# Distal and non-distal NIP theories

Pierre Simon

January 24, 2011

**Abstract**

We study one way in which stable phenomena can exist in an $NIP$ theory. We start by defining a notion of 'pure instability' that we call 'distality' in which no such phenomenon occurs. O-minimal theories and the p-adics for example are distal. Next, we try to understand what happens when distality fails. Given a type $p$ over a sufficiently saturated model, we extract, in some sense, the stable part of $p$ and define a notion of stable-independence which is implied by non-forking and has bounded weight.

As an application, we show that the expansion of a model by traces of externally definable sets from some adequate indiscernible sequence eliminates quantifiers.

## Contents

# 1 Introduction

We study one way in which stability and order can interact in an $NIP$ theory. More precisely, we are interested in the situation where stability and order are intertwined. We start by giving some very simple examples illustrating what we mean.

Consider $M_0 \models \text{DLO}$. A type of $S_1(M_0)$ is determined by a cut in $M_0$ and two types corresponding to different cuts are orthogonal. If we take now $M_1$ a model of some o-minimal theory, it is true that a 1-type is determined by a cut, but in general, types that correspond to different cuts are not orthogonal. However this is true over indiscernible sequences in the following sense: assume $\langle a_t : t < \omega + \omega \rangle \subset M_1$ is an indiscernible sequence. By $NIP$, the sequences of types $\langle \text{tp}(a_t) : t < \omega \rangle$ and $\langle \text{tp}(a_{\omega+t}) : t < \omega \rangle$ converge is $S(M_1)$. Then the two limit types are orthogonal (this follows from dp-minimality, see 2.28). An indiscernible sequence with that property will be called *distal*[1]. A theory is distal if all indiscernible sequences are distal. So any o-minimal theory is distal.

Distality for an indiscernible sequence can be considered as an opposite notion to that of total indiscernibility.

Let now $M_2$ be a model of ACVF (or any other C-minimal structure) and consider an indiscernible sequence $(a_i)_{i<\omega}$ of 1-tuples. Two different behaviors are possible: either the sequence is totally indiscernible, this happens if and

[1]Thanks to Itay Kaplan for suggesting the name.

only if $\mathrm{val}(a_i - a_j) = \mathrm{val}(a_{i'} - a_{j'})$ for all $i \neq j$, $i' \neq j'$, or the sequence is distal. Again, this will follow from the results in Section 2, but could be proved directly. So $M_2$ is neither stable nor distal; the two phenomena exist but do not interact in a single indiscernible sequence of points.

Consider now a fourth structure (a 'colored order') $M_3$ in the language $L_3 = \{\leq, E\}$: $M_3$ is totally ordered by $\leq$ and $E$ defines an equivalence relation, each $E$ class being dense-co-dense with respect to $\leq$. Now an indiscernible sequence of elements from different $E$ classes is neither totally indiscernible nor distal. Given two limit types $p_x$ and $q_y$ of different cuts in such a sequence, the type $p_x \cup q_y$ is consistent with $xEy$ and with $\neg xEy$. Here it is clear that the 'stable part' of a type should be its $E$-class.

The idea behind the work in this paper is that every ordered indiscernible sequence in an $NIP$ theory should look like a colored order: there is an order for which different cuts are orthogonal and a something stable on top of it which does not see the order (see Section 3).

#### 1.0.1 A word about measures

Keisler measures will be used a little in this work, however the reader not familiar with them can skip all parts referring to measures without harm. For this reason, we will be very brief in recalling some facts about them and refer the reader to [6] and [5]. They however give some understanding of the intuition behind some definitions and results. We explain this now.

A Keisler measure (or simply a measure) is a Borel probability measure on a type space $S_x(A)$. Basic definitions for types (non-forking, invariance, coheir, Morley sequence etc.) generalize naturally to measures (see [6] and [5]). Of interest to us is the notion of *generically stable measure*. A measure is generically stable if it is both definable and finitely satisfiable over some small set. Equivalently, its Morley sequence is totally indiscernible. Such measures are defined and studied by Hrushovski, Pillay and the author in [5]. Furthermore, it is shown in [12] that some general constructions give rise to them, and in this sense they are better behaved than the more natural notion of generically stable type.

This paper can be considered as an attempt to understand where generically stable measures come from. What stable phenomena do generically stable measures detect? What does the existence of generically stable measures in some particular theory tell us about types? The first test question was: Can we characterize theories which have non-trivial generically stable measures?

Here “non-trivial” means “non-smooth”: a measure is smooth if it has a unique extension to any bigger set of parameters. This question is answered in Section 2: a theory has a non-smooth generically stable measure if and only if it is not distal.

The main tool at our disposal to link measures to indiscernible sequences is the construction of an average measure of an indiscernible segment (see [5] Lemma 3.4 or [12] Section 3 for a more elaborate construction). Such a measure is always generically stable. The intuition we suggest is that the ‘order’ component of the sequence is evened out in the average measure and only the ‘stable’ component remains.

#### 1.0.2 Organization of the paper and main results

The paper is organized as follows. The first section contains some basic facts about $NIP$ theories and Keisler measures. There is nothing new there, and proofs are omitted. Section 2 studies distal theories. They are defined as theories in which every indiscernible sequence is distal, as explained above. We show that this condition can also be seen through invariant types and generically stable measures. The main results can be summarized by the following theorem.

**Theorem 1.1.** *The following are equivalent:*

- *$T$ is distal,*
- *Any two invariant types that commute are orthogonal,*
- *All generically stable measures are smooth.*

*Furthermore, it is enough to check any one of those conditions in dimension 1.*

As a consequence, o-minimal theories and the p-adics are distal as are more generally any dp-minimal theory with no generically stable type.

Section 3 can be read almost independently of the previous one: it contains a study of the intermediate case of an $NIP$ theory that is neither stable nor distal. We deal with the problem of understanding the ‘stable part’ (or the ‘non-distal’) part of a type. We define a notion of s-domination ($s$ for *stable*), first for points inside an indiscernible sequence and then for any point some $|T|^+$-saturated base $M$. The intuition is that if $a_*$ s-dominates $a$ over $M$, then $a_*$ contains the stable part of $a$ over $M$. We then define a notion of s-independence denoted $a \downarrow_M^s b$ which says (intuitively) that the stable parts of $a$ and $b$ are independent. This is a symmetric notion and is implied by forking-independence. Also, it has bounded weight. We use it to show that two commuting types behave with respect to each other like types in a stable theory (we recover some definability and uniqueness of the non-forking extension). Note that in a distal theory, all those notions are trivial.

As an application, we prove the following 'finite-co-finite theorem' (Theorem 3.24) and give an application of it to the study of externally definable sets.

**Theorem 1.2** (Finite-co-finite theorem)**.** *Let $I = I_1 + I_2 + I_3$ be indiscernible, $I_1$ and $I_3$ being infinite. Assume that $I_1 + I_3$ is $A$-indiscernible and take $\phi(x; a) \in L(A)$, then the set $B = \{b \in I_2 :\vDash \phi(b; a)\}$ is finite or co-finite.*

We end this paper with a short section containing some additional remarks which might lead to future developments. We allow ourselves to state there some results without proofs as it is not clear if they are useful for anything.

Our Bible concerning $NIP$ theories are Shelah's paper's [8], [9], [7], [11] and [10]. We will however use ideas only from the first two. All the basic insights about indiscernible sequences was taken from there (although the important result on shrinking indiscernible sequences originates in [1]).

In fact, we realized after having done most of this work that the idea of 'domination' for indiscernible sequences was already in Shelah's work: in Section 2 of [9] in a slightly different wording and with a very different purpose. The main additional ingredient in Section 3 is the external characterization of domination (3.5) which allows us to say something about points outside of the indiscernible sequence and then to generalize to the invariant type setting.

An important property of stable theories sometimes referred to as the *Shelah reflection principle* says roughly that non-trivial relationships between a realization of a type $p$ and some other point are reflected inside realizations of $p$. Internal concepts (only considering realizations of $p$) often imply external properties (involving the whole structure). For example regularity implies weight one. There is some evidence now that this principle is already true in $NIP$. See [2] for an example (weak stable embeddedness).

In this paper we will use this principle on indiscernible sequences: a property involving only the indiscernible sequence itself or extensions of it usually implies properties of the indiscernible sequence with respect to points outside (the same way total indiscernibitily implies that the trace of every definable set is finite or co-finite). See 2.8 and 3.5.

## 1.1 Preliminaries

### 1.1.1 Basic things

We list here some terminology and basic properties of $NIP$ theories that we will need.

We will often denote sequences of tuples by $I, J, \ldots$. Index sets of families or sequence might be named $\mathcal{I}, \mathcal{J}, \ldots$.

Recall that a theory is $NIP$ if every indiscernible sequence $I$ has a limit type $\lim(I/A)$ over any set $A$ of parameters.

**Assumption :** From now on, until the end of the paper, we work in a $NIP$ theory $T$, in a language $L$.

**Definition 1.3.** Two types $p_x, q_y$ over the same domain $A$ are *orthogonal* if $p_x \cup q_y$ defines a complete type in two variables over $A$.

If $M$ is a $\kappa$-saturated model, $A \subseteq M$ satisfying $|A| < \kappa$, a type $p \in S(M)$ is $A$-invariant if for $a \models p$ and any tuples $b, b' \in M$, $b \equiv_A b' \to ba \equiv b'a$. We will sometimes say simply that $p$ is an invariant type, without specifying $A$. Note that an invariant type has a natural extension to any larger set $B \supset M$ that we will denote by $p|_B$.

A Morley sequence of an invariant type $p$ over some $B \supseteq A$ is a sequence $(a_t)_{t<\omega}$ such that $a_n \models p|_{B \cup a_{<n}}$ for every $n$. All Morley sequences of $p$ over $B$ have the same type over $B$ and are $B$-indiscernible.

If $p_x$ and $q_y$ are two types over $M$ and $p$ is invariant, we can define the product $p_x \otimes q_y$ as the element of $S_{xy}(M)$ defined as $\text{tp}(a, b/M)$ where $b \models q_y$ and $a \models p_x|_{Mb}$. Note that if $q$ is also an invariant type, then $p_x \otimes q_y$ is invariant. In this case, we can also build the product $q_y \otimes p_x$. When the two products are equal, we say that $p$ and $q$ commute.

Note that $\otimes$ is associative. In particular if $p$ and $q$ commute with $r$, then $r$ commutes with $p \otimes q$.

Recall the notion of generically stable type from [6]: an invariant type $p \in S(M)$ is generically stable if it is both definable and finitely satisfiable in some small model $N \subset M$. Equivalently, its Morley sequence is totally indiscernible.

We now move to the very important results concerning shrinking of indiscernibles. We give the statement as in [8, Section 3].

**Definition 1.4.** Let $(a_t)_{t\in\mathcal{I}}$ be an indiscernible sequence. Let $J$ be a convex equivalence relation on $\mathcal{I}$ (*i.e.*, an equivalence relation all of whose classes are convex). If $(t_i)_{i<\alpha}$ and $(s_i)_{i<\alpha}$ are two tuples of elements of $\mathcal{I}$, we write $(t_i)_{i<\alpha} \sim_J (s_i)_{i<\alpha}$ if for every $i < \alpha$, $t_i$ is $J$-equivalent to $s_i$.

**Proposition 1.5** (Shrinking indiscernibles)**.** *Let $A$ be any set of parameters and $(a_t)_{t\in\mathcal{I}}$ be an $A$-indiscernible sequence. Let $c$ be any finite tuple. Let $\phi(x_c; y_0, .., y_{n-1}, t)$ be a formula. There is a convex equivalence relation $J$ on $\mathcal{I}$ with finitely many classes such that for $(t_i)_{i<n} \sim_J (s_i)_{i<n}$ and every $b \in A^{|t|}$, we have $\phi(c; a_{t_0}, .., a_{t_{n-1}}, b) \leftrightarrow \phi(c; a_{s_0}, ..., a_{s_{n-1}}, b)$.*

**Corollary 1.6** (Shrinking indiscernibles 2)**.** *Let $A$ be any set of parameters and $(a_t)_{t\in\mathcal{I}}$ be an $A$-indiscernible sequence. For simplicity assume $\mathcal{I}$ is a complete dense order. Set $\mu = |a_t|$, and let $c$ be any tuple. There is $\mathcal{I}_c \subset \mathcal{I}$ with $|\mathcal{I}_c| \leq \mu + |c| + |T|$ such that: for any $\mathcal{J} \subset I$ convex satisfying $\mathcal{J} \cap \mathcal{I}_c = \varnothing$, then $(a_t)_{t\in\mathcal{J}}$ is indiscernible over $A \cup c \cup \{a_t : t \in \mathcal{I} \setminus \mathcal{J}\}$.*

The corollary above easily adapts to the case of an incomplete order $\mathcal{I}$ by taking $\mathcal{J}$ in the completion of $\mathcal{I}$.

