On dp-minimal ordered structures

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Abstract

We show basic facts about dp-minimal ordered structures. The main results are: dp-minimal groups are abelian-by-finite-exponent, in a divisible ordered dp-minimal group, any infinite set has non-empty interior, and any theory of pure tree is dp-minimal.

Introduction

One of the latest topics of interest in pure model theory is the study of dependent, or NIP, theories. The abstract general study was initiated by Shelah in [14], and pursued by him in [15], [13] and [12]. One of the questions he addresses is the definition of *super-dependent* as an analog of superstable for stable theories. Although, as he writes, he has not completely succeeded, the notion he defines of strong dependence seems promising. In [13] it is studied in detail and in particular, ranks are defined. Those so-called dp-ranks are used to prove existence of an indiscernible sub-sequence in any long enough sequence. Roughly speaking, a theory is strongly dependent if no type can fork infinitely many times, each forking being independent from the previous one. (Stated this way, it is naturally a definition of "strong NTP_2"). Also defined in that paper are notions of minimality, corresponding to the ranks being equal to 1 on 1-types. In [7], Onshuus and Usvyatsov extract from this material the notion of dp-minimality which seems to be the relevant one. A dp-minimal theory is a theory where there cannot be two independent witnesses of forking for a 1-type. It is shown in that paper that a stable theory is dp-minimal if and only if every 1-type has weight 1. In general, unstable, theories, one can link dp-minimality to *burden* as defined by H. Adler ([1]).

Dp-minimality on ordered structures can be viewed as a generalization of weak o-minimality. In that context, there are two main questions to address:

what do definable sets in dimension 1 look like, (*i.e.* how far is the theory from being o-minimal), and what theorems about o-minimality go through. J. Goodrick has started to study those questions in [5], focussing on groups. He proves that definable functions are piecewise locally monotonous extending a similar result from weak-o-minimality.

In the first section of this paper, we recall the definitions and give equivalent formulations. In the second section, we make a few observations on general linearly ordered inp-minimal theories showing in particular that, in dimension 1, forking is controlled by the ordering. The lack of a cell-decomposition theorem makes it unclear how to generalize results to higher dimensions.

In section 3, we study dp-minimal groups and show that they are abelian-by-finite-exponent. The linearly ordered ones are abelian. We prove also that an infinite definable set in a dp-minimal ordered divisible group has non-empty interior, solving a conjecture of A. Dolich.

Finally, in section 4, we give examples of dp-minimal theories. We prove that colored linear orders, orders of finite width and trees are dp-minimal.

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1 Preliminaries on dp-minimality

Definition 1.1. (Shelah) An independence (or inp-) pattern of length κ is a sequence of pairs $(\phi^\alpha(x,y),k^\alpha)_{\alpha<\kappa}$ of formulas such that there exists an array $\langle a_i^\alpha : \alpha<\kappa, i<\lambda\rangle$ for some $\lambda\geq\omega$ such that:

- Rows are k^α-inconsistent: for each $\alpha<\kappa$, the set $\{\phi^\alpha(x,a_i^\alpha) : i<\lambda\}$ is k^α-inconsistent,
- paths are consistent: for all $\eta\in\lambda^\kappa$, the set $\{\phi^\alpha(x,a^\alpha_{\eta(\alpha)}) : \alpha<\kappa\}$ is consistent.

Definition 1.2.
- (Goodrick) A theory is inp-minimal if there is no inp-pattern of length two in a single free variable x.

- (Onshuus and Usvyatsov) A theory is dp-minimal if it is NIP and inp-minimal.

A theory is NTP_2 if there is no inp-pattern of size ω for which the formulas $\phi^\alpha(x,y)$ in the definition above are all equal to some $\phi(x,y)$. It is proven in [2] that a theory is NTP_2 if this holds for formulas $\phi(x,y)$ where x is a single variable. As a consequence, any inp-minimal theory is NTP_2.

We now give equivalent definitions (all the ideas are from [13], we merely adapt the proofs there from the general NIP context to the dp-minimal one).

Definition 1.3. Two sequences $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ are *mutually indiscernible* if each one is indiscernible over the other.

Lemma 1.4. *Consider the following statements:*

1. *T is inp-minimal.*
2. *For any two mutually indiscernible sequences $A = (a_i : i < \omega)$, $B = (b_j : j < \omega)$ and any point c, one of the sequences $(tp(a_i/c) : i < \omega)$, $(tp(b_i/c) : i < \omega)$ is constant.*
3. *Same as above, but change the conclusion to: one the sequences A or B stays indiscernible over c.*
4. *For any indiscernible sequence $A = (a_i : i \in I)$ indexed by a dense linear order I, and any point c, there is i_0 in the completion of I such that the two sequences $(tp(a_i/c) : i < i_0)$ and $(tp(a_i/c) : i > i_0)$ are constant.*
5. *Same as above, but change the conclusion to: the two sequences $(a_i : i < i_0)$ and $(a_i : i > i_0)$ are indiscernible over c.*
6. *T is dp-minimal.*

Then for any theory T, (2), (3), (4), (5), (6) are equivalent and imply (1). If T is NIP, then they are all equivalent.

Proof. (2) $\Rightarrow$ (1): In the definition of independence pattern, one may assume that the rows are mutually indiscernible. This is enough.

