Gentzen's Hauptsatz

Fifth Sixth and Seventh Lectures

26th, 31st August – 2nd September 2004

Consequences of the Hauptsatz (continued) & Remarks

- Subformula Property and Constructivity
- Craig's Interpolation Lemma
- Strong Normalization
- Non Determinism
- Size-expansion during cut-elimination

Subformula property

In LJ, we have constructive proofs for an empty theory:

$$\vdash F$$

So that we have such a result for purely logical proofs, not for mathematical proofs.

How to extend this?

Harrop tehories

Definition: Harrop theories

- *A(n occurrence of a) subformula B of A is said to be strictly positive iff it does not appear on the left of an implication or under a negation;*
- *a Harrop formula is a formula A such that the $\vee$ and $\exists$ connectives never appear as principal connectives of a strictly positive subformula of A;*
- *a Harrop theory is a set of Harrop formulas.*

Theorem/ Exersise: Harrop theories are constructive.

If Γ is a Harrop theory, the following properties hold:

- *if $\Gamma \vdash_{LJ} A \vee B$ then $\Gamma \vdash_{LJ} A$ or $\Gamma \vdash_{LJ} A$;*
- *if $\Gamma \vdash_{LJ} \exists x A$ then there is a term t such that $\Gamma \vdash_{LJ} A[t/x]$.*

We say that Harrop theories are constructive.

Craig's Interpolation Lemma

Craig's Interpolation Lemma

Let A and B be formulas such that $A \vdash_{LK} B$. There exists a formula C such that $A \vdash_{LK} C$ and $C \vdash_{LK} B$ and such that all function or predicate symbols (as well as variables) occurring in C occurs both in A and B.

Strong Normalization

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,RC \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut$$

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,cut \qquad \Gamma', A \vdash \Delta'}{\dfrac{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\vdots\; RC\&LC}\,cut$$

Strong Normalization (2)

$$\cfrac{\cfrac{\cfrac{\overline{A \vdash A}\;ax \qquad \overline{A \vdash A}\;ax}{A \vee A \vdash A, A}\;L\vee}{A \vee A \vdash A}\;RC \qquad \cfrac{\cfrac{\overline{A \vdash A}\;ax \qquad \overline{A \vdash A}\;ax}{A, A \vdash A \wedge A}\;R\wedge}{A \vdash A \wedge A}\;LC}{A \vee A \vdash A \wedge A}\;cut$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{A \vee A \vdash A, A}}{A \vee A \vdash A}\;RC \qquad \cfrac{\overline{A \vdash A} \qquad \overline{A \vdash A}}{A, A \vdash A \wedge A}}{A \vee A, A \vdash A \wedge A}\;cut \qquad \cfrac{\cfrac{\vdots}{A \vee A \vdash A, A}}{A \vee A \vdash A}\;RC}{\cfrac{A \vee A, A \vee A \vdash A \wedge A}{A \vee A \vdash A \wedge A}\;LC}\;cut$$

Size of the cut-free proof

Proposition/Exercise: Size of the Cut Free Proofs

- $S(0, h) = h$
- $S(d + 1, h) = 4^{S(d,h)}$