**Corollary 1.7.** *Let $I = (a_t)_{t\in\mathcal{I}}$ be $A$-indiscernible with $\mathcal{I}$ of cofinality at least $|T|^+$, then for any finite tuple $c$, there is an end segment $I'$ of $I$ that is indiscernible over $Ac$.*

There is a notion of weight that follows from the previous results (usually called 'burden').

**Proposition 1.8.** *Let $(I_i)_{i<|T|^+}$ be mutually indiscernible sequences (over $A$) and let $c$ be a finite tuple. Then there is some $i < |T|^+$ such that $I_i$ is indiscernible over $Ac$.*

**Corollary 1.9.** *Let $M$ be some $\kappa$-saturated model, and let $(p_i)_{i<|T|^+}$ be a family of pairwise commuting invariant types over $M$. Let $p = \bigotimes_{i<|T|^+} p_i$ and $(a_i)_{i<|T|^+} \vDash p$. Let also $q \in S(M)$ be any type and $b \vDash q$. Then there is $i < |T|^+$ such that $(a_i, b) \vDash p_i \otimes q$.*

*Proof.* Build a Morley sequence $\langle (a_i^k)_{i<|T|^+} : 0 < k < \omega \rangle$ of $p$ over everything and set $a_i^0 = a_i$ for each $i$. Then by commutativity, the sequences $(a_i^k)_{k<\omega}$, $i < |T|^+$ are mutually indiscernible. The result follows by Proposition 1.8. $\square$

Observe in particular that if $q$ is an invariant type, taking $b \models q|\{a_i : i < |T|^+\}$, we obtain that there is $i < |T|^+$ such that $p_i$ and $q$ commute. (Those observations come from [8, Section 4].)

### 1.1.2 Dp-minimal theories

We will occasionally mention dp-minimal theories. They are theories for which the notion weight suggested by 1.8 is equal to 1 on 1-types.

**Definition 1.10** (Dp-minimal)**.** An NIP theory $T$ is dp-minimal if for every indiscernible sequence $I$ and 1-tuple $a$, there is a subdivision $I = I_1 + I_2 + I_3$ into convex sets, where $I_2$ is either reduced to a point or empty and $I_1$ and $I_3$ are mutually indiscernible over $a$.

Equivalently, for every two mutually indiscernible sequences $I$ and $J$ and 1-tuple $a$, one of $I$ or $J$ is indiscernible over $a$.

Examples of dp-minimal theories include o-minimal and weakly-o-minimal theories and the p-adics.

### 1.1.3 Measures

As we mentioned in the introduction, we will not recall all definitions concerning measures. Instead, we refer the reader to [6] and [5]. The latter paper contains in particular the definition of a generically stable measure. Also the introduction of [12] contains a concise account of the definitions and basic results we will need, but without proofs.

We however recall the following from [12]:

A measure $\mu \in \mathcal{M}_x(A)$ is *smooth* if it has a unique extension to any $M \subset N$. For any formula $\phi(x,d)$, $d \in \mathfrak{C}$, let $\partial_M \phi$ denote the closed set of $S_x(M)$ of types $p$ such that there are $a, a'$ two realizations of $p$ satisfying $\phi(a,d) \wedge \neg\phi(a',d)$.

**Lemma 1.11.** *The measure $\mu \in \mathcal{M}_x(M)$ is smooth if and only if $\mu(\partial_M \phi) = 0$ for all formulas $\phi(x,d)$, $d \in \mathfrak{C}$.*

### 1.1.4 Indiscernible sequences and cuts

The notation $I = I_1 + I_2$ means that the sequence $I$ is the concatenation of the sequences $I_1$ and $I_2$: $I_1$ is an initial segment of $I$ and $I_2$ the complementary final segment. In this situation, we will say that $(I_1, I_2)$ is a *cut* of $I$.

**Definition 1.12** (Cuts)**.** If $J \subset I$ is a convex subsequence, a cut $c = (I_1, I_2)$ is said to be *interior* to $J$ if $I_1 \cap J$ and $I_2 \cap J$ are infinite.

Two cuts $c = (I_1, I_2)$ and $d = (J_1, J_2)$ of $I$ are *distant* if $J_1 \smallsetminus I_1 \cup J_2 \smallsetminus I_2$ is infinite (*i.e.*, there are infinitely many elements of $I$ between them).

A cut is *non-principal* if both $I_1$ and $I_2^*$ ($I_2$ with the order reversed) have infinite cofinality.

We write $(I_1', I_2') \trianglelefteq (I_1, I_2)$ if $I_1'$ is an end segment of $I_1$ and $I_2'$ an initial segment of $I_2$. A *polarized cut* is a pair $(c, \varepsilon)$ where $c$ is a cut $(I_1, I_2)$ and $\varepsilon \in \{1, 2\}$ is such that $I_\varepsilon$ is infinite. We will write the polarized cut $c^-$ if $\varepsilon = 1$ and $c^+$ if $\varepsilon = 2$.

Given a polarized cut $c^\bullet = ((I_1, I_2), \varepsilon)$ and a set $A$ of parameters, we can define the *limit type* of $c^\bullet$ denoted by $\lim(c^\bullet/A)$ as the limit type of the sequence $I_1$ or $I_2^*$ depending on the value of $\varepsilon$.

If a cut $c$ has a unique polarization, or if we know both polarizations give the same limit type over $A$, we will write simply $\lim(c/A)$.

If $c = (I_1, I_2)$ is a cut, we say that $b$ *fills* the cut $c$ if $I_1 + b + I_2$ is indiscernible.

The following definition is from [8].

**Definition 1.13.** A set $A$ *weakly respects* the cut $c = (I_1, I_2)$ if $c$ has infinite cofinality from both sides and $\lim(c^+/A) = \lim(c^-/A)$. It *respects* $c$ if for every finite $A_0 \subseteq A$, there is $I_1'$ cofinal in $I_1$ and $I_2'$ coinitial in $I_2$ such that $I_1' + I_2'$ is indiscernible over $A_0$.

Note that $\lim(c^\bullet/\mathfrak{C})$ is an invariant type, in fact finitely satisfiable over the sequence $I$. We will simply denote it by $\lim(c^\bullet)$. We can then define its iterates $\lim(c^\bullet)^{(n)}$ and its Morley sequence $\lim(c^\bullet)^{(\omega)}$.

If $c_1$ and $c_2$ are two distinct polarized cuts in an indiscernible sequence $I$ then $\lim(c_1)$ and $\lim(c_2)$ commute: $\lim(c_1)_x \otimes \lim(c_2)_y = \lim(c_2)_y \otimes \lim(c_1)_x$. More precisely $\phi(x, y) \in \lim(c_1)_x \otimes \lim(c_2)_y$ if and only if for some $J_1$ cofinal in $c_1$ and $J_2$ cofinal in $c_2$, $\phi(a, b)$ holds for $(a, b) \in J_1 \times J_2$.

**Definition 1.14** (Polycut)**.** A polycut is a sequence $(c_i)_{i \in \mathcal{I}}$ of pairwise distant cuts.

The definitions given for cuts extend naturally to polycuts: a polarized polycut is a family of polarized cuts. If $c = (c_i)_{i\in\mathcal{I}}$ is a polarized polycut, then we define $\lim(c) = \bigotimes_{i\in\mathcal{I}} \lim(c_i)$. It is a type in variables $(x_i)_{i\in\mathcal{I}}$. A tuple $(a_i)_{i\in\mathcal{I}}$ *fills* $c$ if the sequence $I$ with all the points $a_i$ added in their respective cut is indiscernible. Note that this is stronger than asking that each $a_i$ fills $c_i$.

Two polycuts are *distant* if every cut of one is distant from every cut of the other.

**Definition 1.15** ($I$-independent)**.** Let $I$ be a dense indiscernible sequence, $c_1, .., c_n$ pairwise distant cuts in $I$ and $a_1, .., a_n$ filling those cuts, then $a_1, .., a_n$ are independent over $I$ (or $I$-independent) if the tuple $(a_1, ..., a_n)$ fills the polycut $(c_1, ..., c_n)$.

We will use the notation $a \downarrow_I b$ to mean that $a$ and $b$ are independent over $I$, *i.e.*, that $I \cup \{a\} \cup \{b\}$ remains indiscernible. Note that this is a symmetric notion.

# 2 Distal theories

## 2.1 Indiscernible sequences

By the EM-type of an indiscernible sequence $I = \langle a_i : i \in \mathcal{I}\rangle$, we mean the family $(p_n)_{n<\omega}$, where $p_n \in S(\varnothing)$ is the type of $(a_{\sigma(k)})_{k<n}$ for $\sigma : n \to \mathcal{I}$ any increasing embedding.

**Definition 2.1** (Distal)**.** An indiscernible sequence $I$ is distal if for every dense sequence $J$ of same EM-type as $I$, every distant non-principal cuts $c_1$ and $c_2$ of $J$, if $a$ fills $c_1$ and $b$ fills $c_2$, then $a \downarrow_J b$.

An $NIP$ theory $T$ is *distal* if all indiscernible sequences are distal.

*Remark* 2.2. Equivalently the two types $\lim(c_1/J)$ and $\lim(c_2/J)$ are orthogonal.

*Remark* 2.3. If $I$ is dense and has two distant non-principal cuts $c_1$ and $c_2$, then it is distal if and only if $\lim(c_1/I)$ and $\lim(c_2/I)$ are orthogonal (*i.e.*, there is no need for $J$ in the definition). This is just by compactness.

Actually, it will follow from Lemma 2.8 that the hypothesis that $I$ is dense can be removed.

Exemple 2.4. *Assume $I$ is an indiscernible sequence, $f$ a definable function such that $f(I)$ is totally indiscernible (non constant), then $I$ is not distal. To see this, take $a$ and $b$ in the definition such that $f(a) = f(b)$. See 2.15 for a more general result.*

Exemple 2.5. *In DLO, any two 1-types concentrating on different cuts are orthogonal. It is easy then to check that it is a distal theory. We will see (Corollary 2.28) that in fact any o-minimal theory is distal.*

**Lemma 2.6.** *If $T$ is dp-minimal and $I$ is an indiscernible sequence of points, not totally-indiscernible, then $I$ is distal.*

*Proof.* Let $I + K + J$ be a dense indiscernible sequence and $a$, $b$ filling the cuts $(I, K+J)$ and $(I+K, J)$ respectively. Then $I+K+b+J$ is indiscernible. By dp-minimality, the point $a$ breaks the sequence into three convex sets, at most one of which being finite, such that the two infinite sets are mutually indiscernible over $a$. As the sequence $I + a + K + J$ is indiscernible, not totally indiscernible, there is only one possibility: $I$ must be one of the convex sets, and an other must be $K + b + J$. It follows that $I + a + K + b + J$ is indiscernible. □

**Lemma 2.7.** *Assume $I$ is a dense indiscernible distal sequence, and $c_0, ..., c_{n-1}$ are pairwise distant cuts. If for each $i < n$, $a_i$ fills $c_i$ then the family $(a_i)_{i<n}$ is $I$-independent.*

*Proof.* We prove it by induction on $n$. for $n = 2$, it is the definition of distality. Assume it holds for $n$ and consider a family $(c_i)_{i<n+1}$ and $(a_i)_{i<n+1}$ as in the hypothesis. Let $I' = I \cup \{a_0\}$ (where $a_0$ is inserted in the cut $c_0$). Each cut $c_i$ naturally induces a cut $c'_i$ of $I'$. By the case $n = 2$, for each $0 < i < n+1$, $a_i$ fills $c'_i$. The sequence $I'$ is also distal, so by induction $(a_i)_{0<i<n+1}$ is $I'$-independent. Therefore $(a_i)_{i<n+1}$ is $I$-independent. □

**Lemma 2.8** (External characterization of distality)**.** *A sequence $I$ is distal if and only if the following property holds: If $I' = I_1 + I_2$ is $A$-indiscernible ($I_1$ and $I_2$ without endpoints, EM-*$\mathrm{tp}(I')$=*EM-*$\mathrm{tp}(I)$*), then, if $I_1 + b + I_2$ is indiscernible, it is $A$-indiscernible.*

*Proof.* Assume that $I$ is distal, but the conclusion does not hold. Then there is some $I' = I_1 + I_2$ and formula $\phi(b)$ with parameters from $A \cup I_1 \cup I_2$ which holds and which should not. As $I_1$ and $I_2$ are without endpoints, restricting $I'$ and enlarging $A$, we may assume all the parameters are from $A$. Then, we may freely enlarge $I'$, so assume that it is dense.

