Exposition of Shelah's 950 recounting of types

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Warning: I wrote this text mainly for my own benefit, *i.e.*, to check the details of Shelah's proof. I made it available because I believe that it might be easier to read than Shelah's paper. However it is likely that I have introduced mistakes, inaccuracies etc. Also, as I explain in the end, I did not understand everything.

I have not proofread the text. Itay Kaplan and Zaniar Ghadernejad have pointed out a number of mistakes in the first half of those notes. I thank them for that. Based on their corrections, I estimate that there should be about 72 mistakes left. Also the notations that I use do not match Shelah's.

The reader should be familiar with NIP theories.

Comments/corrections/questions on those notes are very welcome!

In this note, I expose (informally) Shelah's proof from paper 950 that a theory is NIP if and only if there are few types over saturated models, up to conjugancy by an automorphism. The main theorem is the following:

Theorem 0.1. *Let T be countable and NIP; M is a saturated model of T of size κ, where $\kappa = \aleph_\alpha > \beth_\omega$. Then there are at most $\beth_\omega + |\alpha|^{|T|}$ countable types over M up to automorphisms.*

Throughout, we assume that T is countable and we let M be a saturated model of size κ and $\mathfrak{C}$ is a monster model. We define $S_{aut}(M)$ as being the set of countable types over M up to conjugancy by an automorphism of M. Our goal is to show that if T is NIP, then $|S_{aut}(M)| < 2^\kappa$. We will succeed when $\kappa \geq \beth_\omega$ and in fact prove the stronger statement stated above (see 8.5). The technique goes through successive refinement of the type decomposition result of Shelah's paper 900 which we recall below.

Let $f_{T,aut}$ be the function defined on regular cardinals $\kappa = 2^{<\kappa}$ by $f_{T,aut}(\kappa) = |S_{aut}(M)|$ for some saturated M of size κ. Notice some basic facts:

- The function $f_{T,\mathfrak{aut}}$ is bounded if and only if T is stable.
- If T is NIP unstable, then we have $f_{T,\mathfrak{aut}}(\aleph_\alpha) \geq |\alpha + 1|$.

  [Let $(a_i : i < \kappa)$ be an indiscernible sequence, which is not an indiscernible set, then the types $p_\beta = \lim(a_i : i < \aleph_\beta)$, for $\beta < \alpha$, $\aleph_\beta$ regular, are not conjugated. The reason is that the limit types of two indiscernible sequences of different cofinalities commute. The "+1" comes from realized types.]

0.1 The IP case

Proposition 0.2. *Assume that* T *has IP. Let* $\lambda = \mu^+ = 2^\mu$, $\mu < \lambda$ *and* M *a saturated model of size* λ. *Then* $|S_{\mathfrak{aut}}(M)| \geq 2^\lambda$.

Proof. Let $\phi(x; y)$ have IP, with x, y single variables for simplicity. We can find in M a subset A of size μ such that for every $s \subseteq A$, there is $b_s \in M$ satisfying $\phi(A; b_s) = s$.

Call a family $\mathcal{A}$ of subsets of A *boolean independent* if for every two finite and disjoint subsets $G, H \subset \mathcal{A}$, there is $x \in A$ which is in all of the sets in G and in none of the sets in H. By a result of Hausdorff, there is a boolean independent family $\mathcal{A}$ of size $2^\mu = \lambda$.

Now for every subset $W \subset \mathcal{A}$, define a type $p_W \in S(\{b_s : s \subseteq A\})$ by setting $p_W \vdash \phi(x; b_s)^{\langle s \in W \rangle}$. Then the types p_W are consistent and pairwise distinct. Extend each of them to a complete type q_W over M finitely satisfiable in A. We have thus defined 2^λ types $(q_W : W \in \mathcal{W})$.

Assume now that there is a subset $\mathcal{C} \subset \mathcal{W}$ of size $> \lambda$ such that any two types in C are conjugated by an automorphism of M. Fix some $W \in \mathcal{C}$. For every $W' \in \mathcal{C}$, let $\sigma_{W',W} \in \mathrm{Aut}(M)$ send $q_{W'}$ to q_W. Then $\sigma_{W',W}$ maps A to some $A_{W'}$. As there are $\lambda^\mu = \lambda$ subsets of M of size μ, we may assume that $A_{W'}$ is constant for $W' \in \mathcal{C} \setminus \{W\}$. Hence those types are conjugated by an automorphism fixing A, which is impossible (because all those types are finitely satisfiable in A hence fixed by $\mathrm{Aut}(M/A)$). $\square$

1 Decompositions

From now on, we assume that T is NIP (and countable). All the tuples considered are of countable size.

The phrase "$\mathrm{tp}(d/ce) \vdash \mathrm{tp}_\Gamma(d/Ac)$ according to some $\bar\psi = (\psi_\phi, \theta_\phi)$" means that for every $\phi(x, y; z) \in \Gamma$, we have formulas $\psi_\phi(x, y, t)$ and $\theta_\phi(t, y; z)$ such that:

- $\models \psi_\phi(d, c, e)$;
- For each $a \in A$ such that $\models \phi(d, c; a)$, we have $\models \theta_\phi(e, c; a)$;
- $\models (\forall x, z)\psi_\phi(x, c, e) \wedge \theta_\phi(e, c; z) \to \phi(x, c; z)$.

If Γ is all formulas, we omit it.

Let rD be the set of quadruples $\mathbf{x} = (p(x), r(y), q(x, y, x'), \Gamma)$ where:
– Γ is a subset of formulas of the form $\phi(x, y; z) \in L$ (z is any countable variable);
– p is a type over M;
– r is a type over M, finitely satisfiable in some $B_\mathbf{x} \subset M$ of size $< \kappa$;
– q is a type over M, such that if $(d, c, d') \models q$ then :
$\bullet_1$ $d, d' \models p$ and $c \models r$;
$\bullet_2$ $\mathrm{tp}(d/cd') \vdash \mathrm{tp}_\Gamma(d/cM)$ according to some $\bar\psi$;
$\bullet_3$ for every $A \subset M$ of size $< \kappa$, there is some $d_A \in M$ such that $\mathrm{tp}(d_A/dcB_\mathbf{x}) = \mathrm{tp}(d'/dcB_\mathbf{x})$ and $\mathrm{tp}(d_A/Ac) = \mathrm{tp}(d'/Ac)$.

Let $rD^\oplus$ be the set of $\mathbf{x} \in rD$ such that $\Gamma_\mathbf{x} = \{\phi : \phi(x, y; z) \in L\}$.

Note that $\bullet_3$ implies that $c \models r|Md'$ and along with $\bullet_2$ it implies that $\mathrm{tp}(d/cd_A) \vdash \mathrm{tp}_\Gamma(d/cA)$ according to $\bar\psi$.

Proposition 1.1. *Let $\mathbf{x}_1, \mathbf{x}_2 \in rD^\oplus$. Assume that there is an automorphism f of M mapping $B_{\mathbf{x}_1}$ to $B_{\mathbf{x}_2}$ such that $f_*(r_{\mathbf{x}_1}) = f_*(r_{\mathbf{x}_2})$ and $f_*(q_{\mathbf{x}_1} \restriction B_{\mathbf{x}_1}) = f_*(q_{\mathbf{x}_2} \restriction B_{\mathbf{x}_2})$, then $p_{\mathbf{x}_1}$ and $p_{\mathbf{x}_2}$ are conjugate.*

Proof. Let $(d_1, c_1, d_1') \models q_{\mathbf{x}_1}$ and $(d_2, c_2, d_2') \models q_{\mathbf{x}_2}$.

We build by back-and-forth a partial automorphism $f : Md_1c_1 \to Md_2c_2$. Start with $f_0 : B_{\mathbf{x}_1}d_1c_1 \to B_{\mathbf{x}_2}d_2c_2$ given by the assumption. At some stage α we have a partial automorphism $f_\alpha : A_{\alpha,1}d_1c_1 \to A_{\alpha,2}d_2c_2$ extending f_0. Let $a_1 \in M$ and we want to extend f_α to $A_{\alpha,1}a_1d_1c_1$.

Take some $d_1^* \in M$ such that we have both $\mathrm{tp}(d_1^*/d_1c_1) = \mathrm{tp}(d_1'/d_1c_1)$ and $\mathrm{tp}(d_1^*/A_{\alpha,1}a_1c_1) = \mathrm{tp}(d_1'/A_{\alpha,1}a_1c_1)$. Then $\mathrm{tp}(d_1/c_1d_1^*) \vdash \mathrm{tp}(d_1/c_1A_{\alpha,1}a_1)$. Take also $d_2^* \in M$ such that $\mathrm{tp}(d_2^*/d_2c_2) = \mathrm{tp}(d_2'/d_2c_2)$ and $\mathrm{tp}(d_2^*/A_{\alpha,2}c_2) =$

$\mathrm{tp}(d_2', A_{\alpha,1}c_2)$. Then $\mathrm{tp}(A_{\alpha,1}, d_1^*) = \mathrm{tp}(A_{\alpha,2}, d_2^*)$ ($= \mathrm{tp}(A_{\alpha,2}, d_2)$). Therefore we may find $a_2 \in M$ such that $\mathrm{tp}(A_{\alpha,2}, a_2, d_2^*) = \mathrm{tp}(A_{\alpha,1}, a_1, d_1^*)$.

As r_{x_1} and r_{x_2} are finitely satisfiable and conjugate by f_0, we automatically have $\mathrm{tp}(A_{\alpha,2}, a_2, d_2^*, c_2) = \mathrm{tp}(A_{\alpha,1}, a_1, d_1^*, c_1)$. Also, we have $\mathrm{tp}(d_2, d_2^*, c_2) = \mathrm{tp}(d_1, d_1^*, c_1)$ by hypothesis. Since $\mathrm{tp}(d_1/c_1 d_1^*) \vdash \mathrm{tp}(d_1/c_1 A_{\alpha,1} a_1)$, we have $\mathrm{tp}(d_2, c_2, A_{\alpha,2}, a_2) = \mathrm{tp}(d_1, c_1, A_{\alpha,1}, a_1)$ and thus we may extend f_α to $f_{\alpha+1}$ by sending a_1 to a_2. □

Remark 1.2. Let $\mu \leq \kappa$. Up to conjugancy, there are $\leq 2^{<\mu}$ types over M finitely satisfiable in some $B \subset M$ of size $< \mu$ (because we have 2^μ choices of $\mathrm{tp}(B)$, and for each there are $2^{|B|}$ types finitely satisfiable in B by NIP). Hence up to the equivalence defined in the proposition, there are at most 2^μ $\mathbf{x} \in \mathrm{r}D^\oplus$.

