Introduction to model theoretic techniques

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Introductory Workshop: Model Theory, Arithmetic Geometry and Number Theory

Basic definitions

- structure $\mathcal{M} = (M; R_1, R_2, ..., f_1, f_2, ..., c_1, c_2, ...)$
- formula $\exists x \forall y R_1(x, y) \vee \neg R_2(x, y)$
- language L
- satisfaction $\mathcal{M} \models \varphi$
  $\bar{a} \in M, \quad \mathcal{M} \models \psi(\bar{a})$
- theory T: (consistent) set of sentences
- model $\mathcal{M} \models T$
- definable set $\psi(\bar{x}) \longrightarrow \psi(M) = \{\bar{a} \in M^{|\bar{x}|} : \mathcal{M} \models \psi(\bar{a})\}$

  usually definable set means definable *with parameters*:
  $\theta(\bar{x}; \bar{b}) \longrightarrow \theta(M; \bar{b}) = \{\bar{a} \in M^{|\bar{x}|} : \mathcal{M} \models \theta(\bar{a}; \bar{b})\}$

Compactness

Theorem (Compactness theorem)

If all finite subsets of T are consistent, then T is consistent.

Uses of compactness

- Transfer from finite to infinite.
- From infinite to finite:
  Approximate subgroups (see tutorial on multiplicative combinatiorics)
  Szemeredí's theorem (Elek-Szegedy, Towsner ...)
- Obtaining uniform bounds

Understanding definable sets of M

$Th(M)$: set of sentences true in the structure M.

- Elementary equivalence: $M \equiv N$ if $Th(M) = Th(N)$.

  Example: If K and L are two algebraically closed fields of the same characteristic, then $K \equiv L$.
- A theory T is *complete* if it is of the form $Th(M)$.
- Elementary extension: $M \preceq N$ if $M \subseteq N$ and for all $\varphi(\bar{x})$ and $\bar{a} \in M$, $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a})$.

Theorem (Löwenheim-Skolem)

Assume that L is countable, M infinite.

- *Let $\kappa \geq |M|$, then there is an elementary extension $M \prec N$, where $|N| = \kappa$.*
- *If $A \subseteq M$, then there is $M_0 \preceq M$ containing A, $|M_0| = |A| + \aleph_0$.*

• Monster model $\mathcal{U}, \mathbb{C}, \mathbb{M}, \ldots$

Types

Let $B \subset M$ and $\bar{a} \in M^k$.

Definition

The *type* of $\bar{a}$ over B is the set of formulas

$$\{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B^{|\bar{b}|}, M \models \varphi(\bar{a}; \bar{b})\}.$$

Fact

The tuples $\bar{a}, \bar{b} \in \mathcal{U}^k$ have the same type over $B \subset \mathcal{U}$ iff there is an automorphism $\sigma : \mathcal{U} \to \mathcal{U}$ fixing B pointwise such that $\sigma(\bar{a}) = \bar{b}$.

The set of types over B (in a given variable $\bar{x}$) is denoted by $S_{\bar{x}}(B)$. It is a totally disconnected compact space.

Quantifier elimination

Definition

A theory T *eliminates quantifiers* in a language L if every L-formula is equivalent modulo T to a formula without quantifiers.

Examples:

- $Th(\mathbb{C}; 0, 1, +, -, *)$ eliminates quantifiers;
- $Th(\mathbb{R}; 0, 1, +, -, *)$ **does not** eliminate quantifiers:

$$\varphi(x) \equiv \exists y (x^2 = y)$$

- $Th(\mathbb{R}; 0, 1, +, -, *, \leq)$ eliminates quantifiers.

If T eliminates quantifiers and $M, N \models T$, then

$$M \subseteq N \Longrightarrow M \preceq N.$$

Examples

- $(\mathbb{N}; \leq)$;
- $(\mathbb{C}; 0, 1, +, -, *)$;
- $(\mathbb{R}; 0, 1, +, -, *, \leq)$.

Imaginaries

Let $X \subseteq M^k$ be a definable set and $E \subseteq X^2$ a definable equivalence relation. Then X/E is an *imaginary sort* of M.

We say that M *eliminates imaginaries* if every imaginary sort is definably isomorphic to a definable set.