As $I'$ is $A$-indiscernible, for every cut $(I'_1, I'_2)$ of $I'$, there is $b'$ filling it such that $\phi(b')$ holds. There is also $b''$ filling it such that $\neg\phi(b'')$ holds. Fix a large number of such cuts. Then take associated $b'$ and $b''$ alternating, and by distality the sequence formed by adding all those points to $I'$ is still indiscernible. Therefore $\phi(x)$ has infinite alternation number, contradicting $NIP$.

The converse is easy. □

The following technical lemma will be used repeatedly.

**Lemma 2.9** (Strong base change)**.** *Let $I$ be an indiscernible sequence in some model $M$. Let $(c_i)_{i<\alpha}$ be a sequence of pairwise distant non-principal polarized cuts in $I$. For each $i<\alpha$ let $d_i$ fill the cut $c_i$. Then there exist $(d'_i)_{i<\alpha}$ such that $\mathrm{tp}((d'_i)_{i<\alpha}/I) = \mathrm{tp}((d_i)_{i<\alpha}/I)$ and for each $i<\alpha$, $\mathrm{tp}(d'_i/M) = \lim(c_i/M)$*

*Proof.* Assume the result does not hold. Then by compactness, we may assume that $\alpha = n$ is finite and that there is a formula $\phi(x_0,..,x_{n-1}) \in tp((d_i)_{i<n}/I)$ and formulas $\psi_i(x_i) \in \lim(c_i/m)$ for some finite $m \in M^k$ such that $\phi(x_0,..,x_{n-1}) \wedge \bigwedge_i \psi_i(x_i)$ is inconsistent. Let $I_0$ denote the parameters of $\phi$, and assume $I_0 \subseteq m$.

Assume for simplicity that $n = 2$ (the proof for $n > 2$ is the same) and without loss $c_i$ is polarized as $c_i^-$. For $i = 0,1$ take $(J_i, J'_i) \trianglelefteq c_i$ such that $\psi_i$ holds on all elements of $J_i$ and $J_i \cup J'_i$ contains no element of $I_0$. Then $J_0 + J'_0$ and $J_1 + J'_1$ are mutually indiscernible over $I_0$. So for every two cuts $d_0$ and $d_1$ respectively from $J_0 + J'_0$ and $J_1 + J'_1$, we can find points $e_0$ and $e_1$ filling those cuts (even seen as cuts of $I$) such that $\phi(e_0, e_1)$ holds.

Take two cuts $d_0$ and $d_1$ of $I$ such that they are respectively interior to $J_0$ and $J_1$. Fill $d_0$ by $e_0$ and $d_1$ by $e_1$ such that $\phi(e_0, e_1)$ holds. By hypothesis, either $\neg\psi_0(e_0)$ or $\neg\psi_1(e_1)$ holds. Assume $\neg\psi_1(e_1)$ holds. Now forget about $e_0$ and set $I' = I \cup \{e_1\}$. Then $I'$ is indiscernible and we take it as our new $I$. Set $J'_0 = J_0$ and let $J'_1$ be an initial segment of $J_1$ not containing $d_1$ and make the same construction. We obtain new points $(e_0^1, e_1^1)$ that fill the cuts $d_0^1, d_1^1$ of $J'_0$ and $J'_1$ such that $\neg\psi_0(e_0^1) \vee \neg\psi_1(e_1^1)$ holds. Without loss (as we will iterate infinitely many times) again $\neg\psi_1(e_1^1)$ holds.

Iterate this $\omega$ time to obtain a sequence of points $e_1^k$ and cuts $d_1^k$ in $J_1$ such that $I$ with all the points $e_1^k$ added in the cuts $d_1^k$ is indiscernible and $\neg\psi_1(e_1^k)$ holds for all $n$. But $\psi_1(x)$ holds for all $x \in J_1$ so $\psi_1$ has infinite alternation rank, contradicting $NIP$. □

**Corollary 2.10** (Base change)**.** *The notion of being distal is stable both ways under base change: If $I$ is $A$-indiscernible, then $I$ is distal in $T(A)$ if and only if it is distal in $T$.*

*Proof.* Assume $I$ is distal in $T$. Then, by 2.8 it is also distal in $T(A)$.

Conversely, assume $I$ is not distal in $T$. Increase $I$ to some large $A$-indiscernible sequence $I+K+J$ and take $a,b$ such that $I+a+K+J$ and $I+K+b+J$ are indiscernible, but $I+a+K+b+J$ is not. By strong base change, we may assume that $a$ and $b$ realize the limit types over $A$ of the cuts they define. Then $I+a+K+J$ and $I+K+b+J$ are $A$-indiscernible, giving a counter-example to distality in $T(A)$. □

**Lemma 2.11.** *Let $T$ be distal, $I$ and $J$ are two mutually indiscernible sequences. Let $c$ (resp. $d$) be a cut in the interior of $I$ (resp. $J$). Then* $\lim(c)$ *and* $\lim(d)$ *are orthogonal.*

*Proof.* Put $I$ and $J$ together to form one indiscernible sequence of tuples. □

**Definition 2.12** (Weakly linked)**.** Let $\langle (a_i,b_i) : i \in \mathcal{I}\rangle$ an indiscernible sequence of pairs. We say that $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are weakly linked if for every disjoint subsets $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{I}$, $(a_i)_{i\in\mathcal{I}_1}$ and $(b_i)_{i\in\mathcal{I}_2}$ are mutually indiscernible.

**Observation 2.13.**

1. *If $\langle (a_i,b_i) : i \in \mathcal{I}\rangle$ is $A$-indiscernible and $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are mutually indiscernible, then they are mutually indiscernible over $A$.*

2. *If $\langle (a_i,b_i) : i \in \mathcal{I}\rangle$ is $A$-indiscernible and $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are weakly linked, then they are weakly linked over $A$.*

**Lemma 2.14.** *Let $\langle (a_i,b_i) : i \in \mathcal{I}\rangle$ be indiscernible.*

1. *If $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are weakly linked and $(a_i)_{i\in\mathcal{I}}$ is distal, then $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are mutually indiscernible.*

2. *If $(b_i)_{i\in\mathcal{I}}$ is totally indiscernible, then $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are weakly linked.*

*Proof.* (1). Assume $\mathcal{I}$ is dense and pick some finite $\mathcal{I}_2 \subset \mathcal{I}$. Then $(a_i)_{i\notin\mathcal{I}_2}$ is indiscernible over $B = (b_i)_{i\in\mathcal{I}_2}$. By distality, $(a_i)_{i\in\mathcal{I}}$ is indiscernible over $B$. This is enough.

(2). Assume $\mathcal{I}$ is dense and big enough, take $\mathcal{I}_1 \subset \mathcal{I}$ finite and let $A = (a_i)_{i\in\mathcal{I}_1}$. By total indiscernability of $(b_i)_{i\in\mathcal{I}}$, for all but boundedly many indices $j$, $b_j$ realizes $\lim((b_i)_{i\in\mathcal{I}}/A)$. By indiscernability of $\langle (a_i,b_i) : i \in \mathcal{I}\rangle$, this must be true for all $j \notin \mathcal{I}_1$. We get a similar result considering a tuple of points from $(b_i)_{i\in\mathcal{I}}$, so $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are weakly linked. □

**Corollary 2.15.** *Let $\langle (a_i,b_i) : i \in \mathcal{I}\rangle$ be an indiscernible sequence. Assume $(a_i)_{i\in\mathcal{I}}$ is totally indiscernible and $(b_i)_{i\in\mathcal{I}}$ is distal, then $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are mutually indiscernible.*

## 2.2 Invariant types

We prove here a characterization of distality in terms of invariant types.

If $M$ is a $\kappa$-saturated model, by an invariant type over $M$, we mean a type $p \in S(M)$ invariant over some $A \subset M$, $|A| < \kappa$. If $p$ and $q$ are two invariant types over $M$, then we can define the products $p_x \otimes q_y$ and $q_y \otimes p_x$ as explained in the introduction. The types $p$ and $q$ *commute* if those two products are equal.

**Lemma 2.16.** *Assume $T$ is distal. Let $M$ be $\kappa$-saturated and let $p, q \in S(M)$ be invariant types. If $p_x \otimes q_y = q_y \otimes p_x$, then $p$ and $q$ are orthogonal.*

*Proof.* Let $b \vDash q$ and let $N \prec M$ a model of size $< \kappa$ such that $p$ and $q$ are $N$-invariant. Let $I \subset M$ be a Morley sequence of $p$ over $N$. Let $a$ realize $p$, and build $I'$ a Morley sequence of $p$ over $Mab$. The hypothesis implies that $p^{(\omega)}$ and $q$ commute (as $\otimes$ is associative). In particular, $I + I'$ is indiscernible over $b$. By distality, $I + a + I'$ is also $b$-indiscernible. This proves that $tp(a, b/\varnothing)$ is determined.

We can do the same thing adding some parameters to the base, and thus $p$ and $q$ are orthogonal. ☐

**Proposition 2.17.** *The theory $T$ is distal if and only if any two global invariant types $p$ and $q$ that commute are orthogonal.*

*Proof.* Lemma 2.16 gives one implication. Conversely, assume that $T$ is not distal. Then there is a dense indiscernible sequence $I$, two distant non-principal cuts $c_1$ and $c_2$ and $a$ and $b$ filling them such that $a \not\perp_I b$. By strong base change, we may assume that $I \subset M$, for $M$ a large saturated model, and $a \vDash \lim(c_1^-/M)$, $b \vDash \lim(c_2^-/M)$. Then the types $p = \lim(c_1^-/M)$ and $q = \lim(c_2^-/M)$ have the required property. ☐

Consider $p, q \in S(M)$ and assume only that $p$ is invariant. Then $p_x \otimes q_y$ is well defined, but $q_y \otimes p_x$ does not make sense in general. We show now how to define $q_y \otimes p_x$.

**Lemma 2.18.** *Let $M$ be $\kappa$-saturated, $\kappa \geq |T|^+$. Let $p, q \in S(M)$, $p$ being $A$-invariant for some $|A| < \kappa$. Then there is some $B \subset M$, $|B| < \kappa$, such that $A \subseteq B$ and for $b \vDash q$ and any $a, a' \in M$ such that $a, a' \vDash p|_B$, we have* $\operatorname{tp}(a, b/A) = \operatorname{tp}(a', b/A)$.

*Proof.* Let $b \vDash q$.

We try to build inductively a sequence $\langle B_\alpha, a^0_\alpha, a^1_\alpha, \bar{c}_\alpha, \phi_\alpha(x,y;\bar{z}_\alpha) : \alpha < |T|^+\rangle$ such that:
- $B_0 = A$,
- $B_\lambda = \cup_{\alpha<\lambda} B_\alpha$,
- $B_{\alpha+1} = B_\alpha \cup \{a^0_\alpha, a^1_\alpha\}$,
- $a^0_\alpha, a^1_\alpha \models p|_{B_\alpha}$,
- $\bar{c}_\alpha \subset A$,
- $\models \phi_\alpha(b, a^0_\alpha; \bar{c}_\alpha) \wedge \neg\phi_\alpha(b, a^1_\alpha; \bar{c}_\alpha)$.