(2) $\Rightarrow$ (3): Assume $A = \langle a_i : i < \omega\rangle$, $B = \langle b_i : i < \omega\rangle$ and c are a witness to $\neg(3)$. Then there are two tuples $(i_1 < ... < i_n)$, $(j_1 < ... < j_n)$ and a formula $\phi(x; y_1, ..., y_n)$ such that $\models \phi(c; a_{i_1}, ..., a_{i_n}) \wedge \neg\phi(c; a_{j_1}, ..., a_{j_n})$. Take

an $\alpha < \omega$ greater than all the i_k and the j_k. Then, exchanging the i_k and j_k if necessary, we may assume that $\models \phi(c; a_{i_1}, ..., a_{i_n}) \wedge \neg\phi(c; a_{n.\alpha}, ..., a_{n.\alpha+n-1})$. Define $A' = \langle (a_{i_1}, ..., a_{i_n}) \rangle \,\hat{}\, \langle (a_{n.k}, ..., a_{n.k+n-1}) : k \geq \alpha \rangle$. Construct the same way a sequence B'. Then A', B', c give a witness of $\neg(2)$.

(3) $\Rightarrow$ (2): Obvious.

(3) $\Rightarrow$ (5): Let $A = \langle a_i : i \in I \rangle$ be indiscernible and let c be a point. Then assuming (3) holds, for every i_0 in the completion of I, one of the two sequences $A_{<i_0} = \langle a_i : i < i_0 \rangle$ and $A_{>i_0} = \langle a_i : i > i_0 \rangle$ must be indiscernible over c. Take any such i_0 such that both sequences are infinite, and assume for example that $A_{>i_0}$ is indiscernible over c. Let $j_0 = \inf\{i \leq i_0 : A_{>i}$ is indiscernible over c $\}$. Then $A_{>j_0}$ is indiscernible over c. If there are no elements in I smaller than j_0, we are done. Otherwise, if $A_{<j_0}$ is not indiscernible over c, then one can find $j_1 < j_0$ such that again $A_{<j_1}$ is not indiscernible over c. By definition of j_0, $A_{>j_1}$ is not indiscernible over c either. This contradicts (3).

(5) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (2): Assume $\neg(2)$. Then one can find a witness of it consisting of two indiscernible sequences $A = \langle a_i : i \in I \rangle$, $B = \langle b_i : i \in I \rangle$ indexed by a dense linear order I and a point c.

Now, we can find an i_0 in the completion of I such that for any $i_1 < i_0 < i_2$ in I, there are i, i', $i_1 < i < i_0 < i' < i_2$ such that $tp(a_i/c) \neq tp(a_{i'}/c)$. Find a similar point j_0 for the sequence B. Renumbering the sequences if necessary, we may assume that $i_0 \neq j_0$. Then the indiscernible sequence of pairs $\langle (a_i, b_i) : i \in I \rangle$ gives a witness of $\neg$ (4).

(6) $\Rightarrow$ (2): Let A, B, c be a witness of $\neg(2)$. Assume for example that there is $\phi(x, y)$ such that $\models \phi(c, a_0) \wedge \neg\phi(c, a_1)$. Then set $A' = \langle (a_{2k}, a_{2k+1}) : k < \omega \rangle$ and $\phi'(x; y_1, y_2) = \phi(x; y_1) \wedge \neg\phi(x; y_2)$. Then by NIP, the set $\{\phi'(x, \bar{y}) : \bar{y} \in A'\}$ is k-inconsistent for some k. Doing the same construction with B we see that we get an independence pattern of length 2.

(5) $\Rightarrow$ (6): Statement (5) clearly implies NIP (because IP is always witnessed by a formula $\phi(x, y)$ with x a single variable). We have already seen that it implies inp-minimality. □

Standard examples of dp-minimal theories include:

- O-minimal or weakly o-minimal theories (recall that a theory is weakly-o-minimal if every definable set in dimension 1 is a finite union of convex sets),
- C-minimal theories,

- $Th(\mathbf{Z}, +, \leq)$,
- The theory of the p-adics.

We refer the reader to [4] for more details and some proofs.

More examples are given in section 4 of this paper.

2 Inp-minimal ordered structures

Little study has been made yet on general dp-minimal ordered structures. We believe however that there are results to be found already at that general level. In fact, we prove here a few lemmas that turn out to be useful for the study of groups.

We show that, in some sense, forking in dimension 1 is controlled by the order.

We consider $(M, <)$ an inp-minimal linearly ordered structure with no first nor last element. We denote by T its theory, and let $\mathbb{M}$ be a monster model of T.

Lemma 2.1. *Let $X = X_{\bar{a}}$ be a definable subset of $\mathbb{M}$, cofinal in $\mathbb{M}$. Then X is non-forking (over $\emptyset$).*

Proof. If $X_{\bar{a}}$ divides over $\emptyset$, there exists an indiscernible sequence $(\bar{a}_i)_{i<\omega}$, $\bar{a}_0 = \bar{a}$, witnessing this. Every $X_{\bar{a}_i}$ is cofinal in $\mathbb{M}$. Now pick by induction intervals I_k, $k < \omega$, with $I_k < I_{k+1}$ containing a point in each $X_{\bar{a}_i}$. We obtain an inp-pattern of length 2 by considering $x \in X_{\bar{a}_i}$ and $x \in I_k$.