In particular, if μ is strong limit, this is equal to μ.

If $\mathbf{x}, \mathbf{y} \in \mathrm{r}D$, we write $\mathbf{x} \leq \mathbf{y}$ if $p_\mathbf{y}$, $r_\mathbf{y}$ extend $p_\mathbf{x}$, $r_\mathbf{x}$ respectively (*i.e.*, they may contain more variables) and if $\mathrm{tp}(d'_\mathbf{y}, d_\mathbf{y}, c_\mathbf{y}, B_\mathbf{y})$ extends $\mathrm{tp}(d'_\mathbf{x}, d_\mathbf{x}, c_\mathbf{x}, B_\mathbf{x})$, $\mathrm{tp}(d'_\mathbf{y}, c_\mathbf{y}/M)$ extends $\mathrm{tp}(d'_\mathbf{x}, c_\mathbf{x}/M)$ and $\mathrm{tp}(d_\mathbf{y}, c_\mathbf{y}/M)$ extends $\mathrm{tp}(d_\mathbf{x}, c_\mathbf{x}/M)$.

So note that we are not asking for $q_\mathbf{y}$ to extend $q_\mathbf{x}$, only partially. However our hypothesis are sufficient to ensure the implications true for $\mathbf{x}$ remain true for $\mathbf{y}$. More precisely if $\mathrm{tp}(d_\mathbf{x}/c_\mathbf{x} d'_\mathbf{x}) \vdash \mathrm{tp}_{\Gamma_\mathbf{x}}(d_\mathbf{x}/c_\mathbf{x} M)$ according to $\bar{\psi}$, then we also have $\mathrm{tp}(d_\mathbf{y}/c_\mathbf{y} d'_\mathbf{y}) \vdash \mathrm{tp}_{\Gamma_\mathbf{x}}(d_\mathbf{y}/c_\mathbf{y} M)$ according to the same $\bar{\psi}$.

We write $\mathbf{x} \leq^1 \mathbf{y}$ if $\mathbf{x} \leq \mathbf{y}$ and $\Gamma_\mathbf{y}$ contains all formulas $\phi(x_{[d_\mathbf{x}]}, y_{[c_\mathbf{y}]}; z)$.

2 900 decomposition

We recall the statement of the 900 decomposition.

Proposition 2.1. *Let $\mu \leq \kappa$, $\mathrm{cf}(\mu) > |T|$. Let $(d, c) \in \mathfrak{C}$, with $\mathrm{tp}(c/M)$ finitely satisfiable in some $B \subset M$ of size $< \mu$. Then we can increase c to some c', finitely satisfiable in $B' \subset M$ of size $< \mu$ such that for any $A \subset M$ of size $< \mu$, there is some $e_A \in M$ such that $\mathrm{tp}(d/ce_A) \vdash \mathrm{tp}(d/c)$ according to some $\bar{\psi}_A$.*

Note that if $\mathrm{cf}(\mu) > 2^{|T|}$, then we may assume that $\bar{\psi}_A$ is constant.

For a proof, I refer the reader to the notes available on my webpage.

3 Weakly compact

Assume κ is weakly compact. We show $|S_{aut}(M)| \leq \kappa$. For this, we prove density of $rD^{\oplus}$.

Proposition 3.1. *Assume that κ is weakly compact. Let $\mathbf{x} \in rD$, then there is $\mathbf{y} \in rD$ with $\mathbf{x} \leq^1 \mathbf{y}$.*

Proof. Let $(d, c, d') \models q_{\mathbf{x}}$.

First, by 900 decomposition, we can find some c' extending c (so $c' = c^\frown c''$) such that $\mathrm{tp}(c'/M)$ is finitely satisfiable in some small $B' \subset M$ and for every small $A \subset M$, there is some $e_A \in M$ such that $\mathrm{tp}(d/c'e_A) \vdash \mathrm{tp}(d/c'A)$ according to some $\bar{\psi}_A$.

Write M as an increasing union $M = \bigcup_{i<\kappa} A_i$, where $|A_i| < \kappa$ and $B' \subseteq A_0$. Take d_i as in $\bullet_3$, where A there stands for A_i here, and let $e_i = e_{A_i d_i}$. Then $\mathrm{tp}(d/c'e_i) \vdash \mathrm{tp}(d/c'd_iA_i)$ and *a fortiori*, $\mathrm{tp}(d/c'd_ie_i) \vdash \mathrm{tp}(d/c'A_i)$ according to some $\bar{\psi}_i$. By extracting, we may assume that $\bar{\psi}_i = \bar{\psi}$ is constant. By weak compactness, there is an increasing function $f : \kappa \to \kappa$ such that $\mathrm{tp}(d_{f(i)}{}^\frown e_{f(i)}/A_ic'd)$ is increasing. Let $d_*{}^\frown e_*$ realize the union. By construction, $\mathrm{tp}(d_*/M) = \mathrm{tp}(d/M)$, so we may find $e \in \mathfrak{C}$ such that $\mathrm{tp}(d, e/M) = \mathrm{tp}(d_*, e_*/M)$.

Then, there is an increasing $g : \kappa \to \kappa$ such that $\mathrm{tp}(d_{f(g(i))}{}^\frown e_{f(g(i))}/A_ic'de)$ is increasing. Let $d''{}^\frown e''$ realize the union. By construction, $\mathrm{tp}(d, c, d'') = \mathrm{tp}(d, c, d') = q_{\mathbf{x}}|\emptyset$ and $\mathrm{tp}(d''/Mc) = \mathrm{tp}(d'/Mc)$. Also $\mathrm{tp}(d'', e''/M) = \mathrm{tp}(d, e/M)$. Finally, let Γ' consist of all formulas of the form $\phi(x_{[d]}, y_{[c']}; z)$. Then $\mathrm{tp}(d, e/c'd''e'') \vdash \mathrm{tp}_{\Gamma'}(d, e/c'M)$ according to $\bar{\psi}$.

So we set $\mathbf{y} = (\mathrm{tp}(d^\frown e/M), \mathrm{tp}(c'/M), \mathrm{tp}(d^\frown e, c', d''{}^\frown e''/M), \Gamma')$. □

Proposition 3.2. *Let $(\mathbf{x}_k : k < \omega)$ be a sequence of elements of rD such that $\mathbf{x}_k \leq^1 \mathbf{x}_{k+1}$ for every k. Define $\mathbf{x}_\omega$ such that $q_{\mathbf{x}_\omega}$ is an accumulation point of the $q_{\mathbf{x}_k}$'s (and $B_{\mathbf{x}_\omega}$ is $\bigcup B_{\mathbf{x}_n}$). Then $\mathbf{x}_\omega \in rD^{\oplus}$.*

Proof. Write $p_k = p_{\mathbf{x}_k}$ and $p = p_{\mathbf{x}_\omega}$ and similarly r, r_k, q, q_k. Let $(d, c, d') \models q$. Then we have $\mathrm{tp}(d/cd') \vdash \mathrm{tp}(d/cM)$ according to some $\bar{\psi}$. Let $A \subset M$ of size $< \kappa$. Without loss, A contains $B = B_{\mathbf{x}_\omega}$. We need to find e such that $\mathrm{tp}(e/dcB) = \mathrm{tp}(d'/dcB)$ and $\mathrm{tp}(e/cA) = \mathrm{tp}(d'/cA)$.

For each $k < \omega$, we can naturally define d_k, d'_k, c_k as initial segments of d, d', c such that $(d_k, c_k, d'_k) \models q_k|_{B_k}$. The property $\mathbf{x}_k \leq^1 \mathbf{x}_{k+1}$ implies:

$\boxtimes_k$: $\mathrm{tp}(d_k/c_{k+1}d'_{k+1}) \vdash \mathrm{tp}(d_k/c_{k+1}M)$ according to some $\bar{\psi}_k$.

Construct inductively tuples d^*_k in M such that $\mathrm{tp}(d^*_k/d_k c_k B_k) = \mathrm{tp}(d'_k/d_k c_k B_k)$ and $\mathrm{tp}(d^*_k/c_k A_k) = \mathrm{tp}(d'_k/c_k A_k)$, where $A_k = A \cup \{d^*_l : l < k\}$. Then again construct tuples d^{**}_k in M such that $\mathrm{tp}(d^{**}_k/d_k c_k B_k) = \mathrm{tp}(d'_k/d_k c_k B_k)$ and $\mathrm{tp}(d^{**}_k/c_k A'_k) = \mathrm{tp}(d'_k/c_k A'_k)$, where $A'_k = A_\omega \cup \{d^{**}_l : l < k\}$. For $k < l$, define naturally $d^*_{l,k}$ as the initial segment of d^*_l realizing p_k over A_l.

Let $\mathcal{D}$ be a non-principal ultrafilter on ω and let $e \in M$ realize $\lim_{\mathcal{D}}((d^*_k : k < \omega)/A'_\omega)$. We show that $\mathrm{tp}(e/dcB) = \mathrm{tp}(d'/dcB)$ and $\mathrm{tp}(e/cA) = \mathrm{tp}(d'/cA)$. The second point is clear as $B \subseteq A$, $\mathrm{tp}(e/A) = \mathrm{tp}(d'/A)$ and $\mathrm{tp}(c/Md')$ is finitely satisfiable in B.

To show the first point, let e_k the natural initial segment of e. It is enough to show, for every $k < \omega$, that $\mathrm{tp}(e_k/d_k c_k B) = \mathrm{tp}(d'_k/d_k c_k B)$. By $\boxtimes_k$, we have $\mathrm{tp}(d_k/d^{**}_{k+1} c_{k+1}) \vdash \mathrm{tp}(d_k/c_{k+1} A_\omega)$ according to some $\bar{\psi}_k = (\psi_\phi, \theta_\phi)$. Let $\phi(x_k, y_k; x'_k, \bar{b}) \in L(B_k)$ such that $\phi(d_k, c_k; d'_k, \bar{b})$ holds. In particular, for every $l > k$, $\theta_\phi(d^{**}_{k+1}, c_{k+1}; d^*_{l,k}, \bar{b})$ holds.

<u>Claim</u>: The type $\mathrm{tp}(d^{**}_{k+1}, d^*_{l,k}/B_{k+1})$ is constant as $l \geq k+1$ varies, equal to the restriction of $q|_{B_{k+1}}$ to the relevant variables.

