Asymptotics of several-partition Hurwitz numbers

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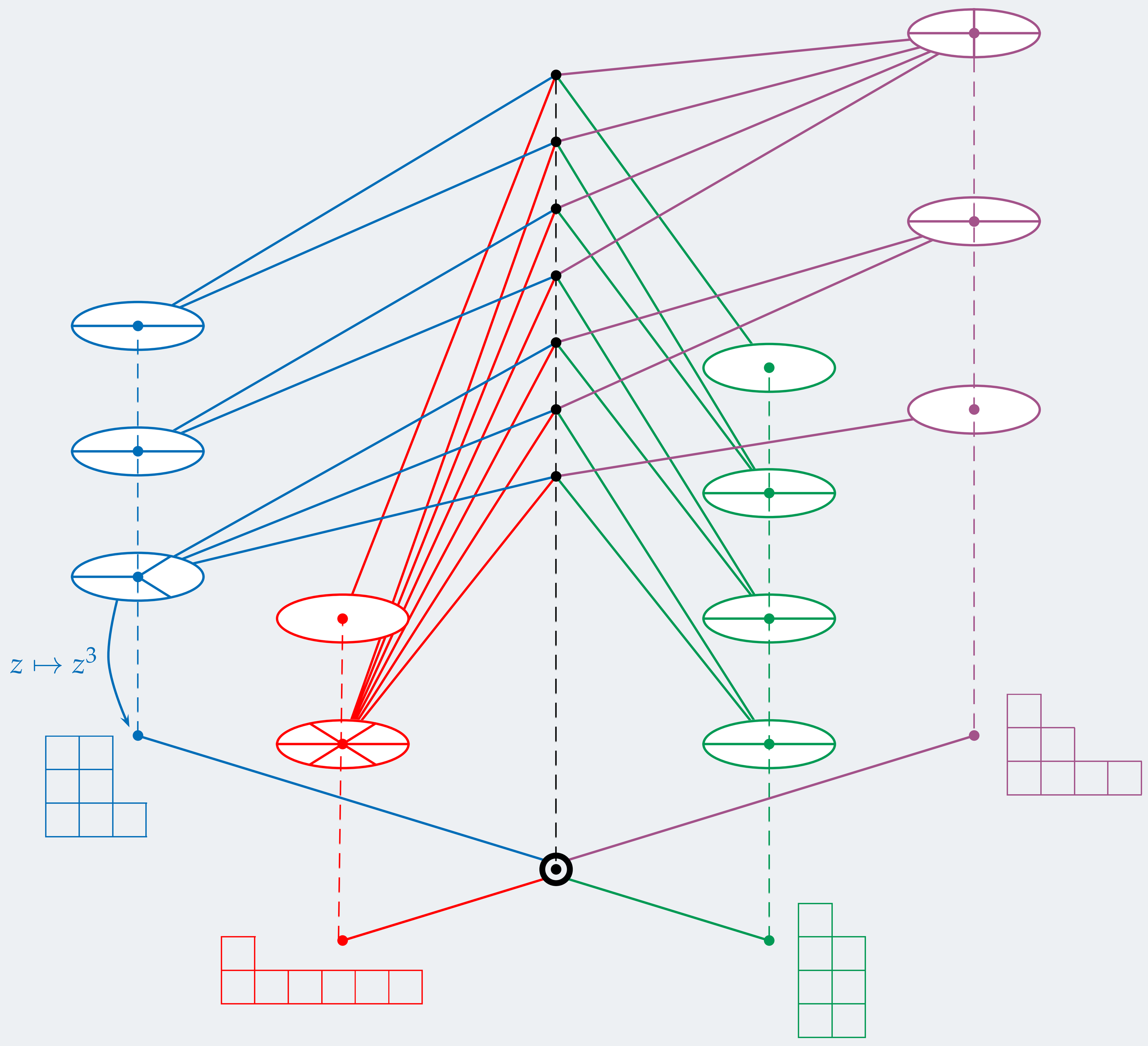
Ramified coverings of the sphere

Hurwitz numbers can be defined as numbers of connected ramified coverings (up to isomorphism) of the sphere with prescribed ramification types (partitions). It leads to a very nice illustration, through **constellations**, of Hurwitz numbers – they will be defined later by means of transitive factorisations in the symmetric group [LZ04]. We have drawn hereafter a constellation corresponding to a ramified covering with genus $g = 2$ and $n = 7$ sheets:



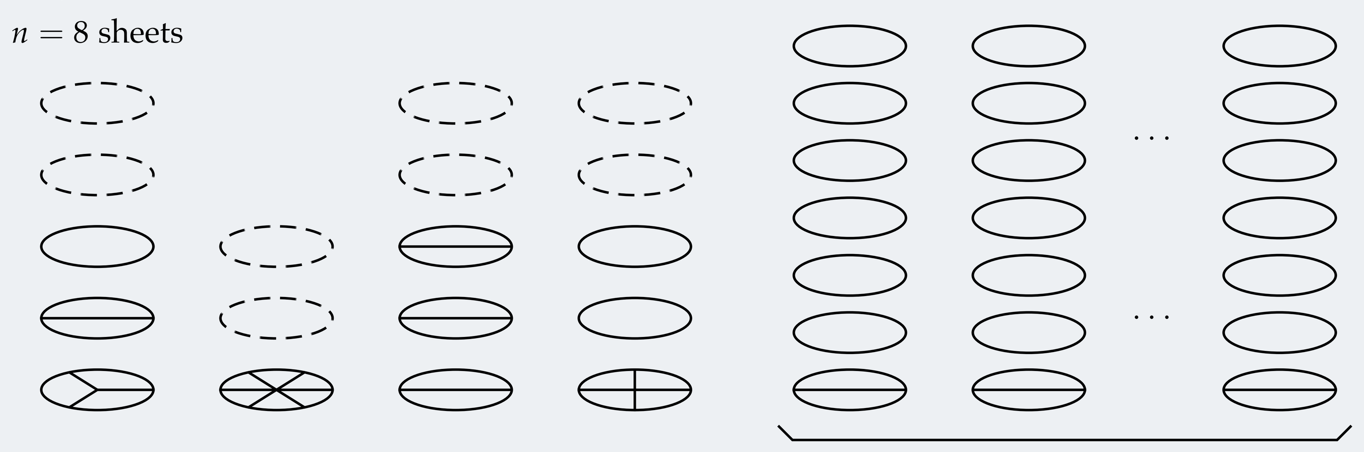
The **genus** of the ramified covering is given by the **Riemann-Hurwitz formula**:

$$2n + 2g - 2 = 2n + 2g' = \sum_{i=1}^{k} |\lambda^i| - \ell(\lambda^i) := r(\lambda^1, \ldots, \lambda^k)$$

where $|\lambda|$ is the size of a partition λ, and $\ell(\lambda)$ is its length. Conversely, given partitions $\lambda^1, \ldots, \lambda^k$ of size less than n, one can prescribe the number n of sheets by completing the partitions, and the genus g by adding

$$T = T_n := 2n + 2g' - r\left(\overrightarrow{\lambda}\right)$$

points with simple ramification (**transpositions** of two sheets). For instance, when $n = 8$ and $g = 2$, if $\lambda^1 = (3,2,1)$, $\lambda^2 = (6)$, $\lambda^3 = (2,2,2)$ and $\lambda^4 = (4,1,1)$, then the completion amounts to the following, with $T = 4$.



Hurwitz numbers and their asymptotics

When one lifts the star graph (with k branches) from the sphere to the covering, one obtains *via* monodromy k permutations of the n sheets. The sphere being simply connected, the product of these permutations is the identity permutation. The factorisation is moreover transitive because the covering is connected.

Definition (Hurwitz numbers as numbers of transitive factorisations in symmetric groups)

Let $n, g \geq 0$ be two integers and $\lambda^1, \cdots, \lambda^k$ be k partitions. Their **Hurwitz number** $h_n^g(\lambda^1, \ldots, \lambda^k)$ is defined as the product of

$$\frac{1}{n!} \prod_{i=1}^{k} \binom{n - |\lambda^i| + m_1(\lambda^i)}{m_1(\lambda^i)}$$

by the number of $(k+T)$-uples $(\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_T) \in (\mathfrak{S}_n)^{k+T}$ satisfying

$$\begin{cases} \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_T = \mathrm{Id}; \\ \text{the subgroup } \langle \sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_T \rangle \text{ is transitive;} \\ \text{all the } \tau_j\text{'s are transpositions and for any } i, \sigma_i \text{ has type } \overline{\lambda^i}; \\ T := 2n + 2g' - r\left(\lambda^1, \ldots, \lambda^k\right). \end{cases}$$