Assume we succeed. Then we may assume that $\phi_\alpha(x,y;\bar{z}_\alpha) = \phi(x,y;\bar{z})$ for all $\alpha$. For $n < \omega$, let $\eta(n) = 0$ if $n$ is even and 1 otherwise. Then the sequence $\langle a'_n = a^{\eta(n)}_n : n < \omega\rangle$ is indiscernible over $A$ but $\phi(b, a'_n; \bar{c}_n)$ holds if and only if $n$ is even. This contradicts $NIP$ (in the form of 1.5).

The construction must therefore stop at some $B_\alpha$, and setting $B = B_\alpha$, we have the required property. □

Let $M$, $p, q$ as in the lemma. Let $b \models q$ and $A \subset M$, $|A| < \kappa$ such that $p$ is $A$-invariant. Let $B$ be given by the lemma. We define $q_y \otimes p_x|_A$ as $\mathrm{tp}_{x,y}(a, b/A)$ where $a \models p|_B$, $a \in M$. By assumption, this does not depend on the choice of $a$. Notice also that it does not depend on $B$. Finally, define $q_y \otimes p_x \in S(M)$ by gluing together the various $q_y \otimes p_x|_A$, $A \subset M$.

Notice that if $q$ was invariant to begin with, then the two definitions of $q_y \otimes p_x$ coincide. Note also that the associativity relation: $p_x \otimes (q_y \otimes r_z) = (p_x \otimes q_y) \otimes r_z$ holds in all possible cases (each product is well defined if and only if at least two of $p, q, r$ are invariant).

The following generalizes 2.16, the proof is the same, using 2.18 to build the Morley sequence $I$ of $p$ inside $M$.

**Lemma 2.19.** *Assume $T$ is distal. Let $M$ be $\kappa$-saturated ($\kappa \geq |T|^+$), $p \in S(M)$ be $A$-invariant for some $A$ of size $< \kappa$ and $q \in S(M)$ be any type. If $p_x \otimes q_y = q_y \otimes p_x$, then $p$ and $q$ are orthogonal.*

## 2.3 Generically stable measures

We prove in this section that distal theories are exactly those theories in which generically stable measures are smooth. We consider this as a justification that distality is a meaningful notion. It was proved in [12] that o-minimal theories and the p-adics have this property. This latter result will be generalized in the next section, where we prove that distality can be checked in dimension 1.

We have two tools at our disposal to link indiscernible sequences of tuples to measures. In one direction, starting with an indiscernible sequence of tuples, we can form the average measure. This construction is defined in [5], extended in [12] and recalled below. In the opposite direction, starting with a generically stable measure $\mu$ (or in fact any invariant measure), we can consider its Morley sequence $\mu^{(\omega)}$. We then want to realize it in some way. We do this by taking smooth extensions; see the proof of 2.23.

Let $I = (a_t)_{t\in[0,1]\cap\mathbb{Q}}$ be an indiscernible sequence indexed by $[0,1]\cap\mathbb{Q}$ (call this an *indiscernible segment*). We can define the *average measure* $\mu$ of $I$ as the global measure defined by $\mu(\phi(x)) = \lambda_0(\{t \in [0,1] : a_t \vDash \phi(x)\})$, where $\lambda_0$ is the Lebesgue measure. That measure is generically stable (in fact definable and finitely satisfiable over $I$).

The *support* of a measure $\mu \in \mathcal{M}(A)$ is the set of weakly-random types for $\mu$, namely the set of types $p \in S(A)$ such that $p \vdash \neg\phi(x)$ for every formula $\phi(x) \in L(A)$ such that $\mu(\phi(x)) = 0$. We will denote it by $S(\mu)$.

**Lemma 2.20.** *Let $\mu$ be the average of the indiscernible segment $I = (a_t)$. Then the support $S(\mu)$ of $\mu$ is exactly the set of limit types of polarized cuts of $I$.*

*Proof.* Let $X$ be the set of limit types of polarized cuts of $I$. We first show that $S(\mu) = \bar{X}$, the closure of $X$. Let $\phi(x) \in L(\mathfrak{C})$ have positive measure. Then by definition of $\mu$, there is a non trivial open interval $J$ of $[0,1]$ such that $\phi(a_t)$ holds for $t \in J$. So for any cut $c$ in that interval, the limit type associated contains $\phi$. Conversely, if $\phi(x)$ is satisfied by some $\lim(c)$, $c$ a cut in $I$, then $\phi(x)$ holds on a subsequence, cofinal in $c$, and therefore has positive measure.

To see that $X$ is closed, take $p \in \bar{X}$. If $I$ is totally indiscernible, then $X$ has one element, so assume this is not the case. Then we can associate to $p$ a cut $c$ of $I$ (the sequence $I$ is ordered by some formula $\phi(x,y) \in L(\mathfrak{C})$ and for every $t$, $p$ must satisfy $\phi(x,a_t) \vee \phi(a_t,y)$). It is then easy to check that $p$ must be either $\lim(c^-)$ or $\lim(c^+)$. ☐

**Proposition 2.21** (Smooth measures imply distality)**.** *Let $I$ be an indiscernible sequence indexed by $[0,1]\cap\mathbb{Q}$, and $\mu$ be the average measure of $I$. Then $\mu$ is smooth if and only if $I$ is distal.*

*Proof.* Assume $\mu$ is not smooth and $I$ is distal. Then there exists a formula $\phi(x,a)$ such that the set of $p \in S(\mathfrak{C})$ such that $p$ neither implies $\phi(x,a)$ nor its negation has positive measure (in other words, $p \in \partial\phi$). We know that the support of $\mu$ is exactly the limit types of cuts in $I$. Therefore, there are infinitely many such cuts $c_i$ in $\partial\phi$. We may also assume the $c_i$ are not principal since the principal cuts form a measure 0 set.

Restricting to some sub-interval of $[0,1]$, we may for example assume that $\phi(x,a)$ holds for no point of $I$. For each index $i$, as $\lim(c_i) \in \partial\phi$, there is $b_i$ filling the cut $c_i$ over $I$ such that $\phi(b_i,a)$ holds. As $I$ is distal, the sequence formed by adding all the $b_i$ to $I$ is still indiscernible. But then the formula $\phi(x,a)$ has infinite alternation number.

Conversely, assume that $I$ is not distal. Then we can find a partition $I = I_1 + I_2 + I_3$ (the $I_i$ without end points other then 0 and 1) and points $b_1, b_2$ such that $I_1+b_1+I_2+I_3$ and $I_1+I_2+b_2+I_3$ are indiscernible, but $I_1+b_1+I_2+b_2+I_3$ is not. By strong base change, we may assume that the types of $b_1$ and $b_2$ over $M$ are the limit types of the cuts they define. There is a formula $\phi$ and $i_k \subset I_k$ and $b_1'$ realizing the same type as $b_1$ over $M$ such that $\phi(i_1,b_1,i_2,b_2,i_3) \wedge \neg\phi(i_1,b_1',i_2,b_2,i_3)$ holds. Then the border $\partial\phi$ of $\phi(i_1,x,i_2,b_2,i_3)$ contains all limit types of cuts between $i_1$ and $i_2$ and has non zero measure. This proves that $\mu$ is not smooth. $\square$

**Corollary 2.22.** *If all generically stable measures are smooth, then $T$ is distal.*

We now show the converse.

**Proposition 2.23.** *If $T$ is distal, then all generically stable measures are smooth.*

*Proof.* Take $\mu$ a generically stable measure over some $|T|^+$-saturated model $N$. The unique global invariant extension of it will also be denoted by $\mu$. Let $a$ be a tuple. Let $\mu'$ be an extension of $\mu$ to $Na$. Take a smooth extension $\mu''$ of $\mu'$ to some $B \supseteq Na$. Let $\langle (B_i,a_i) : i<\omega\rangle$ be a coheir sequence in $\mathrm{tp}(B,a/N)$. Define the measures $\mu^i_{x_i}$ such that $\mu^i$ is smooth over $B_i$ and defined over $B_i$ the same way $\mu''$ is over $B$.

Consider the measure $\lambda_{x_i,i<\omega}$ defined as $\bigotimes_{i<\omega}\mu^i_{x_i}$ (this does not depend on the order of the factors since the $\mu^i$ are generically stable).

<u>Claim</u>: The measure $\lambda_{x_1,x_2,\ldots}$ is a totally indiscernible sequence of measures over $N$.

Note that $\mathrm{tp}(B_2/B_1N)$ is non-forking over $N$. In particular $\mu^2_{x_2}|_{B_1N}$ does not fork over $N$ (as it is finitely satisfiable in $B_2$) so it is the invariant extension of $\mu_{x_2}$ over $B_1N$. Therefore $\mu^1_{x_1}\otimes\mu^2_{x_2}|_N$ is equal to $\mu_{x_1}\otimes\mu_{x_2}|_N$. More generally, $\lambda|_N = \mu^{(\omega)}|_N$. As $\mu$ is generically stable, $\lambda_{x_1,x_2,\ldots}$ is totally indiscernible over $N$.

Now define a measure $\eta_{(x_1,y_1),(x_2,y_2)\dots}$ over $N$, where $y_i$ is a variable of the same size as $B$, by $\eta(\phi(x_1,x_2,..;y_1,y_2,..)) = \lambda(\phi(x_1,x_2,..;B_1,B_2,..))$. By construction, $\eta$ is a measure of an indiscernible sequence. Corollary 2.15 works equally well with measures instead of points, with the same proof, and yields that for any increasing $\sigma:\omega\to\omega$, and any $\phi(x_1,x_2,..;y_1,y_2,..)$,

$$\eta(\phi(x_1,x_2,..;y_{\sigma 1},y_{\sigma 2},..)) = \eta(\phi(x_1,x_2,..;y_1,y_2,..)).$$

Therefore $\mu'|_{Na} = \mu^2|_{Na} = \mu|_{Na}$ and $\text{tp}(a/N)$ and $\mu|_N$ are orthogonal. This proves that $\mu$ is smooth. □

We end this section by giving some type-by-type versions of Proposition 2.23.

**Proposition 2.24.** *Let $M$ be a $\kappa$-saturated model, $p\in S(M)$ orthogonal to all generically stable measures. Let $q\in S(M)$ be an invariant type such that $p_x\otimes q_y = q_y\otimes p_x$, then $p$ and $q$ are orthogonal.*

*Proof.* By associativity of $\otimes$, $p$ and $q^{(\omega)}$ commute. Assume $p$ and $q$ are not orthogonal, let $(a,b)\models p\times q$ such that $\text{tp}(a,b/M)\neq p\otimes q$. Let $\phi(a,b)\in L(M)$ witness this. Without loss, $\phi$ has parameters in $A\subset M$, $|A|<\kappa$, $q$ is $A$-invariant and for every $I,I'\subset M$, Morley sequences of $q$ over $A$ indexed by $\omega$, $\text{tp}(I/Aa)=\text{tp}(I'/Aa)$. Let $I_1$ be a dense countable Morley sequence of $q$ over $A$ inside $M$ and $I_2$ a dense Morey sequence of $q$ over $Mab$. Then $I_1+I_2$ is indiscernible over $Aa$ and $I_1+b+I_2$ is indiscernible over $A$. Let $t$ be a polarized cut inside $I_1$ and $r$ be the limit type of $t$ over $M$.

<u>Claim</u> : $r$ and $p$ are not orthogonal, in fact, there is $c\models r$ such that $\phi(a,c)$ holds.

Assume no such $c$ exists. Then by compactness, there is $\psi(y)\in r$ such that $p(x)\wedge\phi(x,y)\wedge\psi(y)$ is inconsistent. Let $J$ be an interval of $I_1$ on which $\psi$ holds. Let $l_1$ be a cut in $J$. By assumption, there is $c$ filling $l_1$ (over $A$) such that $\phi(a,c)$ holds. Then $\neg\psi(c)$ holds. By saturation, there is $c_1\in M$ filling $l_1$ over $A$ such that $\neg\psi(c_1)$ holds. By assumption on $A$, $I_1\cup\{c\}$ is indiscernible over $a$. So we can go on with another cut $l_2$, and after $\omega$ steps, $\psi(x)$ has infinite alternation rank, contradicting NIP, so the claim is proved.