Examples: $\mathbb{C}$, $\mathbb{R}$ eliminate imaginaries.

Codes of definable sets

End of talk 1.

Let $A \subseteq M$.

- Definable closure $dcl(A)$:
  $e \in dcl(A)$ if there is $\varphi(x; \bar{a}) \in \mathrm{tp}(e/A)$ such that e is the only element in M satisfying $\varphi(x; \bar{a})$.

  Equivalently, $e \in dcl(A)$ if and only if $e = f(\bar{a})$ for some definable function f and tuple $\bar{a}$ of elements of A.
- Algebraic closure $acl(A)$:
  $e \in acl(A)$ if there is $\varphi(x; \bar{a}) \in \mathrm{tp}(e/A)$ such that there are finitely many elements in M satisfying $\varphi(x; \bar{a})$.

Definable types

Definition

A type $\text{tp}(\bar{a}/M)$ is *definable* if for every formula $\varphi(\bar{x};\bar{y})$, there is a formula $d\varphi(\bar{y})$ with parameters in M, such that for any tuple $\bar{b} \in M$:

$$M \models \varphi(\bar{a};\bar{b}) \iff M \models d\varphi(\bar{b})$$

Examples: ACF, $(\mathbb{Q}, \leq)$.
Pushforward $f_* p$.

Stable theories

Definition

A theory T is *stable* if all types over all models of T are definable.

Examples:

- ACF;
- abelian groups;
- DCF_0: differentially closed fields of char 0;
- $\mathrm{SCP}_{p,n}$: separably closed fields.

Some unstable theories:

- $Th(\mathbb{R}, 0, 1, +, -, *, \leq)$;
- valued fields.

Independence (non-forking)

Definition

(T is stable) We say that $\bar{a}$ is independent from $\bar{b}$ over M, or $\text{tp}(\bar{a}/M\bar{b})$ *does not fork* over M, written

$$\bar{a} \underset{M}{\downarrow} \bar{b}$$

if $\text{tp}(\bar{a}/M\bar{b})$ is according to the definition scheme of $\text{tp}(\bar{a}/M)$.

Examples: ACF, divisible torsion free abelian groups.

In stable theories, we can generalize this definition to an arbitrary base set A instead of M.

Some properties of independence

Existence Let $p \in S(A)$ and $A \subseteq B$, then there is $q \in S(B)$ extending p and non-forking over A.

Algebraic closure $c \mathop{\smile\!\!\!\!\!\!\mid}_A c$ if and only if $c \in acl(A)$.

Transitivity $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}, \bar{c}$ iff $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_{A,\bar{b}} \bar{c}$

Symmetry $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}$ iff $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{a}$

Uniqueness if M is a model, $p \in S(M)$ and $M \subseteq B$, then p has a unique non-forking extension to a type over B.
If we replace M by an aribtrary subset A, then p may have up to $2^{\aleph_0}$ non-forking extensions over B.

Stable formulas

Definition

A formula $\varphi(\bar{x};\bar{y})$ has the order property if for every n, we can find tuples $\bar{a}_1,\dots,\bar{a}_n$ and $\bar{b}_1,\dots,\bar{b}_n$ such that:

$$\varphi(\bar{a}_i,\bar{b}_j) \iff i \leq j.$$

Fact

Let M be a structure (in a countable language), $T = Th(M)$ and $M \prec \mathcal{U}$ a monster model. The following are equivalent:

- *T is stable;*
- *no formula $\varphi(\bar{x};\bar{y})$ has the order property;*
- *for any $\bar{a} \in \mathcal{U}$, $B \subset \mathcal{U}$, $\mathrm{tp}(\bar{a}/B)$ is definable;*
- *for any $B \subset \mathcal{U}$, there are at most $|B|^{\aleph_0}$ types over B.*

Example: Separably closed fields.

Geometric stability theory

Definition

A definable set X is *strongly minimal* if any definable subset of X is finite or cofinite.

Examples:

- An infinite set with no structure;
- A k-vector space V;
- An algebraically closed field.

If X is a strongly minimal set, the algebraic closure operator $acl(A)$ satisfies exchange and therefore gives rise to a dimension function $\dim(A)$ on subsets of X.

