Proof and knowledge in Mathematics

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1992

published by Routeledge edited by Michael DETLEFSEN

65-109 Logicism (by Steven J. WAGNER)

92 MATHÉMATIQUES = CLÔTURE APRIORIQUE DES INTUITIONS PRIMITIVES

Imagine an ideal reasoner, beginning with *a* priori intuitions and proceeding by explanatory inference, deduction, and further intuition, without relying on science. Let us say (tendentiously!) that she relies on *strictly mathematical* reasons. (Her mathematics could be called the *a priori closure* of the original intuitions.)

102-103 ÉMERGENCE DE L'ARITHMÉTIQUE : LOGICISME ?

Arithmetical calculation […] seems rationally necessary, with two qualifications. First, what is necessary is some method of calculation or other. Addition, multiplication, and so forth are ways to find cardinalities of certain sets, given simpler cardinality determinations. They take us from statements of number to statements of number. […]

[…]

To summarize, any rational inquiry needs counting. If it needs counting, then calculation will advance it further. The required search for progressively better explanations will, however, drive a calculating inquirer to pure mathematics: first to something like arithmetic, then to set theory. Mathematics at each stage is a collection of theorems deduced from axioms, where the axioms rest on a combination of (non empirical) explanatory power and one's grasp of the concepts that inquiry forced on to develop – such as number and set. This I submit as a form of logicism

110-134 EMPIRICAL INQUIRY AND PROOF (by Shelly STILLWELL)

119 SENS D'UN ÉNONCÉ MATHÉMATIQUE : UNE IMPOSSIBILITÉ

Wittgenstein asserts that a mathematical result "is not designed to express any *experience*, but rather to express the *impossibility* of imagining anything different"

131 RÉSULTAT QUAND ON DIT QU'ON SUIT LA RÈGLE

Wittgenstein says that mathematics yields certain "prophecies." The prophecy a mathematically proven proposition yields "does *not* run, that a man will get *this* result when he follows the rule in making a transformation – but that he will get this result, when we *say* that he is following the rule"

171-207 FOUNDATIONALISM AND FOUNDATIONS OF MATHEMATICS (by Stewart SHAPIRO)

171 DÉFINIR "FONDATIONALISME"

As a first approximation, define *foundationalism*to be the view that it is possible and desirable to place a given branch of mathematics on a completely secure foundation.

182 LANGAGE PRIMITIF *UTILISÉ*

Even in mature foundational studies, […] it is a truism that one cannot continue to formulate formal meta-meta-...-theories. At some point (usually rather quickly) we reach an informal language, a language to be *used*, not studied, and used without the benefit of more theory. At this level, again, Poincaré is correct that "there is no logic and epistemology independent of psychology."

208-250 Brouwerian intuitionism (by Michael DETLEFSEN)

219-222 Épistémologie classique : expérience-inférence-connaissance

though experience may be necessary in order for knowledge to *begin*, it has strictly limited value as a means of *extending* knowledge.

It is this belief in the limited epistemic exploitability of experience that forms the basis of the logic-intensive or representative-intensive view. It takes experience itself to be a relatively unextendable commodity, either because of practical difficulties or because of the costs associated with doing so. It thus sees our epistemic condition as one in which we are allowed a relatively modest budget of experience or intuition to set the epistemic enterprise in motion and in which there is relatively little opportunity for causally prodding or massaging that modest initial budget of experience into a larger fund capable of meeting our epistemic needs. Therefore, we resort instead to *inference*, which, the classical view holds, offers us the epistemic benefits of extension of experience without the attendant costs and difficulties pertaining thereto.

[…] we seek a means of epistemically *projecting* our experience without actually *extending* it, so that our geometrical knowledge need not be bounded by the limitations of size, time, strength, etc. which limit our activities as actual line-drawers, paper-folders, planar-objet-rotaters, etc.

Extending this view beyond geometry to mathematics generally, we arrive at the classical viewpoint, which may be summarized as follows: mathematical knowledge may begin with a type of intuition or practice, but for a variety of reasons (having to do with the practical limitations concerning such things as our susceptibility to pain and the restrictedness of the time, effort, strength, material resources, etc. that we have to invest in such entreprises as the basic constructional activities of mathematics) this experience is insufficiently "plastic" to be pratically extendable to the full variety of propositions over which we should like our knowledge to range; therefore, in place of the relatively impliant *pratical* or *causal massaging* of mathematical *intuition*, we substitute a more pliant scheme of *logical* manipulations of its *contents*.

