Structuring Logic with Sequent Calculus

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Outline of the talk

- Proofs via Natural Deduction
- LK Sequent Calculus
- Examples of Proofs in LK Sequent Calculus
- Cut Elimination Theorem and the Subformula Property
- Symmetry and Non-Constructivism of *LK*
- Introducing Intuitionistic Logic
- Comparison between Intuitionistic and Classical Provability
- Going further: a Taste of Linear Logic

Natural Deduction 1: Rules

$$\frac{}{\Gamma, A \vdash A} \; Axiom$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow I \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A} \forall I \quad (*)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E2$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A \Rightarrow B}{\Gamma \vdash B} \Rightarrow E \qquad \frac{\Gamma \vdash \forall x A}{\Gamma \vdash A[t/x]} \forall E$$

(*) For this rule, $x \notin FV(\Gamma, \Delta)$.

One can consider *transformations of proofs* via the notion of cut: an introduction rule immediately by an elimination rule *on the same connective*.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash A} \quad \dfrac{\dfrac{\Pi_2}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B} \Rightarrow I}{\Gamma \vdash B} \Rightarrow E \quad \longrightarrow \quad \dfrac{\tilde{\Pi}}{\Gamma \vdash B}$$

This system has very good properties:
confluence, strong normalization, ...

Defects of Natural Deduction

- The notion of cut is implicit, there is no explicit rule for the cut;
- ND is satisfying only for a fragment of intuitionistic logic ($\Rightarrow$, $\wedge$, $\forall$).
  Paradoxically, the connectives which are the most interesting for intuitionistic logic are $\vee$ and $\exists$...

$\Rightarrow$ Natural Deduction lacks structure.

Sequent Calculus: Explicit Cut, Rules dedicated to the management of the formulas, Deep left-right symmetry of the system (introduction/elimination rules on the right are replaced by left/right introduction rules).

LK Rules (1)

***Identity Rules** (Axiom and Cut))*

$$\frac{}{A \vdash A}\ \textit{Axiom} \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textit{Cut}$$

***Structural Rules** (Exchange, Weakening and Contraction)*

$$\frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ \textit{LEx} \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ \textit{REx}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{LW} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \textit{RW}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{LC} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \textit{RC}$$

LK Rules (2)

Logical Rules **($\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\forall$, $\exists$)**

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L\neg \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R\neg$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\wedge 1 \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\wedge 2 \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ R\wedge$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ L\vee \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee 1 \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee 2$$

LK Rules (3)

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \Rightarrow B \vdash \Delta_1, \Delta_2}\ L\Rightarrow \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ R\Rightarrow$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ L\forall \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x A, \Delta}\ R\forall \quad (*)$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ L\exists \quad (*) \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\ R\exists$$

(*) For these rules, $x \notin FV(\Gamma, \Delta)$.

Formal Theorems

- $A \vee B \vdash B \vee A$ — Commutativity of disjunction
- $\vdash A \vee \neg A$ — *Tertium non datur*
- $\vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ — Peirce's Law
- $\vdash (\neg\neg A) \Rightarrow A$ — Elimination of Double Negation
- $\vdash \exists x \forall y (P(x) \Rightarrow P(y))$ — The Drinker Property
- $A \vee B \vdash \neg(\neg A \wedge \neg B)$ — An Instance of de Morgan's Laws
- $\vdash (A \Rightarrow B) \vee (B \Rightarrow A)$
- $\vdash \neg\neg(A \vee \neg A)$ — "intuitionistic" *Tertium non datur*
- $(p \Rightarrow q) \vdash (\neg q \Rightarrow \neg p)$
- $(\neg q \Rightarrow \neg p) \vdash (p \Rightarrow q)$

Alternative Rules for $\wedge$ and $\vee$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\wedge' \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\ R\wedge'$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}\ L\vee' \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee'$$

These new rules are called *multiplicative rules* while the original rules of LK are called *additive rules*.

Both sets of rules are equivalent thanks to the structural rules.