Proof: By construction $\mathrm{tp}(d^{**}_{k+1}/d^*_l B) = \mathrm{tp}(d'_{k+1}/d^*_l B) = \mathrm{tp}(d_{k+1}/d^*_l B)$. And also $\mathrm{tp}(d, d^*_l/B_{k+1}) = \mathrm{tp}(d, d'_l/B_{k+1})$ is the restriction of $q|_{B_{k+1}}$ to the relevant variables. Hence the claim follows.

We conclude from the claim, and the fact that $\mathrm{tp}(c_{k+1}/Md')$ is finitely satisfiable in B_{k+1}, that $\theta_\phi(d^{**}_{k+1}, c_{k+1}; e_k, \bar{b})$ holds. Therefore $\phi(d_k, c_k; e_k, \bar{b})$ holds and we are done. $\square$

Theorem 3.3. *Let κ be weakly compact, then for any countable type p over M, there is $x \in rD^{\oplus}$ such that p_x extends p. In particular, there are κ many types over M up to conjugancy.*

4 The partition theorem

Theorem 4.1 (Partition theorem)**.** *Assume that T is countable NIP. Let:*

– $B \subset \mathfrak{C}$;

– $(e_i : i < \kappa)$ *a sequence of tuples of the same length, where*

– $\kappa = cf(\kappa) \geq \beth_\omega(|B| + \aleph_0)$;

– T_1, T_2 *stationary subsets* κ;

– $\Delta \subset L(x_{[e]}, x'_{[e]}; z)$ *finite.*

Then there are two stationary subsets $\mathcal{S}_1 \subseteq T_1$, $\mathcal{S}_2 \subseteq T_2$ and a type $p \in S_\Delta(B)$ such that for $s \in \mathcal{S}_1$, $t \in \mathcal{S}_2$, $s < t$, we have $\mathrm{tp}_\Delta(e_i, e_j/B) = p$.

Proof. Let Δ^1 be finite sets of formulas and $k < \omega$ such that if $(a_i : i < \omega)$ is a (Δ^1, k)-indiscernible sequence over B and $b \in \mathfrak{C}$, then the set $\{i < \omega : \mathrm{tp}_\Delta(a_i, b/B) \neq \mathrm{tp}_\Delta(a_{i+1}, b/B)\}$ is finite. (We can find such a Δ^1 by NIP.)

Let ∂ be the following two-player game: a play lasts ω moves. In the l-th move, the antagonists chooses $\mathcal{X}_l \subseteq \kappa$ a club and the protagonist chooses $s_l \in \mathcal{X}_l \cap T_1$. In the end, the protagonists wins the play if $(e_{s_l} : l < \omega)$ is a (Δ^1, k)-indiscernible sequence over B.

<u>Claim</u>: The protagonists has a winning strategy in the game ∂.

Proof: Assume not. As the game is closed for the protagonists, the antagonist must have a winning strategy $\mathbf{st}_a$. Choose $s_\alpha \in T_1$ by induction on $\alpha < \kappa$ such that: for any initial segment of the play of ∂ in which the antagonists uses $\mathbf{st}_a$ and the protagonists chooses members of $\{s_\beta : \beta < \alpha\}$, the last move of the antagonists is a club $\mathcal{X}$ to which s_α belongs. Letting $\lambda = |S_\omega(B)|$, as $\kappa > \beth_k(\lambda)$, by Erdös-Rado, there is an increasing sequence $(\alpha(i) : i < \omega)$ of ordinals $< \kappa$ such that $(e_{s_{\alpha(i)}} : i < \omega)$ is (Δ^1, k)-indiscernible over B. So the protagonist can play $(e_{s_{\alpha(i)}} : i < \omega)$ and win the game.

Fix a winning strategy $\mathbf{st}$ for ∂. Let $\mathbf{T}$ be the set of initial segments $(s_i : i < n)$ of ∂ played according to $\mathbf{st}$. Let $h : \kappa \to \kappa^{<\omega} \times S_\Delta(B)$ be a bijection and let $E \subseteq \kappa$ be the set of $\delta < \kappa$ such that h induces a bijection from δ to $\delta^{<\omega} \times S_\Delta(B)$. Then E is a club. Fix some $\delta \in E$. Then we can choose a maximal initial segment $\bar{s}_\delta = (s_i : i < n) \in \mathbf{T}$ such that:

– for each $i < n$, $s_i < \delta$;

– for each $i < n - 1$, $\mathrm{tp}_\Delta(e_{s_i}, e_\delta/B) \neq \mathrm{tp}_\Delta(e_{s_{i+1}}, e_\delta/B)$.

Let $f(\delta) = h^{-1}(\mathbf{s}_\delta, \mathrm{tp}_\Delta(e_{s_{n-1}}, e_\delta/B))$. Then $f(\delta) < \delta$. By Fodor's lemma, there is a stationary set $\mathcal{S}_2 \subseteq E \cap T_2$ such that f is constant on $\mathcal{S}_2$ equal to some β. Let $(\bar{s}, p) = h(\beta)$ and define $\mathcal{S}_1 = \{s_* < \kappa : \bar{s}^\frown s_* \in \mathbf{T}\}$. Easily, $\mathcal{S}_1$ is stationary (because by choosing the club $\mathcal{X}$, we can force to play out of some non-stationary set). If $s \in \mathcal{S}_1$, $t \in \mathcal{S}_2$, $s < t$, then $\mathrm{tp}_\Delta(e_s, e_t/B) = p$ as required. □

5 Inaccessible

Proposition 5.1. *Assume that κ is (strongly) inaccessible. Let $\mathbf{x} \in \mathrm{rD}$, then there is $\mathbf{y} \in \mathrm{rD}$ with $\mathbf{x} \leq^1 \mathbf{y}$.*

Proof. Let $(d, c, d') \models q_{\mathbf{x}}$ and write $B = B_{\mathbf{x}}$.

First, by 900 decomposition, we can find some c' extending c (so $c' = c^\frown c''$) such that $\text{tp}(c'/M)$ is finitely satisfiable in some small $B' \subset M$ and for every small $A \subset M$, there is some $e_A \in M$ such that $\text{tp}(d/c'e_A) \vdash \text{tp}(d/c'A)$ according to some constant (ψ_ϕ, θ_ϕ).

Write M as an increasing union $M = \bigcup_{i<\kappa} M_i$, where $|M_i| < \kappa$ and $B' \subseteq M_0$. For each i pick some d_i such that $\text{tp}(d_i/dcB) = \text{tp}(d'/dcB)$ and $\text{tp}(d_i/cM_i) = \text{tp}(d'/cM_i)$, then take $e_i = e_{M_i+d_i}$ as above. Let $e_i^+ = d_i{}^\frown e_i$. As $\kappa > 2^{|B'|+|T|}$, we may assume that $\text{tp}(e_i^+/dc'B')$ is constant. Also, without loss, $e_i^+ \in M_{i+1}$.

Write the set $\{\phi(x_1^+, x_2^+; z) : \phi \in L\}$ as an increasing union of finite sets $\{\Delta_n : n < \omega\}$. By the partition theorem, for each $n < \omega$, we can find stationary sets $\mathcal{S}_n^l$ for $l = 0, 1, 2$ and Δ_n-types $q_{0,n}, q_{1,n}$ over B' such that:

– $\mathcal{S}_{n+1}^l \subseteq \mathcal{S}_n^l$ for each l, n;

– For $l = 0, 1$, $\text{tp}_{\Delta_n}(e_i^+, e_j^+/B') = q_{l,n}$ for any $i < j$, $(i, j) \in \mathcal{S}_n^l \times \mathcal{S}_n^{l+1}$;

Let $\mathcal{D}$ be an ultrafilter on κ extending the club filter and containing each $\mathcal{S}_n^1$. Let $e_\bullet^+ \models Av_{\mathcal{D}}((e_i^+ : i < \kappa)/Mc'd)$. Write $e_\bullet^+ = d_\bullet{}^\frown e_\bullet$. Note that $\text{tp}(d_\bullet/dcB) = \text{tp}(d'/dcB)$ and $\text{tp}(d_\bullet/cM) = \text{tp}(d'/cM)$.

Fix some $i < \kappa$. For each $n < \omega$, pick some $\gamma_n \in \mathcal{S}_n^0 \cap [i, \kappa)$. Let $\gamma = \sup \gamma_n + 1$. Let Λ be the set of finite subsets of $\text{tp}(e_\bullet^+/dc'M_\gamma)$. Then $|\Lambda| < \kappa$. For each $p \in \Lambda$, fix some $\alpha_p \in \mathcal{S}_{|p|}^1 \cap [\gamma, \kappa)$ such that $e_{\alpha_p}^+ \models p$. Finally, for each $n < \omega$, pick some $\beta_n \in \mathcal{S}_n^2$ greater than all α_p's.

Let $\mathcal{D}'$ be an ultrafilter on Λ containing $\{p' \in \Lambda : p' \supseteq p\}$ for every p, and let $g^+ = g_i^+ \models Av_{\mathcal{D}'}((e_{\alpha_p}^+ : p \in \Lambda)/M_i + \{e_{\gamma_n}, e_{\beta_n} : n < \omega\})$, with $g^+ \in M$.

1) $\text{tp}(g^+/dc'B') = \text{tp}(e_\bullet^+/dc'B')$.

Proof: Let $\phi(x, y; x^+, b) \in L(B')$ be such that $\phi(d, c'; e_\bullet^+, b)$ holds. For $\mathcal{D}'$-almost all p, $e_{\alpha_p}^+ \models \phi(d, c'; x^+, b)$. Let n such that $\theta_\phi \in \Delta_n$. We have $\models \psi_\phi(d, c', e_{\beta_n}^+)$ and $\models \theta_\phi(e_{\beta_n}^+, c'; e_{\alpha_p}^+, b)$ for all p. Note that $\text{tp}_{\Delta_n}(e_{\beta_n}^+, e_{\alpha_p}^+/B')$ is constant for $p \in \Lambda$, and this type is equal to $\text{tp}_{\Delta_n}(e_{\beta_n}^+, g^+/B')$. Therefore $\theta_\phi(e_{\beta_n}^+, c'; g^+, b)$ holds. Thus $\phi(d, c'; g^+, b)$ holds.

2) $\text{tp}(d/g^+c') \vdash \text{tp}(d/c'M_i)$ (but according to some different $(\psi'_\phi, \theta'_\phi)$).

Proof: Let $\phi(x, c'; a) \in \text{tp}(d/c'M_i)$. We have some formulas $\psi_\phi(x, c', x^+)$ and $\theta_\phi(x^+, c'; a)$ such that, for all $j \geq i$:

$\boxtimes_1$ $\models \psi_\phi(d, c', e_j^+) \wedge \theta_\phi(e_j^+, c'; a)$;

$\boxtimes_2$ $(\forall x, z)\psi_\phi(x, c', e_j^+) \wedge \theta_\phi(e_j^+, c'; z) \to \phi(x; c'; z)$.

