Measures in NIP Theories

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NIP Theories

Fact

A theory T is NIP iff for all $I = (a_i)_{i<\omega}$ indiscernible and all b, the types $\mathrm{tp}(a_i/b)$ converge to a type $\mathrm{Lim}(I/b)$.

Fact

(NIP) A global type p does not fork over A iff it is $Lstp(A)$-invariant.

In particular : p does not fork over $M \iff p$ is M-invariant.

Invariant Types

Let p_x, q_y be global M-invariant types.

Let $a \models p$, $b \models q|_{\bar{M}a}$.

Define $p_x \rtimes q_y = \mathrm{tp}(a, b/\bar{M})$.

Invariant Types

Let p be M-invariant.

$$p^{(1)} = p$$
$$p^{(n+1)} = p^{(n)} \rtimes p$$

$p^{(\omega)}$ is the *Morley sequence* of p.

Proposition

The M-invariant type p is uniquely determined by $p^{(\omega)} \upharpoonright_M$.

Proof.

Let $b \in \bar{M}$, then $p \upharpoonright_{Mb} = Ev(p^{(\omega)} \upharpoonright_M / Mb)$. □

Invariant Types

Proposition

Let $p \in S(\bar{M})$ be A-invariant, then p is Borel-definable over A.

Proof.

Let $b \in \bar{M}$, $\phi(x; y) \in L$.

$$\begin{aligned} (A_n):\ & \text{There is } (a_1, \ldots, a_n) \models p^{(n)} \text{ such that :} \\ & \models \neg(\phi(a_i; b) \leftrightarrow \phi(a_{i+1}; b)), \text{ for all } i < n, \\ & \models \phi(a_n; b). \\ (B_n):\ & \text{Same, with } \models \neg\phi(a_n; b). \end{aligned}$$

Then $p \models \phi(x; b)$ iff, for some n, (A_n) holds, but (B_{n+1}) does not. □

Invariant Types

Proposition

Assume :

- *For every A, no type over A forks over A,*
- *For every A, Lascar strong types on A coincide with strong types.*

Then, every type over $A = acl(A)$ *extends to an A-invariant type.*

ex. o-minimal, C-minimal (ACVF).

Generically Stable Types

Proposition

Let $p_x \in S(\bar{M})$ be A-invariant. TFAE :

- *p is definable and finitely satisfiable in any $M \supseteq A$,*
- *$p^{(\omega)}$ is totally indiscernible,*
- *For any invariant $q_y \in S(\bar{M})$, $p_x \otimes q_y = q_y \otimes p_x$,*
- *For any $A \subseteq B$, $p|_B$ has a unique global non-forking extension.*

We say that p is *generically stable*.

Keisler measures

Definition

A *Keisler measure* (of arity n) over A is a finitely additive function $\mu : L_n(A) \to [0,1]$.

$$\mu(\phi(x) \wedge \psi(x)) + \mu(\phi(x) \vee \psi(x)) = \mu(\phi(x)) + \mu(\psi(x)),$$
$$\mu(\top) = 1, \mu(\bot) = 0$$

Let $\mathcal{M}_n(A)$ denote the space of Keisler measures on A in n variables. It is a closed subspace of $[0,1]^{L_n(A)}$, so it is compact.

$S_n(A) \subset \mathcal{M}_n(A)$ is a closed subspace.

Keisler measures

Keisler measure on A $\longleftrightarrow$ Regular Borel probability measure on $S_n(A)$.

Keisler measures

For X, Y definable sets, write $X \sim Y$ if $\mu(X \triangle Y) = 0$

Definition

Let $\mu \in \mathcal{M}_n(A)$, a type p is *random* for μ if

$$p \vdash \phi(x) \quad \rightarrow \quad \mu(\phi(x)) > 0.$$

Let $S(\mu)$ be the set of random types for μ.
It is a closed subset of $S_n(A)$.

Proposition

- *Def(A)*$/\sim$ *is bounded.*
- $S(\mu)$ *is bounded.*

Smooth Measures

Definition

A measure μ is *smooth over* M (or *realized in* M), if $\mu|_M$ has a unique extension to any $M \prec N$.

Theorem (Keisler)

(NIP) Let $\mu \in \mathcal{M}(M)$ be a measure. Then there exists an extension $\mu \subset \nu$ to a global measure and $M \prec N$ such that ν is smooth over N.

Smooth Measures

Definition

A global measure is *fim* (over M) if :
for all $\phi(x;y)$, and all $\epsilon > 0$, there is $a_1, \ldots, a_n \in M$ s.t.

$$\text{For all } b \in \bar{M}, |\mu(\phi(x;b)) - Av(a_i)(\phi(x;b))| \leq \epsilon.$$

Where $Av(a_i)$ is the average measure of $(a_1, \ldots, a_n)$:
$Av(a_i) = \frac{1}{n}(tp(a_1/\bar{M}) + \ldots + tp(a_n/\bar{M}))$.

Example

A type is fim iff it is generically stable.

Smooth Measures

Definition

A global measure μ is definable over M if it is M-invariant, and if for all $\phi(x;y)$, and all $\alpha \in [0,1]$, the set $F_\alpha = \{b \in \bar{M} : \mu(\phi(x;b)) \leq \alpha\}$ is a closed set of $S(M)$.

We say the measure is *Borel-definable* If the F_α are Borel subsets of $S(M)$.

Definition

A global measure μ is finitely satisfiable over M if all types in $S(\mu)$ are finitely satisfiable in M.

Proposition

An fim measure is definable and finitely satisfiable.

Proposition

If μ is smooth over M, then μ is fim.

Corollary

A smooth measure is definable and finitely satisfiable.

