when } |Im(\theta)| < \frac{N}{4} + \frac{1}{2}$$

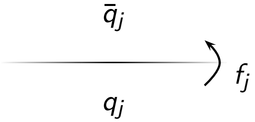
$$T_{N,0} \xrightarrow[L\cosh(\frac{2\pi\theta}{N})\to\infty]{} \varphi^{[+N/2]} \qquad \text{when } Im(\theta) > -\frac{N}{4}$$

- that is compatible with requiring that $q_j(\theta)$ is analytic for $\mathrm{Im}(\theta) < 0$, and $\overline{q_j}(\theta)$ is analytic for $\mathrm{Im}(\theta) > 0$

Introduction of the jump densities

Reality condition , and a choice of gauge ($q_1 = 1$) leaves $N - 1$ unknown functions,

described as a known polynomial asymptotic at large θ plus finite size corrections described by densities $f_j \underset{L+|\theta|\to\infty}{\ll} 1$



$$q_j(\theta) = P_j(\theta) + F_j(\theta)$$

where $\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{f_j(\psi)}{\theta-\psi}\mathrm{d}\psi = \begin{cases} F_j(\theta) & \text{if } \mathrm{Im}(\theta) < 0 \\ \overline{F_j}(\theta) & \text{if } \mathrm{Im}(\theta) > 0 \end{cases}$

Hence $\overline{q_j}^{[+0]} - q_j^{[-0]} \equiv \lim_{\epsilon\to 0} \overline{q_j}^{[+\epsilon]} - q_j^{[-\epsilon]} = -f_j$

Closed equations on the densities

- $T_{a,-1}$ is small because in the determinant, there is a full column of $\overline{q_j}^{[+0]}$ AND a full column of $q_j^{[-0]}$.
  Column substraction and expension w.r.t. these columns give

$$T_{a,-1}\left(\theta - i\frac{N-2a}{4}\right) = \sum_j d_{a,j}(\theta) f_j(\theta)$$

- on the other hand $Y_{a,0} = \frac{T_{a,1}T_{a,-1}}{T_{a+1,0}T_{a-1,0}}$ was expressed by inversion of the middle node equations.
  Hence this system of $N-1$ equations

$$\sum_j d_{a,j}(\theta) f_j(\theta) = e^{L p_a(\theta - i\frac{N-2a}{4})} T_{a,1}^{[a-N/2]} \times \frac{T_{0,0}^{[-N/2+1]}}{T_{0,0}^{[-N/2-1]}} \frac{T_{N,0}^{[N/2-1]}}{T_{N,0}^{[N/2+1]}} \left(\frac{T_{0,0}^{[-3N/2-a+1]}}{T_{0,0}^{[-3N/2+a-1]}}\right)^{*K_N^{[a-1]}} \left(\frac{T_{N,0}^{[3N/2+N-a+1]}}{T_{N,0}^{[3N/2-N+a-1]}}\right)^{*K_N^{[-N+a+1]}}$$

$\rightsquigarrow$ iterative resolution

Regularity conditions

- One of the steps of this iterative solution is to invert the matrix $(d_{a,j})$. When its determinant is zero, that could induce a pole ine the f_j's.
  The requirement that the denominator cancels at the same position (i.e. that f_j's don't have such poles) gives **finite size Bethe equations**
- this procedure gives simple poles to the Y-functions, and it can be explicitely checked that it reproduces χ_{CDD} at $L \longrightarrow \infty$

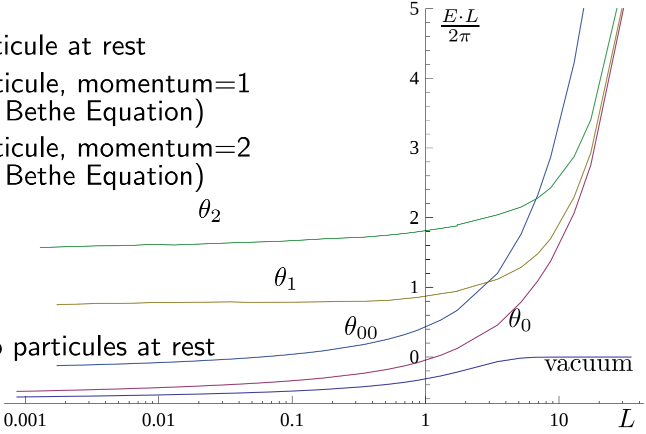
Numerical results

θ_0 : One particle at rest

θ_1 : One particule, momentum=1 ($e^{2i\pi}$ in the Bethe Equation)

