I Upper-continuous results

Definition 14. *$\lambda X.[\![P]\!]_\rho \in \mathcal{P}\ (S \times H) \longmapsto \mathcal{P}\ (S \times H)$ is upper-continuous iff*
For any $\subseteq$-increasing chain x_i, $i \in \mathbb{N}$:
$[\![P]\!]_{\rho[X_i/X]}$ exists and:

$$[\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} = \bigcup_{i\in\mathbb{N}} [\![P]\!]_{\rho[X_i/X]}$$

Lemma 11. *If $\lambda X.[\![P]\!]_\rho$ is upper-continuous then $\lambda X.[\![P[x'/x]]\!]_\rho$ is upper-continuous.*

Proof (Lemma 11). By definition $[\![P[x'/x]]\!]_\rho = \{s, h \mid s[x \mapsto s(x')], h \in [\![P]\!]_\rho\}$. The lemma follows directly:

$$\begin{aligned}
&(\lambda X.[\![P[x'/x]]\!]_\rho)(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![P[x'/x]]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= \{s, h \mid s[x \mapsto s(x')], h \in [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\}\\
&= \{s, h \mid s[x \mapsto s(x')], h \in \bigcup_{i\in\mathbb{N}} [\![P]\!]_{\rho[X_i/X]}\} \text{ (by hyp.)}\\
&= \bigcup_{i\in\mathbb{N}} \{s, h \mid s[x \mapsto s(x')], h \in [\![P]\!]_{\rho[X_i/X]}\}\\
&= \bigcup_{i\in\mathbb{N}} [\![P[x'/x]]\!]_{\rho[X_i/X]}\\
&= \bigcup_{i\in\mathbb{N}} (\lambda X.[\![P[x'/x]]\!]_\rho)(X_i)
\end{aligned}$$

Lemma 12. *If $\lambda X.F$ is upper-continuous and $\lambda X.G$ does not depend on X then $\lambda X.F \cap G$ is upper-continuous.*

Proof (Lemma 12).

$$\begin{aligned}
&(\lambda X.F \cap G)(\bigcup_{i\in\mathbb{N}} X_i)\\
&= F(\bigcup_{i\in\mathbb{N}} X_i) \cap G(\bigcup_{i\in\mathbb{N}} X_i)\\
&= (\bigcup_{i\in\mathbb{N}} (F(X_i))) \cap G(\bigcup_{i\in\mathbb{N}} X_i) \text{ (hyp. 1)}\\
&= (\bigcup_{i\in\mathbb{N}} (F(X_i))) \cap G(X_i) \quad \text{(hyp. 2)}\\
&= (\bigcup_{i\in\mathbb{N}} (F(X_i) \cap G(X_i)) \quad \text{($G(X_i)$ does not depend on i)}\\
&= (\bigcup_{i\in\mathbb{N}} (\lambda X.F \cap G)(X_i)
\end{aligned}$$

□

Lemma 13. *$\lambda X.[\![X_v \wedge E = \texttt{true}]\!]_{[\rho|X_v \mapsto X]}$ is upper-continuous.*

Proof (Lemma 13).
$\lambda X.[\![X_v \wedge E = \texttt{true}]\!]_{[\rho|X_v \mapsto X]}$
$= \lambda X.X \cap [\![E = \texttt{true}]\!]$
since X_v can not occure in $E = \texttt{true}$.
The result then follows by lemma 12. □

Theorem 12. *$\forall P, C$ If $\lambda X.[\![P]\!]_\rho$ is upper-continuous, then $\lambda X.[\![sp(P, C)]\!]_\rho$ is upper-continuous.*

Proof (Theorem 12). Proof by induction on the command C. Since we are working with function from sets to sets the union is always defined. Notice that we use by hypothesis $[\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} = \bigcup_{i\in\mathbb{N}} [\![P]\!]_{\rho[X_i/X]}$ and want to prove that $[\![sp(P,C)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} = \bigcup_{i\in\mathbb{N}} [\![sp(P,C)]\!]_{\rho[X_i/X]}$.

