Type decompositions in NIP theories

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Definition

A formula $\phi(x; y)$ has the independence property if one can find some infinite set B such that for every $C \subseteq B$, there is y_C such that for $x \in B$,

$$\phi(x; y_C) \iff x \in C.$$

A theory is *NIP* if no formula has the independence property.

Example

- *Stable theories,*
- *o-minimal,*
- $\mathbb{Q}_p$,
- *ACVF.*

T is a complete **countable** theory.
$S(M)$: space of types in countably many variables over M.
Recall:

Fact

T is stable if and only if, for all $M \models T$, $|S(M)| \leq |M|^{\aleph_0}$.

(GCH) If T is unstable, then for every κ, there is M of size κ such that $|S(M)| = 2^\kappa = \kappa^+$.

Shelah's idea: instead of counting types, count types up to automorphisms.

Let M be saturated.
$S_{aut}(M)$: quotient of $S(M)$ under the action of $Aut(M)$.
$f(\kappa) = |S_{aut}(M)|$, where M is saturated of size κ. (So f is only defined when $2^{<\kappa} = \kappa$, κ is regular.)

Observations

- *$f(\kappa)$ is bounded iff T is stable. In this case $f(\kappa) \leq 2^{\aleph_0}$.*
- *If T has IP, then $f(\kappa) = 2^\kappa$.*
- *For T =DLO, counting only 1-types instead of countable types, we have:*
  $f_1(\aleph_0) = 6$;
  $f_1(\aleph_\alpha) = 2 \cdot |\alpha| + 6$.

Theorem (Shelah)

If T is NIP, and $\kappa = \aleph_\alpha \geq \beth_\omega$, then

$$f(\kappa) \leq |\alpha|^{\aleph_0} + \beth_\omega.$$

Finitely satisfiable types.

Definition

$p \in S(M)$ is finitely satisfiable in $N \prec M$, if:
$|N| < |M|$;
for every formula $\phi(x; d) \in p$, there is $a \in N$ such that $M \models \phi(a; d)$.

In particular, such a p is invariant under $Aut(M/N)$.

Fact

There are at most $2^{<\kappa} = \kappa$ finitely satisfiable types, up to automorphisms.

In fact, such a p is determined up to automorphisms by $\mathrm{tp}(N)$ and $p^{(\omega)}|_N$.

Types weakly orthogonal to finitely satisfiable types.

Lemma

Let $p \in S(M)$ and $a \models p$. Assume that p is weakly orthogonal to every finitely satisfiable type, then for every small $A \subset M$, there is $e_A \in M$ such that $\mathrm{tp}(a/e_A) \vdash \mathrm{tp}(a/A)$.

In general, given a type $p \in S(M)$, we have to decompose p.

Proposition

(NIP) Let $p \in S(M)$ and $a \models p$. Then there is $b \in \mathfrak{C}$, such that:
– $\mathrm{tp}(b/M)$ is finitely satisfiable in some $N \subset M$;
– for any $A \subset M$, there is $e_A \in M$ with $\mathrm{tp}(a/be_A) \vdash \mathrm{tp}(a/bA)$.

Proof for κ weakly compact

- **Start** with $p \in S(M)$ any type.
- **Extract a finitely satisfiable component** Find $b \in \mathfrak{C}$ such that $\mathrm{tp}(b/M)$ is finitely satisfiable and $\mathrm{tp}(a/bM)$ is weakly orthogonal to $q|Mb$ for any $q \in S(M)$ finitely satisfiable. Hence for every small $A \subset M$, we have some $e_A \in M$ such that $\mathrm{tp}(a/be_A) \vdash \mathrm{tp}(a/bA)$.
- By weak compactness, we may assume that $\mathrm{tp}(e_A/Aab)$ is increasing,
  i.e., there is $e \in \mathfrak{C}$ such that $\mathrm{tp}(e_A/Aab) = \mathrm{tp}(e/Aab)$.
- **Replace** a by $a\hat{\ }e$ and iterate ω times.

Proof for κ weakly compact

In the end, we have extended a to some a' and we have b', e' such that:

- $\mathrm{tp}(b'/M)$ is finitely satisfiable in some small N;
- $a' \equiv_M e'$;
- for any small $A \subset M$, there is $e_A \equiv_{Aa'b'} e'$ such that $\mathrm{tp}(a'/b'e_A) \vdash \mathrm{tp}(a'/b'A)$.

Then $\mathrm{tp}(a'/M)$ is determined up to automorphisms by $\mathrm{tp}(N)$, $q^{(\omega)}|_N$ (where $q = \mathrm{tp}(b'/M)$), $\mathrm{tp}(a'e'/N)$.

Honest definitions

Replace non-orthogonality by commuting.

If p and q are invariant types, we can define $p(x) \otimes q(y)$ as $\mathrm{tp}(a, b/M)$ where $b \models q$ and $a \models p|Mb$.
We say that p and q *commute* if $p(x) \otimes q(y) = q(y) \otimes p(x)$.

Using NIP, there is a way to generalize this definition to the case where only p is invariant and q is any type over M.

Remark: If p and q are weakly-orthogonal, then they commute.

Proposition

(NIP) A type $p \in S(M)$ commutes with every finitely satisfiable type in M if and only if:
For any small $A \subset M$, and formula $\phi(x;y)$, there is a formula $\psi(x;z)$ and $e_A \in M$ such that:

$$\phi(A;a) \subseteq \psi(M;e_A) \subseteq \phi(M;a).$$

Problem: there does not seem to be a corresponding notion of *decomposition*.

Let $p \in S(M)$ and $a \models p$. Let M_p denote the expansion of M obtained by making all the sets $\phi(M; a)$ definable.

Lemma

If M_p is saturated, then p commutes with any type finitely satisfiable in M.

Remark: This generalizes the fact that a definable type commutes with every finitely satisfiable type.

Now let N be any model and $p \in S(N)$. Take a saturated extension $N_p \prec M_{p_0}$. Then we can apply the previous proposition to $p_0 \in S(M)$ and drag the result down to N.

We obtain:

Theorem (Chernikov-S.)

(NIP) Let $p \in S(N)$, and $\phi(x; y)$ a formula. Then there is a formula $\psi(x; z)$ such that for any finite $A \subseteq N$, we can find $e_A \in N$ such that:

$$\phi(A; a) \subseteq \psi(N; e_A) \subseteq \phi(N; a).$$

The same thing is true for a type over an arbitrary set B, instead of model N, with the same proof.

A. Chernikov and P. Simon
Externally definable sets and dependent pairs.

S. Shelah
Dependent theories and the generic pair conjecture.

S. Shelah
Dependent dreams: recounting types.