

# A note on generically stable measures and *fs*g groups

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## Abstract

We prove (Proposition 2.1) that if  $\mu$  is a generically stable measure in an *NIP* theory, and  $\mu(\phi(x, b)) = 0$  for all  $b$  then for some  $n$ ,  $\mu^{(n)}(\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$ . As a consequence we show (Proposition 3.2) that if  $G$  is a definable group with *fs*g in an *NIP* theory, and  $X$  is a definable subset of  $G$  then  $X$  is generic if and only if every translate of  $X$  does not fork over  $\emptyset$ , precisely as in stable groups, answering positively Problem 5.5 from [3]

## 1 Introduction and preliminaries

This short paper is a contribution to the generalization of stability theory and stable group theory to *NIP* theories, and also provides another example where we need to resort to measures to prove statements (about definable sets and/or types) which do not explicitly mention measures. The observations in the current paper can and will be used in the future to sharpen existing results around measure and *NIP* theories (and this is why we wanted to record the

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observations here). Included in these sharpenings will be: (i) replacing average types by generically stable types in a characterization of strong dependence in terms of measure and weight in [6], and (ii) showing the existence of “external generic types” (in the sense of Newelski [5]), over any model, for *fs* groups in *NIP* theories, improving on Lemma 4.14 and related results from [5].

If  $p(x) \in S(A)$  is a stationary type in a stable theory and  $\phi(x, b)$  any formula, then we know that  $\phi(x, b) \in p|_{\mathfrak{C}}$  if and only if  $\models \bigwedge_{i=1, \dots, n} \phi(a_i, b)$  for some independent realizations  $a_1, \dots, a_n$  of  $p$  (for some  $n$  depending on  $\phi(x, y)$ ). Hence  $\phi(x, b) \notin p|_{\mathfrak{C}}$  for all  $b$  implies that (and is clearly implied by) the inconsistency of  $\bigwedge_{i=1, \dots, n} \phi(a_i, y)$  for some (any) independent set  $a_1, \dots, a_n$  of realizations of  $p$ . This also holds for generically stable types in *NIP* theories (as well as for generically stable types in arbitrary theories, with definition as in [7]). In [6], an analogous result was proved for “average measures” in strongly dependent theories. Here we prove it (Proposition 2.1) for generically stable measures in arbitrary *NIP* theories, as well as giving a generalization (Remark 2.2).

The *fs* condition on a definable group  $G$  is a kind of “definable compactness” assumption, and in fact means precisely this in *o*-minimal theories and suitable theories of valued fields (and of course stable groups are *fs*). Genericity of a definable subset  $X$  of  $G$  means that finitely many translates of  $X$  cover  $G$ . Proposition 2.1 is used to show that for  $X$  a definable subset of an *fs* group  $G$ ,  $X$  is generic if and only if every translate of  $X$  does not fork over  $\emptyset$ . This is a somewhat striking extension of stable group theory to the *NIP* environment.

We work with an *NIP* theory  $T$  and inside some monster model  $\mathfrak{C}$ . If  $A$  is any set of parameters, let  $L_x(A)$  denote the Boolean algebra of  $A$ -definable sets in the variable  $x$ . A *Keisler measure* over  $A$  is a finitely additive probability measure on  $L_x(A)$ . Equivalently, it is a regular Borel probability measure on the compact space  $S_x(A)$ . We will denote by  $\mathfrak{M}_x(A)$  the space of Keisler measures over  $A$  in the variable  $x$ . We might omit  $x$  when it is not needed or when it is included in the notation of the measure itself (e.g.  $\mu_x$ ). If  $X$  is a sort, or more generally definable set, we may also use notation such  $L_X(A)$ ,  $S_X(A)$ ,  $\mathfrak{M}_X(A)$ , where for example  $S_X(A)$  denote the complete types over  $A$  which contain the formula defining  $X$  (or which “concentrate on  $X$ ”).

**Definition 1.1.** A type  $p \in S_x(A)$  is *weakly random* for  $\mu_x$  if  $\mu(\phi(x)) > 0$  for any  $\phi(x) \in L(A)$  such that  $p \vdash \phi(x)$ . A point  $b$  is weakly random for  $\mu$  over  $A$  if  $\text{tp}(b/A)$  is weakly random for  $\mu$ .