If $X_{\bar{a}}$ forks over $\emptyset$, it implies a disjunction of formulas that divide, but one of these formulas must be cofinal: a contradiction. □

A few variations are possible here. For example, we assumed that X was cofinal in the whole structure $\mathbb{M}$, but the proof also works if X is cofinal in a $\emptyset$-definable set Y, or even contains an $\emptyset$-definable point in its closure. This leads to the following results.

For X a definable set, let $Conv(X)$ denote the convex hull of X. It is again a definable set.

Porism 2.2. *Let X be a definable set of $\mathbb{M}$ (in dimension 1). Assume $Conv(X)$ is A definable. Then X is non-forking over A.*

Porism 2.3. *Let $M \prec N$ and let p be a complete 1-type over N. If the cut of p over N is of the form $+\infty$, $-\infty$, a^+ or a^- for $a \in M$, then p is non-forking over M.*

Proposition 2.5 generalizes this.

Lemma 2.4. *Let X be an A-definable subset of $\mathbb{M}$. Assume that X divides over some model M, then:*

1. *We cannot find $(a_i)_{i<\omega}$ in M and points $(x_i)_{i<\omega}$ in $X(\mathbb{M})$ such that $a_0 < x_0 < a_1 < x_1 < a_2 < \ldots.$*

2. *The set X can be written as a finite disjoint union $X = \bigcup X_i$ where the X_i are definable over $M \cup A$, and each $Conv(X_i)$ contains no M-point.*

Proof. Easy; (2) follows from (1). □

Proposition 2.5. *Let $A \subset M$, with M, $|A|^+$-saturated, and let $p \in S_1(M)$. The following are equivalent:*

1. *The type p forks over A,*

2. *There exist $a, b \in M$ such that $p \vdash a < x < b$, and a and b have the same type over A,*

3. *There exist $a, b \in M$ such that $p \vdash a < x < b$, and the interval $I_{a,b} = \{x : a < x < b\}$ divides over A.*

Proof. (3) $\Rightarrow$ (1) is trivial.

For (2) $\Rightarrow$ (3), it is enough to show that if $a \equiv_A b$, then $I_{a,b}$ divides over A. Let σ be an A-automorphism sending a to b. Then the tuple $(b = \sigma(a), \sigma(b))$ has the same type as (a, b), and $a < b < \sigma(b)$. By iterating, we obtain a sequence $a_1 < a_2 < \ldots$ such that (a_k, a_{k+1}) has the same type over A as (a, b). Now the sets $I_{a_{2k}, a_{2k+1}}$ are pairwise disjoint and all have the same type over A. Therefore each of them divides over M.

We now prove (1) $\Rightarrow$ (2)

Assume that (2) fails for p. Let $X_{\bar{a}}$ be an M-definable set such that $p \vdash X_{\bar{a}}$. Let $\bar{a}_0 = a, \bar{a}_1, \bar{a}_2, \ldots$ be an A-indiscernible sequence. Note that the cut of p is invariant under all A-automorphisms. Therefore each of the $X_{\bar{a}_i}$ contains a type with the same cut over M as p. Now do a similar reasoning as in Lemma 2.1. □

Corollary 2.6. *Forking equals dividing: for any $A \subset B$, any $p \in S(B)$, p forks over A if and only if p divides over A.*

Proof. By results of Chernikov and Kaplan ([3]), it is enough to prove that no type forks over its base. And it suffices to prove this for one-types (because of the general fact that if $tp(a/B)$ does not fork over A and $tp(b/Ba)$ does not fork over Aa, then $tp(a,b/B)$ does not fork over A).

Assume $p \in S_1(A)$ forks over A. Then by the previous proposition, p implies a finite disjunction of intervals $\bigcup_{i<n}(a_i, b_i)$ with $a_i \equiv_A b_i$. Assume n is minimal. Without loss, assume $a_0 < a_1 < \dots$. Now, as $a_0 \equiv_A b_0$ we can find points a'_i, b'_i, with $(a_i, b_i) \equiv_A (a'_i, b'_i)$ and $a'_0 = b_0$.

Then p proves $\bigcup_{i<n}(a'_i, b'_i)$. But the interval (a_0, b_0) is disjoint from that union, so p proves $\bigcup_{0<i<n}(a_i, b_i)$, contradicting the minimality of n. □

Note that this does not hold without the assumption that the structure is linearly ordered. In fact the standard example of the circle with a predicate $C(x, y, z)$ saying that y is between x and z (see for example [17], 2.2.4.) is dp-minimal.

Lemma 2.7. *Let E be a definable equivalence relation on M, we consider the imaginary sort $S = M/E$. Then there is on S a definable equivalence relation $\sim$ with finite classes such that there is a definable linear order on $S/\sim$.*

Proof. Define a partial order on S by $a/E \prec b/E$ if $\inf(\{x : xEa\}) < \inf(\{x : xEb\})$. Let $\sim$ be the equivalence relation on S defined by $x \sim y$ if $\neg(x \prec y \vee y \prec x)$. Then $\prec$ defines a linear order on $S/\sim$. The proof that $\sim$ has finite classes is another variation on the proof of 2.1. □

From now until the end of this section, we also assume NIP.

Recall that in an NIP theory, if a type p splits over a model M, then it forks over M. In other words, if a, a' have the same type over M, then the formula $\phi(x, a) \triangle \phi(x, a')$ forks over M. (Note that the converse: "if p forks over M, then it splits over M" is true in any theory.)

