F Substitution theorems for $BI^{\mu\nu}$ general

Theorem 10. *If* $[\![Y]\!]_\rho$ *and* $[\![P]\!]_{[\rho|X_v \to [\![Y]\!]_\rho]}$ *exist then*

$$[\![P]\!]_{[\rho|X_v \to [\![Y]\!]_\rho]} = [\![P\{Y/X_v\}]\!]_\rho$$

Proof (Theorem 10). By induction on the formula P.

- Case P as the form $E = E'$, $E \mapsto E_1, e_2$, `false`, `emp`
  $P\{Y/X_v\} = P$ and $\forall \rho.[\![P]\!]_\rho = [\![P]\!]$ so
  $[\![P]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$
  $= [\![P]\!]$
  $= [\![P\{Y/X_v\}]\!]$
  $= [\![P\{Y/X_v\}]\!]_\rho$
- $[\![P \Rightarrow Q]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$
  $= (\top \setminus [\![P]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}) \cup [\![Q]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$
  $= (\top \setminus [\![P\{Y/X_v\}]\!]_\rho) \cup [\![Q\{Y/X_v\}]\!]_\rho$ (ind. hyp.)
  $= [\![P\{Y/X_v\} \Rightarrow Q\{Y/X_v\}]\!]_\rho$
  $= [\![(P \Rightarrow Q)\{Y/X_v\}]\!]_\rho$
- $[\![\exists x.P]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$
  $= \{s, h \mid \exists v \in Val.[s \mid x \mapsto v], h \in [\![P]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}\}$
  $= \{s, h \mid \exists v \in Val.[s \mid x \mapsto v], h \in [\![P\{Y/X_v\}]\!]_\rho\}$ (ind.hyp)
  $= [\![\exists x.(P\{Y/X_v\})]\!]_\rho$
  $= [\![(\exists x.P)\{Y/X_v\}]\!]_\rho$
- $[\![P * Q]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$
  $= \{s, h \mid \exists h_0, h_1\ h_0 \sharp h_1,\ h = h_0.h_1$
  $s, h_0 \in [\![P]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$ and $s, h_1 \in [\![Q]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}\}$
  $= \{s, h \mid \exists h_0, h_1\ h_0 \sharp h_1,\ h = h_0.h_1$
  $s, h_0 \in [\![P\{Y/X_v\}]\!]_\rho$ and $s, h_1 \in [\![Q\{Y/X_v\}]\!]_\rho\}$ (ind. hyp.)
  $= [\![P\{Y/X_v\} * Q\{Y/X_v\}]\!]_\rho$
  $= [\![(P * Q)\{Y/X_v\}]\!]_\rho$
- $[\![P -\!\!* Q]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$
  $= \{s, h \mid \forall h'.$ if $h' \sharp h$ and $s, h' \in [\![P]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$ then
  $s, h.h' \in [\![Q]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}\}$
  $= \{s, h \mid \forall h'.$ if $h' \sharp h$ and $s, h' \in [\![P\{Y/X_v\}]\!]_\rho$ then
  $s, h.h' \in [\![Q\{Y/X_v\}]\!]_\rho\}$ (ind. hyp.)
  $= [\![P\{Y/X_v\} -\!\!* Q\{Y/X_v\}]\!]_\rho$
  $= [\![(P -\!\!* Q)\{Y/X_v\}]\!]_\rho$
- $[\![X_v]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$
  $= [\![Y]\!]_\rho$
  $= [\![X_v\{Y/X_v\}]\!]_\rho$
- $[\![Z_v]\!]_{[\rho|X_v \mapsto [\![Y]\!]_\rho]}$ with $Z_v \neq X_v$
  $= [\![Z_v]\!]_\rho$
  $= [\![Z_v\{Y/X_v\}]\!]_\rho$