Let $\phi_1(x, c', x_1^+) = \psi_\phi$. Then we have $\psi_{\phi_1}(x, c', x^+)$ and $\theta_{\phi_1}(x^+, c'; x_1^+)$ such that for $i \leq j' < j$:

$\boxtimes_3 \models \psi_{\phi_1}(d, c', e_j^+) \wedge \theta_{\phi_1}(e_j^+, c'; e_{j'}^+)$;

$\boxtimes_4$ $(\forall x, x_1^+)\psi_{\phi_1}(x, c', e_j^+) \wedge \theta_{\phi_1}(e_j^+, c'; x_1^+) \to \phi_1(x, c'; x_1^+)$.

By 1), we know that $\models \psi_{\phi_1}(d, c', g^+)$ and $\boxtimes_4$ holds with g^+ instead of e_j^+. Also, taking n large enough so that $\theta_{\phi_1} \in \Delta_n$, we have $\models \theta_{\phi_1}(g^+, c'; e_{\gamma_n}^+)$ since $\theta_{\phi_1}(e_j^+, c'; e_{\gamma_n}^+)$ holds for all $\gamma_n < j \in \mathcal{S}_n^1$. Putting $\boxtimes_4$ and $\boxtimes_2$ together, we see that:

$\boxtimes_{2+4}$ $(\forall x)\psi_{\phi_1}(x, g^+; c') \to \phi(x, c'; a)$.

Hence $\text{tp}(d/g^+c') \vdash \text{tp}(d/c'M_i)$ as required. More precisely, we have $\psi'_\phi(x, c', x^+) = \psi_{\phi_1}(x, c', x^+)$ and $\theta'_\phi = (\exists x_1^+)\theta_{\phi_1}(x^+, c'; x_1^+) \wedge \theta_\phi(x_1^+, c'; z)$.

Thus for each $i < \kappa$, we have defined some tuple g_i^+ such that 1) and 2) above hold. Also by construction, $\text{tp}(g_i^+/M_i) = \text{tp}(e_\bullet^+/M_i)$. But note that we did not prove $\text{tp}(g_i^+/c'M_i) = \text{tp}(e_\bullet^+/c'M_i)$. We now change $e_\bullet^+$ to ensure this. Also, we need to extend d to $d\hat{\ }e$.

So start by picking $e \in \mathfrak{C}$ such that $\text{tp}(d\hat{\ }e/M) = \text{tp}(e_\bullet^+/M)$. Extracting if necessary, we may assume that $\text{tp}(g_i^+/dec'B')$ is constant. Let $\mathcal{F}$ be any ultrafilter on κ containing the club filter, and let $e_*^+ \models \lim_{\mathcal{F}}((g_i^+ : i < \kappa)/Mdec')$. Note that $\text{tp}(c'/Me_*^+)$ does not split over B' (because the types $\text{tp}(g_i^+/M_i)$ are increasing and $\text{tp}(c'/M_ig_i^+)$ does not split over B'). Property 1) above remains true, with the same g_i^+, but replacing $e_\bullet^+$ by e_*^+. Property 2) implies that $\text{tp}(d/e_*^+c') \vdash \text{tp}(d/c'M)$ according to $(\psi'_\phi, \theta'_\phi)$. Finally, for every $i < \kappa$, we have $\text{tp}(g_i^+/M_ic') = \text{tp}(e_*^+/M_ic')$.

Now, we are done. Write $e_*^+ = d_*\hat{\ }e_*$ and $e^+ = d\hat{\ }e$. Then $\text{tp}(d_*/Mc) = \text{tp}(d/Mc)$ and $\text{tp}(d_*/dcB) = \text{tp}(d'/dcB)$. Also $\text{tp}(e^+/M) = \text{tp}(e_*^+/M)$.

Thus we can define $\mathbf{y} = (\text{tp}(e^+/M), \text{tp}(c'/M), \text{tp}(e^+, c', e_*^+/M), \Gamma)$ where Γ is composed of all formulas of the form $\phi(x_{[d]}, y_{[c']}; z)$. □

Propositions 5.1 and 3.2 imply:

Theorem 5.2. *Let κ be inaccessible, then for any countable type p over M, there is $\mathbf{x} \in \mathrm{rD}^\oplus$ such that $p_\mathbf{x}$ extends p.*

Therefore there are at most κ many types over M up to conjugancy.

6 The case $\kappa = \mu^+$

Theorem 6.1. *Let μ be strong limit of uncountable cofinality and $\kappa = \mu^+ = 2^\mu$, then for any countable type p over M, there is $\mathbf{x} \in \mathrm{rD}^\oplus$ with $|B_\mathbf{x}| < \mu$ such that $p_\mathbf{x}$ extends p.*

Therefore there are at most μ many types over M up to conjugancy.

Proof. It is enough to show that given $x \in rD$, with $|B_x| < \mu$, we can find $y \in rD$ with $|B_y| < \mu$ and $x \leq^1 y$.

Let $(d, c, d') \models q_x$.

By 900 decomposition, we can find some c' extending c (so $c' = c^\frown c''$) such that $\mathrm{tp}(c'/M)$ is finitely satisfiable in some $B' \subset M$ of size $< \mu$ and for every $A \subset M$ of size $< \mu$, there is some $e_A \in M$ such that $\mathrm{tp}(d/c'e_A) \vdash \mathrm{tp}(d/c'A)$ according to some $\bar{\psi}$. Now take $A \subset M$ of size μ. Write $A = \bigcup_{i<\mu} A_i$ with $|A_i| < \mu$. For each $i < \mu$, let $e_i = e_{A_i}$ as above. Set $A' = \{e_i : i < \mu\}$ and let $e = e_{A'}$. Then $\mathrm{tp}(d/c'e) \vdash \mathrm{tp}(d/c'A') \vdash \mathrm{tp}(d/c'A)$ according to some $\bar{\psi}'$. Then the proof follows exactly as that of Proposition 5.1. □

7 The case $\kappa < \mu^{+\omega}$

To go beyond μ^+, we need to extend the definition of rD to allow for long indiscernible sequences inside B.

7.1 (μ, κ)-sets

Let $|T| < \mu < \kappa$. A (μ, κ)-set is a pair $\mathbf{f} = (B, \mathbf{I})$ where $B \subset M$ has size $< \mu$, $\mathbf{I} = (I_i : i \in u_{\mathbf{f}})$ and for each $i \in u_{\mathbf{f}}$, $I_i = (a^i_\alpha : \alpha < \kappa_i)$ is an indiscernible sequence of countable tuples with $\mu \leq \kappa_i < \kappa$. We furthermore impose that κ_i is regular for all i and $u_{\mathbf{f}}$ is countable.

Let $\mathbf{f} = (B, \mathbf{I})$ be a (μ, κ)-set. We define $B^+_{\mathbf{f}} = B \cup \{a^i_\alpha : i \in u_{\mathbf{f}}, \alpha < \kappa_i\}$. Let $h \in \prod_{i \in u_{\mathbf{f}}} \kappa_i$ such that $h(i) \in \kappa_i$. We define $\mathbf{f}_h$ to be the (μ, κ)-set $(B, \mathbf{I}_h)$ where $\mathbf{I}_h = (I_{i,h} : i \in u_{\mathbf{f}})$ and $I_{i,h} = (a^i_\alpha : h(i) \leq \alpha < \kappa_i)$.

Let $\Theta_{\mathbf{f}}$ be the set of cardinals $\lambda < \kappa$ for which there is a sequence in $\mathbf{I}$ of size λ. For any $\lambda \in \Theta_{\mathbf{f}}$, let $u_\lambda \subseteq u_{\mathbf{f}}$ be the set of indices i such that I_i is of size λ and define $\mathbf{I}_\lambda = ((a^i_\alpha)_{i \in u_\lambda} : \alpha < \kappa)$.

We say that a (μ, κ)-set $(B, \mathbf{I})$ is *smooth* over A if:

(S_1) each sequence $\mathbf{I}_\lambda$ is indiscernible over AB;

(S_2) the sequences $\{\mathbf{I}_\lambda : \lambda \in \Theta_{\mathbf{f}}\}$ are mutually indiscernible over AB.

The following lemma allows to add a sequence to a (μ, κ)-set preserving condition (S_1).

Lemma 7.1. *Let $I = (a_i : i < \lambda)$ and $J = (b_i : i < \lambda)$ be two indiscernible sequences with $\lambda > |T|$, then there are two increasing sequences $(s_i : i < \lambda)$*

and $(t_i : i < \kappa)$ *of ordinals* $< \lambda$ *such that the sequence* $(a_{s_i} {}^\frown b_{t_i} : i < \lambda)$ *is indiscernible.*

Proof. We first build some $(a^*_n : n < \omega)$ and $(b^*_n : n < \omega)$ such that for each $n < \omega$, we have $a^*_n \models \lim(I/IJa^*_{>n}b^*_{>n})$ and $a^*_n \models \lim(J/IJa^*_{\geq n}b^*_{>n})$. Then $(a^*_n {}^\frown b^*_n : n < \omega)$ is indiscernible over IJ. We then build by induction two sequences $(s_i : i < \lambda)$ and $(t_i : i < \lambda)$ such that $(a_{s_i} {}^\frown b_{t_i} : i < \lambda) + (a^*_n {}^\frown b^*_n : n < \omega)$ is indiscernible.

Assume we have chosen $(s_i, t_i : i < i(*))$. Let $s_* = \sup\{s_i : i < i(*)\}$, $t_* = \sup\{t_i : i < i(*)\}$ and let $X = \{a_{s_i} : i < i(*)\}$. As $b^*_0 \models \lim(J/Xa^*_{\geq 0}b^*_{>0})$, there is some $t_{i(*)} \geq t_*$ such that $\mathrm{tp}(b_{t_{i(*)}}/Xa^*_{\geq 0}b^*_{>0}) = \mathrm{tp}(b^*_0/Xa^*_{\geq 0}b^*_{>0})$. Similarly as $a^*_0 \models \lim(I/Xa^*_{>0}b^*_{>0})$, we can find $s_{i(*)} \geq s_*$ such that $\mathrm{tp}(a_{s_{i(*)}}/Xa^*_{>0}b^*_{>0}) = \mathrm{tp}(a^*_0/Xa^*_{>0}b^*_{>0})$.