Gentzen's Hauptsatz

The Cut rule is admissible in LK.

In fact, Gentzen's result was more than simply a proof of *admissibility of the cut* since he gave *an explicit procedure to eliminate the cuts from a proof*: starting with a proof with cuts, we can step by step transform it into a cut-free proof, and this procedure is *algorithmic*.

Subformula Property

Subformula Property

A provable Sequent can be proved using only subformulas of the formulas appearing in the sequent.
(A cut-free proof only makes use of subformulas of the root sequent)

It reduces the search space a lot!

Symmetry of LK (1)

Sequents are now of the form: $\vdash' \Gamma$.

Implication is a defined connective: $A \Rightarrow B \equiv \neg A \vee B$

Negation only appears on atomic formulas, thanks to de Morgan's laws:

$$\begin{array}{ll} \neg(A \vee B) \equiv (\neg A \wedge \neg B) & \neg\forall x A \equiv \exists x \neg A \\ \neg(A \wedge B) \equiv (\neg A \vee \neg B) & \neg\exists x A \equiv \forall x \neg A \end{array}$$

More precisely, when writing $\neg A$, we will always mean the *negation normal form* of this formula for the obviously terminating and confluent rewriting system:

$$\begin{array}{l|l} \neg(A \vee B) \to (\neg A \wedge \neg B) & \neg\forall x A \to \exists x \neg A \\ \neg(A \wedge B) \to (\neg A \vee \neg B) & \neg\exists x A \to \forall x \neg A \\ \neg\neg A \to A & \end{array}$$

Identity Rules

$$\dfrac{}{\vdash' A, \neg A}\ \textit{Axiom} \qquad \dfrac{\vdash' A, \Gamma \quad \vdash' \neg A, \Delta}{\vdash' \Gamma, \Delta}\ \textit{Cut}$$

Structural Rules

$$\dfrac{\vdash' \Gamma, B, A, \Delta}{\vdash' \Gamma, A, B, \Delta}\ \textit{Ex} \qquad \dfrac{\vdash' \Gamma}{\vdash' A, \Gamma}\ W \qquad \dfrac{\vdash' A, A, \Gamma}{\vdash' A, \Gamma}\ C$$

Logical Rules

$$\dfrac{\vdash' A, \Gamma \quad \vdash' B, \Gamma}{\vdash' A \wedge B, \Gamma}\ \wedge \qquad \dfrac{\vdash' A, \Gamma}{\vdash' A \vee B, \Gamma}\ \vee 1 \qquad \dfrac{\vdash' B, \Gamma}{\vdash' A \vee B, \Gamma}\ \vee 2$$

$$\dfrac{\vdash' A, \Gamma}{\vdash' \forall x A, \Gamma}\ \forall \quad (*) \qquad \dfrac{\vdash' A[t/x], \Gamma}{\vdash' \exists x A, \Gamma}\ \exists$$

Non-Constructivism of *LK*

Proposition

There exist two irrational numbers a, b such that a^b is rational.

Proof

Consider the irrational number $\sqrt{2}$. Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not.
In the first case we are done taking $a = b = \sqrt{2}$ while in the latter we set a to be $\sqrt{2}^{\sqrt{2}}$ and b to be $\sqrt{2}$ and obtain
$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2.$

□

The peculiarity with this proof is that, having completed the proof, we have no evidence about the irrationality of $\sqrt{2}^{\sqrt{2}}$ (it is actually irrational, but the proof of this fact is much more complicated).

LK ***is symmetric but non-constructive.***

Intuitionism

All began with Brouwer who rejected the excluded-middle principle.

Why?
A view of mathematics *centered on the mathematician* so that the formula A is understood as "I know that A" or more precisely as "I have a proof of A". With this in mind, the logical connectives and the logical rules must be reconsidered.

In particular, the disjunction $A \vee B$ means "I have a proof of A or I have a proof of B" ... and the excluded middle is no more a suitable logical principle since $A \vee \neg A$ means that we always have a proof of a formula or of its negation... which is a very strong requirement.