– Case C is $x := E$ then

$$
\begin{aligned}
&\lambda X.[\![sp(P,C)]\!]_\rho(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![\exists x'.\ P[x'/x] \wedge x = E[x'/x]]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= \{s,h \mid \exists v \in Val.[s \mid x \mapsto v], h \in [\![P[x'/x] \wedge x = E[x'/x]]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\}\\
&= \{s,h \mid \exists v \in Val.\\
&\quad [s \mid x \mapsto v], h \in [\![P[x'/x]]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} \text{ and}\\
&\quad [s \mid x \mapsto v], h \in [\![x = E[x'/x]]\!]\}\\
&= \{s,h \mid \exists v \in Val.\\
&\quad [s \mid x \mapsto v], h \in \bigcup_{i\in\mathbb{N}} [\![P[x'/x]]\!]_{\rho[X_i/X]} \text{ and}\\
&\quad [s \mid x \mapsto v], h \in [\![x = E[x'/x]]\!])\} \qquad \text{(lem. 11+ hyp.)}\\
&= \{s,h \mid \exists v \in Val.[s \mid x \mapsto v], h \in \bigcup_{i\in\mathbb{N}} [\![P[x'/x] \wedge x = E[x'/x]]\!]_{\rho[X_i/X]}\}\\
&= \bigcup_{i\in\mathbb{N}} \{s,h \mid \exists v \in Val.[s \mid x \mapsto v], h \in [\![P[x'/x] \wedge x = E[x'/x]]\!]_{\rho[X_i/X]}\}\\
&= \bigcup_{i\in\mathbb{N}} [\![\exists x'.\ P[x'/x] \wedge x = E[x'/x]]\!]_{\rho[X_i/X]}\\
&= \bigcup_{i\in\mathbb{N}} (\lambda X.[\![sp(P,C)]\!]_\rho(X_i))
\end{aligned}
$$

– Case C is $x := E.i$ then

$$
\begin{aligned}
&\lambda X.[\![sp(P,C)]\!]_\rho(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![\exists x'.\ P[x'/x] \wedge x = (E[x'/x]).i]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= [\![\exists x'.\ P[x'/x] \wedge (\exists x_1, x_2.(E[x'/x] \hookrightarrow x_1, x_2) \wedge x = x_i)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= \{s,h \mid \exists v \in Val.\\
&\quad [s \mid x \mapsto v], h \in [\![P[x'/x] \wedge (\exists x_1, x_2.(E[x'/x] \hookrightarrow x_1, x_2) \wedge x = x_i)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\}\\
&= \{s,h \mid \exists v \in Val.\\
&\quad [s \mid x \mapsto v], h \in [\![P[x'/x]]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} \text{ and}\\
&\quad [s \mid x \mapsto v], h \in [\![\exists x_1, x_2.(E[x'/x] \hookrightarrow x_1, x_2) \wedge x = x_i]\!])\}\\
&= \{s,h \mid \exists v \in Val.\\
&\quad [s \mid x \mapsto v], h \in \bigcup_{i\in\mathbb{N}} [\![P[x'/x]]\!]_{\rho[X_i/X]} \text{ and}\\
&\quad [s \mid x \mapsto v], h \in [\![\exists x_1, x_2.(E[x'/x] \hookrightarrow x_1, x_2) \wedge x = x_i]\!])\} \qquad \text{(lem. 11 + hyp.)}\\
&= \{s,h \mid \exists v \in Val.\\
&\quad [s \mid x \mapsto v], h \in \bigcup_{i\in\mathbb{N}} [\![P[x'/x] \wedge (\exists x_1, x_2.(E[x'/x] \hookrightarrow x_1, x_2) \wedge x = x_i)]\!]_{\rho[X_i/X]}\}\\
&= \bigcup_{i\in\mathbb{N}} \{s,h \mid \exists v \in Val.\\
&\quad [s \mid x \mapsto v], h \in [\![P[x'/x] \wedge (\exists x_1, x_2.(E[x'/x] \hookrightarrow x_1, x_2) \wedge x = x_i)]\!]_{\rho[X_i/X]}\}\\
&= \bigcup_{i\in\mathbb{N}} [\![\exists x'.\ P[x'/x] \wedge (\exists x_1, x_2.(E[x'/x] \hookrightarrow x_1, x_2) \wedge x = x_i)]\!]_{\rho[X_i/X]}\\
&= \bigcup_{i\in\mathbb{N}} (\lambda X.[\![sp(P,C)]\!]_\rho(X_i))
\end{aligned}
$$