As this holds for any cut $t$, if $I_1$ is an indiscernible segment of average $\mu$, then $\partial\phi(a,y)$ has $\mu$-measure one, so $\mu$ and $p$ are not orthogonal. □

**Proposition 2.25.** *Let $p$ be a global invariant type, then $p$ is distal if and only if $p^{(\omega)}$ is orthogonal to all generically stable measures.*

*Proof.* The proof of 2.23 shows that if $p$ is non-orthogonal to a generically stable measure, then it is not distal. (Take in the proof $a$ realizing $p$ and instead of taking a coheir sequence $(B_i)$ take a non-forking indiscernible sequence $(B_i)$ such that the corresponding sequence $(a_i)$ is a Morley sequence of $p$.)

Conversely, assume $p$ is $M$-invariant and not distal. Let $I$ be a Morley segment of $p$ over $M$ and $\mu$ the average measure of $I$. Then by strong base change, $p^{(\omega)}$ and $\mu$ are not orthogonal. □

## 2.4 Reduction to dimension 1

The goal of this section is to prove the following theorem.

**Theorem 2.26.** *If all sequences of 1-tuples are distal, then $T$ is distal.*

We first give an informal proof using measures. Assume all sequences of 1-tuples are distal and consider a generically stable measure $\mu$. Then looking at the proof of 2.23 we see that $\mu$ is orthogonal to all 1-types. Then by induction, adding the points one-by-one, $\mu$ is orthogonal to every $n$-type. However, to be made rigorous this proof seems to require the fact that no type forks over its base. To avoid this hypothesis and the use of measures, we give a purely combinatorial proof.

So we start with a witness of non-distality of the following form:

- An indiscernible sequence $I = (a_i)_{i\in\mathcal{I}}$ with $\mathcal{I} = [0,1]$ for simplicity,
- A tuple $b = (b_j)_{j<n}$, some $l \in (0,1)$ and tuple $a$ such that:
  - $a$ fills the cut $l^+$ of $A$,
  - $I$ is $b$-indiscernible,
  - $I$ with $a_l$ replaced by $a$ is not indiscernible over $b$.

We make some simplifications. First let $m < n$ be the first integer such that $b' = b_{<m}$ satisfies the requirements in place of $b$. We can add $b_{<m-1}$ as parameters to the base (by base change, or equivalently we can replace $a_i$ by $a_i' = a_i + b_{<m-1}$) and replace $b$ by $b_{m-1}$. Therefore, we may assume that $b$ is a 1-tuple. Next, adding again some parameters to the base, we may assume that for $i \in \mathcal{I}$, $\mathrm{tp}(a/b) \neq \mathrm{tp}(a_i/b)$.

The goal of the construction that follows is to reverse the situation of $a$ and $b$, *i.e.*, to construct an indiscernible sequence starting with $b$ that is not distal, the non-distality being witnessed by $a$ (or a conjugate of it).

Step 1: Derived sequence

Let $r = \mathrm{tp}(a, b)$. We construct a new sequence $(a'_i)_{i\in\mathcal{I}}$ such that:

- $a'_i$ fills the cut $i^+$ of $A$,
- $tp(a'_i, b) = r$ for each $i$,
- The sequence $\langle (a_i, a'_i) : i \in \mathcal{I}\rangle$ is $b$-indiscernible.

This is possible by indiscernability of $(a_i)_{i\in\mathcal{I}}$ over $b$ (choose the $a_i$ filling the cuts and then extract). The indiscernible sequence $(a'_i)_{i\in\mathcal{I}}$ is called a *derived sequence* of $(a_i)_{i\in\mathcal{I}}$ with respect to $b$.

Step 2: Constructing an array

Using Lemma 2.9 we can iterate this construction to obtain an array $\langle a_i^n : i \in \mathcal{I}, n < \omega\rangle$ and sequence $\langle b_n : n < \omega\rangle$ such that:

- $a_i^0 = a_i$ for each $i$,
- For each $i \in \mathcal{I}$, $0 < n < \omega$, the tuple $a_i^n$ realizes the limit type of the cut $i^+$ of $A$ over $\langle b_k, a_i^k : i \in \mathcal{I}, k < n\rangle$,
- For each $0 < n < \omega$, $tp(b_n, (a_i^n)_{i\in\mathcal{I}}/A) = tp(b, (a'_i)_{i\in\mathcal{I}}/A)$.

Claim: For every $\eta : \mathcal{I}_0 \subset \mathcal{I} \to \omega$ injective, the sequence $\langle a_i^{\eta(i)} : i \in \mathcal{I}_0\rangle$ is indiscernible, of same EM-type as $A$.

*Proof.* Easy, by construction. □

Expanding and extracting, we may assume that the sequence of rows $\langle b_n + (a_i^n)_{i\in\mathcal{I}} : 0 < n < \omega\rangle$ is indiscernible and that $(\langle (a_i^n) = 0 < n < \omega\rangle)_{i\in\mathcal{I}}$ is indiscernible over the sequence $(b_n)_{n<\omega}$.

Step 3: Conclusion

Claim: The sequences $(b_n)_{n<\omega}$ and $\langle (a_i^n)_{i\in\mathcal{I}} : 0 < n < \omega\rangle$ are weakly linked (definition 2.12).

*Proof.* Assume for example $\phi(b_n, a_i^k)$ holds for all $i \in \mathcal{I}$ and $k < n$, then choosing $\eta$ as in the first Claim such that the truth value of "$\eta(i) < n$" alternates infinitely often, we show that $\phi(b_n, a_i^k)$ must hold also for $k > n$. □

Choose an increasing map $\eta : \omega \to \mathcal{I}$, then the sequences $(b_n)_{n<\omega}$ and $(a^n_{\eta(n)})_{n<\omega}$ are weakly linked but not mutually indiscernible. This contradicts Lemma 2.14 and finishes the proof of Theorem 2.26.

**Corollary 2.27.** *If all generically stable measures in dimension 1 are smooth, then all generically stable measures are smooth.*

This generalizes results of [12] where this was proved under additional assumptions.

**Corollary 2.28.** *If $T$ is dp-minimal and has no generically stable type (in $M$), then it is distal. In particular o-minimal theories and the p-adics are distal.*

*Proof.* Recall from 2.6 that in a dp-minimal theory, any indiscernible sequence of 1-tuples is either distal or totally indiscernible. □

# 3 Domination in non-distal theories

We have now two extreme notions for indiscernible sequences: distality and total indiscernability. We want to understand the intermediate case. This part is essentially independent of the previous one but is of course motivated by it. We first concentrate on indiscernible sequences, and then adapt the results to invariant types, where statements become simpler. A last sub-section gives an application to externally definable sets.

The reader might find it useful to have in mind the example of a colored order as defined in the introduction while reading this section.

We will sometimes work with saturated indiscernible sequences, as defined below.

**Definition 3.1** (Saturated sequence)**.** An indiscernible sequence of $\alpha$-tuples is saturated if it is indexed by an $(|T|+|\alpha|)^+$-saturated dense linear order without end points.

A tuple $\bar{a} = (a_t)_{t\in\mathcal{I}}$ (resp. polycut), is *small* if $|\mathcal{I}| \leq |T| + |\alpha|$ where each $a_t$ is an $\alpha$-tuple.

In this section, all the polycuts are implicitly assumed to be non-principal (*i.e.*, every cut in them is of infinite cofinality from both sides). If $c$ and $c'$ are two polycuts, we write $c \ll c'$ if for every two cuts $(I_1, I_2)$ in $c$ and $(J_1, J_2)$ in $c'$, we have $I_1 \subset J_1$ and $J_1 \setminus I_1$ is infinite.

If $\bar{a}$ fills a polycut $c$ of $I$, an extension $J \supseteq I$ is *compatible with* $\bar{a}$ if $\bar{a}$ also fills a polycut of $J$.

We fix a global $A$-invariant type $p \in S_\alpha(\mathfrak{C})$, for some small parameter set $A$. The indiscernible sequences we will consider will be Morley sequences of $p$. This is not a real restriction since every indiscernible sequence is a Morley sequence of some invariant type.

The following is the main definition of this section.

**Definition 3.2** (Domination)**.** Let $I$ be a dense indiscernible Morley sequence of $p$ over $A$, $a \models p|_{AI}$ and $c$ a cut of $I$ filled by a dense sequence $\bar{a}_* = \langle a_t : t \in \mathcal{I}\rangle$ of $\alpha$-tuples. We say that $\bar{a}_*$ dominates $a$ over $(I, A)$ if: For every cut $d$ of $I$ distant from $c$, and $\bar{b}$ a dense sequence filling $d$, we have in the sense of $T(A)$:

$$\bar{b} \downarrow_I \bar{a}_* \Rightarrow \bar{b} \downarrow_I \bar{a}.$$

We say that $\bar{a}_*$ *strongly dominates* $\bar{a}$ over $(I, A)$ if for every $I \subseteq J$ compatible with $\bar{a}_*$ over $A$ and such that $a \models p|_{AJ}$, $\bar{a}_*$ dominates $\bar{a}$ over $J$.

Exemple 3.3. *Let $T$ be the theory of colored orders, as defined in the introduction. Let $p$ be an $A$-invariant type of an element of a new color. Let $I + a$ be a Morley sequence of $p$ over $A$. Let $c$ be a cut in $I$. If $a_*$ fills $c$, then $a_*$ dominates $a$ over $(I, A)$ if and only if $a$ and $a_*$ are of the same color.*

**Proposition 3.4.** *Let $I + a$ be a saturated Morley sequence of $p$ over $A$, $c$ a cut of $I$ then there is a sequence $\bar{a}_*$ of $\alpha$-tuples such that $\bar{a}_*$ fills $c$ and $\bar{a}_*$ strongly dominates $a$ over $(I, A)$.*

*Proof.* Let $I_0$ be any dense Morley sequence of $p$ over $A$ indexed by a linear order $\mathcal{I}_0$ of size $|T| + |\alpha|$. Let $a_0$ realize $p$ over $A + I_0$. Take a cut $c_0$ in $I_0$ and any dense sequence $\bar{a}^0_*$ filling that cut. Assume that $\bar{a}^0_*$ does not strongly dominate $a_0$ over $(I, A)$. Then there is some dense $I'_1 \supset I_0$ compatible with $a^0_*$ such that $I'_1 + a_0$ is a Morley sequence of $p$ over $A$, some cut $c'_1$ of $I'_1$ distant from $c_0$ and $\bar{b}_1$ a dense sequence filling $c'_1$ such that, over $A$ we have:

$$\bar{b}_1 \downarrow_{I'_1} \bar{a}^0_* \text{ but } \bar{b}_1 \not\downarrow_{I'_1} a.$$

Keeping only the relevant parameters, we may assume that $I'_1$ is of size $|T| + |\alpha|$. Let $J \subset I'_1$ denote the subsequence of $I'_1$ lying between the two cuts $c_0$ and $c'_1$. Assume $c'_1 \ll c_0$. Let $a^1_*$ be the sequence $\bar{b}_1 + J + \bar{a}^0_*$. Let also $I_1$ be $I'_1 \smallsetminus J$ and $c_1$ be the cut of $I'_1$ induced by $a^1_*$. If $a^1_*$ does not strongly dominate

$a_0$ over $I_1$, we can iterate this construction. By NIP, this must stop at some step $\beta < (|T|+|\alpha|)^+$. As $I$ is saturated, we can find an embedding $\pi : I_\beta \hookrightarrow I$ such that the cut $c$ induces the cut $\pi(c_\beta)$ on $\pi(I_\beta)$. Then $\pi$ extends to an elementary embedding fixing $A$, sending $a_0$ to $a$ and $\bar{a}_*^\beta$ to some $\bar{a}_*$ filling $c$. By construction $\bar{a}_*$ strongly dominates $a$ over $(I, A)$.

$\square$

Using shrinking of indiscernibles (1.6) it is not hard to show that in general if $I + a$ is a Morley sequence of $p$ and $\bar{a}_*$ strongly dominates $a$ over $(I, A)$, then there is some $I_0 \subset I$ of size $|T| + |\alpha|$ such that $\bar{a}_*$ already strongly dominates $a$ over $I_0$.

