G μ and ν coincide

Theorem 11. $\boxed{\begin{array}{l}\mu X_v.\, P \equiv \neg\nu X_v.\neg(P\{\neg X_v\}) \\ \nu X_v.\, P \equiv \neg\mu X_v.\neg(P\{\neg X_v\})\end{array}}$

Proof (Theorem 11). First we recall the definition of equivalence between formulae $P \equiv Q$ iff $\forall\rho.(\llbracket P\rrbracket_\rho = \llbracket Q\rrbracket_\rho) \vee (\llbracket P\rrbracket_\rho$ and $\llbracket Q\rrbracket_\rho)$ both do not exist.

In this proof, we write $\top$ for $S \times H$, we sometime write C^c for $\top \setminus C$.

Let $B = \llbracket \nu X_v.P\rrbracket_\rho$ and $A = \llbracket \mu X_v.\neg(P\{\neg X_v/X_v\})\rrbracket_\rho$, we want to prove that (B exists $\Rightarrow$ A exists and $B^c = A$) , and (A exists $\Rightarrow$ B exists and $A^c = B$).

If B exists, then

- $B = \mathrm{gfp}^{\subseteq}_{\emptyset} \lambda X.\ \llbracket P\rrbracket_{[\rho|X_v \to X]}$, so
- B is the biggest such that $B = \llbracket P\rrbracket_{[\rho|X_v \to B]}$, so
- B^c is the smallest such that $\top \setminus B^c = \llbracket P\rrbracket_{[\rho|X_v \to \top\setminus B^c]}$, so
- B^c is the smallest such that $B^c = \top \setminus \llbracket P\rrbracket_{[\rho|X_v \to \top\setminus B^c]}$, so
- $B^c = \mathrm{lfp}^{\subseteq}_{\emptyset} \lambda X.\ \top \setminus \llbracket P\rrbracket_{[\rho|X_v \to \top\setminus X]}$, then from Th. 10
- $B^c = \mathrm{lfp}^{\subseteq}_{\emptyset} \lambda X.\ \top \setminus \llbracket P\{\neg X_v/X_v\}\rrbracket_{[\rho|X_v \to X]}$, which is
- $B^c = \mathrm{lfp}^{\subseteq}_{\emptyset} \lambda X.\ \llbracket \neg P\{\neg X_v/X_v\}\rrbracket_{[\rho|X_v \to X]}$, so
- A exists and $B^c = A$

If A exists, then

- $A = \mathrm{lfp}^{\subseteq}_{\emptyset} \lambda X.\ \llbracket \neg P\{\neg X_v/X_v\}\rrbracket_{[\rho|X_v \to X]}$, so
- $A = \mathrm{lfp}^{\subseteq}_{\emptyset} \lambda X.\ \top \setminus \llbracket P\{\neg X_v/X_v\}\rrbracket_{[\rho|X_v \to X]}$, then from Th. 10
- $A = \mathrm{lfp}^{\subseteq}_{\emptyset} \lambda X.\ \top \setminus \llbracket P\rrbracket_{[\rho|X_v \to \top\setminus X]}$, which is
- A is the smallest such that $A = \top \setminus \llbracket P\rrbracket_{[\rho|X_v \to \top\setminus A]}$, so
- A is the smallest such that $\top \setminus A = \llbracket P\rrbracket_{[\rho|X_v \to \top\setminus A]}$, so
- A^c is the biggest such that $A^c = \llbracket P\rrbracket_{[\rho|X_v \to A^c]}$, so
- $A^c = \mathrm{gfp}^{\subseteq}_{\emptyset} \lambda X.\ \llbracket P\rrbracket_{[\rho|X_v \to X]}$, so
- B exists and $A^c = B$

For the case $\mu X_v.P \equiv \neg\nu.\neg(P\{\neg X_v/X_v\})$, we proceed the same way. $\square$