Type Inference with structural Subtyping: the faithful description of an efficient constraint solver

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Talk Outline

- Structural subtyping
- Solving constraints
- Simplifying constraints
- Experimental results
Structural subtyping

- Solving constraints
- Simplifying constraints
- Implementation

Structural subtyping
Structural subtyping

What is subtyping?

- A partial order $\leq$ on types and a subsumption rule:
  \[
  \frac{\Gamma \vdash e : t_1 \quad t_1 \leq t_2}{\Gamma \vdash e : t_2}
  \]

- Widely used: object-oriented languages, static analysis, ...

- The definition of the subtyping order itself varies, depending on the application.
Structural subtyping: an example

Ground types

\[ t ::= \text{num} \mid t \rightarrow t \mid t \times t \]
Structural subtyping: an example

Atoms

\[ a \in \text{real} \quad \leq A \quad \text{complex} \quad \leq A \quad \text{fermat} \quad \leq A \quad \text{naturals} \]

Ground types

\[ t ::= \text{num } a \mid t \rightarrow t \mid t \times t \]

Subtyping order

\[
\begin{align*}
& a \leq_a a' \\
& \text{num } a \leq \text{num } a'
\end{align*}
\]

\[
\begin{align*}
& t'_1 \leq t_1 \\
& t_2 \leq t'_2 \\
& t_1 \rightarrow t_2 \leq t'_1 \rightarrow t'_2
\end{align*}
\]

\[
\begin{align*}
& t_1 \leq t'_1 \\
& t_2 \leq t'_2 \\
& t_1 \times t_2 \leq t'_1 \times t'_2
\end{align*}
\]
In structural subtyping, comparable types must have the same shape and can only differ by their atomic leaves.

- A poset of atoms \( a \in (\mathcal{A}, \leq_{\mathcal{A}}) \), supposed to be a lattice.
- A set of type constructors \( c \in \mathcal{C} \).
  
  Every type constructor is given with an arity, each parameter must be either covariant or contravariant.

- **Ground types** are defined by the following grammar:

\[
t ::= a \mid c(t_1, \ldots, t_n)
\]
Structural subtyping (2/2)

The subtyping order $\leq$ is defined by lifting the atomic order $\leq_\mathcal{A}$ along the type structure:

- If $c$'s $i^{th}$ parameter is covariant then $t_i \leq t'_i$
- If $c$'s $i^{th}$ parameter is contravariant then $t'_i \leq t_i$

$$a \leq_\mathcal{A} a' \quad \implies \quad a \leq a'$$
$$c(t_1, \ldots, t) \leq c(t'_1, \ldots, t')$$

We also define $t \approx t'$ (read: $t$ has the same shape as $t'$):

$$a \approx a' \quad \implies \quad \forall i \ t_i \approx t'_i \quad \implies \quad c(t_1, \ldots, t) \approx c(t'_1, \ldots, t')$$

$\approx$ is the symmetric transitive closure of $\leq$. 
## Structural subtyping

Type inference for structural subtyping has been widely studied

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<td>Mitchell</td>
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<td>Type inference is equivalent to constraint resolution</td>
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<td>Rehof</td>
<td>1997</td>
<td>The first-order theory of structural subtyping is decidable</td>
</tr>
<tr>
<td>Kuncak and Rinard</td>
<td>2003</td>
<td></td>
</tr>
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Our work

However, most of the implementation techniques remains in folklore. This paper proposes:

- A complete account about efficient techniques for constraint resolution in the case of structural subtyping.
- A generic implementation of an efficient constraint solver for structural subtyping.
Type inference systems for subtyping are constraint-based:

- Every piece of code is described by a constrained type scheme, which is computed by a syntactic analysis:

\[
\begin{align*}
\sigma & ::= \forall \bar{\alpha}[C].\tau \quad \text{(scheme)} \\
\tau & ::= \alpha \mid a \mid c(\tau_1, \ldots, \tau) \quad \text{(type)} \\
C & ::= \tau \leq \tau' \mid C_1 \land C_2 \mid \exists \alpha.C \quad \text{(constraint)}
\end{align*}
\]

- It is well-typed if and only if the type scheme has an instance, i.e. its constraint is satisfiable

