Externally definable sets in NIP theories

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Let $A \subset \mathcal{U}$ be any set (big or small).

An *externally definable* subset of A is a $\phi(A; b) \subseteq A^k$, $b \in \mathcal{U}$.

An *internally definable* subset of A is a $\psi(A; d) \subseteq A^k$, $d \in A$.

Problem

Can we understand externally de nable sets in terms of internally de nable ones?

Stable T

Ideal situation: any externally de nable subset of A is internally de nable.
(A is *stably-embedded*.)

DLO

An externally de nable set of M is a nite union of convex subsets.
$\implies$ tame

NIP

According to the philosophy that *NIP* is \stable + DLO", it should be tame too.

From now on, T is NIP.

Theorem (Honest de nitions)

Let $A \subset \mathcal{U}$ and $\phi(x; b) \in L(\mathcal{U})$.
There is $\psi(x; z)$ such that for any finite $A_0 \subseteq \phi(A; b)$, we can find $d \in A$ such that

$$A_0 \subseteq \psi(A; d) \subseteq \phi(A; b).$$

Another way to say the same thing.

Let $L_{\mathbf{P}} = L \cup \{\mathbf{P}(x)\}$.
Let $A \subseteq M$, (M, A): expansion of M to $L_{\mathbf{P}}$ setting $\mathbf{P}(M) = A$.

Theorem (Honest de nitions)

Let $A \subset M$ and $\phi(x; b) \in L(M)$.
There is an extension $(M, A) \prec (M', A')$ and $\psi(x; d) \in L(A')$ such that:

$$\psi(A; d) = \phi(A; b)$$
$$\psi(A'; d) \subseteq \phi(A'; b)$$

Corollary (Weak stable-embeddedness, Guingona)

Let $A \subset M$ and $\phi(x; b) \in L(M)$.
There is an extension $(M, A) \prec (M', A')$ and $\psi(x; d) \in L(A')$ such that:

$$\psi(A; d) = \phi(A; b)$$

Examples

$A = I$, a small indiscernible sequence.
Assume (for simplicity) that I is ordered by some $\emptyset$-de nable $<_I$ and the order is dense Dedekind complete.

Theorem (Baldwin-Benedikt)

I is stably embedded.

Proof.

Let $(M, I) \prec (M', I')$. Then I' is an indiscernible sequence.
By weak stable-embeddedness, it is enough to consider the case of some $\phi(I; b)$, $b \in I'$.
By indiscernability, the set $\phi(I'; b)$ is de nable using $=$ and the ordering $<_I$. We conclude by Dedekind completeness. □

Examples

$A = M$, a model.

Let M^{Sh} be the expansion of M obtained by adding a predicate for every externally de nable subset of M^k.

Theorem (Shelah)

M^{Sh} has elimination of quantifiers and is NIP.

Proof.

Let $M \prec N$ and $\phi(x_1, x_2; b) \in L(N)$. Take $(N, M) \prec (N', M')$ and $\psi(x_1, x_2; d) \in L(M')$ such that:

$$\psi(A; d) = \phi(A; b)$$
$$\psi(A'; d) \subseteq \phi(A'; b)$$

Let $\theta(x_1; d) = (\exists x_2)\psi(x_1, x_2; d)$. Then $\theta(M; d)$ coincides with the rst projection of $\phi(M; b)$. □

Expansions

We consider the following situation: M is NIP, we name some subset $A \subset M$ by a new predicate $\mathbf{P}(x)$.

Problem

Give sufficient conditions for the pair (M, A) to be NIP.

Results/special cases established by Berenstein, Boxall, Dolich, Günaydin, Hieronymi, Onshuus.

De nition

An $L_{\mathbf{P}}$-formula is *bounded* if it is of the form

$$(\forall x_1 \in \mathbf{P})(\exists x_2 \in \mathbf{P}) \cdots (\forall x_n \in \mathbf{P})\phi(x;y),$$

where $\phi(x;y)$ in an L-formula.

We say that the theory of (M, A) is *bounded* if all $L_{\mathbf{P}}$-formulas are equivalent to a bounded one.

Theorem

Assume that M is NIP, A_{ind} is NIP and the theory of the pair (M, A) is bounded, then (M, A) is NIP.

Corollary

If M is NIP, $A \prec M$ and the theory of (M, A) is bounded, then (M, A) is NIP.

Uniformity

Theorem (Uniformity of honest de nitions)

Let $\phi(x; y) \in L$. Then there is some $\psi(x; z) \in L$ such that: For every $A \subset \mathcal{U}$, and $b \in \mathcal{U}$, for every finite $A_0 \subseteq \phi(A; b)$, there is $d \in A$ such that

$$A_0 \subseteq \psi(A; d) \subseteq \phi(A; b).$$

Corollary (UDTFS)

Let $\phi(x; y) \in L$, then there is $\psi(x; z) \in L$ such that for every finite set A and every $b \in \mathcal{U}$, there is $d \in A$ with

$$\phi(A; b) = \psi(A; d).$$

Remark: We have to assume that the full theory is NIP. The UDTFS conjecture is still open for an NIP formula $\phi(x; y)$ in a (possibly) independent theory.

The proof uses compactness and a theorem of Alon-Kleitman and Matousek:

(p, q)-Theorem (special case)

If $\phi(x; y)$ in NIP and $q < \omega$ is big enough, then there is N such that for any nite family $\mathcal{B} = \{\phi(x; b_i) : i < n\}$ if any q sets from $\mathcal{B}$ intersect, then there is an N-point set in $\mathcal{U}$ intersecting all sets of $\mathcal{B}$.

Distal theories

Distal theories are \completely unstable" NIP theories.

De nition

An NIP theory T is *distal* if for every indiscernible sequence $I + b + J$ (I and J in nite sequences) and set A, if

$$I + J \text{ is indiscernible over } A,$$

then

$$I + b + J \text{ is indiscernible over } A.$$

Examples

O-minimal theories, $\mathbb{Q}_p$ are distal.

Theorem (Strong honest de nitions for distal theories)

Let $\phi(x; y) \in L$*. Then there is* $\psi(x; z) \in L$ *such that: for any* $b \in \mathcal{U}^{|\bar{y}|}$ *and* <u>*finite*</u> $A_0 \subseteq \phi(x; b)$*,*
there is $d \in A_0$ *such that:*

$$A_0 \subseteq \psi(x; d),$$
$$\psi(x; d) \to \phi(x; b).$$

Corollary (UDTFS for distal theories)

Let $\phi(x; y) \in L$*, there there is* $\theta(x; z) \in L$ *and* N *such that for any* $b \in \mathcal{U}^{|\bar{y}|}$ *and finite* $A \subset \mathcal{U}$*, there is some* $A_0 \subseteq A$ *of size* $\leq N$ *with:*

$$\mathrm{tp}_\theta(b/A_0) \vdash \mathrm{tp}_\phi(b/A).$$

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