We briefly recall some definitions and properties of Keisler measures, refer-

ring the reader to [4] for more details.

If  $\mu \in \mathfrak{M}_x(\mathfrak{C})$  is a global measure and  $M$  a small model, we say that  $\mu$  is  $M$ -invariant if  $\mu(\phi(x, a) \Delta \phi(x, a')) = 0$  for every formula  $\phi(x, y)$  and  $a, a' \in \mathfrak{C}$  having the same type over  $M$ . Such a measure admits a Borel defining scheme over  $M$ : For every formula  $\phi(x, y)$ , the value  $\mu(\phi(x, b))$  depends only on  $\text{tp}(b/M)$  and for any Borel  $B \subset [0, 1]$ , the set  $\{p \in S_y(M) : \mu(\phi(x, b)) \in B \text{ for some } b \models p\}$  is a Borel subset of  $S_y(M)$ .

Let  $\mu_x \in \mathfrak{M}(\mathfrak{C})$  be  $M$ -invariant. If  $\lambda_y \in \mathfrak{M}(\mathfrak{C})$  is any measure, then we can define the *invariant extension* of  $\mu_x$  over  $\lambda_y$ , denoted  $\mu_x \otimes \lambda_y$ . It is a measure in the two variables  $x, y$  defined in the following way. Let  $\phi(x, y) \in L(\mathfrak{C})$ . Take a small model  $N$  containing  $M$  and the parameters of  $\phi$ . Define  $\mu_x \otimes \lambda_y(\phi(x, y)) = \int f(p) d\lambda_y$ , the integral ranging over  $S_y(N)$  where  $f(p) = \mu_x(\phi(x, b))$  for  $b \in \mathfrak{C}$ ,  $b \models p$  (this function is Borel by Borel definability). It is easy to check that this does not depend on the choice of  $N$ .

If  $\lambda_y$  is also invariant, we can also form the product  $\lambda_y \otimes \mu_x$ . In general it will not be the case that  $\lambda_y \otimes \mu_x = \mu_x \otimes \lambda_y$ .

If  $\mu_x$  is a global  $M$ -invariant measure, we define by induction:  $\mu_{x_1 \dots x_n}^{(n)}$  by  $\mu_{x_1}^{(1)} = \mu_{x_1}$  and  $\mu_{x_1 \dots x_{n+1}}^{n+1} = \mu_{x_{n+1}} \otimes \mu_{x_1 \dots x_n}^{(n)}$ . We let  $\mu_{x_1 x_2 \dots}^{(\omega)}$  be the union and call it the *Morley sequence* of  $\mu_x$ .

Special cases of  $M$ -invariant measures include definable and finitely satisfiable measures. A global measure  $\mu_x$  is *definable* over  $M$  if it is  $M$ -invariant and for every formula  $\phi(x, y)$  and open interval  $I \subset [0, 1]$  the set  $\{p \in S_y(M) : \mu(\phi(x, b)) \in I \text{ for some } b \models p\}$  is open in  $S_y(M)$ . The measure  $\mu$  is *finitely satisfiable* in  $M$  if  $\mu(\phi(x, b)) > 0$  implies that  $\phi(x, b)$  is satisfied in  $M$ . Equivalently, any weakly random type for  $\mu$  is finitely satisfiable in  $M$ .

**Lemma 1.2.** *Let  $\mu_x \in \mathfrak{M}_x(\mathfrak{C})$  be definable over  $M$ , and  $p(x) \in S_x(\mathfrak{C})$  be weakly random for  $\mu$ . Let  $\phi(x_1, \dots, x_n)$  be a formula over  $\mathfrak{C}$ . Suppose that  $\phi(x_1, \dots, x_n) \in p^{(n)}$ . Then  $\mu^{(n)}(\phi(x_1, \dots, x_n)) > 0$ .*