– Case C is $E.1 := E'$ then

$$\begin{aligned}
&\lambda X.[\![sp(P,C)]\!]_\rho(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![\exists x_1 \exists x_2.(E \mapsto E', x_2) * ((E \mapsto x_1, x_2) -\!\!* P)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= \{s,h \mid \exists v_1, v_2 \in Val.[s \mid x_i \mapsto v_i], h \in [\![E \mapsto E', x_2]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} * ((E \mapsto x_1, x_2) -\!\!* P))\}\\
&= \{s,h \mid \exists v_1, v_2 \in Val.\exists h_0, h_1\ h_0 \sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x_i \mapsto v_i], h_0 \in [\![E \mapsto E', x_2]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} \text{ and}\\
&\quad [s \mid x_i \mapsto v_i], h_1 \in [\![(E \mapsto x_1, x_2) -\!\!* P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\}\\
&= \{s,h \mid \exists v_1, v_2 \in Val.\exists h_0, h_1.\ h_0 \sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x_i \mapsto v_i], h_0 \in [\![E \mapsto E', x_2]\!] \text{ and}\\
&\quad \forall h_0'.\begin{array}{l}\text{if } h_1 \sharp h_0' \text{ and } [s \mid x_i \mapsto v_i], h_0' \in [\![E \mapsto x_1, x_2]\!]\\ \text{then}[s \mid x_i \mapsto v_i], h_0'.h_1 \in [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\end{array}\}\\
&= \{s,h \mid \exists v_1, v_2 \in Val.\exists h_0, h_1.\ h_0 \sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x_i \mapsto v_i], h_0 \in [\![E \mapsto E', x_2]\!] \text{ and}\\
&\quad \forall h_0'.\begin{array}{l}\text{if } h_1 \sharp h_0' \text{ and } [s \mid x_i \mapsto v_i], h_0' \in [\![E \mapsto x_1, x_2]\!]\\ \text{then}[s \mid x_i \mapsto v_i], h_0'.h_1 \in \bigcup_{i\in\mathbb{N}} [\![P]\!]_{\rho[X_i/X]}\end{array}\}
\end{aligned}$$

here we have a $\forall h_o'$ but for the if branch to be satifyied there is only one h_0' that could work: $[[\![E]\!]^s \mapsto v_1, v_2]$ so we can delete it and we also know that this is disjoint from h_1 from the previous conditions.

$$\begin{aligned}
&= \{s,h \mid \exists v_1, v_2 \in Val.\exists h_0, h_1.\ h_0 \sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x_i \mapsto v_i], h_0 \in [\![E \mapsto E', x_2]\!] \text{ and}\\
&\quad \text{if } [\![E]\!]^s \text{ exists}\\
&\quad \text{then}[s \mid x_i \mapsto v_i], [[\![E]\!]^s \mapsto v_1, v_2].h_1 \in \bigcup_{i\in\mathbb{N}} [\![P]\!]_{\rho[X_i/X]}\}\\
&= \bigcup_{i\in\mathbb{N}} \{s,h \mid \exists v_1, v_2 \in Val.\exists h_0, h_1.\ h_0 \sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x_i \mapsto v_i], h_0 \in [\![E \mapsto E', x_2]\!] \text{ and}\\
&\quad \text{if } [\![E]\!]^s \text{ exists}\\
&\quad \text{then}[s \mid x_i \mapsto v_i], [[\![E]\!]^s \mapsto v_1, v_2].h_1 \in [\![P]\!]_{\rho[X_i/X]}\}\\
&= \bigcup_{i\in\mathbb{N}} \{s,h \mid \exists v_1, v_2 \in Val.\exists h_0, h_1.\ h_0 \sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x_i \mapsto v_i], h_0 \in [\![E \mapsto E', x_2]\!] \text{ and}\\
&\quad \forall h_0'.\begin{array}{l}\text{if } h_1 \sharp h_0' \text{ and } [s \mid x_i \mapsto v_i], h_0' \in [\![E \mapsto x_1, x_2]\!]\\ \text{then}[s \mid x_i \mapsto v_i], h_0'.h_1 \in [\![P]\!]_{\rho[X_i/X]}\end{array}\}\\
&= \bigcup_{i\in\mathbb{N}} [\![\exists x_1 \exists x_2.(E \mapsto E', x_2) * ((E \mapsto x_1, x_2) -\!\!* P)]\!]_{\rho[X_i/X]}\\
&= \bigcup_{i\in\mathbb{N}} (\lambda X.[\![sp(P,C)]\!]_\rho(X_i))
\end{aligned}$$

– Case C is $E.2 := E'$ then
almost the same as the previous one...