- $[\![\mu X_v.P]\!]_{[\rho|X_v\mapsto[\![Y]\!]_\rho]}$
  $= \mathrm{lfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[[\rho|X_v\mapsto[\![Y]\!]_\rho]|X_v\mapsto Z]}$
  $= \mathrm{lfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[\rho|X_v\mapsto Z]}$
  $= \mathrm{lfp}_\emptyset^\subseteq\ \lambda X.[\![P]\!]_{[\rho|X_v\mapsto X]}$
  $= [\![\mu X_v.P]\!]_\rho$
  $= [\![(\mu X_v.P)\{Y/X_v\}]\!]_\rho$
- $[\![\mu Z_v.P]\!]_{[\rho|X_v\mapsto[\![Y]\!]_\rho]}$ with $X_v \neq Z_v$
  $= \mathrm{lfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[[\rho|X_v\mapsto[\![Y]\!]_\rho]|Z_v\mapsto Z]}$
  $= \mathrm{lfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[[\rho|Z_v\mapsto Z]|X_v\mapsto[\![Y]\!]_\rho]}$
  $= \mathrm{lfp}_\emptyset^\subseteq\ \lambda Z.[\![P\{Y/X_v\}]\!]_{[\rho|Z_v\mapsto Z]}$ (ind. hyp.)
  $= [\![\mu Z_v.(P\{Y/X_v\})]\!]_\rho$
  $= [\![(\mu Z_v.P)\{Y/X_v\}]\!]_\rho$
- $[\![\nu X_v.P]\!]_{[\rho|X_v\mapsto[\![Y]\!]_\rho]}$
  $= \mathrm{gfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[[\rho|X_v\mapsto[\![Y]\!]_\rho]|X_v\mapsto Z]}$
  $= \mathrm{gfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[\rho|X_v\mapsto Z]}$
  $= \mathrm{gfp}_\emptyset^\subseteq\ \lambda X.[\![P]\!]_{[\rho|X_v\mapsto X]}$
  $= [\![\nu X_v.P]\!]_\rho$
  $= [\![(\nu X_v.P)\{Y/X_v\}]\!]_\rho$
- $[\![\nu Z.P]\!]_{[\rho|X_v\mapsto[\![Y]\!]_\rho]}$ with $X \neq Z$
  $= \mathrm{gfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[[\rho|X_v\mapsto[\![Y]\!]_\rho]|Z_v\mapsto Z]}$
  $= \mathrm{gfp}_\emptyset^\subseteq\ \lambda Z.[\![P]\!]_{[[\rho|Z_v\mapsto Z]|X_v\mapsto[\![Y]\!]_\rho]}$
  $= \mathrm{gfp}_\emptyset^\subseteq\ \lambda Z.[\![P\{Y/X_v\}]\!]_{[\rho|Z_v\mapsto Z]}$ (ind. hyp.)
  $= [\![\nu Z.(P\{Y/X_v\})]\!]_\rho$
  $= [\![(\nu Z.P)\{Y/X_v\}]\!]_\rho$
- $[\![P[E'/x]]\!]_{[\rho|X_v\mapsto[\![Y]\!]_\rho]}$
  $= \{s, h \mid [s \mid x \mapsto [\![E']\!]^s], h \in [\![P]\!]_{[\rho|X_v\mapsto[\![Y]\!]_\rho]}\}$
  $= \{s, h \mid [s \mid x \mapsto [\![E']\!]^s], h \in [\![P\{Y/X_v\}]\!]_\rho\}$ (ind. hyp.)
  $= [\![P\{Y/X_v\}[E'/x]]\!]_\rho$
  $= [\![(P[E'/x])\{Y/X_v\}]\!]_\rho$
  $\square$

Lemma 8. *If $[\![Y]\!]_\rho$ exists*

$$sp(P, C)\{Y/X_v\} = sp(P\{Y/X_v\}, C)$$

Proof (Lemma 8). Proof by induction on C.

- Case $x := E$
  $(sp(P, C)\{Y/X_v\})$
  $= (\exists x'.P[x'/x] \wedge x = E[x'/x])\{Y/X_v\}$
  $= (\exists x'.(P\{Y/X_v\})[x'/x] \wedge x = E[x'/x])$
  $= sp(P\{Y/X_v\}, C)$
- Case $x := E.i$
  $(sp(P, C)\{Y/X_v\})$
  $= (\exists x'.P[x'/x] \wedge x = (E[x'/x].i))\{Y/X_v\}$
  $= (\exists x'.(P\{Y/X_v\})[x'/x] \wedge x = (E[x'/x].i))$
  $= sp(P\{Y/X_v\}, C)$