Lemma 2.8. *(NIP). Let $p \in S_1(\mathbb{M})$ be a type inducing an M-definable cut, then p is definable over M.*

Proof. We know that p does not fork over M, so by NIP, p does not split over M. Let M_1 be an $|M|^+$-saturated model containing M. Then the restriction of p to M_1 has a unique M-invariant extension. Therefore by NIP, it has a unique global extension that does not fork over M. This in turn implies by 2.5 that $p|_{M_1}$ has a unique global extension inducing the same cut as p, in particular it has a unique heir.

Therefore p is definable, and being M-invariant, p is definable over M. $\square$

The next lemma states that members of a uniformly definable family of sets define only finitely many "germs at $+\infty$".

Lemma 2.9. *(NIP). Let $\phi(x,y)$ be a formula with parameters in some model M_0, x a single variable. Then there are $b_1, ..., b_n$ such that for every b, there is $\alpha \in \mathbb{M}$ and k such that the sets $\phi(x,b) \wedge x > \alpha$ and $\phi(x,b_k) \wedge x > \alpha$ are equal.*

Proof. Let E be the equivalence relation defined on tuples by bEb' iff $(\exists\alpha)(x > \alpha \rightarrow (\phi(x,b) \leftrightarrow \phi(x,b')))$. Let b, b' having the same type over M_0. By NIP, the formula $\phi(x,b)\triangle\phi(x,b')$ forks over M_0. By Lemma 2.1, this formula cannot be cofinal, so b and b' are E-equivalent. This proves that E has finitely many classes. $\square$

If the order is dense, then this analysis can be done also locally around a point a with the same proof:

Lemma 2.10. *(NIP + dense order). Let $\phi(x,y)$ be a formula with parameters in some model M_0, x a single variable. Then there exists n such that: For any point a, there are $b_1, ..., b_n$ such that for all b, there is $\alpha < a < \beta$ and k such that the sets $\phi(x,b) \wedge \alpha < x < \beta$ and $\phi(x,b_k) \wedge \alpha < x < \beta$ are equal.*

3 Dp-minimal groups

We study inp-minimal groups. Note that by an example of Simonetta, ([16]), not all such groups are abelian-by-finite. It is proven in [6] that C-minimal groups are abelian-by-torsion. We generalize the statement here to all inp-minimal theories.

Proposition 3.1. *Let G be an inp-minimal group. Then there is a definable normal abelian subgroup H such that G/H is of finite exponent.*

Proof. Let A, B be two definable subgroups of G. If $a \in A$ and $b \in B$, then there is $n > 0$ such that either $a^n \in B$ or $b^n \in A$. To see this, assume $a^n \notin B$ and $b^n \notin A$ for all $n > 0$. Then, for $n \neq m$, the cosets $a^m B$ and $a^n B$ are distinct, as are $A.b^m$ and $A.b^n$. Now we obtain an independence pattern of length two by considering the sequences of formulas $\phi_k(x) =$ "$x \in a^k B$" and $\psi_k(x) =$ "$x \in A.b^k$".

For $x \in G$, let $C(x)$ be the centralizer of x. By compactness, there is k such that for $x, y \in G$, for some $k' \leq k$, either $x^{k'} \in C(y)$ or $y^{k'} \in C(x)$. In particular, letting $n = k!$, x^n and y^n commute.

Let $H = C(C(G^n))$, the bicommutant of the nth powers of G. It is an abelian definable subgroup of G and for all $x \in G$, $x^n \in H$. Finally, if H contains all n powers then it is also the case of all conjugates of H, so replacing H by the intersection of its conjugates, we obtain what we want. $\square$

Now we work with ordered groups.

Note that in such a group, the convex hull of a subgroup is again a subgroup.

Lemma 3.2. *Let G be an inp-minimal ordered group. Let H be a definable subgroup of G and let C be the convex hull of H. Then H is of finite index in C.*

Proof. We may assume that H and C are $\emptyset$-definable. So without loss, assume $C = G$.

If H is not of finite index, there is a coset of H that forks over $\emptyset$. All cosets of H are cofinal in G. This contradicts Lemma 2.1. $\square$

Proposition 3.3. *Let G be an inp-minimal ordered group, then G is abelian.*

Proof. Note that if $a, b \in G$ are such that $a^n = b^n$, then $a = b$, for if for example $0 < a < b$, then $a^n < a^{n-1}b < a^{n-2}b^2 < ... < b^n$.

For $x \in G$, let $C(x)$ be the centralizer of x. We let also $D(x)$ be the convex hull of $C(x)$. By 3.2, $C(x)$ is of finite index in $D(x)$. Now take $x \in G$ and $y \in D(x)$. Then xy is in $D(x)$, so there is n such that $(xy)^n \in C(x)$. Therefore $(yx)^n = x^{-1}(xy)^n x = (xy)^n$. So $xy = yx$ and $y \in C(x)$. Thus $C(x) = D(x)$ is convex.

Now if $0 < x < y \in G$, then $C(y)$ is a convex subgroup containing y, so it contains x, and x and y commute. $\square$

This answers a question of Goodrick ([5] 1.1).

Now, we assume NIP, so G is a dp-minimal ordered group. We denote by G^+ the set of positive elements of G.

Let $\phi(x)$ be a definable set (with parameters). For $\alpha \in G$, define $X_\alpha = \{g \in G^+ : (\forall x > \alpha)(\phi(x) \leftrightarrow \phi(x+g))\}$. Let H_α be equal to $X_\alpha \cup -X_\alpha \cup \{0\}$.

