Planning for the rest of the Course:

- 17/08/04 (today) Gentzen's LK Sequent Calculus
- 19/08/04 Intuitionism
- 24/08/04 Completeness for LK (+ Compactness and Löwenheim-Skolem) by Proof-Theoretical Means
- 26/08/04 Hauptsatz (Cut Elimination) and its Applications
- 31/08/04 Linear Logic: Introduction
- 02/09/04 Linear Logic: Focalization Theorem
- 07/09/04 Linear Logic: Proofs Nets I
- 09/09/04 Linear Logic: Proofs Nets II
- 14/09/04 Computation as Proof Search
- 16/09/04 Computation as Proof Normalization
- 21/09/04 Interactivity: Playing with Proofs.
- 23/09/04 ...

Gentzen's *LK* Sequent Calculus

Second Lecture

17th August 2004

Previous Lecture

- Hilbert Proof Systems
- Natural Deduction
- Sequent Calculus

Today

Begin the detailed study of Sequent Calculus:

- Examples of proofs in Sequent Calculus
- Relationships between the three groups of rules
- Symmetry and Non-Constructivism of *LK*
- The question of Completeness and of Cut-Elimination

LK Rules (1)

Identity Rules

$$\frac{}{A \vdash A}\ \textit{Axiom} \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textit{Cut}$$

Structural Rules

$$\frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ \textit{LEx} \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ \textit{REx}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{LW} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \textit{RW}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{LC} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \textit{RC}$$

LK Rules (2)

Logical Rules

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \; L\neg \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \; R\neg$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; L\wedge 1 \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; L\wedge 2 \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \; R\wedge$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \; L\vee \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \; R\vee 1 \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \; R\vee 2$$

LK Rules (3)

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \Rightarrow B \vdash \Delta_1, \Delta_2}\ L\Rightarrow \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ R\Rightarrow$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ L\forall \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x A, \Delta}\ R\forall \quad (*)$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ L\exists \quad (*) \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\ R\exists$$

(*) For these rules, $x \notin FV(\Gamma, \Delta)$.

Formal Theorems

- $A \lor B \vdash B \lor A$ — Commutativity of disjunction
- $\vdash A \lor \neg A$ — *Tertium non datur*
- $\vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ — Peirce's Law
- $\vdash (\neg\neg A) \Rightarrow A$ — Elimination of Double Negation
- $\vdash \exists x \forall y (P(x) \Rightarrow P(y))$ — The Drinker Property
- $A \lor B \vdash \neg(\neg A \land \neg B)$ — An Instance of de Morgan's Laws
- $\vdash (A \Rightarrow B) \lor (B \Rightarrow A)$
- $\vdash \neg\neg(A \lor \neg A)$ — "intuitionistic" *Tertium non datur*
- $(p \Rightarrow q) \vdash (\neg q \Rightarrow \neg p)$
- $(\neg q \Rightarrow \neg p) \vdash (p \Rightarrow q)$

Relationhips between the rules

- Identity Rules/Structural Rules
- Logical Rules/Structural Rules
- Logical Rules/Identity Rules

Alternative Rules for $\wedge$ and $\vee$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \; L\wedge' \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2} \; R\wedge'$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2} \; L\vee' \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \; R\vee'$$

These new rules are called *multiplicative rules* while the original rules of LK are called *additive rules*.

One-sided Sequent Calculus (1)

Sequents are now of the form: $\vdash' \Gamma$.

Implication is a defined connective: $A \Rightarrow B \equiv \neg A \vee B$

Negation only appears on atomic formulas, thanks to de Morgan's laws:

$$\begin{array}{ll} \neg(A \vee B) \equiv (\neg A \wedge \neg B) & \neg\forall x A \equiv \exists x \neg A \\ \neg(A \wedge B) \equiv (\neg A \vee \neg B) & \neg\exists x A \equiv \forall x \neg A \end{array}$$

More precisely, when writing $\neg A$, we will always mean the *negation normal form* of this formula for the obviously terminating and confluent rewriting system:

$$\begin{array}{l|l} \neg(A \vee B) \rightarrow (\neg A \wedge \neg B) & \neg\forall x A \rightarrow \exists x \neg A \\ \neg(A \wedge B) \rightarrow (\neg A \vee \neg B) & \neg\exists x A \rightarrow \forall x \neg A \\ \neg\neg A \rightarrow A & \end{array}$$

One-sided Sequent Calculus (2)

Identity Rules

$$\frac{}{\vdash' A, \neg A}\ \textit{Axiom} \qquad \frac{\vdash' A, \Gamma \quad \vdash' \neg A, \Delta}{\vdash' \Gamma, \Delta}\ \textit{Cut}$$

Structural Rules

$$\frac{\vdash' \Gamma, B, A, \Delta}{\vdash' \Gamma, A, B, \Delta}\ \textit{Ex} \qquad \frac{\vdash' \Gamma}{\vdash' A, \Gamma}\ W \qquad \frac{\vdash' A, A, \Gamma}{\vdash' A, \Gamma}\ C$$

Logical Rules

$$\frac{\vdash' A, \Gamma \quad \vdash' B, \Gamma}{\vdash' A \wedge B, \Gamma}\ \wedge \qquad \frac{\vdash' A, \Gamma}{\vdash' A \vee B, \Gamma}\ \vee 1 \qquad \frac{\vdash' B, \Gamma}{\vdash' A \vee B, \Gamma}\ \vee 2$$

$$\frac{\vdash' A, \Gamma}{\vdash' \forall x A, \Gamma}\ \forall \quad (*) \qquad \frac{\vdash' A[t/x], \Gamma}{\vdash' \exists x A, \Gamma}\ \exists$$

Non-Constructivism of *LK*

Proposition

There exist two irrational numbers a, b such that a^b is rational.

Proof

Consider the irrational number $\sqrt{2}$. Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not.
In the first case we are done taking $a = b = \sqrt{2}$ while in the latter we take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ and obtain
$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2.$

□

The peculiarity with this proof is that, having completed the proof, we have no evidence about the irrationality of $\sqrt{2}^{\sqrt{2}}$ (it is actually irrational, but the proof of this fact is much more complicated).

Gödel's Completeness Theorem for Predicate Logic (LK)

We give a brief idea of what is the completeness property, anticipating on next Tuesday lecture:

Gödel's Completeness Theorem

A formula F is valid if and only if $\vdash F$ provable in LK.

Actually, we will prove the contrapositive: if $\vdash F$ is not provable, then we can find a model of the language in which F is not satisfied.

The proof scheme will actually be as follows: we will design a proof search procedure that will search for a proof of $\vdash F$. Since there is no such proof, we cannot end up with an object which is a proof: the resulting object will actually be a failure from which we will build a counter-model for F.

Gentzen's Hauptsatz

In its sequent calculus LK, Gentzen introduced the Cut rule which corresponds to the use of a lemma in a proof. The next step in his paper, and the first important result about Sequent Calculus (Completeness is not a feature of LK but of first-order classical logic...), was to prove that this rule is non necessary: the cut rule is admissible in the sense the we prove exactly the same sequents in LK and in LK without cut.

In fact, Gentzen's result was more than simply a proof of admissibility of the cut since he gave a explicit procedure to eliminate the cuts from a proof which contains such rules: starting with a proof with cuts, we can step by step transform it into a cut-free proof, and this procedure is *algorithmic*.
The heart of the proof uses the relationships between the logical rules and the cut rule, that we mentioned but did not make explicit.