Gödel's Completeness Theorem for *LK*

Fourth Lecture

24th August 2004

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Books:

- *Proofs and Types* – Girard, Lafont, Taylor @
- *Proof Theory and Logical Complexity* – Girard @
- *Logic for Computer Science* – Jean Gallier @
- *Proof Theory* – Gaisi Takeuti

Tutorial Papers and Lecture Notes:

- *Introduction to Proof Theory* – Samuel Buss @
- *Sequent Calculus and the Specification of Computation* – Dale Miller @
- *Linear Logic, its Syntax and Semantics* – Jean-Yves Girard @

Research Papers:

- *Investigations into Logical Deduction* – Gerhard Gentzen
- *Linear Logic* – Jean-Yves Girard, TCS87 @
- *On the Meaning of Logical Rules, I* – Jean-Yves Girard @

Completeness and Soundness

Two important questions:

- Are all theorems true? *Soundness*
- Are all the true formulas provable? *Completeness*

What do we mean by "*true*"?

- Soundness is easily proved by induction on the deduction rules.
- Completeness is more complicated.

Intuition of the proof

Actually, we will prove the contrapositive: if $\vdash F$ is not provable, then we can find a model of the language in which F is not satisfied, that is F is not valid.

The proof scheme will actually be the following: we will design a proof search procedure that will search for a proof of $\vdash F$. Since there is no such proof, we cannot end up with an object which is a proof: the resulting object will actually be a failure from which we will build a counter-model for F by correctly choosing the truth values for the predicates in order to falsify all formulas in the base sequent.

Derivation Trees

Improper Rules

$$\frac{\boxed{\text{Open}}}{L;\Gamma;\Phi}\ \textit{Open} \qquad \frac{L;\Gamma;\Phi}{\boxed{\text{Root}}}\ \textit{Root} \qquad \frac{\boxed{\text{Failure}}}{L;\emptyset;\emptyset}\ \textit{Failure}$$

Classification Rules

$$\frac{l, L;\Gamma;\Phi}{L; l,\Gamma;\Phi}\ \textit{Literal} \qquad \frac{L;\Gamma;\Phi,\exists xA}{L;\exists xA,\Gamma;\Phi}\ \textit{Existential}$$

Logical Rules

$$\frac{}{A,\neg A, L;\Gamma;\Phi}\ \textit{axiom} \qquad \frac{}{L;\top,\Gamma;\Phi}\ \top \qquad \frac{L;A,\Gamma;\Phi \quad L;B,\Gamma;\Phi}{L;A\wedge B,\Gamma;\Phi}\ \wedge$$

$$\frac{L;A,B,\Gamma,\Phi}{L;A\vee B,\Gamma,\Phi}\ \vee \qquad \frac{L;A[y/x],\Gamma;\Phi}{L;\forall xA,\Gamma;\Phi}\ \forall \qquad \frac{L;A[t/x];\Phi,\exists xA}{L;\emptyset;\exists xA,\Phi}\ \exists$$

$$
\begin{array}{c l}
\hline
P(y_1), \neg P(y_0), P(y_0), \neg P(t_0) \;;\; \emptyset \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \textit{axiom} \\
\hline
 & \textit{literal} \\
\neg P(y_0), P(y_0), \neg P(t_0) \;;\; P(y_1) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \textit{literal} \\
P(y_0), \neg P(t_0) \;;\; \neg P(y_0), P(y_1) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \lor \\
P(y_0), \neg P(x_0) \;;\; (\neg P(y_0) \lor P(y_1)) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \forall \\
P(y_0), \neg P(x_0) \;;\; \forall y(\neg P(y_0) \lor P(y)) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \exists \\
P(y_0), \neg P(x_0) \;;\; \emptyset \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \textit{literal} \\
\neg P(x_0) \;;\; P(y_0) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \textit{literal} \\
\emptyset \;;\; \neg P(x_0), P(y_0) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \lor \\
\emptyset \;;\; (\neg P(x_0) \lor P(y_0)) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \forall \\
\emptyset \;;\; \forall y(\neg P(x_0) \lor P(y)) \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \exists \\
\emptyset \;;\; \emptyset \;;\; \exists x \forall y(\neg P(x) \lor P(y)) & \\
\hline
 & \textit{existential} \\
\emptyset \;;\; \exists x \forall y(\neg P(x) \lor P(y)) \;;\; \emptyset &
\end{array}
$$