Then the sequence $(a_{s_i} {}^\frown b_{t_i} : i \leq i(*)) + (a^*_n {}^\frown b^*_n : 0 < n < \omega)$ is indiscernible. On the other hand, we know that the sequence $(a^*_n {}^\frown b^*_n : n < \omega)$ is indiscernible over IJ. It follows that $(a_{s_i} {}^\frown b_{t_i} : i \leq i(*)) + (a^*_n {}^\frown b^*_n : n < \omega)$ is indiscernible, as required. □

This lemma generalizes at once to less than λ sequences with the same proof.

Lemma 7.2. *Let* λ *a regular cardinal, and let* $\mathbf{I} = (I_i : i < \theta)$ *be a family of sequences, each of size* λ*. Assume* $\theta < \lambda$ *and write* $I_j = (a^j_i : i < \lambda)$*. Then there are increasing sequences* $(s^j_i : i < \lambda)$*, for* $j < \theta$ *such that the sequence* $((a^j_{s^j_i})_{j<\theta} : i < \lambda)$ *is indiscernible.*

Lemma 7.3. *Let* A *of size* $< \mu$*, and assume that* $(B, \mathbf{I})$ *satisfies* (S_1) *over* $\emptyset$*, then there is* $h \in \prod_{i \in u_f} \kappa_i$ *such that* f_h *is smooth over* A*.*

Proof. Using the assumption (S_1), and replacing $\mathbf{I}$ by $(\mathbf{I}_\lambda : \lambda \in \Theta_\lambda)$, we may assume that the sequences in $\mathbf{I}$ are of different length.

Renumbering the sequences, assume that $u_f = \theta$ is a countable ordinal and than $\kappa_i < \kappa_j$ for $i < j$. For $\beta < \theta$, let $f|\beta$ be the (μ, κ)-set $(B, (I_i : i < \beta))$. We show by induction on β that there is $h \in \prod_{i<\beta} \kappa_i$ such that $(f|\beta)_h$ is smooth over A.

For limit β, take the supremum. This works as $\mathrm{cf}(\kappa_i) = \kappa_i > \omega$ for each κ_i. Assume we have such an h for β. The set $A' = A + B^+_{f|\beta}$ has size $< \kappa_\beta$. Hence by NIP, there is h_β such that $(a^i_\alpha : h_\beta \leq \alpha < \kappa_i)$ is indiscernible over A'. Let $C = (a^i_\alpha : h_\beta \leq \alpha \leq h_\beta + \omega)$. Then C is countable. By induction, there is h' such that the (μ, κ)-set $(\mathbf{f}|\beta)_{h'}$ is smooth over $A + C$. Without loss $h'(i) \geq h(i)$ for all i. Extend h' by setting $h'(\beta) = h_\beta$. Then $(\mathbf{f}|\beta + 1)_{h'}$ is smooth over A. ☐

7.1.1 Counting types

Say that two (μ, κ)-sets $(B, \mathbf{I})$ and $(B', \mathbf{I}')$ are very similar if $B = B'$ and every limit type of a sequence in $\mathbf{I}$ is equal to the limit type of a sequence in $\mathbf{I}'$. We say that $\mathbf{f}$ and $\mathbf{f}'$ are similar if there is an automorphism σ of M such that $\sigma(\mathbf{f})$ is very similar to $\mathbf{f}'$).

Lemma 7.4. *Assume that μ is strong limit and $\kappa = \aleph_\alpha$. Then there are at most $\mu + |\alpha|^{\aleph_0}$ similarity classes of (μ, κ)-sets.*

Proof. First, we may restrict ourselves to smooth (μ, κ)-sets, since every (μ, κ)-set is very similar to a smooth one. Then to describe a (μ, κ)-set up to conjugancy, we only need to give tp$(B/\emptyset)$, the length of each sequence ($|\alpha|^{\aleph_0}$ choices) and the joint EM-types of them over B ($|T|^{|B|} \leq \mu$ possibilities). ☐

7.2 sD_μ

Definition 7.5. Let sD_μ be the set of quadruples $\mathbf{x} = (p(x), r(y), q(x, y, x'), \Gamma) \in rD$ such that there is a (μ, κ)-set $(B, \mathbf{I})$ with:

$\bullet_4$ $y = (y_i : i < \nu_\mathbf{x})$, $(c_i : i < \nu_\mathbf{x}) \models y$, and for each $i < \nu_\mathbf{x}$, tp$(c_i/c_{<i}M)$ is either finitely satisfiable in B, or is equal to the limit type of one of the sequences in $\mathbf{I}$.

Let $sD^\oplus_\mu = sD_\mu \cap rD^\oplus$. We define $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \leq^1 \mathbf{y}$ for sD_μ as we did for rD.

Remark 7.6. If μ is strong limit, then having fixed $\mathbf{f}$, fixing $\nu = \omega$, the number of possibilities for r is $\leq \mu$.

7.3 Partition theorem, again

Our first task is to extend the partition theorem to work for (μ, κ)-sets.

Theorem 7.7 (Partition theorem II)**.** *Assume that T is countable NIP. Let:*

– $\aleph_0 < \mu$ *and* $\kappa = \mathrm{cf}(\kappa) \geq \beth_\omega(\mu)$*;*

– $\mathbf{f} = (B, \mathbf{I})$ *a (μ, κ)-set, smooth over $\emptyset$, $\Theta_\mathbf{f}$ is finite. For $h \in \prod_{i \in u_\mathbf{f}} \kappa_i$, let B^+_h be $B \cup \{a^i_\alpha : \alpha \in [h(i), \kappa_i)\}$;*

– $(e_i : i < \kappa)$ *a sequence of countable tuples;*
– T_1, T_2 *stationary subsets* κ*;*
– $\Delta \subset L(x_{[e]}, x'_{[e]}; z)$ *finite.*

Then there are two stationary subsets $S_1 \subseteq T_1$*,* $S_2 \subseteq T_2$*,* $h \in \prod_{i \in u_f} \kappa_i$ *and a type* $p \in S_\Delta(B_h^+)$ *such that for* $s \in \mathcal{S}_1$*,* $t \in \mathcal{S}_2$*,* $s < t$*, we have* $\mathrm{tp}_\Delta(e_i, e_j/B_h^+) = q$ *and* $\mathbf{f}_h$ *is smooth over* $e_i{}^\frown e_j$*.*

Proof. Let Δ^1 be a finite set of formulas and $k < \omega$ such that if $(a_i : i < \omega)$ is a (Δ^1, k)-indiscernible sequence over some C and $b \in \mathfrak{C}$, then the set $\{i < \omega : \mathrm{tp}_\Delta(a_i, b/C) \neq \mathrm{tp}_\Delta(a_{i+1}, b/C)\}$ is finite.

By smoothness of $\mathbf{f}$, replacing $\mathbf{I}$ by $(\mathbf{I}_\lambda : \lambda \in \Theta_\mathbf{f})$, we may assume that $u = u_\mathbf{f}$ is finite. Then $\prod \kappa_i$ has cofinality $< \kappa$ so let $\mathcal{F}$ be a cofinal set of size $< \kappa$. For each $i < u$, let s_i be the limit type of I_i. Let J be a Morley sequence of $\bigotimes_{i<u} s_i$ over M and set $C = B \cup J$.

For $h \in \mathcal{F}$, let $\eth_h$ be the following two-player game: a play lasts ω moves. In the l-th move, the antagonists chooses $\mathcal{X}_l \subseteq \kappa$ a club and the protagonist chooses $s_l \in \mathcal{X}_l \cap T_1$. In the end, the protagonists wins the play if $(e_{s_l} : l < \omega)$ is a (Δ^1, k)-indiscernible sequence over C and $\mathbf{f}_h$ is smooth over it. Note that this implies that $(e_{s_l} : l < \omega)$ is a (Δ^1, k)-indiscernible sequence over B_h^+.

<u>Claim</u>: For some h, the protagonists has a winning strategy in the game $\eth_h$.

Proof: Assume not. As each $\eth_h$ is closed for the protagonist, the antagonist must have a winning strategy $\mathbf{st}_h$. Choose $s_\alpha \in \mathcal{S}$ by induction on $\alpha < \kappa$ such that: for any $h \in \mathcal{F}$ and any initial segment of the play of $\eth_h$ in which the antagonists uses $\mathbf{st}_h$ and the protagonists chooses members of $\{s_\beta : \beta < \alpha\}$, the last move of the antagonists is a club $\mathcal{X}$ to which s_α belongs.

Letting $\lambda = |S_\omega(C)|$, as $\kappa > \beth_k(\lambda)$, by Erdös-Rado, there is an increasing sequence $(\alpha(i) : i < \omega)$ of ordinals $< \kappa$ such that $(e_{\alpha(i)} : i < \omega)$ is (Δ^1, k)-indiscernible over B. Then there is h such that $\mathbf{f}_h$ is smooth over it. So the protagonist can play $(e_{\alpha(i)} : i < \omega)$ and win the game $\eth_h$.

Fix $h \in \mathcal{F}$ and a winning strategy $\mathbf{st}$ for $\eth_h$. The end of the proof is then exactly as in Theorem 4.1. □

7.4 The density theorem

Definition 7.8. Let (d, c), where $\mathrm{tp}(c/M)$ is finitely satisfiable in a set of size $< \kappa$. Let $\lambda < \kappa$. We say that (d, c) is λ-good if for every subset $A \subset M$ of size λ, and every formula $\phi(x_{[d]}, c; z)$, there is $e \in M$ such that $\text{tp}(d/ce) \vdash \text{tp}_\phi(d/cA)$ according to some (ψ_ϕ, θ_ϕ).

Proposition 7.9. *Assume that μ is strong limit, $\mu < \kappa < \mu^{+\omega}$. Let $\mathbf{x} \in sD_\mu$, then there is $\mathbf{y} \in sD_\mu$ with $\mathbf{x} \leq^1 \mathbf{y}$.*

Write $\kappa = \mu^{+n}$.

We will prove the proposition in a number of steps. We start with $(d, c, d') \models q_{\mathbf{x}}$. Our first task is to find c' extending c such that (d, c') is κ-good, and $\text{tp}(c'/M)$ satisfies the requirements in the definition of sD_μ. First take $\mathbf{f} = (B, \mathbf{I})$ obtained by trimming $\mathbf{f_x}$ so as to make it smooth. Then replace $\mathbf{f}$ by $(B, (\mathbf{I}_i : 0 < i < n))$ where $\mathbf{I} = \mathbf{I}_{\mu^{+i}}$. Set $\kappa_i = \mu^{+i}$.