*Proof.* We will carry out the proof in the case where  $\mu$  is definable (over  $M$ ), which is anyway the case we need. Note that  $p^{(m)}$  is  $M$ -invariant for all  $m$ . The proof of the lemma is by induction on  $n$ . For  $n = 1$  it is just the definition of weakly random. Assume true for  $n$  and we prove for  $n + 1$ . So suppose  $\phi(x_1, \dots, x_n, x_{n+1}) \in p^{(n+1)}$ . This means that for  $(a_1, \dots, a_n)$  realizing  $p^{(n)}|M$ ,  $\phi(a_1, \dots, a_n, x) \in p$ . So as  $p$  is weakly random for  $\mu$ ,  $\mu(\phi(a_1, \dots, a_n, x)) = r > 0$ . So as  $\mu$  is  $M$ -invariant,  $\text{tp}(a'_1, \dots, a'_n/M) = \text{tp}(a_1, \dots, a_n/M)$  implies

$\mu(\phi(a'_1, \dots, a'_n, x)) = r$  and thus also  $r - \epsilon < \mu(\phi(a'_1, \dots, a'_n, x))$  for any small positive  $\epsilon$ . By definability of  $\mu$  and compactness there is a formula  $\psi(x_1, \dots, x_n) \in tp(a_1, \dots, a_n/A)$  such that  $\models \psi(a'_1, \dots, a'_n)$  implies  $0 < r - \epsilon < \mu(\phi(a'_1, \dots, a'_n, x))$ . By induction hypothesis,  $\mu^{(n)}(\psi(x_1, \dots, x_n)) > 0$ . So by definition of  $\mu^{(n+1)}$  we have that  $\mu^{(n+1)}(\phi(x_1, \dots, x_n, x_{n+1})) > 0$  as required.  $\square$

A measure  $\mu_{x_1, \dots, x_n}$  is *symmetric* if for any permutation  $\sigma$  of  $\{1, \dots, n\}$  and any formula  $\phi(x_1, \dots, x_n)$ , we have  $\mu(\phi(x_1, \dots, x_n)) = \mu(\phi(x_{\sigma.1}, \dots, x_{\sigma.n}))$ . A special case of a symmetric measure is given by powers of a generically stable measure. The following is Theorem 3.2 of [4]:

**Fact 1.3.** *Let  $\mu_x$  be a global  $M$ -invariant measure. Then the following are equivalent:*

1.  $\mu_x$  is both definable and finitely satisfiable (necessarily over  $M$ ),
2.  $\mu_{x_1, \dots, x_n}^{(n)}|_M$  is symmetric for all  $n < \omega$ ,
3. for any global  $M$ -invariant Keisler measure  $\lambda_y$ ,  $\mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x$ ,
4.  $\mu$  commutes with itself:  $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$ .

If  $\mu_x$  satisfies one of those properties, we say it is generically stable.

If  $\mu \in \mathfrak{M}_x(A)$  and  $D$  is a definable set such that  $\mu(D) > 0$ , we can consider the *localisation* of  $\mu$  at  $D$  which is a Keisler measure  $\mu_D$  over  $A$  defined by  $\mu_D(X) = \mu(X \cap D)/\mu(D)$  for any definable set  $X$ .

We will use the notation  $Fr(\theta(x), x_1, \dots, x_n)$  to mean

$$\frac{1}{n} |\{i \in \{1, \dots, n\} : \models \theta(x_i)\}|.$$

The following is a special case of Lemma 3.4. of [4].

**Proposition 1.4.** *Let  $\phi(x, y)$  be a formula over  $M$  and fix  $r \in (0, 1)$  and  $\epsilon > 0$ . Then there is  $n$  such that for any symmetric measure  $\mu_{x_1, \dots, x_{2n}}$ , we have*

$$\mu_{x_1, \dots, x_{2n}}(\exists y (|Fr(\phi(x, y), x_1, \dots, x_n) - Fr(\phi(x, y), x_{n+1}, \dots, x_{2n})| > r)) \leq \epsilon.$$

## 2 Main result

**Proposition 2.1.** *Let  $\mu_x$  be a global generically stable measure. Let  $\phi(x, y)$  be any formula in  $L(\mathfrak{C})$ . Suppose that  $\mu(\phi(x, b)) = 0$  for all  $b \in \mathfrak{C}$ . Then there is  $n$  such that  $\mu^{(n)}(\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$ .*