– Case C is $x :=$ `cons`(E_1, E_2) then

$$
\begin{aligned}
&\lambda X.[\![sp(P,C)]\!]_\rho(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![\exists x'.(P[x'/x] * (x \mapsto E_1[x'/x], E_2[x'/x]))]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= \{s,h \mid \exists v'_x \in Val.[s \mid x' \mapsto v'_x], h \in [\![(P[x'/x] * (x \mapsto E_1[x'/x], E_2[x'/x]))]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\}\\
&= \{s,h \mid \exists v'_x \in Val.\exists h_0,h_1.\ h_0\sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x' \mapsto v'_x], h \in [\![P[x'/x]]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} \text{ and}\\
&\quad [\![x \mapsto E_1[x'/x], E_2[x'/x]]\!]\}\\
&= \{s,h \mid \exists v'_x \in Val.\exists h_0,h_1.\ h_0\sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x' \mapsto v'_x], h \in \bigcup_{i\in\mathbb{N}} [\![P[x'/x]]\!]_{\rho[X_i/X]} \text{ and}\\
&\quad [\![x \mapsto E_1[x'/x], E_2[x'/x]]\!]\} \text{ (lem. 11+hyp.)}\\
&= \bigcup_{i\in\mathbb{N}} \{s,h \mid \exists v'_x \in Val.\exists h_0,h_1.\ h_0\sharp h_1,\ h = h_0.h_1 \text{ and}\\
&\quad [s \mid x' \mapsto v'_x], h \in [\![P[x'/x]]\!]_{\rho[X_i/X]} \text{ and}\\
&\quad [\![x \mapsto E_1[x'/x], E_2[x'/x]]\!]\}\\
&= \bigcup_{i\in\mathbb{N}} [\![\exists x'.(P[x'/x] * (x \mapsto E_1[x'/x], E_2[x'/x]))]\!]_{\rho[X_i/X]}\\
&= \bigcup_{i\in\mathbb{N}} (\lambda X.[\![sp(P,C)]\!]_\rho(X_i))
\end{aligned}
$$

– Case C is `dispose`(E) then

$$
\begin{aligned}
&\lambda X.[\![sp(P,C)]\!]_\rho(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![\exists x_1,x_2.\ ((E \mapsto x_1,x_2) -\!\!* P)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= \{s,h \mid \exists v_1,v_2.[s \mid x_i \mapsto v_i], h \in [\![(E \mapsto x_1,x_2) -\!\!* P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\}\\
&= \{s,h \mid \exists v_1,v_2.\forall h'.\begin{array}{l}\text{if } h\sharp h',\ \text{and } [s \mid x_i \mapsto v_i], h' \in [\![E \mapsto x_1,x_2]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\ \text{then } [s \mid x_i \mapsto v_i], h.h' \in [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\end{array}\}\\
&= \{s,h \mid \exists v_1,v_2.\forall h'.\begin{array}{l}\text{if } h\sharp h',\ \text{and } [s \mid x_i \mapsto v_i], h' \in [\![E \mapsto x_1,x_2]\!]\\ \text{then } [s \mid x_i \mapsto v_i], h.h' \in \bigcup_{i\in\mathbb{N}} [\![P]\!]_{\rho[X_i/X]}\end{array}\} \qquad \text{(hyp.)}\\
&= \{s,h \mid \exists v_1,v_2.\begin{array}{l}\text{if } [\![E]\!]^s \text{ exists and } [\![E]\!]^s \notin dom(h)\\ \text{then } [s \mid x_i \mapsto v_i], h.[[\![E]\!]^s \mapsto v_1,v_2] \in \bigcup_{i\in\mathbb{N}} [\![P]\!]_{\rho[X_i/X]}\end{array}\}\\
&\quad \text{(same operation as in the case of } E.i := E')\\
&= \bigcup_{i\in\mathbb{N}} \{s,h \mid \exists v_1,v_2.\begin{array}{l}\text{if } [\![E]\!]^s \text{ exists and } [\![E]\!]^s \notin dom(h)\\ \text{then } [s \mid x_i \mapsto v_i], h.[[\![E]\!]^s \mapsto v_1,v_2] \in [\![P]\!]_{\rho[X_i/X]}\end{array}\}\\
&= \bigcup_{i\in\mathbb{N}} [\![\exists x_1,x_2.\ ((E \mapsto x_1,x_2) -\!\!* P)]\!]_{\rho[X_i/X]}\\
&= \bigcup_{i\in\mathbb{N}} (\lambda X.[\![sp(P,C)]\!]_\rho(X_i))
\end{aligned}
$$

– Case C is $C_1; C_2$ then:

$sp(P,C) = sp(sp(P,C_1),C_2)$
$\lambda X.[\![sp(P,C_1)]\!]_\rho$ is upper-continuous by induction hypothesis on C_1 for P and $\lambda sp(P,C)$ is upper-continuous by induction hypothesis on C_2 for $sp(P,C_1)$.