Then H_α is a definable subgroup of G and if $\alpha < \beta$, H_α is contained in H_β. Finally, let H be the union of the H_α for $\alpha \in G$, it is the subgroup of *eventual periods* of $\phi(x)$.

Now apply Lemma 2.9 to the formula $\psi(x, y) = \phi(x - y)$. It gives n points $b_1, ..., b_n$ such that for all $b \in G$, there is k such that $b - b_k$ is in H. This implies that H has finite index in G.

If furthermore G is densely ordered, then we can do the same analysis locally. This yields a proof of a conjecture of A. Dolich: in a dp-minimal divisible ordered group, any infinite set has non empty interior. As a consequence, a dp-minimal divisible definably complete ordered group is o-minimal.

We will make use of two lemmas from [5] that we recall here for convenience.

Lemma 3.4 ([5], 3.3)**.** *Let G be a densely ordered inp-minimal group, then any infinite definable set is dense in some non trivial interval.*

In the following lemma, $\overline{G}$ stands for the completion of G. By a definable function f into $\overline{G}$, we mean a function of the form $a \mapsto \inf \phi(a; G)$ where $\phi(x; y)$ is a definable function. So one can view $\overline{G}$ as a collection of imaginary sorts (in which case it naturally contains only *definable* cuts of G), or understand $f : G \to \overline{G}$ simply as a notation.

Lemma 3.5 (special case of [5], 3.19)**.** *Let G be a densely ordered group, $f : G \to \overline{G}$ be a definable partial function such that $f(x) > 0$ for all x in the domain of f. Then for every interval I, there is a sub-interval $J \subseteq I$ and $\epsilon > 0$ such that for $x \in J \cap \mathrm{dom}(f)$, $|f(x)| \geq \epsilon$.*

Theorem 3.6. *Let G be a divisible ordered dp-minimal group. Let X be an infinite definable set, then X has non-empty interior.*

Proof. As before, $I_{a,b}$ denotes the open interval $a < x < b$, and τ_b is the translation by $-b$.

Let $\phi(x)$ be a formula defining X.

By Lemma 3.4, there is an interval I such that X is dense in I. By Lemma 2.10 applied to $\psi(x; y) = \phi(y + x)$ at 0, there are $b_1, ..., b_n \in M$ such that for all $b \in M$, there is $\alpha > 0$ and k such that $|x| < \alpha \to (\phi(b + x) \leftrightarrow \phi(b_k + x))$.

Taking a smaller I and X, if necessary, assume that for all $b \in I \cap X$, we may take $k = 1$.

Define $f : x \mapsto \sup\{y : I_{-y,y} \cap \tau_{b_1} X = I_{-y,y} \cap \tau_x X\}$, it is a function into $\overline{M}$, the completion of M. By Lemma 3.5, there is $J \subset I$ such that, for all $b \in J$, we have $|f(b)| \geq \epsilon$.

Fix $\nu < \frac{\epsilon}{2}$ and $b \in J$ such that $I_{b-2\epsilon,b+2\epsilon} \subseteq J$ (taking smaller ϵ if necessary). Set $L = I_{b-\nu,b+\nu}$ and $Z = L \cap X$. Assume for simplicity $b = 0$. Easily, if $g_1, g_2 \in Z$, then $g_1 + g_2 \in Z \cup (G \setminus L)$ and $-g_1 \in Z$ (because the two points 0 and g_1 in Z have isomorphic neighborhoods of size ϵ). So Z is a group interval: it is the intersection with $I_{b-\nu,b+\nu}$ of some subgroup H of G. Now if $x, y \in L$ satisfy that there is $\alpha > 0$ such that $I_{-\alpha,\alpha} \cap \tau_x X = I_{-\alpha,\alpha} \cap \tau_y X$, then $x \equiv y$ modulo H. It follows that points of L lie in finitely many cosets modulo H. Assume Z is not convex, and take $g \in L \setminus Z$. Then for each $n \in \mathbf{N}$, the point g/n is in L and the points g/n define infinitely many different cosets; a contradiction.

Therefore Z is convex and X contains a non trivial interval. □

Corollary 3.7. *Let G be a dp-minimal ordered group. Assume G is divisible and definably complete, then G is o-minimal.*

Proof. Let X be a definable subset of G. By 3.6, the (topological) border Y of X is finite.

Let $a \in X$, then the largest convex set in X containing a is definable. By definable completeness, it is an interval and its end-points must lie in Y. As Y is finite, X is a finite union of (closed or open) intervals. □

4 Examples of dp-minimal theories

We give examples of dp-minimal theories, namely: linear orders, order of finite width and trees.

We first look at linear orders. We consider structures of the form $(M, \leq, C_i, R_j)$ where $\leq$ defines a linear order on M, the C_i are unary predicates ("colors"), the R_j are binary monotone relations (that is $x_1 \leq x R_j y \leq y_1$ implies $x_1 R_j y_1$).

The following is a (weak) generalization of Rubin's theorem on linear orders (see [10], or [9]).

Proposition 4.1. *Let $(M, \leq, C_i, R_j)$ be a colored linear order with monotone relations. Assume that all $\emptyset$-definable sets in dimension 1 are coded by a color and all monotone $\emptyset$-definable binary relations are represented by one of the R_j. Then the structure eliminates quantifiers.*

Proof. The result is obvious if M is finite, so we may assume (for convenience) that this is not the case.