*Moreover,  $n$  depends only on  $\phi(x, y)$  and not on  $\mu$ .*

*Proof.* Let  $\mu_x$  be a global generically stable measure and  $M$  a small model over which  $\phi(x, y)$  is defined and such that  $\mu_x$  is  $M$ -invariant. Assume that  $\mu(\phi(x, b)) = 0$  for all  $b \in \mathfrak{C}$ . For any  $k$ , define

$$W_k = \{(x_1, \dots, x_n) : \exists y(\wedge_{i=1..k} \phi(x_i, y))\}.$$

This is a definable set. We want to show that  $\mu^{(n)}(W_n) = 0$  for  $n$  big enough. Assume for a contradiction that this is not the case.

Let  $n$  be given by Proposition 1.4 for  $r = 1/2$  and  $\epsilon = 1/2$ . Consider the measure  $\lambda_{x_1, \dots, x_{2n}}$  over  $M$  defined as being equal to  $\mu^{(2n)}$  localised on the set  $W_{2n}$  (by our assumption, this is well defined). As the measure  $\mu^{(2n)}$  is symmetric and the set  $W_{2n}$  is symmetric in the  $2n$  variables, the measure  $\lambda$  is symmetric. Let  $\chi(x_1, \dots, x_{2n})$  be the formula “ $(x_1, \dots, x_{2n}) \in W_{2n} \wedge \forall y(|Fr(\phi(x, y), x_1, \dots, x_n) - Fr(\phi(x, y), x_{n+1}, \dots, x_{2n})| \leq 1/2)$ ”. By definition of  $n$ , we have  $\lambda(\exists y(|Fr(\phi(x, y), x_1, \dots, x_n) - Fr(\phi(x, y), x_{n+1}, \dots, x_{2n})| > 1/2)) \leq 1/2$ . Therefore  $\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) > 0$ .

As  $\mu$  is  $M$ -invariant, we can write

$$\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) = \int_{q \in S_{x_1, \dots, x_n}(M)} \mu^{(n)}(\chi(q, x_{n+1}, \dots, x_{2n})) d\mu^{(n)},$$

where  $\mu^{(n)}(\chi(q, x_{n+1}, \dots, x_{2n}))$  stands for  $\mu^{(n)}(\chi(a_1, \dots, a_n, x_{n+1}, \dots, x_{2n}))$  for some (any) realization  $(a_1, \dots, a_n)$  of  $q$ . As  $\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) > 0$ , there is  $q \in S_{x_1, \dots, x_n}$  such that

(\*)  $\mu^{(n)}(\chi(q, x_{n+1}, \dots, x_{2n})) > 0$ .

Fix some  $(a_1, \dots, a_n) \models q$ . By (\*), we have  $(a_1, \dots, a_n) \in W_n$ . So let  $b \in \mathfrak{C}$  such that  $\models \bigwedge_{i=1..n} \phi(a_i, b)$ . Again by (\*), we can find some  $(a_{n+1}, \dots, a_{2n})$  weakly random for  $\mu^{(n)}$  over  $Ma_1 \dots a_n b$  and such that

(\*\*)  $\models \chi(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n})$ .

In particular, for  $j = n+1, \dots, 2n$ ,  $a_j$  is weakly random for  $\mu$  over  $Mb$  hence  $\models \neg \phi(a_j, b)$ . But then  $|Fr(\phi(x, b); a_1, \dots, a_n) - Fr(\phi(x, b); a_{n+1}, \dots, a_{2n})| = 1$ . This contradicts (\*\*).  $\square$

*Remark 2.2.* The proof above adapts to showing the following generalization: Let  $\mu_x$  be a global generically stable measure,  $\phi(x, y)$  a formula in  $L(\mathfrak{C})$ . Let  $\Sigma(x)$  be the partial type (over the parameters in  $\phi$  together with a small model over which  $\mu$  is definable) defining  $\{b : \mu(\phi(x, b)) = 0\}$ . Then for some  $n$ :  $\mu^{(n)}(\exists y(\Sigma(y) \wedge \phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$ .