θ_2 : One particule, momentum=2 ($e^{4i\pi}$ in the Bethe Equation)

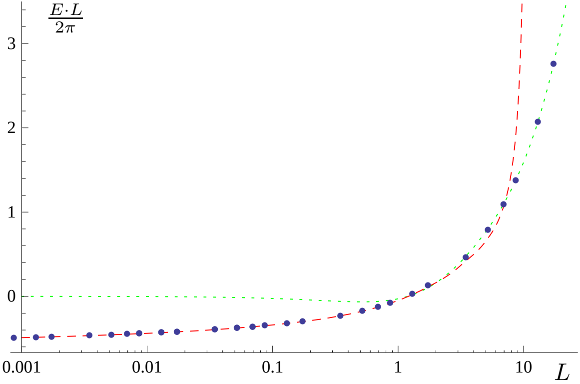
$\theta_{0,0}$: Two particules at rest



Energies of vacuum, of mass gap and of some excited states as functions of L, at $N = 3$

Checks for mass gap

- The μ term (Lüscher correction) is reproduced from the imaginary part of the Bethe root (green dotted line).



- The first logarithmic corrections, in the conformal limit (red dashed line), are also reproduced.

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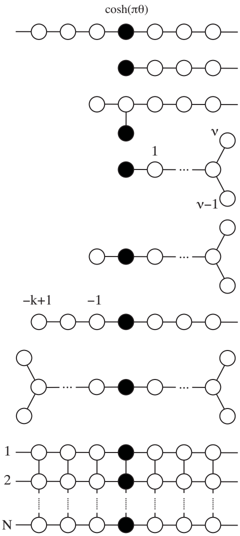
PCF is not over

- The numerics still deserve to be improved
  $\rightsquigarrow$ better precision, study of $N \geq 4$, states outside the $U(1)$ sector
- the choice of the contour isn't completely understood yet
- It could be interesting to understand the UV limit in terms of these densities
- The $N \to \infty$ limit also requires some investigations...

Other models having a “known” Y-system



O(4) σ-model

SU(2) Chiral Gross-Neveu

O(3) σ-model

Sine-Gordon with $\beta^2=8\pi(1-1/\nu)$

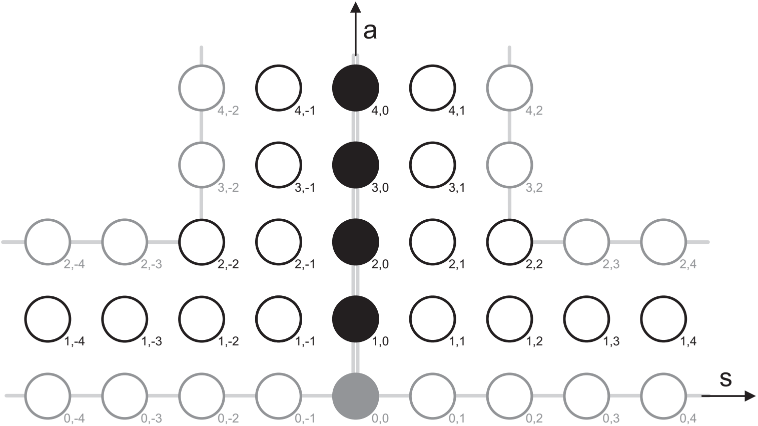
Super-sine-Gordon

Current-current perturbation of $SU_k(2)$ WZW

Sausage models

SU(N+1) Principal Chiral Field

AdS/CFT Y-system



AdS/CFT Y-system

- non relativistic dispersion relation : $\epsilon_a(u) = a + \frac{2ig}{x^{[+a]}} - \frac{2ig}{x^{[+a]}}$ where $\frac{u}{g} = x + \frac{1}{x}$
- the mapping $u \mapsto x$ has zhukowski cuts
- the hirota equation has a wronskian solution, which explains quite well the analyticity strips
- infinite number of “middle nodes”
- different reality conditions

Conclusion

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Thanks !

Thank you