Constructivism: a proof must provide a way to build an object that represents the property we proved.

What Disjunction?

We saw two possible rules for disjunction on the right:

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

and

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

Which one shall we choose for intuitionistic logic?

LJ Sequent Calculus

Identity Rules

$$\frac{}{A \vdash A}\ \textit{axiom} \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2, A \vdash \Xi}{\Gamma_1, \Gamma_2 \vdash \Xi}\ \textit{cut}$$

Structural Rules

$$\frac{\Gamma_1, B, A, \Gamma_2 \vdash \Xi}{\Gamma_1, A, B, \Gamma_2 \vdash \Xi}\ \textit{LEx} \qquad \frac{\Gamma \vdash \Xi}{\Gamma, A \vdash \Xi}\ \textit{LW} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ \textit{RW} \qquad \frac{\Gamma, A, A \vdash \Xi}{\Gamma, A \vdash \Xi}\ \textit{LC}$$

Logical Rules

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash}\ L\neg \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ R\neg$$

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2, B \vdash \Xi}{\Gamma_1, \Gamma_2, A \Rightarrow B \vdash \Xi}\ L \Rightarrow \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ R \Rightarrow$$

LJ Rules (2)

$$\frac{\Gamma, A \vdash \Xi}{\Gamma, A \wedge B \vdash \Xi}\ L\wedge 1 \qquad \frac{\Gamma, B \vdash \Xi}{\Gamma, A \wedge B \vdash \Xi}\ L\wedge 2 \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ R\wedge$$

$$\frac{\Gamma, A \vdash \Xi \quad \Gamma, B \vdash \Xi}{\Gamma, A \vee B \vdash \Xi}\ L\vee \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ R\vee 1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ R\vee 2$$

$$\frac{\Gamma, A[t/x] \vdash \Xi}{\Gamma, \forall x A \vdash \Xi}\ L\forall \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ R\forall \quad (*)$$

$$\frac{\Gamma, A \vdash \Xi}{\Gamma, \exists x A \vdash \Xi}\ L\exists \quad (**) \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ R\exists$$

(*) For this rule, $x \notin FV(\Gamma)$.
(**) For this rule, $x \notin FV(\Gamma, \Xi)$.

Thanks to cut-elimination we have:

Disjunction Property

If $\vdash_{LJ} A \vee B$*, then* $\vdash_{LJ} A$ *or* $\vdash_{LJ} B$

Existence Property

If $\vdash_{LJ} \exists x A$*, then there exists a term* t *such that* $\vdash_{LJ} A[t/x]$

LJ is constructive but non-symmetric.

Correspondence between classical and intuitionistic provability (1)

LJ is clearly weaker than LK: $\Gamma \vdash_{LJ} A$ implies $\Gamma \vdash_{LK} A$

Can we make more precise the relation between the two notions of provability?

We will see that LJ can be considered not to be weaker than LK but finer!

Correspondence between classical and intuitionistic provability (2)

Remember that in LJ, contraction is not available on the right of $\vdash$ but it is freely available on the left.

$A \vee \neg A$ is not provable in LJ but $\neg\neg(A \vee \neg A)$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ Axiom}{\vdash A, \neg A}\ R\neg}{\vdash A, A \vee \neg A}\ R\vee}{\vdash A \vee \neg A, A \vee \neg A}\ R\vee}{\vdash A \vee \neg A}\ RC$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ Axiom}{A \vdash A \vee \neg A}\ R\vee}{\neg(A \vee \neg A), A \vdash}\ L\neg}{\neg(A \vee \neg A) \vdash \neg A}\ R\neg}{\neg(A \vee \neg A) \vdash A \vee \neg A}\ R\vee}{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}\ L\neg}{\neg(A \vee \neg A) \vdash}\ LC}{\vdash \neg\neg(A \vee \neg A)}\ R\neg$$

Correspondence between classical and intuitionistic provability (3)

Gödel Translation

The idea of the intuitionistic proof of $\neg\neg(A \vee \neg A)$ is to send the formula to the left so that it is possible to use left contraction. The occurrence of the double negation $\neg\neg$ precisely allows to cross twice the $\vdash$ and to use left contraction.