### 3 Generics in *fsg* groups

Let  $G$  be a definable group, without loss defined over  $\emptyset$ . We call a definable subset  $X$  of  $G$  left (right) generic if finitely many left (right) translates of  $X$  cover  $G$ , and a type  $p(x) \in S_G(A)$  is left (right) generic if every formula in  $p$  is. We originally defined ([2])  $G$  to have “finitely satisfiable generics”, or to be *fsg*, if there is some global complete type  $p(x) \in S_G(\mathfrak{C})$  of  $G$  every left  $G$ -translate of which is finitely satisfiable in some fixed small model  $M$ .

The following summarizes the situation, where the reader is referred to Proposition 4.2 of [2] for (i) and Theorem 7.7 of [3] and Theorem 4.3 of [4] for (ii), (iii), and (iv).

**Fact 3.1.** *Suppose  $G$  is an *fsg* group. Then*

(i) *A definable subset  $X$  of  $G$  is left generic iff it is right generic, and the family of nongeneric definable sets is a (proper) ideal of the Boolean algebra of definable subsets of  $G$ ,*

(ii) *There is a left  $G$ -invariant Keisler measure  $\mu \in \mathfrak{M}_G(\mathfrak{C})$  which is generically stable.*

(iii) *Moreover  $\mu_x$  from (ii) is the unique left  $G$ -invariant global Keisler measure on  $G$  as well as the unique right  $G$ -invariant global Keisler measure on  $G$ ,*

(iv) *Moreover  $\mu$  from (ii) is generic in the sense that for any definable set  $X$ ,  $\mu(X) > 0$  iff  $X$  is generic.*

Remember that a definable set  $X$  (or rather a formula  $\phi(x, b)$  defining it) forks over a set  $A$  if  $\phi(x, b)$  implies a finite disjunction of formulas  $\psi(x, c)$  each of which divide over  $A$ , and  $\psi(x, c)$  is said to divide over  $A$  if for some  $A$ -indiscernible sequence  $(c_i : i < \omega)$  with  $c_0 = c$ ,  $\{\phi(x, c_i) : i < \omega\}$  is inconsistent.

**Proposition 3.2.** *Suppose  $G$  is *fsg* and  $X \subseteq G$  a definable set. Then  $X$  is generic if and only if for all  $g \in X$ ,  $g \cdot X$  does not fork over  $\emptyset$  (if and only if for all  $g \in G$ ,  $X \cdot g$  does not fork over  $\emptyset$ ).*

*Proof.* Left to right: It suffices to prove that any generic definable set  $X$  does not fork over  $\emptyset$ , and as the set of nongenerics forms an ideal it is enough to prove that any generic definable set does not divide over  $\emptyset$ . This is carried out in (the proof of) Proposition 5.12 of [3].

Right to left: Assume that  $X$  is nongeneric. We will prove that for some  $g \in G$ ,  $g \cdot X$  divides over  $\emptyset$  (so also forks over  $\emptyset$ ).

Let  $\mu_x$  be the generically stable  $G$ -invariant global Keisler measure given by Fact 3.1. Let  $M_0$  be a small model such that  $\mu$  does not fork over  $M_0$  (namely, as  $\mu$  is generic, every generic formula does not fork over  $M_0$ ) and  $X$  is definable over  $M_0$ . Let  $\phi(x, y)$  denote the formula defining  $\{(x, y) \in G \times G : x \in y \cdot X\}$ . So  $\phi$  has additional (suppressed) parameters from  $M_0$ . Note that for  $b \in G$ ,  $\phi(x, b)$  defines the set  $b \cdot X^{-1}$ . As  $X$  is nongeneric, so is  $X^{-1}$  so also  $b \cdot X^{-1}$  for all  $b \in G$ . Hence, as  $\mu$  is generic,  $\mu(\phi(x, b)) = 0$  for all  $b$ . By Proposition 2.1, for some  $n$   $\mu^{(n)}(\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$ . Let  $p$  be any weakly random type for  $\mu$  (which in this case amounts to a global generic type, which note is  $M_0$ -invariant). So by Lemma 1.2 the formula  $\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y)) \notin p^{(n)}$ . Let  $(a_1, \dots, a_n)$  realize  $p^{(n)}|M_0$ . Then  $(a_1, \dots, a_n)$  extends to an  $M_0$ -indiscernible sequence  $(a_i : i = 1, 2, \dots)$ , a Morley sequence in  $p$  over  $M_0$ , and  $\models \neg \exists y(\phi(a_1, y) \wedge \dots \wedge \phi(a_n, y))$ . So in particular  $\{\phi(a_i, y) : i = 1, 2, \dots\}$  is inconsistent. Hence the formula  $\phi(a_i, y)$  divides over  $M_0$ , so also divides over  $\emptyset$ . But  $\phi(a_1, y)$  defines the set  $a_1 \cdot X$ , so  $a_1 \cdot X$  divides over  $\emptyset$  as required.  $\square$