- Type inference is reduced to constraint solving
Example

\[ \text{let } f = \lambda x. \lambda y (\text{bind } s = x + y \text{ in} \] 
\[ \quad \text{bind } p = x \times y \text{ in} \] 
\[ \quad (p + s, p - s)) \]

\[ \forall \alpha \left[ \exists \alpha_1 \alpha_2 \alpha_3. (\alpha_1 = \alpha_2 \rightarrow \alpha_3 \land \alpha_1 \leq \alpha \right] \]
\[ \land \exists \alpha_4 \alpha_5 \alpha_6. (\alpha_4 = \alpha_5 \rightarrow \alpha_6 \land \alpha_4 \leq \alpha_3 \land \beta_1 \gamma_1. (\gamma_1 = \text{num } \beta_1 \land \alpha_2 \leq \gamma_1 \land \alpha_5 \leq \gamma_1) \land \exists \beta_2 \gamma_2. (\gamma_2 = \text{num } \beta_2 \land \alpha_2 \leq \gamma_2 \land \alpha_5 \leq \gamma_2) \land \exists \beta_3 \gamma_3. (\gamma_3 = \text{num } \beta_3 \land \gamma_1 \leq \gamma_3 \land \gamma_2 \leq \gamma_3) \land \exists \beta_4 \gamma_4. (\gamma_4 = \text{num } \beta_4 \land \gamma_1 \leq \gamma_4 \land \gamma_2 \leq \gamma_4 \land \exists \alpha_7 \alpha_8 \gamma_5. (\gamma_3 \leq \alpha_7 \land \gamma_4 \leq \alpha_8 \land \gamma_5 = \alpha_7 \times \alpha_8 \land \gamma_5 \leq \alpha_6) \right) \]
Example

\[
\text{let } f = \lambda x.\lambda y(\text{bind } s = x + y \text{ in} \\
\quad \text{bind } p = x \times y \text{ in} \\
\quad (p + s, p - s))
\]

\[
\forall \alpha [\exists \beta. (\alpha = \text{num } \beta \to \text{num } \beta \to \text{num } \beta \times \text{num } \beta)] . \alpha
\]

\[
\forall \beta. \text{num } \beta \to \text{num } \beta \to \text{num } \beta \times \text{num } \beta
\]
Solving constraints
Overview

The solving procedure consists in three steps

- **Unification**: discovering sharing between type structures
- **Expansion and decomposition**: making type structure explicit and decomposing inequalities
- **Solving the resulting atomic problem**.
Example: unification (1/2)

Two unification algorithms are simultaneously performed:

- One for type equality (=)
- One for type structure (≈)

\[
\begin{align*}
\alpha_1 & = \alpha_2 \rightarrow \alpha_3 \\
\alpha_1 & \leq \alpha
\end{align*}
\]

\[
\frac{\langle \alpha_1 = \alpha_2 \rightarrow \alpha_3 \rangle \approx \alpha}{\alpha_1 \leq \alpha}
\]
Example: unification (2/2)

\[
\begin{align*}
\langle \alpha_1 = \alpha_2 \rightarrow \alpha_3 \rangle & \approx \alpha \\
\alpha_1 & \leq \alpha \\
\langle \alpha_4 = \alpha_5 \rightarrow \alpha_6 \rangle & \approx \alpha_3 \\
\alpha_4 & \leq \alpha_3 \\
\langle \gamma_5 = \alpha_7 \times \alpha_8 \rangle & \approx \alpha_6 \\
\gamma_5 & \leq \alpha_6 \\
\langle \gamma_1 = \text{num} \beta_1 \rangle & \approx \langle \gamma_2 = \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \text{num} \beta_4 \rangle \approx \\
\alpha_2 & \approx \alpha_5 \approx \alpha_7 \approx \alpha_8 \\
\alpha_2, \alpha_5 & \leq \gamma_1, \gamma_2 \land \gamma_1, \gamma_2 & \leq \gamma_3, \gamma_4 \land \gamma_3 & \leq \alpha_7 \land \gamma_4 \leq \alpha_8 \\
\beta_1 & \approx \beta_2 \approx \beta_3 \approx \beta_4
\end{align*}
\]
Solving constraints