– Case C is $if\ \ E\ \ then\ \ C_1\ \ else\ \ C_2$ then

$$
\begin{aligned}
&\lambda X.[\![sp(P,C)]\!]_\rho(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![sp(P \wedge E = \mathtt{true}, C_1) \vee sp(P \wedge E = \mathtt{false}, C_2)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= [\![sp(P \wedge E = \mathtt{true}, C_1)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} \cup [\![sp(P \wedge E = \mathtt{false}, C_2)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\\
&= (\bigcup_{i\in\mathbb{N}} [\![sp(P \wedge E = \mathtt{true}, C_1)]\!]_{\rho[X_i/X]}) \cup (\bigcup_{i\in\mathbb{N}} [\![sp(P \wedge E = \mathtt{false}, C_2)]\!]_{\rho[X_i/X]}) \text{ (by ind. hyp)}\\
&= \bigcup_{i\in\mathbb{N}} (([\![sp(P \wedge E = \mathtt{true}, C_1)]\!]_{\rho[X_i/X]}) \cup ([\![sp(P \wedge E = \mathtt{false}, C_2)]\!]_{\rho[X_i/X]}))\\
&= \bigcup_{i\in\mathbb{N}} ([\![sp(P \wedge E = \mathtt{true}, C_1) \vee sp(P \wedge E = \mathtt{false}, C_2)]\!]_{\rho[X_i/X]})\\
&= \bigcup_{i\in\mathbb{N}} ([\![sp(P,C)]\!]_{\rho[X_i/X]})
\end{aligned}
$$

– Case C is `skip` then

$sp(P,C) = P$ so $\lambda X.[\![sp(P,C)]\!]_\rho$ is upper-continuous by hypothesis.

– Case C is *while* E *do* C_1 then

$$
\begin{aligned}
&\lambda X.[\![sp(P,C)]\!]_\rho(\bigcup_{i\in\mathbb{N}} X_i)\\
&= [\![(\mu Y_v.sp(Y_v \wedge E = \mathtt{true}, C_1) \vee P) \wedge\ (E = \mathtt{false})]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]} \quad \text{(def. of sp. } Y \notin \rho[\bigcup_{i\in\mathbb{N}} X_i/X] + Y_v \notin P)\\
&= ([\![(\mu Y_v.sp(Y_v \wedge E = \mathtt{true}, C_1) \vee P)]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}) \cap [\![E = \mathtt{false}]\!] \text{ (since } Y_v \text{ not in } E)\\
&= \mathrm{lfp}^{\subseteq}_{\emptyset} \lambda Y.\ [\![sp(Y_v \wedge E = \mathtt{true}, C_1) \vee P]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]}\\
&\quad \cap [\![E = \mathtt{false}]\!] \quad \text{(def of } \mu)\\
&= \mathrm{lfp}^{\subseteq}_{\emptyset}(\lambda Y. \begin{array}{l}[\![sp(Y_v \wedge E = \mathtt{true}, C_1)]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]}\\ \cup [\![P]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]}\end{array})\\
&\quad \cap [\![E = \mathtt{false}]\!]\\
&= \mathrm{lfp}^{\subseteq}_{\emptyset}(\lambda Y. \begin{array}{l}[\![sp(Y_v \wedge E = \mathtt{true}, C_1)]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]}\\ \cup [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}\end{array})\\
&\quad \cap [\![E = \mathtt{false}]\!] \quad \text{(since } Y_v \notin P)
\end{aligned}
$$

let $G = \lambda X.(\lambda Y.\ [\![sp(Y_v \wedge E = \mathtt{true}, C_1)]\!]_{[\rho|Y_v \mapsto Y]} \cup [\![P]\!]_\rho)$

we have $G(\bigcup_{i\in\mathbb{N}} X_i) = (\lambda Y.\ [\![sp(Y_v \wedge E = \mathtt{true}, C_1)]\!]_{[\rho|Y_v \mapsto Y][\bigcup_{i\in\mathbb{N}} X_i/X]} \cup [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]})$

and by chose of Y we can rewrite it as:

