G Simplifications on [/]

Theorem 11. $\boxed{\text{If } \begin{array}{c} P \text{ is v-closed} \\ z \notin Var(P) \end{array} \text{ then } P[z/y] \equiv P\{[z/y]\} \wedge is(z)}$

Proof (Th. 11). From the generalized variable renaming theorem Th. 6, $[\![P\{[z/y]\}]\!] = \{s,h \mid s^\bullet, h \in [\![P]\!]\}$,
so $[\![P\{[z/y]\} \wedge is(z)]\!] = \{s,h \mid [s \mid y \to s(z)], h \in [\![P]\!]\}$,
which is $[\![P[z/y]]\!]$. □

Some more simplifications:
Remember that we have in Lemma 1 $[\![E\{E'/x\}]\!]^s = [\![E]\!]^{[s|x \to [\![E']\!]^s]}$ if $[\![E']\!]^s$ exists.

- $\boxed{(E_1 = E_2)[E'/x] \equiv (E_1\{E'/x\} = E_2\{E'/x\}) \wedge is(E')}$:

$$\begin{aligned} & [\![(E_1\{E'/x\} = E_2\{E'/x\}) \wedge is(E')]\!] \\ &= \{s,h \mid [\![E_1\{E'/x\}]\!]^s = [\![E_2\{E'/x\}]\!]^s \text{ and} [\![E']\!]^s \text{ exists}\} \\ &= \{s,h \mid [\![E_1]\!]^{[s|x \to [\![E']\!]^s]} = [\![E_2]\!]^{[s|x \to [\![E']\!]^s]}\} \\ &= \{s,h \mid [\![E_1]\!]^{[s|x \to [\![E']\!]^s]} = [\![E_2]\!]^{[s|x \to [\![E']\!]^s]}\} \\ &= \{s,h \mid [s \mid x \to [\![E']\!]^s], h \in [\![E_1 = E_2]\!]\} \\ &= [\![(E_1 = E_2)[E'/x]]\!] \end{aligned}$$

- $\boxed{(E \mapsto E_1, E_2)[E'/x] \equiv (E\{E'/x\} \mapsto E_1\{E'/x\}, E_2\{E'/x\}) \wedge is(E')}$:

$$\begin{aligned} & [\![(E\{E'/x\} \mapsto E_1\{E'/x\}, E_2\{E'/x\}) \wedge is(E')]\!] \\ &= \{s,h \mid dom(h) = \{[\![E\{E'/x\}]\!]^s\} \text{ and } h([\![E\{E'/x\}]\!]^s) = \langle [\![E_1\{E'/x\}]\!]^s, [\![E_2\{E'/x\}]\!]^s \rangle \\ &\qquad \text{and } [\![E']\!]^s \text{ exists}\} \\ &= \{s,h \mid dom(h) = \{[\![E]\!]^{[s|x \mapsto [\![E']\!]^s]}\} \\ &\qquad \text{and } h([\![E]\!]^{[s|x \mapsto [\![E']\!]^s]}) = \langle [\![E_1]\!]^{[s|x \mapsto [\![E']\!]^s]}, [\![E_2]\!]^{[s|x \mapsto [\![E']\!]^s]} \rangle\} \\ &= \{s,h \mid [s \mid x \mapsto [\![E']\!]^s], h \in [\![E \mapsto E_1, E_2]\!]\} \\ &= [\![(E \mapsto E_1, E_2)[E'/x]]\!] \end{aligned}$$

- $\boxed{\texttt{false}[E'/x] \equiv \texttt{false}}$:

$$\begin{aligned} & [\![\texttt{false}[E'/x]]\!] \\ &= \{s,h \mid [s \mid x \to [\![E']\!]^s], h \in [\![\texttt{false}]\!]\} \\ &= \emptyset \\ &= [\![\texttt{false}]\!] \end{aligned}$$

- $\boxed{(P \Rightarrow Q)[E'/x] \equiv P[E'/x] \Rightarrow Q[E'/x]}$:

$$\begin{aligned} & [\![(P \to Q)[E'/x]]\!]_\rho \\ &= \{s,h \mid [s \mid x \to [\![E']\!]^s], h \in [\![P \Rightarrow Q]\!]_\rho\} \\ &= \{s,h \mid [s \mid x \to [\![E']\!]^s], h \in ((\top \setminus [\![P]\!]_\rho) \cup [\![Q]\!]_\rho)\} \\ &= ((\top \setminus \{s,h \mid [s \mid x \to [\![E']\!]^s], h \in [\![P]\!]_\rho\}) \cup \{s,h \mid [s \mid x \to [\![E']\!]^s], h \in [\![Q]\!]_\rho\}) \\ &= ((\top \setminus [\![P[E'/x]]\!]_\rho) \cup [\![Q[E'/x]]\!]_\rho) \\ &= [\![P[E'/x] \Rightarrow Q[E'/x]]\!]_\rho \end{aligned}$$