Example: expansion and decomposition

\[
\langle \alpha_1 = \alpha_2 \rightarrow \alpha_3 \rangle \approx \langle \alpha = \alpha_9 \rightarrow \alpha_{10} \rangle
\]

\[
\alpha_1 \leq \alpha
\]

\[
\langle \alpha_4 = \alpha_5 \rightarrow \alpha_6 \rangle \approx \alpha_3 \approx \alpha_{10}
\]

\[
\alpha_3 \leq \alpha_{10} \land \alpha_4 \leq \alpha_3
\]

\[
\langle \gamma_5 = \alpha_7 \times \alpha_8 \rangle \approx \alpha_6
\]

\[
\gamma_5 \leq \alpha_6
\]

\[
\langle \gamma_1 = \text{num} \beta_1 \rangle \approx \langle \gamma_2 = \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \text{num} \beta_4 \rangle \approx
\]

\[
\alpha_2 \approx \alpha_5 \approx \alpha_7 \approx \alpha_8 \approx \alpha_9
\]

\[
\alpha_9 \leq \alpha_2 \land \alpha_5 \leq \gamma_1, \gamma_2 \land \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \land \gamma_3 \leq \alpha_7 \land \gamma_4 \leq \alpha_8
\]

\[
\beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4
\]
Example: expansion and decomposition

\[ \langle \alpha_1 = \alpha_2 \rightarrow \alpha_3 \rangle \approx \langle \alpha = \alpha_9 \rightarrow \alpha_{10} \rangle \]

\[ \langle \alpha_4 = \alpha_5 \rightarrow \alpha_6 \rangle \approx \langle \alpha_3 = \alpha_{11} \rightarrow \alpha_{12} \rangle \approx \langle \alpha_{10} = \alpha_{13} \rightarrow \alpha_{14} \rangle \]

\[ \alpha_3 \leq \alpha_{10} \land \alpha_4 \leq \alpha_3 \]

\[ \langle \gamma_5 = \gamma_7 \times \gamma_8 \rangle \approx \langle \gamma_6 \approx \alpha_{12} \approx \alpha_{14} \rangle \]

\[ \gamma_5 \leq \alpha_6 \land \alpha_{12} \leq \alpha_{14} \land \alpha_6 \leq \alpha_{12} \]

\[ \langle \gamma_1 = \text{num} \beta_1 \rangle \approx \langle \gamma_2 = \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \text{num} \beta_4 \rangle \approx \]

\[ \alpha_2 \approx \alpha_5 \approx \alpha_7 \approx \alpha_8 \approx \alpha_9 \approx \alpha_{11} \approx \alpha_{13} \]

\[ \alpha_9 \leq \alpha_2 \land \alpha_{13} \leq \alpha_{11} \land \alpha_{11} \leq \alpha_5 \land \alpha_2, \alpha_5 \leq \gamma_1, \gamma_2 \land \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \land \gamma_3 \leq \alpha_7 \land \gamma_4 \leq \alpha_8 \]

\[ \beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4 \]
Complexity

This strategy is theoretically exponential, but has a linear complexity under the hypothesis of types of bounded height.

We have to simplify constraints throughout the solving process:

► To improve its efficiency,
► To obtain human readable results.
Simplifying constraints
Simplifying constraints

Overview

Simplification is a subtle problem:

- It must be correct!
- It cannot be time consuming.

Simplification can be performed:

- Throughout the expansion process, in order to reduce the potential number of expanded variables.
- At the atomic level: useful in presence of let-polymorphism.