Smooth Measures

Definition

Let $\mu_{(x,y)}$ be a measure in two variables.
The two variables x and y are *separated* if, for all $\phi(x)$ and $\psi(y)$:

$$\mu(\phi(x) \wedge \psi(y)) = \mu(\phi(x)).\mu(\psi(y)).$$

Proposition

Let $\mu_x \in \mathcal{M}(M)$ be smooth over M, and let $\nu_y \in \mathcal{M}(M)$ be any measure.
Then there is a unique $\lambda_{(x,y)} \in \mathcal{M}(M)$ extending μ_x and ν_y and such that the variables x and y are separated.

Borel definability

Proposition

Let $\mu \in \mathcal{M}(M)$, and take $\phi(x;y)$ and $\epsilon > 0$. There is $p_1, \ldots, p_n \in S(M)$ such that :

$$\text{For all } b \in M, |\mu(\phi(x;b)) - Av(p_i)(\phi(x;b))| \leq \epsilon.$$

Proof.

Let $\mu \subset \nu$ a smooth extension of μ to some $M \prec N$.
Take $x_1, \ldots, x_n \in N$ given by the previous theorem for ν.
Let $p_i = tp(x_i/M)$. □

Corollary

Any M-invariant measure is Borel-definable over M.

Product of Measures

Let μ_x, ν_y be global M-invariant measures. Then we can define $(\mu \rtimes \nu)_{(x,y)}$ by :

$$\mu \rtimes \nu(\phi(x,y)) = \int_{p \in S_x(M)} \nu(\phi(p,y)) d\mu.$$

Where $\nu(\phi(p,y)) = \nu(\phi(a,y))$, for any $a \in \bar{M}$, $a \models p$.

If μ is M-invariant, define :

$$\mu^{(1)} = \mu$$
$$\mu^{(n+1)} = \mu^{(n)} \rtimes \mu$$

Let $\mu_{(x_1,x_2,\dots)}$ be a measure in ω variables.

Definition

μ is an *indiscernible sequence* (over A) if, for all $i_1 < i_2 < \cdots < i_n$, $j_1 < j_2 < \cdots < j_n$, all formula $\phi \in L(A)$, we have :

$$\mu(\phi(x_{i_1}, \dots, x_{i_n})) = \mu(\phi(x_{j_1}, \dots, x_{j_n})).$$

Indiscernible sequences

Let $\mu_{(x_1,x_2,\ldots,y)}$ be a measure in $\omega+1$ variables over a set A.

Proposition

Assume that μ, restricted to the variables $(x_1, x_2, \ldots)$, is an indiscernible sequence. Assume that $(x_i)_{i<\omega}$ and y are separated. Then, for all formula $\phi(x; y)$, the sequence $\mu(\phi(x_i, y))$ converges.

Indiscernible sequences

If μ is a global M-invariant measure, then $\mu^{(\omega)}$ is an indiscernible sequence.

The analogues of results for types hold :

- An M-invariant measure μ is uniquely determined by $\mu^{(\omega)}|_M$,
- For any $b \in \bar{M}$, $\mu|_{Mb} = Ev(\mu^{(\omega)}/Mb)$.

Generically stable measures

Proposition

Let μ_x be an M-invariant global measure. TFAE :

- μ *is definable and finitely satisfiable,*
- $\mu^{(\omega)}$ *is totally indiscernible,*
- $\mu_x \rtimes \nu_y = \nu_y \rtimes \mu_x$ *for all invariant measures* ν_y*,*
- μ *is fim,*
- *For all* $M \subset N$, $\mu|_N$ *has a unique global non-forking extension.*

Example

Proposition

Let $p \in S(A)$ be a type, non forking over A.
Then, there exists a global A-invariant Keisler measure μ extending p.

Definition

A type $p \in S(A)$ is fsg if it has a global extension $p' \in S(\bar{M})$ s.t. for any $|A|^+$-saturated model N containing A, and every formula $\phi(x;b)$ such that $p' \models \phi(x;b)$, there is $a \in p(N)$ s.t. $\models \phi(a;b)$.

Proposition

For $p \in S(A)$, non-forking over A, the following are equivalent :

- *p is fsg*
- *The invariant measure μ is generically stable.*

G-invariant measures

Let G be a definable group.

Definition

The group G is *definably amenable* if G admits a global G-invariant Keisler measure.

Examples :

- G abelian
- G stable and connected

G-invariant measures

Proposition

Assume $\mu \in \mathcal{M}(M)$ is G-invariant. Then, μ extends to a global generically stable $G(M)$-invariant measure μ'.

In particular, $Stab(\mu') = \{g \in G : g.\mu' = \mu'\}$ is a type definable subgroup of G.

Proposition

Assume μ is a generically stable G-invariant measure, then μ is the unique G-invariant measure on G.

Application

Let G be an abelian group, assume G has no non trivial type-definable subgroup. Then G has an invariant generically stable type.

G-invariant measures

Definition

A group G it *f.s.g.* if there is a global type p and a small model M_0 such that every translate of p is finitely satisfiable in M_0.

Proposition

An f.s.g. group admits a G-invariant generically stable Keisler measure.
In particular, it is the unique G-invariant measure on G.

The o-minimal case

Let T be an o-minimal theory.

Fact

In dimension 1, any atomless measure is smooth.

Proposition

Any generically stable measure is smooth.

The o-minimal case

Theorem

Let G be a definable, definably compact group, then G is f.s.g. In particular, it has a unique G-invariant Keisler measure, which is moreover smooth.

Proposition

Let T be an o-minimal expansion of a real closed field, $\mathbf{R}$ a model of T, expansion of the standard model.
Take any Borel measure on $\mathbf{R}^n$. Then the Keisler measure defined by it is smooth.