### 3.0.1 External characterization and base change

Similarly to what we did in the distal case, we give an external characterization of domination.

**Proposition 3.5** (External characterization of domination)**.** *Let $I$ be a saturated Morley sequence of $p$ over $A$, $a \vDash p_{AI}$. Let $\bar{a}_*$ fill a cut $c$ of $I$ over $A$ such that $\bar{a}_*$ strongly dominates $a$ over $(I, A)$. Let also $d \in \mathfrak{C}$. Assume:*

*There is a partition $I = J_1 + J_2 + J_3 + J_4$ such that $J_2$ and $J_4$ are infinite, $c$ in interior to $J_2$, $J_2 \cup \{\bar{a}_*\}$ is indiscernible over $Ad + J_1 + J_3 + J_4$ and $J_4$ is a Morley sequence of $p$ over $Ad + J_1 + J_2 + J_3$.*

*Then $a \vDash p|_{AId}$.*

*Proof.* Let $I$, $a$, $\bar{a}_*$, $d$, $J_1, ..., J_4$ as in the statement of the proposition.

Assume $a_0$ does not realize $p$ over $AId$. So there is $\phi(d, \bar{i}; x) \in L(A)$, $(\bar{i} \subset I)$ a formula satisfied by $a_0$ that should not be. Incorporating $\bar{i}$ in $d$ and changing the partition, we may assume that $\bar{i} = \varnothing$. Pick a sequence of cuts of $I_2$ $c_0 \ll c_1 \ll ....$ Let $\langle a_*^k : k < \omega\rangle$ fill the polycut $\langle c_k : k < \omega\rangle$ over $Ad \cup \{J_l : l \neq 2\}$.

We may find some $a_0 \vDash p|AI \cup \{a_*^k : k \neq 0\}$ such that $\phi(d; a_0)$ holds and $a_*^0$ strongly dominates $a_0$ over $(I, A)$.

(Why ? Let $J_* \subset J_2$ be of size $|T| + |\alpha|$ such that $\bar{a}_*$ strongly dominates $a$ over $(J_1 + J_* + J_3 + J_4, A)$. Then, by saturation, take some $J_*^0 \subset J_2$ such that $\mathrm{tp}(J_*^0, \bar{a}_*^0) = \mathrm{tp}(J_*, \bar{a}_*)$. Now look for $a_0$ such that

– $\mathrm{tp}(J_*^0, \bar{a}_*^0, a_0/A) = \mathrm{tp}(J_*, \bar{a}_*, a)$,
– $a_0 \vDash p|AI \cup \{\bar{a}_*^k : k \neq 0\}$ and
– $\phi(d; a_0)$ holds.

To see this is possible, take some finite condition $\theta(x) \wedge \psi(x) \wedge \phi(d;x)$ where $\theta(\bar{j}^0, \bar{a}^0_*; x)$, $\bar{j}^0 \subset J^0_*$ and $\psi(\bar{i}, \bar{c}; x) \in p|AI \cup \{\bar{a}^k_* : k \neq 0\}$ with $\bar{i} \subset I$ and $\bar{c} \subset \{\bar{a}^k_* : k \neq 0\}$. By indiscernability of $J_2 \cup \{\bar{a}^k_* : k < \omega\}$, there is an automorphism $\sigma$ fixing $AdJ_1J_3J_4$ sending the parameters $\bar{j}^0$ to the corresponding points in $J_*$ and $\bar{i}, \bar{c}$ to points in $I$. Then $\sigma^{-1}(a)$ satisfies the conditions considered.)

Let $K_1$ realize an infinite Morley sequence of $p$ over everything considered so far. Let $I_1 = I + K_1 \cup \{a^k_* : k > 1\}$. As above, we may find $a_1 \models p|AI_1$ such that $a^1_*$ strongly dominates $a_1$ over $(I_1, A)$ and $\phi(d; a_1)$ holds. Now as $a_0 \downarrow_{I_1} a^1_*$, we have $a_0 \downarrow_{I_1} a_1$. We iterate this construction building an indiscernible sequence $I_\omega = I + K_1 + K_2 + ....$ and points $\langle a_k : k < \omega \rangle$ filling the cuts between the $K_i$'s and independent over $I_\omega$ such that $\phi(d; a_k)$ holds for each $k$. As by assumption $\neg\phi(d;x)$ holds for every $x \in I_\omega$, $\phi$ has infinite alternation rank, contradicting NIP. $\square$

**Proposition 3.6** (Base change). *Let $p$ be $A$ invariant and $A \subset B$. If $I$ is a saturated Morley sequence of $p$ over $B$, $a \models p|BI$ and $\bar{a}_*$ fills a cut of $I$ in the sense of $T(B)$, then if $\bar{a}_*$ strongly dominates $a$ over $(I, A)$ it does so over $(I, B)$.*

*Proof.* Assume that $\bar{a}_*$ fills a cut $c$ of $I$ in the sense of $T(B)$ and dominates $a$ over $(I, A)$. Then let $\bar{d}$ fill a cut $c'$ of $I$ over $B$ with $c'$ distant from $c$. Assume that $\bar{d} \downarrow_I \bar{a}_*$ over $B$. Then $\bar{d}B$ satisfies ♈. By domination over $(I, A)$, $a \models p|I \cup dB$. This proves that $\bar{a}_*$ dominates $a$ over $(I, B)$. This remains true if we first increase $I$ so $\bar{a}_*$ strongly dominates $a$ over $(I, B)$. $\square$

## 3.1 Domination for types

We now have all we need to state domination results for types over $|T|^+$-saturated models, instead of cuts in indiscernible sequences.

We work over a fixed $\kappa$-saturated model $M$. By an *invariant type* we mean here a type over $M$, invariant over some $A \subset M$ of size less than $\kappa$.

For the following definition, recall the construction of $p_x \otimes q_y$ when $q$ is invariant (Lemma 2.18 and the paragraph following it).

**Definition 3.7** (Distant). Let $p, q \in S(M)$ be two types, assume that at least one of them is invariant, then we say that $p$ and $q$ are distant if they commute: $p_x \otimes q_y = q_y \otimes p_x$. If $a, b \in \mathfrak{C}$, we wil say that $a$ and $b$ are distant over $M$ if $\mathrm{tp}(a/M)$ and $\mathrm{tp}(b/M)$ are.

Given two distant types $p, q \in S(M)$ and $a \vDash p$, $b \vDash q$ we say that $a$ and $b$ are independent over $M$ if $\mathrm{tp}(a, b/M) = p \otimes q$. We write $a \downarrow_M b$. This is a symmetric relation.

**Definition 3.8** (S-domination)**.** Let $p \in S(M)$ be invariant, $a \vDash p$. A tuple $b$ s-dominates $a$ over $M$ if:

For every invariant type $r \in S(M)$ distant from $p$ and $q$, and $d \vDash r$, if $d \downarrow_M b$, then $d \downarrow_M a$.

The reader might be concerned by the fact that this definition depends on the choice of $\kappa$ (taking a smaller $\kappa$ we have less invariant types to check). However, we will see later that we get an equivalent definition if we add in the condition that $r$ is invariant over a subset of size $\aleph_0$.

Exemple 3.9. *Taking again the example of a colored order, if $p$ and $q$ are two invariant types (of tuples), $\bar{a} \vDash p$ and $\bar{b} \vDash q$, then $\bar{b}$ s-dominates $\bar{a}$ over $M$ if and only if, for every point $a_0$ in $\mathrm{rg}(\bar{a})$, there is a point $b_0$ in $\mathrm{rg}(\bar{b}) \cup M$ of the same color.*

### 3.1.1 The moving-away lemma

**Lemma 3.10.** *Let $p \in S(M)$ be any type, and $a \vDash p$. Then there is some $a_*$ dominating $a$ over $M$.*

*Proof.* This is similar to 3.4. Start with some $a_*$, if it does not dominate $a$, there is an invariant type $r$ distant from $a_*$ and $a$ over $M$ and $b \vDash r|Ma_*$ such that $b \not\downarrow_M a$. Replace $a_*$ by $a_* b$ and iterate. By NIP, this construction must stop after less than $(|T| + |a|)^+$ steps. □

However for applications we will also need to show that we can find such a dominating tuple distant from any given type.

**Lemma 3.11.** *Let $I \subset M$ be a dense indiscernible sequence of $\alpha$-tuples and $(I_i)_{i<\lambda}$ a family of distinct initial segments of $I$, with $\lambda \geq |T| + |\alpha|$. For $i < \alpha$, let $p_i = \lim(I_i/M)$. Then given a type $q \in S(M)$, there is $i < \lambda$ such that $p_i$ is distant from $q$.*

*Proof.* Observe that the types $p_i$ pairwise commute. Then use 1.9. □

**Lemma 3.12.** *Let $p, q \in S(M)$, be types of $\alpha$-tuples ($|\alpha| < \kappa$) with $p$ invariant over some $A$. Let $a \models p$. Then there is $r \in S(M)$ invariant over some $B$ of size $\aleph_0$, distant from $p$ and $q$ and $\bar{b} \models r$ such that $|\bar{b}| \leq |T| + |\alpha|$ and $\bar{b}$ s-dominates $a$ over $M$.*

*Proof.* By the proof of 3.4 we can find $I'_0$ a dense Morley sequence of $p$ over $A$ of size $|T| + |\alpha|$, $\bar{a}'_*$ such that $a \models p|AI'_0$, $\bar{a}'_*$ fills a cuts $c$ of $I'_0$ and $\bar{a}'_*$ strongly dominates $a$ over $I'_0$. Let $\bar{b}'$ be the sequence $I'_0 \cup \bar{a}'_*$ where $\bar{a}'_*$ is placed in its cut.

Let $I \subset M$ be a saturated Morley sequence of $p$ over $A$, let $c$ be a non-principal polarized cut of $I$ of cofinality $\aleph_0$ such that $\lim(c)$ is distant from $r$ (using 3.11). We may find some $\bar{b} \equiv_{Aa} \bar{b}'$ such that $\bar{b}$ fills the cut $\lim(c)$ of $I$. Let also $I_0, \bar{a}_*$ be such that $(\bar{b}, I_0, \bar{a}_*) \equiv (\bar{b}', I'_0, a'_*)$.

Let $I_\infty$ realize an infinite Morley sequence of $p$ over everything. The strong base change lemma (2.9) works equally well if instead of considering points $d_i$ filling the cuts $c_i$, we take sequences $\bar{d}_i$. We apply this modified version with $M$ as set of parameters, $I + I_\infty$ as indiscernible sequence, $\bar{d}_0 = \bar{b}$ and $\bar{d}_1 = a$. We conclude that we may assume that $\bar{b}$ is a Morley sequence of $\lim(c)$ over $M$.

Set $r = \lim(c)$ and let $B \subset M$ be of size $\aleph_0$ such that $r$ is $B$-invariant.

Let $d$ realize any type $s \in S(M)$ distant from $p$ and $q$. Assume that $d \downarrow_M \bar{b}$. Let $C \subset M$ be a subset of size $< \kappa$ such that $p, q$ and $r$ are invariant over $C$. Let $I' \subset M$ be a Morley sequence $p$ over $C$. Then by invariance of $r$, $\bar{b} + I'$ is a Morley sequence of $p$ over $C$ and by 3.6 $a_*$ strongly dominates $a$ over $(I, C)$. The hypothesis of 3.5 are satisfied with $J_1 = J_3 = \varnothing$, $J_2 = I_0$ and $J_4 = I'$. We conclude that $a \models p|Cd$. As this is true for every small $C$, $d$ and $a$ are independent over $M$. This proves that $\bar{b}$ s-dominates $a$ over $M$. □

Note that $a_*$ as constructed in the previous lemma has the following additional property:

For every $d \in \mathfrak{C}$ such that $\mathrm{tp}(d/Ma_*)$ does not fork over $M$, and such that $a_*d$ commutes with $p$, then $a \downarrow_M d$.

This assumption is satisfied in particular when $d$ is distant from $a$ and $a_*$ and $a_* \downarrow_M d$.