We are not interested by optimality (in size) of simplification techniques.
Polarities
[Fuh & Mishra (1989), Trifonov & Smith, Pottier]

Polarities assigned to type variables in a type scheme gives some information about their use:

A \underline{negative} type variable represents an \underline{input} of the expression
A \underline{positive} type variable represents an \underline{output} of the expression

Example:

\[ \forall \alpha_1 \alpha_2 \beta_1 \beta_2 [\alpha_1 \leq \beta_1 \land \alpha_2 \leq \beta_1 \land \alpha_2 \leq \beta_2]. \alpha_1 \rightarrow \alpha_2 \rightarrow \beta_1 \times \beta_2 \]

A \underline{negative} type variable can receive new \underline{lower bounds}
A \underline{positive} type variable can receive new \underline{upper bounds}
Chains reduction
[Eifrig et al. (1995), Aiken and Fähndrich (1996)]

Polarities can be exploited throughout expansion to eliminate variables:

A non-positive variable can be unified with its unique upper bound
A non-negative variable can be unified with its unique lower bound

Example:

\[ \forall \alpha_1 \alpha_2 \beta_1 \beta_2 [\alpha_1 \leq \beta_1 \land \alpha_2 \leq \beta_1 \land \alpha_2 \leq \beta_2]. \alpha_1 \rightarrow \alpha_2 \rightarrow \beta_1 \times \beta_2 \]

is equivalent to

\[ \forall \alpha_1 \alpha_2 \beta_1 \beta_2 [\alpha_1 \leq \beta_1 \land \alpha_2 \leq \beta_1]. \alpha_1 \rightarrow \alpha_2 \rightarrow \beta_1 \times \alpha_2 \]
Example: reducing a chain

\[\alpha \approx \langle \alpha_1 = \alpha_2 \rightarrow \alpha_3 \rangle\]
\[\alpha_1 \leq \alpha\]

\[\alpha_3 \approx \langle \alpha_4 = \alpha_5 \rightarrow \alpha_6 \rangle\]
\[\alpha_4 \leq \alpha_3\]

\[\alpha_6 \approx \langle \gamma_5 = \alpha_7 \times \alpha_8 \rangle\]
\[\gamma_5 \leq \alpha_6\]

\[\langle \gamma_1 = \text{num} \beta_1 \rangle \approx \langle \gamma_2 = \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \text{num} \beta_4 \rangle \approx \]
\[\alpha_2 \approx \alpha_5 \approx \alpha_7 \approx \alpha_8\]
\[\alpha_2, \alpha_5 \leq \gamma_1 \wedge \alpha_2, \alpha_5 \leq \gamma_2 \wedge \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \wedge \gamma_3 \leq \alpha_7 \wedge \gamma_4 \leq \alpha_8\]

\[\beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4\]
Example: propagating polarities

\[ \langle \alpha = \alpha_1 = \alpha_2 \rightarrow \alpha_3 \rangle \]

\[ \frac{\langle \alpha_4 = \alpha_5 \rightarrow \alpha_6 \rangle \approx \alpha_3}{\alpha_4 \leq \alpha_3} \]

\[ \frac{\langle \gamma_5 = \alpha_7 \times \alpha_8 \rangle \approx \alpha_6}{\gamma_5 \leq \alpha_6} \]

\[ \langle \gamma_1 = \text{num } \beta_1 \rangle \approx \langle \gamma_2 = \text{num } \beta_2 \rangle \approx \langle \gamma_3 = \text{num } \beta_3 \rangle \approx \langle \gamma_4 = \text{num } \beta_4 \rangle \approx \langle \gamma_5 = \text{num } \beta_5 \rangle \approx \langle \gamma_6 = \text{num } \beta_6 \rangle \approx \langle \gamma_7 = \text{num } \beta_7 \rangle \approx \langle \gamma_8 = \text{num } \beta_8 \rangle \]

\[ \frac{\alpha_2 \approx \alpha_5 \approx \alpha_7 \approx \alpha_8}{\alpha_2, \alpha_5 \leq \gamma_1 \land \alpha_2, \alpha_5 \leq \gamma_2 \land \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \land \gamma_3 \leq \alpha_7 \land \gamma_4 \leq \alpha_8} \]

\[ \beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4 \]
Example: reducing a chain

\[
\langle \alpha = \alpha_2 \rightarrow \alpha_3 \rangle
\]

\[
\alpha_3 \approx \langle \alpha_4 = \alpha_5 \rightarrow \alpha_6 \rangle
\]

\[
\alpha_4 \leq \alpha_3
\]

\[
\alpha_6 \approx \langle \gamma_5 = \alpha_7 \times \alpha_8 \rangle
\]

\[
\gamma_5 \leq \alpha_6
\]