224-225 Induction via les preuves-expériences

Mathematical knowledge is […] essentially a form of constructional activity, with the consequence that extension of that knowledge must take the form of extension of that activity, rather than a mere, actionally disembodied, logical extrapolation of its contents. This, at any rate, is the anti-classical kernel of Brouwerian epistemology. […]

[…] In contradistinction to the classical model of epistemic growth, then, Brouwerian epistemology does not present the prover as reflecting on contents, generating new from old by this logical reflection, and thence transferring the warrant for the old to the new (by appeal to the warrant-preservingness of the modes of contentual analysis employed). Rather – and it is hard to overemphasize the importance of this difference to the present discussion – the mathematician transforms old proof-experiences or proof-activities into new ones and thus witnesses the extension of her knowledge to new propositions when such a proposition emerges as the content of the newly created proof-experience. For once such experience exists, knowledge may be extended to whatever its content is. What logical relation the content of this newly created experience might beat to that of the old is a matter of secondary concern. For knowledge-extension proceeds not by the logical extraction of new propositions from ones already known, but rather by the phenomenological transformation of one proof-experience into another – the next content emerging as *the content of* the new experience produced by this transformation. Mathematical inference or proof thus follows the path of the possibilities relating mathematical activities, rather than the chain of connections determined by some logico-linguistic analysis of the (propositional) contents of such activities, as classical epistemology maintains.

227 Position brouwerienne

Brouwer could […] sum up his criticism of classical logic as an instrument for determining which propositions are capable of being the contents of an intuitionistic proof-experience by saying that "there are intuitionist structures which cannot be fitted into any classical logical frame, and there are classical arguments not applying to any introspective image" […]. The first part of this claim emphasizes the inaccuracy borne of the *incompleteness* of the classical instrument, while the second emphasizes that which results from its *unsoundness*. If the principles of classical logic were to be amended in such a way as to eliminate these deficiencies of incompleteness and unsoundness, then one would have an apt logical instrument; that is, an accurate device for determining which propositions are potential contents for intuitionistic proof-experiences. However, such a device could still serve only to *identify* those propositions that are capable of intuitionistic justification – which is a very different thing from (and epistemically inferior to) actually supplying such justification.

Such, at any rate, is our understanding of the Brouwerian standpoint, which is strikingly at odds with the usual version of intuitionism presented in the literature. On the usual version, the critique of excluded middle is presented as the centerpiece of the intuitionist's concerns and the crux of his criticism of classical mathematics. Our view differ from this in two ways. First it suggests that the question “What logic is the logic of mathematics?” (and particularly the subquestion “Does the law of excluded middle belong to the logic of mathematics?”) is of secondary importance. **The more fundamental question is “What role does *any* logic (including the “right” one) have to play in the construction of intuitionistic proofs?”** Judged from this vantage, **the question “Which logic is the logic of mathematics?” can only be regarded as misleading**.

228 Épistémologie classique : un garant préservé par l'inférence

That [basic] motivation [of classical epistemology], it will be remembered, had its basis in the conviction that intuition or experience is a relatively scarce epistemic commodity – that it is not readily accessible in sufficient quantities to beings subject to the practical limitations (e.g. of strength, size, sensitivity to heat, flammability, etc.) that we are. Therefore, the classicist seeks a way of liberating knowledge from its meager intuitional or experimental origins. His answer is the logic-intensvie or representation-intensive stratagem. On this stratagem, warrant is identified with some property (e.g. certainty, or certainty plus such thing as *a priori* status) that is relatively insensitive to the fine points of the cognitive mode of a warrant and focuses more on its content. As a result, it (i.e. warrant) becomes the sort of thing that can be passed on by techniques of inferences that preserve relatively few of the details of the cognitive mode under which the premises of the inference are presented as warranted.

230 Épistémolgie de l'*action*

there is somehow something of greater value in a kind of knowledge that brings with it a capacity to *do* something than in a kind of knowledge which consists solely in an intellectual “acknowledgment” or “acceptance” of a proposition. **Genuine knowledge** – so the idea would go – **enlivens and enables**. **It moves to action**. It is more than just the doffing of one's intellectual hat to a proposition. **Practical knowledge therefore penetrates to a level of our cognitive being to which theoretical or purely intellectual knowledge typically does not**. [...]

In an epistemology thus dominated by a practical conception of knowledge, it should come as no surprise that such accoutrements of the theoretical or scientific conception of knowledge as the use of logical inference and the axiomatic method are devaluated, and concern for the convertibility (or, to use the term that we have been using, the “transformability”) of one activity or practical capacity into another is put in their place. Thus, on the epistemology being sketched here, an area of mathematical thought (the correlate) is to be thought of as a body of actions organized by a scheme of actional connections reflecting some sort of practical disposition to pass from one *act* to another, rather than a body of truths organized by a network of logical relations. Likewise, in place of a plan for epistemic growth which sees it as a march from one intellectual “acceptance” to another via the steady logical exploitation of the propositions thus accepted, towards a goal of “complete” acceptance (that is, acceptance of the compete set of truths pertaining to the subject-matter of the science in question), there is a course of practical development which is seen as consisting in the practical transformation of one act into another in such a way as to bring one's overall mathematical activity into closer conformity to a network or “stream” of actions which is taken to represent the ideal of an abundant mathematical *life*.