- Case $E.1 := E'$
  $(sp(P, C)\{Y/X_v\})$
  $= (\exists x_1, x_2.(E \mapsto E', x_2) * ((E \mapsto x_1, x_2) -\!\!* P)\{Y/X_v\}$
  $= (\exists x_1, x_2.(E \mapsto E', x_2) * ((E \mapsto x_1, x_2) -\!\!* (P\{Y/X_v\}))$
  $= sp(P\{Y/X_v\}, C)$
- Case $E.2 := E'$
  $(sp(P, C)\{Y/X_v\})$
  $= (\exists x_1, x_2.(E \mapsto E', x_2) * ((E \mapsto x_1, x_2) -\!\!* P)\{Y/X_v\}$
  $= (\exists x_1, x_2.(E \mapsto x_1, E') * ((E \mapsto x_1, x_2) -\!\!* (P\{Y/X_v\}))$
  $= sp(P\{Y/X_v\}, C)$
- Case $x := \texttt{cons}(E_1, E_2)$
  $(sp(P, C)\{Y/X_v\})$
  $= \exists x'.(P[x'/x] * (x \mapsto E_1[x'/x], E_2[x'/x]))\{Y/X_v\}$
  $= \exists x'.((P\{Y/X_v\})[x'/x] * (x \mapsto E_1[x'/x], E_2[x'/x]))$
  $= sp(P\{Y/X_v\}, C)$
- Case $dispose(E)$
  $(sp(P, C)\{Y/X_v\})$
  $= \exists x_1, x_2.\ ((E \mapsto x_1, x_2) -\!\!* P)\{Y/X_v\}$
  $= \exists x_1, x_2.\ ((E \mapsto x_1, x_2) -\!\!* (P\{Y/X_v\}))$
  $= sp(P\{Y/X_v\}, C)$
- Case $C_1; C_2$
  $(sp(P, C)\{Y/X_v\})$
  $= sp(sp(P, C_1), C_2)\{Y/X_v\}$
  $= sp(sp(P, C_1)\{Y/X_v\}, C_2)$ (ind.hyp)
  $= sp(sp(P\{Y/X_v\}, C_1), C_2)$ (ind.hyp)
  $= sp(P\{Y/X_v\}, C)$
- Case $if\ E\ then\ C_1\ else\ C_2$
  $(sp(P, C)\{Y/X_v\})$
  $= (sp(P \wedge E = \texttt{true}, C_1) \vee sp(P \wedge E = \texttt{false}, C_2))\{Y/X_v\}$
  $= sp(P \wedge E = \texttt{true}, C_1)\{Y/X_v\} \vee sp(P \wedge E = \texttt{false}, C_2)\{Y/X_v\}$
  $= sp(P\{Y/X_v\} \wedge E = \texttt{true}, C_1) \vee sp(P\{Y/X_v\} \wedge E = \texttt{false}, C_2)$ (ind. hyp.)
  $= sp(P\{Y/X_v\}, C)$
- Case $\texttt{skip}$
  $(sp(P, C)\{Y/X_v\})$
  $= P\{Y/X_v\}$
  $= sp(P\{Y/X_v\}, C)$
- Case $while\ E\ do\ C_1$
  Case X_v not free in P
  $(sp(P, C)\{Y/X_v\})$
  $= ((\mu X_v.sp(X_v \wedge E = \texttt{true}, C_1) \vee P)\ \wedge\ (E = \texttt{false}))\{Y/X_v\}$
  $= ((\mu X_v.sp(X_v \wedge E = \texttt{true}, C_1) \vee P)\ \wedge\ (E = \texttt{false}))$
  $= ((\mu X_v.sp(X_v \wedge E = \texttt{true}, C_1) \vee P\{Y/X_v\})\ \wedge\ (E = \texttt{false}))$
  $= sp(P\{Y/X_v\}, C)$
  Case X_v free in P

$(sp(P, C)\{Y/X_v\})$
$= ((\mu Z_v.sp(Z_v \wedge E = \texttt{true}, C_1) \vee P) \ \wedge \ (E = \texttt{false}))\{Y/X_v\}$
$= ((\mu Z_v.sp(Z_v \wedge E = \texttt{true}, C_1) \vee P\{Y/X_v\}) \ \wedge \ (E = \texttt{false}))$
$= ((\mu W_v.sp(W_v \wedge E = \texttt{true}, C_1) \vee P\{Y/X_v\}) \ \wedge \ (E = \texttt{false}))$
$= sp(P\{Y/X_v\}, C)$
We have the egality between the case Z_v and W_v since $[\![Y]\!]_\rho$ exists so there is no free formula variables in Y, so neither Z_v or W_v can become bounded in some Y placed in P.

□