Definition: Gödel Translation

- $A^\star = \neg\neg A$ *for A atomic;*
- $(A \wedge B)^\star = A^\star \wedge B^\star$;
- $(\forall x A)^\star = \forall x A^\star$;
- $(\neg A)^\star = \neg A^\star$;
- $(A \Rightarrow B)^\star = A^\star \Rightarrow B^\star$;
- $(A \vee B)^\star = \neg\neg(A^\star \vee B^\star)$;
- $(\exists x A)^\star = \neg\neg\exists x A^\star$.

Correspondence between classical and intuitionistic provability (4)

Theorem

$\Gamma \vdash_{LK} A$ *iff* $\Gamma^\star \vdash_{LJ} A^\star$

Lemma

$\vdash_{LK} A \Leftrightarrow A^\star$

Definition

A is said to be stable when $\vdash_{LJ} \neg\neg A \Rightarrow A$.

Lemma

For all formula A, $A^\star$ is stable.

Lemma

If $\Gamma \vdash_{LK} \Delta$ then $\Gamma^\star, \neg\Delta^\star \vdash_{LJ}$.

Correspondence between classical and intuitionistic provability (5)

In what sense can we say that LJ is finer (subtler?) than LK?

An intuitionistic logician cannot necessarily prove a formula A when a classical mathematician can, *BUT* he can find another formula ($A^\star$) that he is able to prove and that the classical mathematician cannot distinguish from the previous one.

In particular, in intuitionistic logic, the use of excluded middle (or contraction on the right) shall be *explicitly mentioned in the formula* thanks to the use of double negation

Is it possible to be even more drastic with structural rules?

Linear Logic

Does Classical Logic allow to model everything?

Let us remove all the structural rules!

The two alternative presentations of disjunction and of conjunction are no more equivalent.

We have two different conjunctions and two different disjunctions!

We need to recover the contraction and weakening rules, but in a controlled way.

LL Sequent Calculus

Identity Rules:

$$\frac{}{\vdash A^{\perp}, A}\ ax \qquad \frac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ cut$$

Structural Rule:

$$\frac{\vdash \Gamma, B, A, \Delta}{\vdash \Gamma, A, B, \Delta}\ Ex$$

Logical Rules:

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ \parr \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ \otimes$$

$$\frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \& G, \Gamma}\ \& \qquad \frac{\vdash F, \Gamma}{\vdash F \oplus G, \Gamma}\ \oplus 1 \qquad \frac{\vdash G, \Gamma}{\vdash F \oplus G, \Gamma}\ \oplus 2$$

$$\frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ \bot \qquad \frac{}{\vdash \top, \Gamma}\ \top$$

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ ? \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ !$$

$$\frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ ?W \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ ?C$$

Important Characteristics of Linear Logic

- Thanks to the additional structure put in the sequents themselves, we can capture more things at the logical level and not at the term level like in classical logic;
- The control on structural rules allows a careful study of cut-elimination, which via Curry-Howard corresponds to execution of a functional program;
- Thanks to the richness of the sequents it is possible to consider the sequents as storing a state of the computation in a process of proof-search (logic programming paradigm).
- Lots of other directions...

Conclusion

The rules that at first seemed to be the less significant in logic (at such an extent that they are missing in Natural Deduction) are eventually crucial in the proof theoretic analysis of logic. Indeed, it is by controlling these rules that we can choose the focus we want to put on logic and the level of detail we desire: Controlling the structural rules, we can *zoom* and catch more details of the proofs.

From the computer science point of view, the very structured object that a LL proof is allows for various uses and applications.

Less formally and more informally, an interest of this study of structure is that we came from a logical study driven by the notion of *truth* and that now we can do logic as study of geometrical properties of proofs, the logical character being assured by some formal requirement such as cut-elimination, symmetrical properties...