**Corollary 3.13.** *Let $p, q \in S(M)$ be any two types of $|\alpha|$-tuples ($|\alpha| < \kappa$) and let $a \models M$. Then there is $a_*$ a tuple of length $\leq |T| + |\alpha|$, distant from $q$ over $M$ and such that $a_*$ s-dominates $a$ over $M$. Furthermore, we may assume that $\mathrm{tp}(a_*/M)$ is invariant over a subset of size $\aleph_0$.*

*Proof.* By 3.10, there is some $a_{**}$ s-dominating $a$ over $M$. By Lemma 3.12, there is a tuple $a_*$ s-dominating $a_{**}$ over $M$ with the required size, whose type over $M$ is invariant over a subset of size $\aleph_0$ and distant from $q$.

We check that $a_{**}$ s-dominates $a$ over $M$. Let $r \in S(M)$ be an invariant type distant from $a_{**}$ and $a$. Let $b \models r$ with $b \downarrow_M a_{**}$. By Lemma 3.12, there is $b_*$ be given by with $q = \mathrm{tp}(a\hat{\ }a_*\hat{\ }a_{**}/M)$. Composing by an automorphism over $Mb$, we may further assume that $b_* \downarrow_M a_{**}$. Then as $a_{**}$ s-dominates $a_*$ over $M$, we have $b_* \downarrow_M a_*$ and as $a_*$ s-dominates $a$ over $M$, $b_* \downarrow_M a$. By the property of $b_*$ this implies $b \downarrow_M a$. □

**Lemma 3.14** (Transitivity of s-domination)**.** *Let $a \in \mathfrak{C}$ and let $a_*$ s-dominate $a$ over $M$. Let also $a_{**}$ s-dominate $a_*$ over $M$. Then $a_{**}$ s-dominates $a$ over $M$.*

*Proof.* Let $d \in \mathfrak{C}$ be distant from $a$ and $a_{**}$ with $d \downarrow_M a_{**}$. By 3.13, let $d_*$ s-dominate $d$ over $M$ and distant from $a\hat{\ }a_*\hat{\ }a_{**}$. Composing by an automorphism over $Md$, we may assume that $d_* \downarrow_M a_{**}$. Then we have $d_* \downarrow_M a_*$ and $d_* \downarrow_M a$ and finally $d \downarrow_M a$. □

Exemple 3.15. *If $p \in S(M)$ is generically stable, and $a \models p$, then $a$ is s-dominated by itself. In the opposite situation, if $p$ is invariant and its Morley sequence is distal, then $a$ is s-dominated by the empty set.*

### 3.1.2 Stable independence

**Definition 3.16** (S-independence)**.** Let $p, q$ be any types over $M$, let $a \models p$ and $b \models q$. We say that $a$ and $b$ are *s-independent* over $M$ and write $a \downarrow^s_M b$ if there is a tuple $a_*$ s-dominating $a$ and distant from $b$ such that $a_* \downarrow_M b$.

Note that if $a$ and $b$ are distant, then $a \downarrow^s_M b$ if and only if $a \downarrow_M b$.

**Proposition 3.17** (Symmetry of s-independence)**.** *S-independence is symmetric: if $a$ and $b$ realize invariant types, then $a \downarrow^s_M b$ if and only if $b \downarrow^s_M a$ if and only if there are $a_*$, $b_*$ s-dominating $a$ and $b$ respectively, distant form each other such that $a_* \downarrow_M b_*$.*

*Proof.* It is enough to prove the last equivalence. To see right to left, let $a_{**}$ s-dominate $a_*$ and be distant from $b_*$ and $b$ over $M$. Assume also that $a_{**} \downarrow_M b_*$, then by 3.14, $a_{**}$ s-dominates $a$ over $M$ and is independent from $b_*$ over $M$.

Conversely, assume that $a \downarrow^s_M b$. Let $a_*$ be s-dominating $a$ and distant from $a$ and $b$ such that $a \downarrow_M a_*$. We can find a tuple $b'_*$ dominating $b$ distant from $a, a_*$ and $b$. As $a_* \downarrow_M b$, there is $b_* \equiv_{Ma} b'_*$ such that $a_* \downarrow_M b_*$. □

**Proposition 3.18** (Weight is bounded). *Let $(b_i)_{i<|T|^+}$ be a sequence of tuples such that $b_i \downarrow_M b_{<i}$ for each $i$, and $a \in \mathfrak{C}$. Then there is $i < |T|^+$ such that $a \downarrow^s_M b_i$.*

*Proof.* By Lemma 3.13, we can find a family $(b^*_i)_{i<|T|^+}$ such that: For each $i < |T|^+$, $b^*_i$ realizes an invariant type $r_i$ distant from $q := \mathrm{tp}(a/M)$ and $r_j$, $j \neq i$, $b^*_i$ s-dominates $b_i$ over $M$ and $b^*_i \downarrow_M b^*_{<i}$. By Corollary 1.9, there is $i < |T|^+$ such that $\mathrm{tp}(b^*_i, a/M) = r_i \otimes q$. By definition, $a \downarrow^s_M b_i$. □

The following special case of this proposition makes no reference to s-domination.

**Corollary 3.19.** *Let $q \in S(M)$ be $A$-invariant and, for $i < |T|^+$, let $p_i \in S(M)$ be an invariant type. Assume that $p_i$ commutes with $q$, for each $i$. Let $(b_i) \models \bigotimes p_i$ and $a \models q$. Then there is $i < |T|^+$ such that $\mathrm{tp}(b_i, a/N) = p_i \otimes q$.*

**Corollary 3.20.** *Let $a, b \in \mathfrak{C}$ such that $a \not\downarrow^s_M b$, then $\mathrm{tp}(b/Ma)$ forks over $M$.*

*Proof.* Otherwise, we could find a global $M$-invariant extension $\tilde{p}$ of $\mathrm{tp}(b/Ma)$. Take $(a_i)_{i<|T|^+}$ be a sequence of realizations of $\mathrm{tp}(a/M)$ with $a_0 = a$ and $a_i \downarrow^s_M a_{<i}$ for each $i$. By invariance, if $b_* \models \tilde{p}$ over everything, for each $i < |T|^+$, $\mathrm{tp}(b_*, a_i/M) = \mathrm{tp}(b_*, a/M)$ and $b_* \not\downarrow^s_M a_i$. This contradicts Proposition 3.18. □

**Corollary 3.21.** *Let $a$ and $b$ realize two distant types over $M$, then $\mathrm{tp}(a/Mb)$ forks over $M$ if and only if $\mathrm{tp}(b/Ma)$ forks over $M$ if and only if $a \downarrow_M b$.*

**Proposition 3.22.** *Let $p \in S(M)$ be an invariant type and $q \in S(M)$ be distant from $p$. Let $I = (a_i)_{i<\omega}$ be a Morley sequence of $p$ over $M$ and $b \models q$. Then $\lim(I/Mb) = p|_{Mb}$.*

*Proof.* This follows easily from Proposition 3.18 by making the sequence $I$ of large cardinality. □

Exemple 3.23 (ACVF). *The reader might wonder what those results say in ACVF. As mentioned in the introduction, this example is not very relevant to this work, since the o-minimal and stable components are already separated.*

*More precisely, let $p \in S(M)$ be an invariant type of a field element. By* **??**, *Corollary 12.14, there are definable functions $f$ and $g$ respectively into the residue field $k$ and the value group $\Gamma$ such that letting $p_k = f_*(p)$ and $p_\Gamma = g_*(p)$, we have:*

*For any $a \vDash p$ and $b \in \mathfrak{C}$, $\mathrm{tp}(a/Mb) = p|_{Mb}$ if and only if $\mathrm{tp}(f(a)/Mb) = p_k|_{Mb}$ and $\mathrm{tp}(g(a)/Mb) = p_\Gamma|_{Mb}$.*

*Take such an invariant type $p$ and $a \vDash p$. Then $a$ is s-dominated by $f(a)$ since if $b \in \mathfrak{C}$ is distant from $a$ over $M$, then by distality of $\Gamma$, $\mathrm{tp}(b/M)$ and $\mathrm{tp}(g(a)/M)$ are orthogonal.*

## 3.2 The finite-co-finite theorem and application

We prove now the indiscernible sequence analog of Proposition 3.18. We could prove it by embedding the sequence $I$ in a saturated model and applying 3.18 or by reproducing the proof of that proposition in the context of indiscernible sequences. There are approximately the same amount of details to check in both cases, so we give here the second approach and leave the first as exercise to the reader.

**Theorem 3.24** (Finite-co-finite theorem)**.** *Let $I = I_1 + I_2 + I_3$ be indiscernible, $I_1$ and $I_3$ being infinite. Assume that $I_1 + I_3$ is $A$-indiscernible and take $\phi(x;a) \in L(A)$, then the set $B = \{b \in I_2 :\vDash \phi(b;a)\}$ is finite or co-finite.*

*Proof.* Assume this does not hold. Then by compactness and indiscernability, we may assume that $I$ is a very large saturated sequence of finite tuples and for every convex subset $K \subset I$, there is $a_K \in \mathfrak{C}$ such that, for $b \in I$ we have $\phi(b;a_K) \Leftrightarrow b \in J$ and $I \setminus K$ is indiscernible over $a_K$.

Pick a partition $I = J_1 + I_0 + J_2$, all pieces being infinite with no end points. Consider a cut $c_1$ of $I$ interior to $I_0$ filled by some $b_1$. Let $d_1$ be a small polycut of $I$ interior to $J_2$ and filled by $\bar{b}'_1$ such that $\bar{b}'_1$ strongly dominates $b_1$ over $J = J_1 + J_2$. Now take some $c_2 \ll c_1$ interior to $I_0$ and filled by some $b_2$ such that $b_2 \downarrow_I \bar{b}'_1$. By domination we also have $b_2 \downarrow_I b_1$. Take a tuple $\bar{b}'_2$ strongly dominating $b_2$ over $J$ and filling a polycut $d_2 > d_1$ such that $\bar{b}'_2 \downarrow_I \bar{b}'_1$. Iterate to have an increasing sequence of cuts $\langle c_i : i < |T|^+\rangle$, point $b_i$, $i < |T|^+$ filling them independently over $I$, an increasing sequence of polycuts $\langle d_i : i < |T|^+\rangle$ filled independently by $\bar{b}'_i$, each $\bar{b}'_i$ strongly dominating the corresponding $b_i$ over $J$.

By assumption, there is some point $a \in \mathfrak{C}$ such that $J$ is indiscernible over $a$, $\phi(b;a)$ holds for each $b \in I_0 \cup \{b_i : i < |T|^+\}$ and $\neg\phi(b;a)$ holds for $b \in J$. Let $J'_2$ be the sequence $J_2$ with all the $\bar{b}'_i$, $i < |T|^+$ added in their respective cuts. Let also $I' = J_1 + I_0 + J'_2$. This is an indiscernible sequence. By shrinking of indiscernible (1.6), there is $J''_2 \subseteq J_2$ obtained by removing at most $|T|$ of the tuples $\bar{b}'_i$ from $J'_2$ such that $J_1 + J''_2$ is indiscernible over $a$. Without loss, assume we have not removed the tuple $\bar{b}'_0$. We know that $\bar{b}'_0$ dominates $b_0$ over $J$. Let $e$ and $g$

denote the respective cuts of $b_0$ and $\bar{b}_0'$ over $J$. Then $\bar{b}_0' \vDash \lim(g/J + a)$. By the external characterization of domination (Proposition 3.5), $b_0 \vDash \lim(e/J + a)$. So $b_0 \vDash \neg\phi(x;a)$ contradicting the initial hypothesis. □

We now give an application of this result to externally definable sets.

We will use the following notation: if $M \vDash T$, $M \prec N$ is an elementary extension and $A \subseteq N$ containing $M$, then $M_{[A]}$ is the structure with universe $M$ with language composed of a predicate for every subset of $M^l$ (any $l$) of the form $\phi(M;\bar{c})$, $\bar{c} \in A^k$ for any $\phi(\bar{x};\bar{y}) \in L(M)$, interpreted the obvious way.

Shelah proved in [9] that $M_{[\mathfrak{C}]}$ eliminates quantifiers. We refer the reader to [2] for a slightly different approach, that we will use here. If $p \in S(M)$ is any type and $a \vDash p$, then it is not true in general that $M_{[a]}$ eliminates quantifiers (see [2] for a counterexample). However it is conjectured that $M_{[I]}$ does, where $I$ is a coheir sequence starting with $a$. We prove a special case of this when $p$ is interior to $M$. See the definition below.