\[
\langle \gamma_1 = \text{num } \beta_1 \rangle \approx \langle \gamma_2 = \text{num } \beta_2 \rangle \approx \langle \gamma_3 = \text{num } \beta_3 \rangle \approx \langle \gamma_4 = \text{num } \beta_4 \rangle \approx \alpha_2 \approx \alpha_5 \approx \alpha_7 \approx \alpha_8
\]

\[
\alpha_2, \alpha_5 \leq \gamma_1 \land \alpha_2, \alpha_5 \leq \gamma_2 \land \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \land \gamma_3 \leq \alpha_7 \land \gamma_4 \leq \alpha_8
\]

\[
\beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4
\]
Example: reducing a chain

\[ \langle \alpha = \alpha_2 \rightarrow \alpha_3 \rangle \]

\[ \langle \alpha_3 = \alpha_5 \rightarrow \alpha_6 \rangle \]

\[ \alpha_6 \approx \langle \gamma_5 = \alpha_7 \times \alpha_8 \rangle \]

\[ \gamma_5 \leq \alpha_6 \]

\[ \langle \gamma_1 = \text{num} \beta_1 \rangle \approx \langle \gamma_2 = \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \text{num} \beta_4 \rangle \approx \]

\[ \alpha_2 \approx \alpha_5 \approx \alpha_7 \approx \alpha_8 \]

\[ \alpha_2, \alpha_5 \leq \gamma_1, \alpha_2, \alpha_5 \leq \gamma_2, \gamma_1 \leq \gamma_3, \gamma_4 \leq \gamma_3 \leq \alpha_7 \land \gamma_4 \leq \alpha_8 \]

\[ \beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4 \]
Simplifying constraints

Reducing two chains

\[ \langle \alpha = \alpha_2 \rightarrow \alpha_3 \rangle \]

\[ \langle \alpha_3 = \alpha_5 \rightarrow \alpha_6 \rangle \]

\[ \langle \alpha_6 = \alpha_7 \times \alpha_8 \rangle \]

\[ \langle \gamma_1 = \text{num } \beta_1 \rangle \approx \langle \gamma_2 = \text{num } \beta_2 \rangle \approx \langle \gamma_3 = \text{num } \beta_3 \rangle \approx \langle \gamma_4 = \text{num } \beta_4 \rangle \approx \]

\[ \alpha_2 \approx \alpha_5 \approx \alpha_7 \approx \alpha_8 \]

\[ \alpha_2, \alpha_5 \leq \gamma_1, \gamma_2 \wedge \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \wedge \gamma_3 \leq \alpha_7 \wedge \gamma_4 \leq \alpha_8 \]

\[ \beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4 \]
Simplifying constraints

Example: reducing two chains

\[ \langle \alpha = \alpha_2 \rightarrow \alpha_3 \rangle \]

\[ \langle \alpha_3 = \alpha_5 \rightarrow \alpha_6 \rangle \]

\[ \langle \alpha_6 = \alpha_7 \times \alpha_8 \rangle \]

\[ \langle \gamma_1 = \text{num} \beta_1 \rangle \approx \langle \gamma_2 = \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \alpha_7 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \alpha_8 = \text{num} \beta_4 \rangle \approx \langle \alpha_2 \rangle \approx \langle \alpha_5 \rangle \approx \alpha_2, \alpha_5 \leq \gamma_1, \gamma_2 \land \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \]

\[ \beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4 \]
Simplifying constraints

Example: expanding variables

\[ \langle \alpha = \alpha_2 \rightarrow \alpha_3 \rangle \]

\[ \langle \alpha_3 = \alpha_5 \rightarrow \alpha_6 \rangle \]

\[ \langle \alpha_6 = \alpha_7 \times \alpha_8 \rangle \]

\[ \langle \gamma_1 = \text{num} \beta_1 \rangle \approx \langle \gamma_2 = \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \alpha_7 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \alpha_8 = \text{num} \beta_4 \rangle \approx \]

\[ \langle \alpha_2 = \text{num} \beta_5 \rangle \approx \langle \alpha_5 = \text{num} \beta_6 \rangle \]

\[ \alpha_2, \alpha_5 \leq \gamma_1, \gamma_2 \wedge \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \]

\[ \beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4 \approx \beta_5 \approx \beta_6 \]
Simplifying constraints