First, we can use 900 decomposition to obtain c' such that (d, c') is λ-good for all $\lambda < \mu$. But then, automatically (d, c') is μ-good as in the proof of Theorem 6.1. So we may assume that (d, c) is μ-good.

Let $\lambda = \mu^{+n(*)}$ be the least cardinal for which (d, c) is not λ-good. Then λ is regular. Let $A \prec M$ of size λ and $\phi(x, y; z)$ witness that (d, c) is not λ-good.

Write $I_{n(*)} = (b_i : i < \lambda)$ and write A as an increasing union $A = \bigcup_{i<\lambda} A_i$ with $|A_i| < \lambda$. We may assume that B and all the ranges of the sequences $\{I_i : 0 < i < n(*)\}$ are included in A_0 and that $b_i \in A_{i+1}$ for all $i < \lambda$. For each $i < \lambda$, we can find e_i such that $\text{tp}(d/ce_i) \vdash \text{tp}(d/cA_i e_{<i})$ according to some (ψ_ϕ, θ_ϕ), which we may take to be constant. Define formulas $\phi_1 = \psi_\phi(x, y, t)$ and $\theta_1 = \theta_\phi(t, y; z)$ (where $|t| = |e_i|$). Then again set $\phi_2 = \psi_{\phi_1}(x, y, t)$, $\theta_2 = \theta_{\phi_1}(t, y; t')$. For $l = 1, 2$, let $\tilde{\theta}_l(x, x'; y) = \theta_{\phi_l}(x, y; x')$ and set $\Delta = \{\tilde{\theta}_l : l = 1, 2\}$.

Step 1: Building an indiscernible sequence $(e'_i : i < \lambda)$.

For each $n(*) \leq i < n$, let p_i be the limit type of the sequence I_i. Let $C \subset \mathfrak{C}$ be a Morley sequence of $\bigotimes_i p_i$ over Mdc.

Apply the Partition theorem 7.7 to the sequence $(e_i : i < \lambda)$, the set Δ of formulas and the (μ, κ) set $D = (B \cup C, (I_i : i < n(*)))$. We obtain stationary sets $\mathcal{S}_0$, $\mathcal{S}'_1$, some $h'_1 \in \prod_{i<n(*)} \kappa_i$ and a type $q_0 \in S_\Delta(D^+_{h'_1})$. Then apply it again, with T_1 there equal to $\mathcal{S}'_1$ to obtain stationary sets $\mathcal{S}_1 \subseteq \mathcal{S}'_1$ and $\mathcal{S}_2$, some $h_1 \in \prod_{i<n(*)} \kappa_i$, which we may take to be $\geq h'_1$ and a type $q_1 \in S_\Delta(D^+_{h_1})$. Trimming the sequence, we may assume that $\mathcal{S}_l \supseteq \{\omega\cdot\alpha+3k+l : k < \omega, \alpha < \lambda\}$, for $l = 0, 1, 2$.

Let $h_2 \in \prod_{n(*)<i<n} \kappa_i$ be such that $(B, (I_i : n(*) < i < n))$ is smooth over $A \cup \{e_i : i < \lambda\}$. Let also $h_* \in \prod_{i<n, i\neq n(*)} \kappa_i$ be $h_1 \times h_2$ and define B^+_* as

the union of $B \cup C$ and the ranges of the sequences $\{I_{i,h_*} : i < n, i \neq n(*)\}$. By construction, there is q'_0 such that for any $i \in \mathcal{S}_0$ and $i < j \in \mathcal{S}_1$, $q'_0 = \mathrm{tp}_\Delta(e_i, e_j/B^+_*)$, and define similarly q'_1.

Lemma 7.10. *There is an increasing sequence* $(\alpha_\epsilon : \epsilon < \lambda)$ *of limit ordinals* $< \lambda$ *such that: for every* n *and every finite* $\Delta \subset L(x_{[e],i} : i < \omega)$ *and for every* $0 \leq \epsilon_0 < \cdots < \epsilon_n$ *we can find* $\beta_l \in [\alpha_{\epsilon_l}; \alpha_{\epsilon_{l+1}}) \cap \mathcal{S}_1$ *for* $l < n$ *such that* $(e_{\beta_0}, \ldots, e_{\beta_{n-1}})$ *is a* Δ*-indiscernible sequence.*

Proof. For each such pair (Δ, n), define a game $\partial_{\Delta,n}$ with n moves. In the m-th move, the antagonist chooses an ordinal $\beta_m < \lambda$ which is greater than $\sup\{\gamma_k : k < m\}$ and the protagonist chooses $\gamma_m \in [\beta_m, \lambda) \cap \mathcal{S}_1$. In the end, the protagonist wins if $(e_{\gamma_0}, \ldots, e_{\gamma_{m-1}})$ is a Δ-indiscernible sequence. This game is determined as it is finite, hence there is a winning strategy $\mathbf{st}_{\Delta,n}$ for either the protagonist or the antagonist. Let $E = \{\delta < \lambda$, δ limit & in any initial segment of a game $\partial_{\Delta,n}$ in which all moves made are $< \delta$, the strategy $\mathbf{st}_{\Delta,n}$ gives a next move which is $< \delta\}$. Then E is a club. Let $(\alpha_\epsilon : \epsilon < \lambda)$ list $E \cap \mathcal{S}_1$ in increasing order.

<u>Claim</u>: The protagonist wins each game $\partial_{\Delta,n}$.

Assume that the protagonist looses $\partial_{\Delta,n}$. There is a finite subsequence of $E \cap \mathcal{S}_1$ of size n which is Δ-indiscernible. Then the protagonist can playing that subsequence in the game $\partial_{\Delta,n}$ where the antagonist uses $\mathbf{st}_{\Delta,n}$.

Then the sequence $(\alpha_\epsilon : \epsilon < \lambda)$ has the required properties. □

Let $(\alpha_\epsilon : \epsilon < \lambda)$ as given by the lemma.

We define a structure A^+ as follows:

- The universe of A^+ is $A \cup \{e_i : i < \lambda\}$;
- The language is $L^\oplus = L \cup \{P(z), R(x_{[e]}; z), S_1(x_{[e]}), S_2(x_{[e]})\}$, where $|z| = 1$. (If $|e|$ is infinite, then really we mean that we have for example $\{R(x'; z) : x' \subset x_{[e]} \text{ finite}\}$, but we will ignore this.)
- We interpret $P(z)$ as A, $R(x_{[e]}; z)$ as $\{(e_i, a) : a \in A_i\}$ and $S_l(x_{[e]})$ as $\{e_i : i \in \mathcal{S}_l\}$, for $l \in \{1, 2\}$.

By the lemma and compactness, there is $A^\oplus \subset M$ an elementary extension of A^+ of size λ and a sequence $(e'_\epsilon : \epsilon < \lambda)$ of elements of $A^\oplus$ such that, for $\epsilon < \lambda$:

$\odot_0$ $A^\oplus \models R(e'_\epsilon; a)$ for each $a \in A_\alpha$, where $\alpha < \alpha_\epsilon$;

$\odot_1$ The sequence $(e'_\epsilon : \epsilon < \lambda)$ is indiscernible;

$\odot_2$ The type $\mathrm{tp}_{L^\oplus}(e'_\epsilon / A^+ \cup B^+)$ is finitely satisfiable in $\{e_i : i \in [\alpha_\epsilon, \alpha_{\epsilon+1})\} \cap \mathcal{S}_1$.

Claim 1: We have $\models \psi_{\phi_1}(d, c, e'_\epsilon)$.

Take $\alpha \in [\alpha_{\epsilon+1}, \lambda) \cap \mathcal{S}_2$. Then for all $i \in [\alpha_\epsilon, \alpha_{\epsilon+1}) \cap \mathcal{S}_1$, we have $\theta_{\phi_2}(e_\alpha, c; e_i)$. There is $\gamma < \kappa_{n(*)}$ such that the sequence $(b_i : \gamma \leq i < \kappa_{n(*)})$ is indiscernible over $\{a_i : i \in [\alpha_\epsilon, \alpha]\} + A_0 + C$. Define $h \in \prod_{i<n} \kappa_i$ by $h(n(*)) = \gamma$ and $h(i) = h_*(i)$ otherwise. Then for any $i \in [\alpha_\epsilon, \alpha_{\epsilon+1}) \cap \mathcal{S}_1$, the (μ, κ)-set f_h is smooth over $e_i e_\alpha$. Also $\mathrm{tp}_\Delta(e_i, e_\alpha / B^+_h)$ is constant as i varies in $[\alpha_\epsilon, \alpha_{\epsilon+1}) \cap \mathcal{S}_1$. Therefore, by $\odot_2$:
$\models \theta_{\phi_2}(e_\alpha, c; e'_i)$.

The claim then follows from the fact $\models \psi_{\phi_2}(d, e_\alpha; c)$.

Claim 2: For any $\alpha < \alpha_\epsilon$, $\alpha \in \mathcal{S}_0$, we have $\models \theta_{\phi_1}(e'_\epsilon, c; e_\alpha)$.

For every $\alpha_\epsilon \leq i < \alpha_{\epsilon+1}$, we have $\models \theta_{\phi_1}(e_i, c; e_\alpha)$. Also the type $\mathrm{tp}_\Delta(e_i, e_\alpha / B)$ is constant as i varies. Hence the claim follows.

Claim 3: We have $\mathrm{tp}(d/ce'_\epsilon) \vdash \mathrm{tp}_\phi(d/cA_i)$ for any $i < \alpha_\epsilon$.

We know that $d \models \psi_{\phi_1}(x, c, e'_\epsilon)$. On the other hand, by Claim 2 we have $\psi_{\phi_1}(x, c, e'_\epsilon) \vdash \phi_1(x, c; e_i)$. And then by construction of ϕ_1 and e_i, $\phi_1(x, c; e_i) \vdash \mathrm{tp}_\phi(d/cA_i)$. The claim follows.

Let $\theta_*(t, c; z) = (\exists t')\theta_{\phi_1}(t, c; t') \wedge \theta_\phi(t', c; z)$ so that:

$\boxdot$ $\psi_{\phi_1}(x, c, e'_\epsilon) \wedge \theta_*(e'_\epsilon, c; z) \to \phi(x, c; z)$.

Step 2: $\lim((e'_i)_{i<\lambda}/Mc)$ and $\mathrm{tp}(d/Mc)$ are not weakly-orthogonal.

