On Shelah 900

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We give an account of the type decomposition theorem of [Sh :900]. This is exactly Shelah's proof, extracted from the paper.

Let T be an NIP theory, κ a cardinal.

Theorem 0.1. *Let $|T| \leq \theta < cf(\kappa)$, and M a κ-saturated model. Let $\bar{d} \in^{\theta\geq} \mathbb{C}$, then there exists $C \in [\mathbb{C}]^\theta$ such that, letting $p = tp(\bar{d}/MC)$:*

- *$tp(C/M)$ is B-invariant for some $B \subset M$ of cardinality $< \kappa$.*
- *For every $A \subset M$ of cardinality $< \kappa$, there is $D \subset M$, $|D| = \theta$, such that $p|_{DC} \vdash p|_{AC}$.*

We let M and $\bar{d}$ be as above.

Lemma 0.2. *There does not exist a sequence $(\bar{c}_\alpha = (c^0_\alpha, c^1_\alpha))_{\alpha<\theta^+}$ of finite tuples such that, for all $\alpha < \theta^+$:*

1. *$tp(c^0_\alpha/MC_\alpha) = tp(c^1_\alpha/MC_\alpha)$,*
2. *$tp(\bar{c}_\alpha/MC_\alpha)$ is B_α-invariant for some $B_\alpha \subseteq M$ of size κ,*
3. *$tp(\bar{d}, c^\alpha_0/MC_\alpha) \neq tp(\bar{d}, c^\alpha_1/MC_\alpha)$,*

Where $C_\alpha = \bigcup_{\beta<\alpha}\{\bar{c}_\beta\}$.

Proof. Assume such a sequence exists. Notice that by the second assumption, for all $\alpha < \theta^+$, the tuples c^α_0 and c^α_1 have the same type over $M \cup \bigcup_{\beta\neq\alpha}\{\bar{c}_\beta\}$.

Without loss of generality, we may assume that there is a formula $\phi(\bar{x}; y, z)$ such that for all $\alpha < \theta^+$, there is $d_\alpha \in M \cup C_\alpha$ with $\phi(\bar{d}; c^0_\alpha, d_\alpha) \wedge \neg\phi(\bar{d}; c^1_\alpha, d_\alpha)$. For each α, let $f(\alpha) < \alpha$ be such that $d_\alpha \in M \cup C_{f(\alpha)}$.

By Fodor's lemma, there is $S \subset \theta^+$ cofinal and $\beta < \theta^+$ such that $f(\alpha) = \beta$ for all $\alpha \in S$.

Now, by the second hypothesis, for any function $\eta \in^S \{0, 1\}$, the sequence $(c^\alpha_{\eta(\alpha)})$ has the same type as (c^α_0) over $M \cup C_\beta$ (because each c^α_0, c^α_1 realizes the unique invariant extension of $tp(c^\alpha_1/M)$ to $M \cup C_\alpha$). Thus, the type $\bigwedge_{\alpha\in S} \phi(\bar{x}; c^\alpha_0, d_\alpha)^{\langle\eta(\alpha)=0\rangle}$ is consistent. This contradicts NIP. □

Returning to the theorem, construct by induction a maximal sequence $(\bar{c}_\alpha)$ satisfying the properties of the lemma. This construction must stop at some $\lambda < \theta^+$. We can thus find a $B \subset M$, $|B| < \kappa$ containing all the B_α. Let C be the union of the $\bar{c}_\alpha$, and let $p = tp(\bar{d}/MC)$.

Lemma 0.3. *Let $q \in S(MC)$ be D-invariant for some $D \subset M$ of cardinality $< \kappa$. Then q is weakly orthogonal to p.*

Proof. Suppose not. Then there are c_0, c_1 realizations of q such that $tp(c_0/MC\bar{d}) \neq tp(c_1/MC\bar{d})$. One of these types, say $tp(c_0/MC\bar{d})$, is not the invariant extension of q (note that as M is κ-saturated, the invariant extension is uniquely defined). Now let c_1' realize the invariant extension of q to $MCc_0\bar{d}$. We thus have :

– $tp(c_0/MC) = tp(c_1'/MC)$,
– $tp(c_0, c_1'/MC)$ is D-invariant,
– $tp(c_0/MC\bar{d}) \neq tp(c_1'/MC\bar{d})$.

This contradicts the maximality of C. □

Let $A \subset M$ of cardinality $< \kappa$. By the previous lemma, p is orthogonal to all types q finitely satisfiable in A. Let $\phi(\bar{x}, y)$ be a formula with parameters in C ($lg(\bar{x}) = lg(\bar{d})$). By compactness, there is $\psi(\bar{x}) \in p$ such that any $\bar{e} \models \psi(\bar{x})$, any $q \in S(MC)$ finitely satisfiable in A, and any $c \models q$, we have

$$\models \phi(\bar{d}, c) \leftrightarrow \phi(\bar{e}, c).$$

This in particular applies to any $c \in A$.

Letting $\phi(\bar{x}, y)$ vary, let $D \subset M$ be of size θ, such that $D \cup C$ contains all the parameters of the corresponding $\psi(x)$. Then D satisfies the conclusion of the theorem.

This finishes the proof.