We classify such sets X according to the behavior of *acl*:

Disintegrated $acl(A) = \bigcup_{a \in A} acl(\{a\})$;

Locally modular $\dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B)$ for A, B closed, $\dim(A \cap B) \geq 1$;

Not locally modular The condition above does not hold.

Definition

A type $p(x)$ is minimal if for every formula $\varphi(x)$, either $p(\mathcal{U}) \cap \varphi(\mathcal{U})$ or $p(\mathcal{U}) \setminus \varphi(\mathcal{U})$ is finite.

If $p(x)$ is a minimal type, then as in the case of strongly minimal formulas, one considers the algebraic closure operator $acl(A)$ on subsets $A \subset p(\mathcal{U})$ and the associated dimension function.

End of talk 2.

VC-dimension

Let X be a set and $\mathcal{C} \subseteq \mathfrak{P}(X)$ a family of subsets of X.
Let $A \subseteq X$, then $\mathcal{C}$ *shatters* A if $\mathcal{C} \cap A = \mathfrak{P}(A)$.

Definition

The family $\mathcal{C}$ has *VC-dimension* d if it shatters some subset $A \subseteq X$ of size d, but no subset of size $d+1$.
If $\mathcal{C}$ shatters subsets of arbitrary large (finite) size, we say that it has infinite VC-dimension.

Examples: The family of intervals of $(\mathbb{R}, \leq)$ has VC-dimension 2.
The family of half-spaces of $\mathbb{R}^2$ has VC-dimension 3.

Define the *shatter function* $\pi_{\mathcal{C}}$ of $\mathcal{C}$ as

$$\pi_{\mathcal{C}}(n) = \max_{A \subseteq X, |A| \leq n} |\mathcal{C} \cap A|.$$

Note that $\pi_{\mathcal{C}}(n) = 2^n$ if and only if VC-dim$(\mathcal{C}) \geq n$.

Fact (Sauer-Shelah lemma)

Either :

- $\pi_{\mathcal{C}}(n) = 2^n$ *for all* n *(infinite VC-dimension)*

or

- $\pi_{\mathcal{C}}(n) = O(n^d)$ *(one can take* $d =$*VC-dim*$(\mathcal{C})$*).*

The *VC-density* of $\mathcal{C}$ defined as the infimum of r such that $\pi_{\mathcal{C}}(n) = O(n^r)$ is often more meaningful than the VC-dimension.

NIP theories

Let M be a structure and $T = Th(M)$.

$$\varphi(\bar{x};\bar{y}) \longrightarrow \mathcal{C}_\varphi = \{\varphi(M;\bar{b}) : \bar{b} \in M^{|\bar{y}|}\} \subseteq \mathfrak{P}(M^{|\bar{x}|}).$$

Definition

The formula $\varphi(\bar{x};\bar{y})$ is *NIP* (No Independence Property) if the family $\mathcal{C}_\varphi$ has finite VC-dimension.
The theory T is NIP if all formulas are.

In other words, the formula $\varphi(\bar{x};\bar{y})$ has IP if for all n, one can find $\bar{a}_1, \dots, \bar{a}_n \in M^{|\bar{x}|}$ and a family $(\bar{b}_J : J \in \mathfrak{P}(\{1, \dots, n\}))$ such that:

$$M \models \varphi(\bar{a}_i;\bar{b}_J) \iff i \in J.$$

Examples:

- The formula $x \leq y$, where $\leq$ is a linear order is NIP;
- The formula $x|y$ (x divides y) in $\mathbb{N}$ has IP.
- Every stable theory is NIP;
- $Th(\mathbb{R}; 0, 1, +, -, *, \leq)$ is NIP;
- Some theories of valued fields: ACVF, $Th(\mathbb{Q}_p)$ are NIP.

Lemma (VC-duality)

A formula $\varphi(\bar{x}; \bar{y})$ is NIP if and only if the opposite formula $\varphi^{opp}(\bar{y}; \bar{x})$ is NIP.