We prove the theorem by back-and-forth. Assume that M is ω-saturated and take two tuples $\bar{x} = (x_1, ..., x_n)$ and $\bar{y} = (y_1, ..., y_n)$ from M having the same quantifier free type.

Take $x_0 \in M$; we look for a corresponding y_0. Notice that $\leq$ is itself a monotone relation, a finite boolean combinations of colors is again a color, a positive combination of monotone relations is again a monotone relation, and if xRy is monotone $\phi(x, y) = \neg yRx$ is monotone. By compactness, it is enough to find a y_0 satisfying some finite part of the quantifier-free type of x_0; that is, we are given

- One color C such that $M \models C(x_0)$,
- For each k, monotone relations R_k and S_k such that $M \models x_0 R_k x_k \wedge x_k S_k x_0$.

Define $U_k(x) = \{t : tR_k x_k\}$ and $V_k(x) = \{t : xS_k t\}$. The $U_k(x)$ are initial segments of M and the $V_k(x)$ final segments. For each k, k', either $U_k(x_k) \subseteq U_{k'}(x_{k'})$ or $U_{k'}(x_{k'}) \subseteq U_k(x_k)$. Assume for example $U_k(x_k) \subseteq U_{k'}(x_{k'})$, then this translates into a relation $\phi(x_k, x_{k'})$, where $\phi(x, y) = (\forall t)(tR_k x \rightarrow tR_{k'} y)$. Now $\phi(x, y)$ is a monotone relation itself. The assumptions on $\bar{x}$ and $\bar{y}$ therefore imply that also $U_k(y_k) \subseteq U_{k'}(y_{k'})$.

The same remarks hold for the final segments V_k.

Now, we may assume that $U_1(x_1)$ is minimal in the $U_k(x_k)$ and $V_l(x_l)$ is minimal in the $V_k(x_k)$. We only need to find a point y_0 satisfying $C(x)$ in the intersection $U_1(y_1) \cap V_l(y_l)$.

Let $\psi(x, y)$ be the relation $(\exists t)(C(t) \wedge tR_1 y \wedge xR_l t)$. This is a monotone relation. As it holds for (x_0, x_l), it must also hold for (y_0, y_l), and we are done. □

The following result was suggested, in the case of pure linear orders, by John Goodrick.

Proposition 4.2. *Let $\mathcal{M} = (M, \leq, C_i, R_j)$ be a linearly ordered infinite structure with colors and monotone relations. Then $Th(\mathcal{M})$ is dp-minimal.*

Proof. By the previous result, we may assume that $T = Th(\mathcal{M})$ eliminates quantifiers. Let $(x_i)_{i \in I}$, $(y_i)_{i \in I}$ be mutually indiscernible sequences of n-tuples, and let $\alpha \in M$ be a point. We want to show that one of the following holds:

- For all $i, i' \in I$, x_i and $x_{i'}$ have the same type over α, or

- for all $i, i' \in I$, y_i and $y_{i'}$ have the same type over α.

Assume that I is dense without end points.

By quantifier elimination, we may assume that $n = 1$, that is the x_i and y_i are points of M. Without loss, the (x_i) and (y_i) form increasing sequences. Assume there exists $i < j \in I$ and R a monotone definable relation such that $M \models \neg\alpha R x_i \wedge \alpha R x_j$. By monotonicity of R, there is a point i_R of the completion of I such that $i < i_R \rightarrow \neg\alpha R x_i$ and $i > i_R \rightarrow \alpha R x_i$.

Assume there is also a monotone relation S and an i_S such that $i < i_S \rightarrow \neg\alpha S y_i$ and $i > i_S \rightarrow \alpha S y_i$.

For points x, y define $I(x, y)$ as the set of $t \in M$ such that $M \models \neg t R x \wedge t R y$. This is an interval of M. Furthermore, if $i_1 < i_2 < i_3 < i_4$ are in I, then the intervals $I(x_{i_1}, x_{i_2})$ and $I(x_{i_3}, x_{i_4})$ are disjoint. Define $J(x, y)$ the same way using S instead of R.

Take $i_0 < i_R < i_1 < i_2 < \ldots$ and $j_0 < i_S < j_1 < j_2 < \ldots$. For $k < \omega$, define $I_k = I(x_{i_{2k}}, x_{i_{2k+1}})$ and $J_k = J(y_{j_{2k}}, y_{j_{2k+1}})$. The two sequences (I_k) and (J_k) are mutually indiscernible sequences of disjoint intervals. Furthermore, we have $\alpha \in I_0 \cap J_0$. By mutual indiscernibility, $I_i \cap J_j \neq \emptyset$ for all indices i and j, which is impossible.

We treated the case when α was to the left of the increasing relations R and S. The other cases are similar. □

An ordered set $(M, \leq)$ is of *finite width*, if there is n such that M has no antichain of size n.

Corollary 4.3. *Let $\mathcal{M} = (M, \leq)$ be an infinite ordered set of finite width, then $Th(\mathcal{M})$ is dp-minimal.*

Proof. We can define such a structure in a linear order with monotone relations: see [11]. More precisely, there exists a structure $P = (P, \prec, R_j)$ in which $\prec$ is a linear order and the R_j are monotone relations, and there is a definable relation $O(x, y)$ such that the structure (P, O) is isomorphic to $(M, \leq)$.