By the initial assumption on A, there is a global type $p_1(z)$, finitely satisfiable in A such that, letting $p'_1 = p_1|Mc$ and $p'_2(x) = \mathrm{tp}(d/Mc)$, both $p'_1(z) \wedge p'_2(x) \wedge \phi(x, c; z)$ and $p'_1(z) \wedge p'_2(x) \wedge \neg\phi(x, c; z)$ are consistent. Without loss, assume that $p_1|Mdc \vdash \phi(d, c; z)$. Let $\mathcal{D}$ be an ultrafilter on $A^{|z|}$ whose limit type is p_1.

Note that by the minimality assumption on λ (or equivalently, the existence of the e_i's), no $A_i^{|z|}$ is in $\mathcal{D}$. For each $b \in A^{|a|}$, let $\epsilon(b)$ be the minimal $\epsilon < \lambda$ such that $b \in A_\epsilon^{|a|}$.

We can find $(e', a') \in \mathfrak{C}$ realizing the type $s(t_{[e]}, z_{[a]}) = \{\zeta(t, z) : \zeta(t, z) \in L(Mc), \{b \in A^{|a|} :\models \zeta(e_{\epsilon(b)}, b)\} \in \mathcal{D}\}$. Note:

$\boxtimes_1$ We have $\models \theta_*(e', c; a')$;

$\boxtimes_2$ The type $\mathrm{tp}(e'/Mc)$ is the limit type of the sequence $(e'_\epsilon : \epsilon < \lambda)$.

Let $(a'_1, e'_1) \models s(t, z) \wedge \neg\phi(d; z)$. Then by $\boxdot$ and $\boxtimes_1$, we must have: $\models \neg\psi_{\phi_1}(d, c, e'_1)$.

On the other hand, for every $\epsilon < \lambda$, we have $\models \psi_{\phi_1}(d, c, e'_\epsilon)$, hence we have $\models \psi_{\phi_1}(d, c, e')$. Thus we conclude that $\lim((e'_i)_{i<\lambda}/Mc)$ and $\mathrm{tp}(d/Mc)$ are not weakly-orthogonal.

Step 3: Getting (d, c) to be κ-good.

We let $c' = c^\frown e'_1{}^\frown e'_2$ where e'_2 realizes $\lim((e'_\epsilon)_{\epsilon<\lambda})$ over everything. Then $\mathrm{tp}(e'_1/Mc) = \mathrm{tp}(e'_2/Mc)$, but $\mathrm{tp}(d/ce'_1) \neq \mathrm{tp}(d/ce'_2)$. We update $\mathbf{f}$ by adding the sequence $(e'_\epsilon : \epsilon < \lambda)$ to it.

Then we iterate the construction. This must end after less than $|T|^+$ steps and then the (d, c) we obtain is κ-good.

Step 4: Conclusion.

We assume that (d, c) is κ-good. Then for any $A \subset M$, there is $e_A \in M$ such that $\mathrm{tp}(d/e_A A) \vdash \mathrm{tp}(d/cA)$ according to some (ψ_ϕ, θ_ϕ), which we can take to be constant. Now we follow the proof of Proposition 5.1, except that we use the second partition theorem instead of the first one.

This ends the proof of Proposition 7.9.

Theorem 7.11. *Assume that $\kappa = \mu^{+n}$ for some integer n and μ strong limit of uncountable cofinality, then for any countable type p over M, there is $\mathbf{x} \in sD^{\oplus}_{\mu}$ such that $p_{\mathbf{x}}$ extends p.*

Therefore there are at most μ many types over M up to conjugancy.

8 The general case

What goes wrong in the proof above when $\kappa \geq \mu^{+\omega}$? Two things: first, the number of sequences in $\mathbf{I}$ will become infinite, and second the first λ for which (d, c) is not λ-good will not necessarily be regular (in general, it is either regular or of countable cofinality).

We now deal with those two issues.

Let $uD^{\oplus}_{\mu}$ be the set of triples $(p(\bar{x}), r(\bar{y}), q(\bar{x}, \bar{y}, \bar{x}'), \Gamma) \in sD_{\mu}$ where $\bar{x} = (x_i : i < w_x)$ such that:

$\bullet_5$ for every formula $\phi_1(\bar{x}, \bar{y}; z) \equiv \phi(x_\rho, y_\nu; z)$, there is a *duplicate* $\phi_2(\bar{x}, \bar{y}; z) \equiv \phi(x_{\rho_1}, y_{\nu_1}; z)$ such that $\phi_2 \in \Gamma$.

By a duplicate of $\phi_1(\bar{x}, \bar{y}; z)$, we mean a formula $\phi_2(\bar{x}, \bar{y}; z)$ such that:

– there are $\rho_1, \rho_2 \in {}^n w_x$ and $\nu_1, \nu_2 \in {}^m w_x$ for some n, m;

– $\phi_1(\bar{x}, \bar{y}; z) \equiv \phi(x_{\rho_1}, y_{\nu_2}; z)$ and $\phi_2(\bar{x}, \bar{y}; z) \equiv \phi(x_{\rho_2}, y_{\nu_2}; z)$;

– for every $b \in M$, we have $\models \phi(d_{\rho_1}, c_{\nu_1}; b) \leftrightarrow \phi(d_{\rho_2}, c_{\nu_2}; b)$.

If $\mathbf{x} \leq \mathbf{y}$ are in sD_{μ}. We write $\mathbf{x} \leq^2 \mathbf{y}$ if for every formula $\phi_1(x_{[c_x]}, y_{[c_y]}; z)$ there is a duplicate $\phi_2(x_{[c_y]}, y_{[c_y]}; z)$ which is in $\Gamma_{\mathbf{y}}$.

When ϕ is a formula of the form $\phi(x, c; z)$ we define $\mathrm{tp}_{\pm\phi}(d/cM)$ as the set $\{\phi(x, c; b) : b \in M \text{ and } \models \phi(d, c; b)\}$.

8.1 The density theorem

To obtain λ regular, we replace the notion of λ-good with a local one.

Definition 8.1. Let (d, c), where $tp(c/M)$ is finitely satisfiable in a set of size $< \kappa$. Let $\lambda < \kappa$. Let $\psi(x, c, b) \in \mathrm{tp}(d/cM)$ and $\phi(x, y; z)$ a formula. We say that (d, c) is (λ, ψ, ϕ)-good if for every subset $A \subset M$ of size λ, there is $e = e_1 \frown \ldots \frown e_k \in M$ and $\phi_*(x, y; z_1 \ldots z_k)$ a finite boolean combination of instances of $\phi(x, y; z)$ such that $\models \phi_*(d, c; e)$ and $\psi(x, c, b) \wedge \phi_*(x, c; e) \vdash \mathrm{tp}_{\pm\phi}(d/cM)$.

The point of this definition is the following:

Lemma 8.2. *For any ψ and ϕ, the least λ for which (d, c) is not (λ, ψ, ϕ)-good is regular.*

Proof. Let λ be singular, and assume that (d, c) is (λ, ψ, ϕ)-good for every $\mu < \lambda$. Let $A \subset M$ of size λ and write $A = \bigcup_{i < cf(\lambda)} A_i$ where $|A_i| < \lambda$. For each $i < cf(\lambda)$, fix some finite e_i such that $\psi(x, c, b) \wedge \mathrm{tp}_{\pm\phi}(d/c, e_i) \vdash \mathrm{tp}_{\pm\phi}(d/cM)$.

Then take a finite e such that $\psi(x, c, b) \wedge \mathrm{tp}_{\pm\phi}(d/c, e) \vdash \mathrm{tp}_{\pm\phi}(d/c\{e_i : i < \mathrm{cf}(\lambda)\})$. Then also $\psi(x, c, b) \wedge \mathrm{tp}_{\pm\phi}(d/c, e) \vdash \mathrm{tp}_{\pm\phi}(d/cA)$ as required. □

Proposition 8.3. *Let* $\mathbf{x} = (p, r, q, \emptyset) \in uD_\mu$ *and* $\mathbf{f_x} = (B, \mathbf{I})$, $\mathbf{I} = (I_i : i < u)$.

Let $(d, c, d') \models q$. *Let* $\psi(x, c; b) \in \mathrm{tp}(d/cM)$ *and* $\phi(x, y; z) \in L$. *Assume that there is* $A \subset M$ *of size* $\lambda = \mathrm{cf}(\lambda) < \kappa$ *such that:*

$\odot_1$ *for no finite* $p_0 \subset \mathrm{tp}_{\pm\phi}(d/cM)$ *do we have* $p_0(x) \wedge \psi(x, c; b) \vdash \mathrm{tp}_{\pm\phi}(d/cA)$*;*

$\odot_2$ *there is* $\phi_*(x, c; \bar{z})$ *a finite boolean combination of instances of* ϕ *such that for all* $A_0 \subset M$ *of size* $< \lambda$*, there is* e *such that* $d \models \phi_*(x, c; e)$ *and* $\phi_*(x, c; e) \wedge \psi(x, c; b) \vdash \mathrm{tp}_{\pm\phi}(d/cA_0)$.

Then there is an indiscernible sequence $(e'_i : i < \lambda)$ *in* M *such that the types* $\lim((e'_i)_{i<\lambda}/cM)$ *and* $\mathrm{tp}_{\pm\phi}(d/cM) \cup \{\psi(x, c; b)\}$ *are not weakly-orthogonal.*

Proof. Let $\tilde{u} \subseteq u$ be a finite subset such that for every i in the range of ν, $\mathrm{tp}(c_i/c_{<i}M)$ is either finitely satisfiable in B or is the limit type of a sequence in $\{I_i : i \in \tilde{u}\}$.

We follow very closely Steps 1+2 of the proof of Proposition 7.9. In fact, this is easier.

So we start with $A = \bigcup_{i<\lambda} A_i$. For each $i < \lambda$ we have some finite e_i such that $d \models \phi_*(x, c; e_i)$ and $\psi(x, c; b) \wedge \phi_*(d, c; e_i) \vdash \mathrm{tp}_{\pm\phi}(d/cA_i)$. We impose the same conditions on the A_i's as in 7.9.

In Step 1, we care only about the (μ, κ)-set $(B, (I_i : i \in \tilde{u}))$. The formulas ψ_ϕ's are conjunction of ψ and boolean combination of instances of ϕ. Then the proof goes through. We obtain an indiscernible sequence e'_i such that $\psi(x, c; b) \wedge \phi_*(x, c; e'_i) \vdash \mathrm{tp}_{\pm\phi}(d/cA_i)$ for all $i < \lambda$, and $d \models \phi_*(x, c; e'_i)$.

