On geometric homogeneous structures

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From permutation groups to model theory, in honor of Dugald Macpherson
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Introduction

M is an ω-categorical structure (with elimination of quantifiers in a relational language L).

- $f_n(M)$=number of orbits of $Aut(M)$ on unordered sets of size n = number of non-isomorphic substructures of M of size n.

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  = number of non-isomorphic substructures of M of size n.

Examples:

- $M = (\mathbb{Q}; \leq)$: $f_n = 1$
- $M = (S^1, C(x, y, z))$: circular order; $f_n = 1$
- random/Rado graph: (G, R); $f_n \sim 2^{\binom{n}{2}}/n! \approx c^{n^2}$
- $(M; \leq_1, \leq_2)$, homogeneous permutation: $f_n = n! \approx c^{n \log n}$
- C-relation/tree: $f_n \approx c^n$.

Theorem (Cameron)

- *The function $f_n(M)$ is non-decreasing with n.*
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Theorem (Macpherson '85)

There is some $c > 1$ such that if M is primitive either $f_n(M) = 1$ for all n, or $f_n(M) \geq c^n/p(n)$ for some polynomial p.

Macpherson obtains $c \approx 1.148$. This was improved to $c \approx 1.324$ by Merola (2001). The result is false for $c > 2$.

Conjecture (Macpherson)

One can take $c = 2$.

Cameron and Macpherson have observed that all known (primitive) structures with f_n of exponential growth are tree-like or order-like and have raised the question of classifying them.

NIP structures

• A structure M has the independence property if there is a formula $\phi(\bar{x};\bar{y})$ and sets of tuples $A = \{\bar{a}_i : i < \omega\}$ and $B = \{\bar{b}_j : j < \omega\}$ such that $\phi(\bar{x};\bar{y})$ induces a random bipartite graph on $A \times B$.
Otherwise, we say M is NIP.

• By the Sauer-Shelah theorem, a finitely homogeneous M is NIP if and only if the number of types over a finite set A is bounded by $p(|A|)$ for some polynomial $p(X)$.

Theorem (Macpherson)

- *If M has the independence property, then there is a polynomial p of degree at least 2 such that $f_n \geq 2^{p(n)}$.*
- *If M is finitely homogeneous and there is some $\epsilon > 0$ such that $f_n > 2^{n^{1+\epsilon}}$, then M has the independence property.*

NIP STRUCTURES

Conjecture / Hope

NIP finitely homogeneous structures are classifiable. At least, there are only countably many of them.

The stable case (more generally ω-categorical ω-stable) is well understood by work of Zilber, Lachlan, Cherlin, Harrington, Hrushovski, Evans.

Key theorem: construction of linear orders

Theorem

Let M be ω-categorical, NIP and unstable. Then (over parameters) there is a definable equivalence relation E on M with infinitely many classes, and a definable linear order on the quotient M/E.

Theorem

If M is primitive, then one of the following occurs:

- $f_n \geq 2^n/p(n)$ *for some polynomial* p*;*
- *M is one of the 5 reducts of DLO;*
- *M is strictly stable (stable, not ω-stable).*

Corollary

1. *Macpherson's conjecture is true for finitely homogeneous structures.*
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Theorem

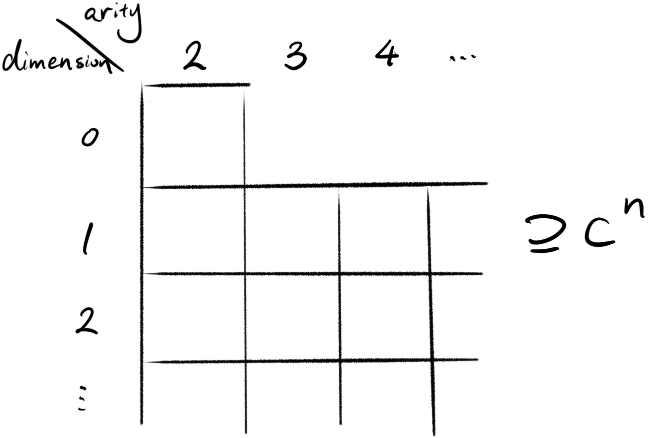
Assume $f_n(M) = o(c^n)$ *for some* $c < 1.57$*, then there is a reduct* M^* *of* M *such that:*

- M^* *is* ω*-stable;*
- $f_n(M) = f_n(M^*)$ *for all* n.