The result therefore follows from the previous one. □

We now move to trees. A tree is a structure $(T, \leq)$ such that $\leq$ defines a partial order on T, and for all $x \in T$, the set of points smaller than x is linearly ordered by $\leq$. We will also assume that given $x, y \in T$, the set of points smaller than x and y has a maximal element $x \wedge y$ (and set $x \wedge x = x$). This is not actually a restriction, since we could always work in an imaginary sort to ensure this.

Given $a, b \in T$, we define the open ball $B(a; b)$ of center a containing b as the set $\{x \in T : x \wedge b > a\}$, and the closed ball of center a as $\{x \in T : x \geq a\}$.

Notice that two balls are either disjoint or one is included in the other.

Lemma 4.4. *Let $(T, \leq)$ be a tree, $a \in T$, and let D denote the closed ball of center a. Let $\bar{x} = (x^1, ..., x^n) \in (T \setminus D)^n$ and $\bar{y} = (y^1, ..., y^m) \in D^m$. Then $tp(\bar{x}/a) \cup tp(\bar{y}/a) \vdash tp(\bar{x} \cup \bar{y}/a)$.*

Proof. A straightforward back-and-forth, noticing that $tp(\bar{x}/a) \cup tp(\bar{y}/a) \vdash tp_{qf}(\bar{x} \cup \bar{y}/a)$ (quantifier-free type). □

We now work in the language $\{\leq, \wedge\}$, so a sub-structure is a subset closed under $\wedge$.

Proposition 4.5. *Let $A = (a_0, ..., a_n)$, $B = (b_0, ..., b_n)$ be two sub-structures from T. Assume:*

1. *A and B are isomorphic as sub-structures,*
2. *for all i, j such that $a_i \geq a_j$, $tp(a_i, a_j) = tp(b_i, b_j)$.*

Then $tp(A) = tp(B)$.

Proof. We do a back-and-forth. Assume T is ω-saturated and A, B satisfy the hypothesis. We want to add a point a to A. We may assume that $A \cup \{a\}$ forms a sub-structure (otherwise, if some $a_i \wedge a$ is not in $A \cup \{a\}$, add first this element).

We consider different cases:

1. The point a is below all points of A. Without loss a_0 is the minimal element of A (which exists because A is closed under $\wedge$). Then find a b such that $tp(a_0, a) = tp(b_0, b)$. For any index i, we have: $tp(a_i, a_0) = tp(b_i, b_0)$ and $tp(a, a_0) = tp(b, b_0)$. By Lemma 4.4, $tp(a_i, a) = tp(b_i, b)$.

2. The point a is greater than some point in A, say a_1, and the open ball $\mathfrak{a} := B(a_1; a)$ contains no point of A.

Let $\mathcal{A}$ be the set of all open balls $B(a_1; a_i)$ for $a_i > a_1$. Let n be the number of balls in $\mathcal{A}$ that have the same type p as $\mathfrak{a}$. Then $tp(a_1)$ proves that there are at least $n + 1$ open balls of type p of center a_1. Therefore, $tp(b_1)$ proves the same thing. We can therefore find an open ball $\mathfrak{b}$ of center b_1 of type p that contains no point from B. That ball contains a point b such that $tp(b_1, b) = tp(a_1, a)$. Now, if a_i is smaller than a_1, we have $tp(a_i, a_1) = tp(b_i, b_1)$ and $tp(a_1, a) = tp(b_1, b)$, therefore by Lemma 4.4, $tp(a, a_i) = tp(b, b_i)$.

The fact that we have taken b in a new open ball of center b_1 ensures that $B \cup \{b\}$ is again a sub-structure and that the two structures $A \cup \{a\}$ and $B \cup \{b\}$ are isomorphic.

3. The point a is between two points of A, say a_0 and a_1 ($a_0 < a_1$), and there are no points of A between a_0 and a_1.

Find a point b such that $tp(a_0, a_1, a) = tp(b_0, b_1, b)$. Then if i is such that $a_i > a$, we have $a_i \geq a_1$ and again by Lemma 4.4, $tp(a_i, a) = tp(b_i, b)$. And same if $a_i < a$.

□

Corollary 4.6. *Let $A \subset T$ be any subset. Then $\bigcup_{(a,b,c)\in A^3} tp(a,b,c) \vdash tp(A)$.*

Proof. Let A_0 be the substructure generated by A. By the previous theorem the following set of formulas implies the type of A_0:

- the quantifier-free type of A_0,
- the set of 2-types $tp(a, b)$ for $(a, b) \in A_0^2$, $a \leq b$.

We need to show that those formulas are implied by the set of 3-types of elements of A. We may assume A is finite.

First, the knowledge of all the 3-types is enough to construct the structure A_0. To see this, start for example with a point $a \in A$ maximal. Knowing the 3-types, one knows in what order the $b \wedge a, b \in A$ are placed. Doing this for all such a, enables one to reconstruct the tree A_0.

Now take $m_1 = a \wedge b$, $m_2 = c \wedge d$ for $a, b, c, d \in A$ such that $m_1 \leq m_2$. The points m_1 and m_2 are both definable using only 3 of the points a, b, c, d, say a, b, c. Then $tp(a, b, c) \vdash tp(m_1, m_2)$. □

The previous results are also true, with the same proofs, for colored trees.