We explain Step 2, which is now weaker, namely we show that $s = \lim((e'_i)_{i<\lambda}/Mc)$ and $p = \mathrm{tp}_{\pm\phi}(d/Mc) \cup \{\psi(x, c; b)\}$ are not weakly-orthogonal.

First note that $s_z \otimes p_x \vdash \phi_*(x, c; z)$. We show now that $s(z) \cup p(x) \cup \{\neg\phi_*(x, c; z)\}$ is also consistant. If not, then by compactness, there is some finite $p_0 \subset \mathrm{tp}_{\pm\phi}(d/cM)$ and finite $s_0 \subset s$ such that $s_0(z) \wedge p_0(x) \wedge \psi(x, c; b) \vdash \phi_*(x, c; z)$. In particular, for $\epsilon < \lambda$ big enough so that $e'_\epsilon \models s_0(z)$, we have $p_0(x) \wedge \psi(x, c; b) \vdash \phi_*(x, c; e_\epsilon)$. Hence $p_0(x) \wedge \psi(x, c; b) \vdash \mathrm{tp}_{\pm\phi}(d/cA)$ which contradicts $\odot_1$. □

Note that in the conclusion, we have "$\mathrm{tp}_{\pm\phi}(d/cM)$" and not "$\mathrm{tp}(d/cM)$". So we cannot use the proposition to increase c to some c' as we did in the proof of Proposition 7.9. Instead, we will first increase d to $d' = d\hat{\ }d''$ where d'' is a duplicate of d (or rather of the relevant finite subtuple) which has the same ϕ-type over cM and then we can increase c.

Proposition 8.4. *Let $\kappa > \mu$ be regular, and $\mu \geq \beth_\omega$ strong limit, singular. Let $\mathbf{x} \in \mathrm{uD}_\mu$, then there is some $\mathbf{y} \in \mathrm{uD}_\mu$ with $\mathbf{x} \leq^2 \mathbf{y}$.*

Proof. Let $(d, c, d') \models q_{\mathbf{x}}$ where $d = (d_i : i < \omega)$, with the d_i's being singletons and $c = (c_i : i < \nu)$ as in the definition of sD_μ, ν is an ordinal. Write $\mathbf{f}_{\mathbf{x}}$ as $(B, \mathbf{I})$ and $\mathbf{I} = (I_i : i < u)$.

Let $(x_i : i < \omega + \omega)$ be variables of size 1 and for every $n < \nu + \omega$, y_n is a variable of size ω. Let $\mathcal{L}$ be the set of formulas of the form $\phi(\bar{x}_\rho, \bar{y}_\nu; z)$, $\rho \in {}^{<\omega}(\omega + \omega)$, $\nu \in {}^{<\omega}(\nu + \omega)$, z a finite tuple. List $\mathcal{L}$ as $(\phi_n : n < \omega)$ such that every formula appears infinitely many times.

We build inductively finite tuples $c_{\omega+n}$ and $\tilde{d}_n = (d_{n,1}, \dots, d_{n,k_n})$, and formulas $(\psi_\rho(x_\rho; \bar{y}_{\nu_\rho}) : \rho \in {}^{<\omega}(\omega+\omega))$. At stage n, let $d' = d^\frown\tilde{d}_0{}^\frown \cdots {}^\frown\tilde{d}_{n-1}$, which we write as $d' = (d_i : i < \omega + N)$ (so $N = \sum_{l<n} k_l$) and $c' = c^\frown c_\omega{}^\frown \cdots {}^\frown c_{\omega+n-1}$. Adding dummy elements, we may assume that each c_k is indexed by ω.

Let $\phi(\bar{x}_\rho, \bar{y}_\nu; z) = \phi_n$. If the range of ρ is not in $\omega + N$, or the range of ν is not in $\omega + n$, then do nothing. Otherwise it makes sense to consider the formula $\phi(d', c'; z)$ and the type $\mathrm{tp}_{\pm\phi}(d'/c'M)$. If the formula ψ_ρ has not yet be defined, then set $\psi_\rho = \top$. Let $\psi(\bar{x}_\rho, \bar{y}_\nu) = \psi_\rho$. If (d', c') is (κ, ψ, ϕ)-good, then do nothing. Otherwise, let λ be minimal such that (d', c') is not (κ, ψ, ϕ)-good. Then λ is regular.

<u>Case 1</u>: $\lambda < \mu$.

Let $A \subset M$ of size λ witness that (d', c') is not (λ, ψ, ϕ)-good. Then as in 900, there is a global type $s(z)$ finitely satisfiable in A such that $\psi(x, c') \wedge \mathrm{tp}_{\pm\phi}(d'/c'M) \wedge s(z) \wedge \phi(x, c; z)^{\mathbf{t}}$ is consistant for $\mathbf{t} = 0, 1$.

Let $n_k = \lg(\rho)$ and take $\tilde{d}_n = (d_{n,1}, \dots, d_{n,k_n}) \in \mathfrak{C}$ such that:

$\odot_1$ $\mathrm{tp}_{\pm\phi}(\tilde{d}_n/c'M) = \mathrm{tp}_{\pm\phi}(d_\rho/c'M)$;

$\odot_2$ $\tilde{d}_n \models \psi(\bar{x}_\rho, c')$;

$\odot_3$ $\psi(\tilde{d}_n, c') \wedge \mathrm{tp}_{\pm\phi}(d'/c'M) \wedge s(z) \wedge \phi(\tilde{d}_n, c; z)^{\mathbf{t}}$ is consistant for $\mathbf{t} = 0, 1$.

Let $\mathbf{t} \in \{0, 1\}$ be such that $s|M\tilde{d}_n c' \vdash \phi(\tilde{d}_n, c'; z)^{\mathbf{t}}$. Let c_0^- realize $s(z)|Mc' \cup \{\phi(\tilde{d}_n, c'; z)^{1-\mathbf{t}}\}$ and let $c_1^- \models s|Mc'\tilde{d}_n, c_0^-$. Finally set $c_{\nu+n} = c_0^-{}^\frown c_1^-$ and $\psi_{(N+1,\dots,N+k_n)} = \psi \wedge \phi(x_{(N+1,\dots,N+k_n)}, c'; c_1^-)^{\mathbf{t}} \wedge \phi(x_{(N+1,\dots,N+k_n)}, c'; c_0^-)^{1-\mathbf{t}}$.

<u>Case 2</u>: $\lambda \geq \mu$.

By definition of λ, for every $A \subset M$ of size $< \lambda$, there is a finite $p_0 \subset \mathrm{tp}_{\pm\phi}(d'/c'M)$ such that $p_0(x) \wedge \psi(x, c') \vdash \mathrm{tp}_{\pm\phi}(d'/c'M)$. As $\lambda > |T|$, we can assume that $|p_0|$ is constant and more precisely that there is some formula $\phi_*(x, c'; \bar{z})$, a boolean combinations of formulas $\phi(x, c'; z_i)$ such that for each

$A \subset M$ of size $< \lambda$, there is $e_A \in M$ with $d \models \phi(x, c'; e_A)$ and $\phi(x, c'; e_A) \wedge \psi(x, c') \vdash \mathrm{tp}_{\pm\phi}(d'/c'M)$.

Hence we can apply Proposition 8.3 and we obtain an indiscernible sequence $(e'_i : i < \lambda)$ such that if $s(z)$ denotes the limit type of $(e'_i : i < \lambda)$, then $s(z) \wedge \mathrm{tp}_{\pm\phi}(d'/c'M) \wedge \psi(x, c') \wedge \phi_*(x, c'; z)^t$ is consistant for $t = 0, 1$. Then end as in Case 1.

In the end, we obtain a tuple $d' = (d_i : i < \omega + \omega)$ a family $c' = (c_i : i < \nu + \omega)$ of countable tuples and formulas $\{\psi_\rho\}$. Let $\phi(x_\rho, y_\nu; z)$ be a formula. Let n be such that it is equal to ϕ_n, n big enough. If (d', c') is not $(\kappa, \psi_\rho, \phi)$-good, then at stage n, we have created some tuples d_{ρ_1}, c_n and a formula ψ_{ρ_1}. Let $\phi^1 = \phi(x_{\rho_1}, y_\nu; z)$ and n_1 big enough so that $\phi^1 = \phi_{n_1}$. Then again, if (d', c') is not $(\kappa, \psi_\rho, \phi)$, then at stage n_1 we have created tuples d_{ρ_2}, c_{n_1} and a formula ψ_{ρ_2}. As in 900, NIP implies that this must stop at some finite stage (at each step, the formula ψ_{ρ_k} witnesses an extra spliting).

Therefore we find ρ_* such that $\mathrm{tp}_{\pm\phi}(d_\rho/c'M) = \mathrm{tp}_{\pm\phi}(d_{\rho_*}/c'M)$ and a duplicate $\phi_*(x_{\rho_*}, y_\nu; z)$ of ϕ such that (d', c') is $(\kappa, \psi_{\rho_*}, \phi_*)$-good.

To obtain $\mathfrak{y}$ such that $\mathfrak{x} \leq^2 \mathfrak{y}$ all we have to do is to obtain the type $q_{\mathfrak{y}}$. For this, we proceed essentially as in Proposition 5.1. The only difference is that we sometimes have to replace a formula $\phi(x, y; z)$ by a duplicate to get (ψ_ϕ, θ_ϕ). This poses no difficulty. □

Theorem 8.5. *Assume $\kappa = \aleph_\alpha > \beth_\omega$. Then for every $\mathfrak{x} \in \mathrm{uD}_\mu$, there is $\mathfrak{y} \in \mathrm{uD}^\oplus_\mu$ such that $\mathfrak{x} \leq \mathfrak{y}$.*

In particular, there are at most $\beth_\omega + |\alpha|^{|T|}$ countable types over M up to automorphisms.

Proof. For the first point, all we have left to do is prove that countable limit of a $\leq^2$-increasing chain of elements of uD_μ is in $\mathrm{uD}^\oplus_\mu$. This is done by adapting the proof of Proposition 3.2.

The second point then follows from Lemma 7.4 and Remark 7.6. □

9 What is left?

Shelah goes on to prove some applications of those decompositions. Namely, he shows that any type over M is the limit of an indiscernible sequence of size κ inside M, gives a criterion for saturation in terms of realizing limit types of indiscernible sequences, and proves the generic pair conjecture. However those results require a stronger decomposition than the one we proved here. Namely we would need in the definition of say rD to replace "$\text{tp}(d/cd') \vdash \text{tp}(d/cM)$" by "$\text{tp}(d/cd') \vdash \text{tp}(d/cd'M)$". However, I do not understand the proof of this stronger decomposition. The problem is in adapting Proposition 5.1.