– $\boxed{(\exists x.P)[E/x] \equiv (\exists x.P) \wedge is(E)}$:

$$
\begin{aligned}
&[\![(\exists x.P)[E/x]]\!]_\rho \\
&= \{s, h \mid [s \mid x \to [\![E]\!]^s], h \in [\![\exists x.P]\!]_\rho\} \\
&= \{s, h \mid \exists v.[s \mid x \to [\![E]\!]^s \mid x \to v], h \in [\![P]\!]_\rho\} \\
&= \{s, h \mid \exists v.[s \mid x \to v], h \in [\![P]\!]_\rho \text{ and } [\![E]\!]^s \text{ exists}\} \\
&= [\![(\exists x.P) \wedge is(E)]\!]_\rho
\end{aligned}
$$

– $\boxed{\text{If } \begin{array}{l} y \notin Var(E) \\ x \neq y \end{array} \text{ then } (\exists y.P)[E/x] \equiv \exists y.(P[E/x])}$:

First, notice that if $y \notin Var(E)$, then $[\![E]\!]^s = [\![E]\!]^{[s|y \to v]}$.

$$
\begin{aligned}
&[\![(\exists y.P)[E/x]]\!]_\rho \\
&= \{s, h \mid [s \mid x \to [\![E]\!]^s], h \in [\![\exists x.P]\!]_\rho\} \\
&= \{s, h \mid \exists v.[s \mid x \to [\![E]\!]^s \mid y \to v], h \in [\![P]\!]_\rho\} \\
&= \{s, h \mid \exists v.[s \mid y \to v \mid x \to [\![E]\!]^s], h \in [\![P]\!]_\rho\} \\
&= \{s, h \mid \exists v.[s \mid y \to v \mid x \to [\![E]\!]^{[s|y \to v]}], h \in [\![P]\!]_\rho\} \\
&= \{s, h \mid \exists v.[s \mid y \to v], h \in [\![P[E/x]]\!]_\rho\} \\
&= [\![\exists y.(P[E/x])]\!]_\rho
\end{aligned}
$$

– $\boxed{\texttt{emp}[E'/x] \equiv \texttt{emp} \wedge is(E')}$:

$$
\begin{aligned}
&= [\![\texttt{emp} \wedge is(E')]\!] \\
&= \{s, h \mid h = [] \text{ and } [\![E']\!]^s \text{ exists}\} \\
&= \{s, h \mid [s \mid x \mapsto [\![E']\!]^s], h \in [\![\texttt{emp}]\!]\} \\
&= [\![\texttt{emp}[E'/x]]\!]
\end{aligned}
$$

– $\boxed{(P * Q)[E'/x] \equiv P[E'/x] * Q[E'/x]}$:

$$
\begin{aligned}
&= [\![P[E'/x] * Q[E'/x]]\!]_\rho \\
&= \{s, h \mid \exists h_0, h_1.\ h_0 \sharp h_1, h_0.h_1 = h, s, h_0 \in [\![P[E'/x]]\!]_\rho \text{ and } s, h_1 \in [\![Q[E'/x]]\!]_\rho\} \\
&= \{s, h \mid \exists h_0, h_1.\ h_0 \sharp h_1, h_0.h_1 = h, [s \mid x \mapsto [\![E']\!]^s], h_0 \in [\![P]\!]_\rho \text{ and } [s \mid x \mapsto [\![E']\!]^s], h_1 \in [\![Q]\!]_\rho\} \\
&= \{s, h \mid [s \mid x \mapsto [\![E']\!]^s], h \in [\![P * Q]\!]_\rho\} \\
&= [\![(P * Q)[E'/x]]\!]_\rho
\end{aligned}
$$

– $\boxed{(P \mathrel{-\!\!*} Q)[E'/x] \equiv P[E'/x] \mathrel{-\!\!*} Q[E'/x]}$:

$$
\begin{aligned}
&= [\![P[E'/x] \mathrel{-\!\!*} Q[E'/x]]\!]_\rho \\
&= \{s, h \mid \forall h'.\text{ if } h' \sharp h \text{ and } s, h' \in [\![P[E'/x]]\!]_\rho \text{ then } s, h.h' \in [\![Q[E'/x]]\!]_\rho\} \\
&= \{s, h \mid \forall h'.\text{ if } h' \sharp h \text{ and } [s \mid x \mapsto [\![E']\!]^s], h' \in [\![P]\!]_\rho \text{ then } [s \mid x \mapsto [\![E']\!]^s], h.h' \in [\![Q]\!]_\rho\} \\
&= \{s, h \mid [s \mid x \mapsto [\![E']\!]^s], h \in [\![P \mathrel{-\!\!*} Q]\!]_\rho\} \\
&= [\![(P \mathrel{-\!\!*} Q)[E'/x]]\!]_\rho
\end{aligned}
$$