It is proven in [8] that theories of trees are NIP. We give a more precise result.

Proposition 4.7. *Let $\mathcal{T} = (T, \leq, C_i)$ be a colored tree. Then $Th(\mathcal{T})$ is dp-minimal.*

Proof. We will use criterion (3) of 1.4: if $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ are mutually indiscernible sequences and $\alpha \in T$ is a point, then one of the sequences (a_i) and (b_j) is indiscernible over α.

We will always assume that the index sets (I and J) are dense linear orders without end points.

1) We start by showing the result assuming the a_i and b_j are points (not tuples).

We classify the indiscernible sequence (a_i) in 4 classes depending on its quantifier-free type.

I The sequence (a_i) is monotonous (increasing or decreasing).

II The a_i are pairwise incomparable and $a_i \wedge a_j$ is constant equal to some point β.

If (a_i) is in none of those two cases, consider indices $i < j < k$. Note that it is not possible that $a_i \wedge a_j < a_i \wedge a_k$, so there are two cases left to consider:

III $a_i \wedge a_k = a_i \wedge a_j$. Then let $a'_i = a_i \wedge a_j$ (this does not depend on j, $j > i$). The a'_i form an increasing indiscernible sequence.

IV $a_i \wedge a_k < a_i \wedge a_j$. Then $a'_j = a_i \wedge a_j$ is independent of the choice of i ($i < j$) and (a'_j) is a decreasing indiscernible sequence.

Assume (a_i) lands in case **I**. Consider the set $\{x : x < \alpha\}$. If that set contains a non-trivial subset of the sequence (a_i), we say that α *cuts* the sequence. If this is not the case, then the sequence (a_i) stays indiscernible over α. To see this, assume for example that (a_i) is increasing and that α is greater than all the a_i. Take two sets of indices $i_1 < ... < i_n$ and $j_1 < ... < j_n$ and a $k \in I$ greater than all those indices. Then $tp(a_{i_1}, ..., a_{i_n}/a_k) = tp(a_{j_1}, ..., a_{j_n}/a_k)$. Therefore by Lemma 4.4, $tp(a_{i_1}, ..., a_{i_n}/\alpha) = tp(a_{j_1}, ..., a_{j_n}/\alpha)$.

In case **II**, note that if (a_i) is not α-indiscernible, then there is $i \in I$ such that α lies in the open ball $B(\beta; a_i)$ (we will also say that α *cuts* the sequence (a_i)). This follows easily from Proposition 4.5.

In the last two cases, if (a_i) is α-indiscernible, then it is also the case for (a'_i). Conversely, if (a'_i) is α-indiscernible, then α does not cut the sequence (a'_i). From 4.5, it follows easily that (a_i) is also α-indiscernible. We can therefore replace the sequence (a_i) by (a'_i) which belongs to case **I**.

Going back to the initial data, we may assume that (a_i) and (b_j) are in case **I** or **II**. It is then straightforward to check that α cannot cut both sequences.

For example, assume (a_i) is increasing and (b_j) is in case **II**. Then define β as $b_i \wedge b_j$ (any i, j). If α cuts (b_j), then $\alpha > \beta$. But (a_i) is β-indiscernible. So β does not cut (a_i). The only possibility for α to cut (a_i) is that β is smaller that all the a_i and the a_i lie in the same open ball of center β as α. But then the a_i lie in the same open ball of center β as one of the b_j. This contradicts mutual indiscernability.

2) Reduction to the previous case. We show that if $(a_i)_{i \in I}$ is an indiscernible sequence of n-tuples and $\alpha \in T$ such that (a_i) is not α-indiscernible, then there is an indiscernible sequence $(d_i)_{i \in I}$ of points of T in $dcl((a_i))$ such that (d_i) is not α-indiscernible.

First, by 4.6, we may assume that $n = 2$. Write $a_i = (b_i, c_i)$ and define $m_i = b_i \wedge c_i$. We again study different cases:

1. The m_i are all equal to some m.

As (a_i) is not α-indiscernible, necessarily, $\alpha > m$ and the ball $B(m; \alpha)$ contains one b_i (resp. c_i). Then take $d_i = b_i$ (resp. $d_i = c_i$) for all i.

2. The m_i are linearly ordered by $<$ and no b_i nor c_i is greater than all the m_i.

Then the balls $B(m_i; b_i)$ and $B(m_i; c_i)$ contain no other point from $(b_i, c_i, m_i)_{i \in I}$. Then, α must cut the sequence (m_i) and one can take $d_i = m_i$ for all i.

3. The m_i are linearly ordered by $<$ and, say, each b_i is greater than all the m_i.

Then each ball $B(m_i; a_i)$ contains no other point from $(b_i, c_i, m_i)_{i \in I}$. If α cuts the sequence m_i, than again one can take $d_i = m_i$. Otherwise, take a point γ larger than all the m_i but smaller than all the d_i. Applying 4.4 with a there replaced by γ, we see that (b_i) cannot be α-indiscernible. Then take $d_i = b_i$ for all i.

4. The m_i are pairwise incomparable.

The the sequence (m_i) lies in case **II**, **III** or **IV**. The open balls $B(m_i; b_i)$ and $B(m_i; c_i)$ cannot contain any other point from $(b_i, c_i, m_i)_{i \in I}$. Considering the different cases, one sees easily that taking $d_i = m_i$ will work.

This finishes the proof.

□

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