Example: decomposing inequalities

\[ \langle \alpha = \alpha_2 \rightarrow \alpha_3 \rangle \]

\[ \langle \alpha_3 = \alpha_5 \rightarrow \alpha_6 \rangle \]

\[ \langle \alpha_6 = \alpha_7 \times \alpha_8 \rangle \]

\[ \langle \gamma_1 \geq \text{num} \beta_1 \rangle \approx \langle \gamma_2 \geq \text{num} \beta_2 \rangle \approx \langle \gamma_3 = \alpha_7 = \text{num} \beta_3 \rangle \approx \langle \gamma_4 = \alpha_8 = \text{num} \beta_4 \rangle \approx \langle \alpha_2 = \text{num} \beta_5 \rangle \approx \langle \alpha_5 = \text{num} \beta_6 \rangle \]

\[ \alpha_2, \alpha_5 \leq \gamma_1, \gamma_2 \land \gamma_1, \gamma_2 \leq \gamma_3, \gamma_4 \]

\[ \beta_1 \approx \beta_2 \approx \beta_3 \approx \beta_4 \approx \beta_5 \approx \beta_6 \]

\[ \beta_5, \beta_6 \leq \gamma_1, \gamma_2 \land \gamma_1, \gamma_2 \leq \beta_3, \beta_4 \]
Summary of expansion

- On this example, expansion introduces only 2 variables.
- In the absence of simplification, they would have been 25.
- It remains to simplify the atomic graph.
Example: polarised closure
[Fuh & Mishra (1989)]

Polarised closure removes non-polar variables in the atomic graph, and keeps only paths from negative variables to positive ones.
Example: minimisation

Two \(\beta_6\) \(\beta_5\) negative variables with the same \(\beta_4\) \(\beta_3\) successors are equivalent positive predecessors.

\[
\begin{align*}
\beta_6 & = \beta_5 \\
\beta_4 & = \beta_3 \\
\beta_6 & = \beta_4 \\
\beta_5 & = \beta_3
\end{align*}
\]
Example: final result

\[
\begin{align*}
\langle \alpha = \alpha_2 \rightarrow \alpha_3 \rangle \\
\langle \alpha_3 = \alpha_5 \rightarrow \alpha_6 \rangle \\
\forall \alpha. [\exists \ldots]. \alpha \\
\forall \alpha[\exists \beta_5.(\alpha = \text{num } \beta_5 \rightarrow \text{num } \beta_5 \rightarrow \text{num } \beta_5 \times \text{num } \beta_5)]. \alpha - \\
\forall \beta_5. \text{num } \beta_5 \rightarrow \text{num } \beta_5 \rightarrow \text{num } \beta_5 \times \text{num } \beta_5 \\
\langle \alpha_2 = \alpha_5 = \alpha_7 = \alpha_8 = \text{num } \beta_5 \rangle \\
\langle \beta_5 = \beta_6 = \beta_3 = \beta_4 \rangle
\end{align*}
\]
Implementation
The Dalton Library

- The **Dalton Library** is a real-size implementation in Objective Caml of these algorithms.
  
  http://cristal.inria.fr/~simonet/soft/dalton/

- It comes as a functor parametrized by a series of modules describing the client’s type system.

- It has been used within the **Flow Caml System**, an information flow analyzer for the Caml Language.
  
  http://cristal.inria.fr/~simonet/soft/flowcaml/
### Experimental results

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<th>Typing the code of Caml Light library</th>
<th>Compiler</th>
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<tr>
<td>A.s.t. nodes</td>
<td>14002</td>
<td>22996</td>
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<tr>
<td>Unification solver</td>
<td>0.346 s</td>
<td>0.954 s</td>
</tr>
<tr>
<td>Structural subtyping solver</td>
<td>0.966 s</td>
<td>2.213 s</td>
</tr>
<tr>
<td></td>
<td>2.79</td>
<td>2.31</td>
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</table>
Conclusion

- An **efficient** implementation type inference engine for structural subtyping
- whose correctness is **proved**.