Indiscernible sequences

Definition

Let $(I, <_I)$ be a linear order and $A \subset M$. A sequence $(a_i : i \in I)$ of tuples of M is *indiscernible* over A if for all $i_1 <_I \cdots <_I i_k$ and $j_1 <_I \cdots <_I j_k$, we have

$$\mathrm{tp}(a_{i_1} \ldots a_{i_k}/A) = \mathrm{tp}(a_{j_1} \ldots a_{j_k}/A).$$

Fact (Ramsey+Compactness)

Given any sequence $(a_i : i < \omega)$ *of tuples and a linear order* $(I, <_I)$*, there is an indiscernible sequence* $(b_i : i \in I)$ *in* $\mathcal{U}$ *such that for any* $i_1 <_I \cdots <_I i_k$ *if*

$$\mathcal{U} \models \varphi(b_{i_1}, \ldots, b_{i_k}),$$

then there are $j_1 < \cdots < j_k < \omega$ *such that*

$$\mathcal{U} \models \varphi(a_{j_1}, \ldots, a_{j_k}).$$

Lemma

T is NIP if and only if for any indiscernible sequence $(a_i : i < \omega)$ and any model M, the sequence of types $(\mathrm{tp}(a_i/M) : i < \omega)$ converges.

More generally:

Lemma

The theory T is NIP if and only if for any set $A \subseteq \mathcal{U}$, any sequence of types over A has a converging subsequence.

Theorem

If all formulas $\varphi(x; \bar{y})$, x a singleton, are NIP, then T is NIP.

o-minimality

Assume that the language L contains a distinguished binary relation $\leq$ which defines a linear order on M.

Definition

The structure $(M, \leq, \ldots)$ is o-minimal if any definable subset of M is a finite union of intervals and points.

Fact

Assume that M is o-minimal, $a, b \in M \cup \{\pm\infty\}$ and let $f : (a, b) \to M$ be a definable function, then there are

$$a = a_0 < a_1 < \cdots < a_k = b$$

such that for each i, $f|_{(a_i, a_{i+1})}$ is either constant or a continuous monotonic bijection to an interval.

Fact (Cell decomposition)

Assume that M is o-minimal, then any definable subset of M^k is a finite union of cells.

Uniform finiteness

Fact

Let M be o-minimal. Let $\phi(x, \bar{y})$ *be a formula, then there is some integer n such that any* $\phi(x, \bar{b})$, $\bar{b} \in M$, *defines a union of at most n intervals.*

Corollary

Assume that M is o-minimal, then any structure elementarily equivalent to M is o-minimal. Hence o-minimiality is a property of the theory $Th(M)$.

Examples of o-minimal structures

- $\mathbb{R}$, with the field structure;
- $\mathbb{R}_{exp}$: the field $\mathbb{R}$ with the exponential function;
- $\mathbb{R}_{an}$: the field $\mathbb{R}$ along with restricted analytic functions;
- $\mathbb{R}_{an,exp}$.

Back to definable types

Let $p \in S_{\bar{x}}(\mathcal{U})$ be definable over a model $M \prec \mathcal{U}$. Recall that this means that we have a mapping

$$\varphi(\bar{x};\bar{y}) \longrightarrow d_p\varphi(\bar{y}), \qquad d_p\varphi(\bar{y}) \in L_M$$

such that for all $\bar{b} \in \mathcal{U}^{|\bar{y}|}$;

$$\varphi(\bar{x};\bar{b}) \in p \iff \mathcal{U} \models d_p\varphi(\bar{b}).$$

Product of definable types

Let $p(x)$ and $q(y)$ in $S(M)$ be definable, then one can define the product $p \otimes q(x, y)$ as $\mathrm{tp}(a, b/M)$, where

$$b \models q \text{ and } a \models p|Mb.$$

A Morley sequence of p over M is a sequence $(a_i : i < \omega)$ such that:

$$a_0 \models p \upharpoonright M \qquad a_{k+1} \models p \upharpoonright Ma_0...a_k.$$

Such a sequence is indiscernible over M.

Generically stable types

Definition

A type $p \in S(M)$ is *generically stable* if:

• p is definable;

• some/any Morley sequence $(a_i : i < \omega)$ of p is totally indiscernible (*i.e.*, every permutation of it is indiscernible).

Fact

A generically stable type commutes with any definable type.

Example: (ACVF) the generic type of a closed ball.

End of talk 3.