Towards a classification of NIP homogeneous structures

Two parameters of interest:

- The "pseudo-arity": minimal arity of a structure in which M can be interpreted.
  (I expect this to be equal the minimal arity of a Ramsey expansion of M.)
- The dimension: number of independent linear orders.


arity
dimension
2
3
4
...
0
1
2
⋮
$\geq c^n$

Classification of (pseudo-)binary structures

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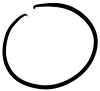
Theorem

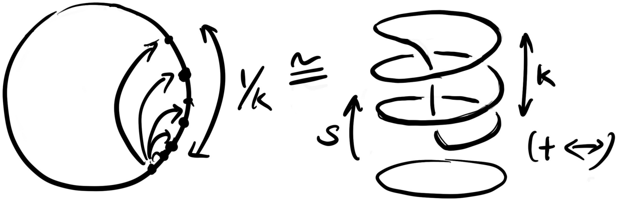
Given $n < \omega$, there are finitely many structures M (up to bi-definability) which are ω-categorical, NIP, rank 1, primitive and have at most n 4-types.
One can give an explicit list.

arity 2 dimension 1

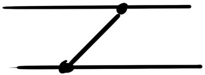
$(+ \leftrightarrow)$



$(+ \leftrightarrow)$

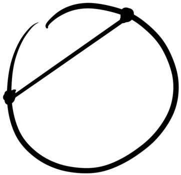


arity 2 dimension 2



$(M, \leq_1, \leq_2)$

$(M, \leq_1, \leq_2, R)$



+ finite covers

$\vdots$

Some applications

- Homogeneous multi-orders.

Proposition

If $(M; \leq_1, \ldots, \leq_n)$ is a primitive homogeneous multi-order such that no two orders are equal, up to reversal, then M is the Fraïssé limit of sets with n orders.

(The case $n = 2$ was done by Cameron, and $n = 3$ by Braunfeld.)

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- Reducts: one can easily determine the reducts of any structure in the catalog. There are always finitely many.
  - Reducts of $(M; \leq_1, \leq_2)$ were classified by Linman and Pinsker: there are 39 of them.
  - Quotients of circular orders by an equivalence relation with finite classes have no non-trivial proper reducts.

Sketch of the proof

Step 1 Linear orders interpretable in a binary structure are very constrained.

Proposition

Let $(V; \leq)$ be a transitive ω-categorical linear order. Assume that one cannot define (with parameters) a C-structure with convex, bounded, balls on V. Then any $\emptyset$-definable closed subset of V^n is a boolean combination of inequalities between variables $x_i \leq x_j$.

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Conjecture

The same is true if V does not admit a $\emptyset$-definable C-structure with convex bounded balls.

Sketch of the proof

Step 2 If M is unstable, we find a linear order on some $V \subseteq M$, definable over parameters. By glueing together all definable linear orders in the structure, we obtain a finite cover $\tilde{M}$ of M composed of linear and circular orders.

Sketch of the proof

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Step 3 Any extra structure on $\tilde{M}$ can be defined *locally* by stable relations. Because of the rank 1 assumption, such relations have to be finite equivalence relations.
By standard topological arguments, local finite equivalence relations are classified by finite covers of the space.

Speculations on higher arities

- Possible structures of closed definable sets on ω-categorical linear orders can be completely classified.
- In arity 3, there are interpretable trees, but the structure on their branches is binary. One can then prove similar things as in the binary cases with trees instead of linear orders.
- In arity n, one can find a tree with structure on the branches of arity $n-1$ and one can classify the structures by induction.
- Some of those ideas might be interesting outside of the NIP context.
  Question: If M is finitely homogeneous and does not have a formula $\phi(\bar{x}, \bar{y}, \bar{z})$ which induces a (non-degenerate) C-structure on an infinite set, is M interpretable in a ternary structure?