Systematic Derivation Trees

Definition: Systematic Derivation Tree

Let us consider ϵ an enumeration of all terms in the language.
A ***systematic derivation tree*** *for a sequent $L; \Gamma; \Phi$ is a derivation tree such that:*

- *the axiom rule is applied as soon as it is possible. That means that in a branch of a systematic derivation tree, there cannot be two opposite literals in the left component of the sequent except in the upmost sequent which must be followed by an axiom rule;*
- *when the $\exists$ rule is used, the bound variable must be instantiated with the first term (according to ϵ that has not yet been used in the instantiation of this formula lower in the derivation tree.*

Example II

$$
\begin{array}{c l}
\hline
\neg P(t_k), P(y_{k-1}), \ldots, \neg P(t_1), P(y_0), \neg P(t_0) \;\;;\;\; P(y_k) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \textit{axiom} \\
\hline
P(y_{k-1}), \ldots, \neg P(t_1), P(y_0), \neg P(t_0) \;\;;\;\; \neg P(t_k), P(y_k) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \textit{literal} \\
\hline
\vdots & \\
\hline
P(y_1), \neg P(t_1), P(y_0), \neg P(t_0) \;\;;\;\; \emptyset \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \exists \\
\hline
\neg P(t_1), P(y_0), \neg P(t_0) \;\;;\;\; P(y_1) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \textit{literal} \\
\hline
P(y_0), \neg P(t_0) \;\;;\;\; \neg P(t_1), P(y_1) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \textit{literal} \\
\hline
P(y_0), \neg P(t_0) \;\;;\;\; (\neg P(t_1) \vee P(y_1)) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \vee \\
\hline
P(y_0), \neg P(t_0) \;\;;\;\; \forall y(\neg P(t_1) \vee P(y)) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \forall \\
\hline
P(y_0), \neg P(t_0) \;\;;\;\; \emptyset \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \exists \\
\hline
\neg P(t_0) \;\;;\;\; P(y_0) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \textit{literal} \\
\hline
\emptyset \;\;;\;\; \neg P(t_0), P(y_0) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \textit{literal} \\
\hline
\emptyset \;\;;\;\; (\neg P(t_0) \vee P(y_0)) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \vee \\
\hline
\emptyset \;\;;\;\; \forall y(\neg P(t_0) \vee P(y)) \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \forall \\
\hline
\emptyset \;\;;\;\; \emptyset \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) & \exists \\
\hline
\emptyset \;\;;\;\; \exists x \forall y(\neg P(x) \vee P(y)) \;\;;\;\; \emptyset & \textit{existential}
\end{array}
$$

Corollaries: Löwenheim-Skolem and Compactness Theorems

Löwenheim-Skolem Theorem

If a formula F has a model, it has a model which is finite or denumerable.

Compactness Theorem

If a set $\mathcal{S}$ of formulas is such that all its finite subsets are satisfiable then $\mathcal{S}$ itself is satisfiable.

Up to now, we had only finite sequents and thus we could not assert anything about infinite theories.
We will generalize the setting of the previous proof in order to prove Completeness also for infinite theories so that as a corollary, we will have the Compactness Theorem.

Definition: Systematic ω-Derivation Tree

We add the following rule:

$$\frac{L; \Gamma, \neg F; \Phi}{L; \Gamma; \Phi}\ \omega \qquad for \quad F \in \mathcal{S}$$