$G(\bigcup_{i\in\mathbb{N}} X_i) = (\lambda Y.\ [\![sp(Y_v \wedge E = \mathtt{true}, C_1)]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]} \cup [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]})$

and $G(X_i) = (\lambda Y.\ [\![sp(Y_v \wedge E = \mathtt{true}, C_1)]\!]_{[\rho[X_i/X]|Y_v \mapsto Y]} \cup [\![P]\!]_{\rho[X_i/X]})$

we can rewrite the former formula as:

$= (\text{lfp}^{\subseteq}_{\emptyset}(G(\bigcup_{i\in\mathbb{N}} X_i))) \cap [\![E = \texttt{false}]\!]$

$= (\bigcup_{n\in\mathbb{N}} (G(\bigcup_{i\in\mathbb{N}} X_i))^n(\emptyset)) \cap [\![E = \texttt{false}]\!]$ $\quad (G(\bigcup_{i\in\mathbb{N}} X_i)$ u-c on Y + Tarski)

$= (\bigcup_{n\in\mathbb{N}} (\bigcup_{i\in\mathbb{N}} G(X_i))^n(\emptyset)) \cap [\![E = \texttt{false}]\!]$ $\quad$ (G u-c on X)

$= (\bigcup_{i\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} ((G(X_i))^n(\emptyset)) \cap [\![E = \texttt{false}]\!]$

$= \bigcup_{i\in\mathbb{N}} (\text{lfp}^{\subseteq}_{\emptyset} G(X_i)) \cap [\![E = \texttt{false}]\!]$ $\quad (G(X_i)$ u-c on Y + Tarski)

$= \bigcup_{i\in\mathbb{N}} (\text{lfp}^{\subseteq}_{\emptyset} \lambda Y.\ [\![sp(Y_v \wedge E = \texttt{true}, C_1) \vee P]\!]_{[\rho[X_i/X]|Y_v \mapsto Y]}) \cap [\![E = \texttt{false}]\!]$

$= \bigcup_{i\in\mathbb{N}} ([\![\mu Y_v.sp(Y_v \wedge E = \texttt{true}, C_1) \vee P]\!]_{\rho[X_i/X]}) \cap [\![E = \texttt{false}]\!]$ $\quad$ def of μ)

$= \bigcup_{i\in\mathbb{N}} ([\![(\mu Y_v.sp(Y_v \wedge E = \texttt{true}, C_1) \vee P) \wedge\ (E = \texttt{false})]\!]_{\rho[X_i/X]})$

$= \bigcup_{i\in\mathbb{N}} ([\![sp(P, C)]\!]_{\rho[X_i/X]})$

Complement:

- G is upper-continuous on X:
  recall $G = \lambda X.(\lambda Y.\ [\![sp(Y_v \wedge E = \texttt{true}, C_1)]\!]_{[\rho|Y_v \mapsto Y]} \cup [\![P]\!]_\rho)$
  by hypothesis: $\lambda X.[\![P]\!]_\rho$ is upper-continuous,
  by chose of Y not in ρ we have that $[\![sp(Y_v \wedge E = \texttt{true}, C_1)]\!]_{[\rho|Y_v \mapsto Y]}$ does not depend on X so $\lambda X.[\![sp(Y_v \wedge E = \texttt{true}, C_1)]\!]_{[\rho|Y_v \mapsto Y]}$ is upper-continuous on X
  and so G is upper-continuous on X.
- $G(\bigcup_{i\in\mathbb{N}} X_i)$ upper-continuous on Y:
  recall: $G(\bigcup_{i\in\mathbb{N}} X_i) = (\lambda Y.\ [\![sp(Y_v \wedge E = \texttt{true}, C_1)]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]} \cup [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]})$
  by lemma 13 we have that $\lambda Y.[\![Y_v \wedge E = \texttt{true}]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]}$ is upper-continuous
  and by ind. hyp. we have that this current theorem holds for any subprogram so it hold for C_1 and so $\lambda Y.[\![sp(Y_v \wedge E = \texttt{true}, C_1)]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]}$ is upper-continuous
  by chose of Y $[\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]}$ does not depend on Y
  so $(\lambda Y.\ [\![sp(Y_v \wedge E = \texttt{true}, C_1)]\!]_{[\rho[\bigcup_{i\in\mathbb{N}} X_i/X]|Y_v \mapsto Y]} \cup [\![P]\!]_{\rho[\bigcup_{i\in\mathbb{N}} X_i/X]})$ is upper-continuous
- $G(X_i)$ upper-continuous on Y: Same proof as for $G(\bigcup_{i\in\mathbb{N}} X_i)$

□