We will need some notions from [2] that we recall now. If $X$ is an externally definable subset of $X$ (*i.e.*, a subset of the form $\phi(M,c)$ for some tuple $c \in \mathfrak{C}$), then an *honest definition* of $X$ is a formula $\theta(x,d) \in L(\mathfrak{C})$ such that (1) $\theta(M,d) = X$ and (2) for every formula $\psi(x) \in L(M)$ such that $\psi(M) \subseteq X$ then $\mathfrak{C} \vDash \theta(x) \to \psi(x)$.

**Lemma 3.25.** *If $A \subset \mathfrak{C}$ containing $M$ is such that for every formula $\phi(x;c) \in L(A)$, $\phi(M;c)$ has an honest definition with parameters in $A$, then $M_{[A]}$ eliminates quantifiers.*

*Proof.* Let $\phi(x,y;c) \in L(A)$ and let $\theta(x,y;d) \in L(A)$ be an honest definition of $X := \phi(M;c)$. Let $\pi : M^{|x|+|y|} \to M$ be the projection on the first $|x|$ coordinates. Let $\psi(x;d) = (\exists y)(\theta(x,y;d))$. Then $\psi(M;d) = \pi(X)$: It is clear that $\psi(M;d) \subseteq \pi(X)$, and if $a \in M^{|x|} \setminus \pi(X)$, then the set $\{(x,y) \in M^{|x|+|y|} : y \neq a\}$ contains $X$ and by honesty $\mathfrak{C} \vDash \theta(x,y;d) \to y \neq a$ which gives the reverse inclusion. □

**Definition 3.26.** Let $p$ be an $M$-invariant global type. We say that $p$ is *interior* to $M$ if $p^{(\omega)}$ is both an heir and a co-heir of its restriction to $M$.

An example of an interior type is given by the following situation: Let $I \subset M$ be indiscernible and $c$ a cut interior to $I$ such that $M$ respects $c$. Then the type $p = \lim(c^+)$ is interior to $M$.

**Lemma 3.27.** *Let $p$ be a global $M$-invariant type interior to $M$. Let $I_0 + I_1 + I_2$ be a Morley sequence of $p$ over $M$. For $i < 3$ let $\bar{a}_i \subset I_i$ be a finite tuple. Assume* *that $\bar{a}_1 \models \phi(\bar{x}; \bar{a}_0, \bar{a}_2)$, $\phi \in L(M)$, then there are two tuples $\bar{b}_0, \bar{b}_2 \subset M$ such that $\bar{a}_1 \models \phi(\bar{x}; \bar{b}_0, \bar{b}_2)$.*

*Proof.* First find $\bar{b}_2$ such that $\bar{a}_1 \models \phi(\bar{x}; \bar{a}_0, \bar{b}_2)$ by the coheir hypothesis. Then find $\bar{b}_0$ by the heir hypothesis. □

**Theorem 3.28** (Shelah expansion for interior types)**.** *Let $p$ be a global $M$-invariant type interior to $M$. Let $I$ be a Morley sequence of $p$ over $M$. Then $M_{[I]}$ eliminates quantifiers.*

*Proof.* Take a saturated extension $M_{[I]} \prec N^*$ of size $\kappa > |M|$. The model $N^*$ can be seen as a reduct to the language of $M_{[I]}$ of some $N_{[J]}$ for $M \prec N$ and $J \equiv_M I$, $J$ indiscernible over $N$. Without loss $I = J$. Notice that $N^*$ and $N_{[I]}$ have the same definable sets, in particular $N_{[I]}$ is also saturated.

Claim: There is an indiscernible sequence $I_1 + I_2 \subset N$ such that $N$ respects the cut $c = (I_1, I_2)$ and $I \models \lim(c^+)^{(\omega)}$.

Proof: Write $N = \bigcup_{i<\kappa} A_i$ with $|A_i| < \kappa$. Let $i < \kappa$. By Lemma 3.27 and saturation, we can find sequences $K_i, L_i \subset N$ of order type $\omega$ such that $K_i + I + L_i$ is indiscernible over $A_i$. Let $I_1 = K_1 + K_2 + ...$ and $I_2 = ... + L_2 + L_1$, the sums ranging over $i < \kappa$. The required property is then easy to check.

Let $\phi(x; y)$ be a formula and $a_0 \models p$, $a_0 \in I$. We consider the pair $(M, N)$ and show that $\phi(a_0; M)$ has an honest definition with parameters in $M + I_1 + I_2$.

By the Theorem 3.24 and compactness, there are $\delta, k, n, N$ such that for every finite sequence $J_1 + J_3 + J_2$, such that:
– $J_1$ and $J_2$ are of size at least $n$,
– $J_1 + J_3 + J_2$ is indiscernible,
– $J_1 + J_2$ is $(\delta, k)$-indiscernible over $b$ and
– $\phi(x; b)$ holds on all elements of $J_1$ and $J_2$,
then $\neg\phi(x; b)$ holds on at most $N$ elements of $J_3$.

Let $I_1' \subset I_1$ and $I_2' \subset I_2$ be finite of size $n$ such that $I_1' + I_2'$ is $M$-indiscernible. Consider the formula $\theta(y) \in L(MI)$ such that if $b \models \theta(y)$, then $I_1' + I_2'$ is $(\delta, k)$-indiscernible over $a$, and $\phi(\bar{a}_0; y)$ holds on all elements of $I_1' + I_2'$. Define analogously $\theta_1(y)$ using $\neg\phi$ instead of $\phi$.

Then, for every $b \in M$, $\theta(b)$ holds if and only if $\phi(a_0; b)$ holds. Also, if $b \in N$, and $\theta(b)$ holds, then $\phi(a_0; b)$ holds (Why ? Only finitely many elements $a$ from $I_1 + I_2$, with $I_1' < a < I_2'$ can satisfy $\phi(a; b)$). This easily implies that $\theta$ is an honest definition of $\phi(a_0; M)$.

To conclude the theorem, notice that we can do the same thing replacing $p$ by $p^{(n)}$ for any $n$, which takes care of formulas $\phi(\bar{a}; y)$ with $\bar{a}$ a finite subset of $I$ instead of one element. □

# 4 Additional observations

## 4.1 Sharp theories

Sharp theories are theories where the stable part of types is witnessed by a generically stable type.

**Definition 4.1.** An NIP theory $T$ is *sharp* if for every $|T|^+$-saturated model $M$ and $a \in \mathfrak{C}$, there is $b \in \mathfrak{C}$ whose type over $M$ is generically stable such that $b$ s-dominates $a$ over $M$.

There is an analog for indiscernible sequences.

**Proposition 4.2.** *A theory is sharp if and only if:*

*For every dense indiscernible $I$, and $c$ a non-principal cut interior to $I$, there is some set $D$ and $q \in S(\mathfrak{C})$, generically stable and $D$-definable, such that:*

*– $I$ is indiscernible over $D$,*

*– For every $a$ filling $c$ over $ID$, there is $a_* \vDash q|_{ID}$ such that the following domination condition holds: For every $d \in \mathfrak{C}$ such that $I$ is $d$-indiscernible, if $a_* \vDash q_{IDd}$, then $I \cup \{a\}$ is $d$-indiscernible.*

We will say that in indiscernible sequence $I$ is sharp if there is $q$ as above. Adapting the construction of section 2.4, we can show that it enough to check the criterion in dimension 1.

**Proposition 4.3.** *Assume that every indiscernible sequence of singletons is sharp, then $T$ is sharp.*

This is the case in particular if every sequence of singletons is either distal or totally indiscernible.

**Corollary 4.4.** *Any dp-minimal theory is sharp.*

## 4.2 Notions of weights and minimality

Two notions of weight naturally stem from the approach presented here: the usual weight, or burden, decomposes into a distal weight and a stable one.

**Definition 4.5.** Let $p \in S(A)$ then $p$ has distal weight $\geq \kappa$ if there are a set $\{(b_t^\alpha)_{t \in \omega+\omega} : \alpha < \kappa\}$ of mutually indiscernible sequences over $A$ (each sequence is indiscernible over the union of the others), formulas $\phi_\alpha(x; y_\alpha)$ for $\alpha < \kappa$ and $a \vDash p$ such that, for each $\alpha < \kappa$, $\vDash \phi_\alpha(a; b_t^\alpha)$ holds if and only if $t < \omega$.

Let $\mathcal{I}_1$ be a linear order of type $\omega$, $\mathcal{I}_2$ an order of type $\omega_*$, and $\mathcal{I}_3 = \mathcal{I}_1 + \{*\} + \mathcal{I}_2$. The type $p$ has stable weight $\geq \kappa$ if there is are a set $\{(d_t^\alpha)_{t \in \mathcal{I}_3} : \alpha < \kappa\}$ of mutually indiscernible sequences over $A$ and $a \vDash p$ such that, for each $\alpha < \kappa$, the sequence $(d_t^\alpha)_{t \in \mathcal{I}_1 + \mathcal{I}_2}$ is indiscernible over $Aa$, but $(d_t^\alpha)_{t \in \mathcal{I}_3}$ is not.

A theory is distal if and only if every type has stable weight 0.

A theory is *dw*1 if every type of a singleton has distal weight at most 1. This notion seems similar to o-stability (see [13]) which is defined for linearly ordered theories.

## 4.3 Types over finite sets

Perhaps the most fascinating conjecture about $NIP$ theories is the conjecture of uniform definability of types over finite sets (which says that for every formula $\phi(x, y) \in L$ there is some formula $\psi(x, z) \in L$ such that for any tuple $a$ of size $|y|$ and finite set $B$, there is $c \in B^{|z|}$ satisfying $\vDash \phi(b, a) \leftrightarrow \psi(b, c)$ for any $b \in B^{|x|}$).

This conjecture has be proved for dp-minimal theories by Guingona in [3].

In the case of distal theories, we state a stronger conjecture:

**Conjecture 4.6.** *Let $T$ be distal, and let $\phi(x, y) \in L$. Then there is an integer $N_\phi$ such that for every finite $B \subset \mathfrak{C}$ and tuple $a \in \mathfrak{C}^{|y|}$, there is a subset $B_0 \subset B$ of size at most $N_\phi$ such that* $\mathrm{tp}(a/B_0) \vdash \mathrm{tp}_\phi(a/B)$.

Observe that the converse of the conjecture is easy. In fact, if we restrict ourselves to finite subsets $B$ that are the underlining sets of an indiscernible sequence of tuples, then the property described in the conjecture is easily seen to be equivalent to distality of $T$.

The conjecture has been checked by Guingona in the case of dp-minimal linearly ordered theories (unpublished).

# References

[1] John Baldwin and Michael Benedikt. Stability theory, permutations of indiscernibles, and embedded finite models. *Transactions of the American Mathematical Society*, 352(11):4937–4969, 11 2000.

[2] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs. *preprint*.

[3] Vincent Guingona. On uniform definability of types over finite sets. *Modnet preprint 250*.

[4] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. Stable domination and independence in algebraically closed valued fields.

[5] Ehud Hrushovsk, Anand Pillay, and Pierre Simon. Generically stable and smooth measures in NIP theories. *preprint*.

[6] Ehud Hrushovski and Anand Pillay. On nip and invariant measures. *preprint*.

[7] Saharon Shelah. Strongly dependent theories. *submitted*.

[8] Saharon Shelah. Classification theory for elementary classes with the dependence property—a modest beginning. *Sci. Math. Jpn.*, 59(2):265–316, 2004. Special issue on set theory and algebraic model theory.

[9] Saharon Shelah. Dependent first order theories, continued. *Israel Journal of Mathematics*, 173(1):1–60, 2009.

[10] Saharon Shelah. A dependent dream and recounting types. *preprint*, 2010. 950 in Shelah archive.

[11] Saharon Shelah. Dependent theories and the generic pair conjecture. *Communications in Contemporary Mathematics*, submitted, 2010. math.LO/0702292.

[12] Pierre Simon. Finding generically stable measures. *preprint*.

[13] Victor Verbovskiy. Ordered o-stable groups. *Modnet preprint 194*.