Recall that we called a global type  $p(x)$  of a  $\emptyset$ -definable group  $G$ , left  $f$ -generic if every left  $G$ -translate of  $p$  does not fork over  $\emptyset$ .

We conclude the following (answering positively Problem 5.5 from [3] as well as strengthening Lemma 4.14 of [1]):

**Corollary 3.3.** *Suppose  $G$  is fsg and  $p(x) \in S_G(\mathfrak{C})$ . Then the following are equivalent:*

- (i)  $p$  is generic,
- (ii)  $p$  is left (right)  $f$ -generic,
- (iii) (Left or right)  $Stab(p)$  has bounded index in  $G$  (where left  $Stab(p) = \{g \in G : g \cdot p = p\}$ ).

*Proof.* The equivalence of (i) and (ii) is given by Proposition 3.2 and the definitions. We know from [2], Corollary 4.3, that if  $p$  is generic then  $Stab(p)$  is precisely  $G^{00}$ . Now suppose that  $p$  is nongeneric. Hence there is a definable set  $X \in p$  such that  $X$  is nongeneric. Let  $M$  be a small model over which

$X$  is defined. Note that the *fs**g* property is invariant under naming parameters. Hence  $G$  is *fs**g* in  $Th(\mathfrak{C}, m)_{m \in M}$ . By Proposition 3.2 (as well as what is proved in “Right to left” there), for some  $g \in G$ ,  $g \cdot X$  divides over  $M$ . As  $X$  is defined over  $M$  this means that there is an  $M$ -indiscernible sequence  $(g_\alpha : \alpha < \bar{\kappa})$  (where  $\bar{\kappa}$  is the cardinality of the monster model) and some  $n$  such that  $g_{\alpha_1} \cdot X \cap \dots \cap g_{\alpha_n} \cdot X = \emptyset$  whenever  $\alpha_1 < \dots < \alpha_n$ . This clearly implies that among  $\{g_\alpha \cdot p : \alpha < \bar{\kappa}\}$ , there are  $\bar{\kappa}$  many types, whereby  $Stab(p)$  has unbounded index.  $\square$

## References

- [1] A. Conversano and A. Pillay, Connected components of definable groups and  $\mathcal{o}$ -minimality I, submitted.
- [2] E. Hrushovski, Y. Peterzil, and A. Pillay, Groups, measures, and the *NIP*, Journal of AMS, 21 (2008), 563 - 596.
- [3] E. Hrushovski and A. Pillay, On *NIP* and invariant measures, to appear in J. European Math. Soc.
- [4] E. Hrushovski, A. Pillay, and P. Simon, Generically stable and smooth measures in *NIP* theories, to appear in Transactions AMS.
- [5] L. Newelski, Model theoretic aspects of the Ellis semigroup, to appear in Israel J. Mathematics.
- [6] A. Pillay, Strong dependence, weight and measures, submitted.
- [7] A. Pillay and P. Tanovic, Generic stability, regularity and quasiminimality, *Models, Logics, and Higher-Dimensional Categories: A Tribute to the Work of Mihaly Makkai*, CRM proceedings and Lecture Notes volume 53, AMS, 2011.