A. Hurwitz, 1859–1919

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we set $D_\lambda := \frac{1}{|\mathrm{Aut}\,\lambda|} \prod_{i=1}^{r} \frac{\lambda_i^{\lambda_i}}{\lambda_i!}$ and we define **Hurwitz series** by

$$H^g\left(\overrightarrow{\lambda}, q\right) := \sum_{n=1}^{\infty} \frac{h_n^g(\lambda^1, \ldots, \lambda^k)}{T_n!} q^n \quad \text{and} \quad \widetilde{H}^g\left(\overrightarrow{\lambda}, q\right) := \frac{1}{D_{\lambda^1} D_{\lambda^2} \cdots D_{\lambda^k}} H^g\left(\overrightarrow{\lambda}, q\right).$$

Examples. Let λ be a (non-empty) partition of size n with p parts. Then one has

$$\frac{h_n^0(\lambda)}{(n+p-2)!} = D_\lambda n^{p-3} \quad ; \quad \frac{h_n^1(\lambda)}{(n+p)!} = \frac{D_\lambda}{24}\left(n^p - n^{p-1} - \sum_{i=2}^{p} (i-2)!\, e_i(\lambda)\, n^{p-i}\right).$$

We have obtained the following **asymptotic estimate**, which generalizes previous results of Kazarian [Kaz]:

Theorem (Main result)

For any genus $g \geq 0$, their is a positive constant c_g satisfying, for any partitions $\lambda^1, \ldots, \lambda^k$, the asymptotics

$$\frac{h_n^g\left(\lambda^1, \ldots, \lambda^k\right)}{\left(2n + 2g' - r\left(\overrightarrow{\lambda}\right)\right)!} \sim_{n\to\infty} D_{\lambda^1} D_{\lambda^2} \cdots D_{\lambda^k}\, c_g\, n^{\frac{5}{2}g'} n^{(\ell(\lambda^1)+\cdots+\ell(\lambda^k))} \frac{e^n}{n}.$$

The constants c_g can be computed from the rational numbers $\alpha_{g-1} := c_g 2^{\frac{5g-1}{2}} \Gamma\left(\frac{5g-1}{2}\right)$ by the recursion formula

$$\alpha_{-1} = -1 \quad ; \quad \alpha_g = \frac{25g^2 - 1}{12} + \sum_{\substack{p+q=g-1 \\ p,q \geq 0}} \alpha_p \alpha_q.$$

Remarks. The recursion amounts to saying that the series

$$u(t) := \sum_{g=0}^{\infty} c_g \Gamma\left(\frac{5g-1}{2}\right) t^{-\frac{5g-1}{2}} = \sum_{g=0}^{\infty} \alpha_{g'} (2t)^{-\frac{5g-1}{2}}$$

satisfies the **Painlevé I equation**

$$u(t)^2 + \frac{1}{6}\frac{d^2 u(t)}{dt^2} = 2t.$$

On the other hand, one has the identity $c_g = \sqrt{2}^{g-3} t_g$, where t_g is the **Bender-Gao-Richmond constant** appearing in map asymptotics [BGR08]. Hence,

$$c_0 = \frac{1}{\sqrt{2\pi}} \quad ; \quad c_1 = \frac{1}{48} \quad ; \quad c_2 = \frac{1}{\sqrt{2\pi}} \frac{7}{4320} \quad ; \quad \cdots$$

The algebra $\mathcal{A} = \mathbb{Q}\left[\sum_{n\geq 1} \frac{n^{n-1}}{n!} q^n, \sum_{n\geq 1} \frac{n^n}{n!} q^n\right]$

The exponential generating functions $Y := \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} q^n$ and $Z := \sum_{n=1}^{\infty} \frac{n^n}{n!} q^n$ of one- and two-rooted Cayley trees span an **algebra of power series** $\mathcal{A} = \mathbb{Q}[Y, Z]/(YZ = Z - Y)$ — see [Zvon04]. This algebra has very nice asymptotic properties:

$$\forall i \geq 1, \left(\textit{leading coefficient in } \frac{Y^i}{i}\right) \sim C_{-1} \frac{e^n}{n} \sqrt{n}^{-1} \quad ; \quad \forall k \geq 1, \left(\textit{leading coefficient in } Z^k\right) \sim C_k \frac{e^n}{n} \sqrt{n}^k$$

where $C_{-1} := \frac{1}{\sqrt{2\pi}}$ and $C_k := \frac{1}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)}$. Moreover, any series in $\mathcal{A} = \mathbb{Q}[Y] + \mathbb{Q}[Z]$ is asymptotically determined by its leading coefficient in Z — if there is no Z, apply the differential operator $D := q\frac{\partial}{\partial q}$.

Theorem (Kazarian, Zvonkine)

All Hurwitz series lie in the algebra $\mathcal{A}$, except for $H^1(\emptyset, q) = D^{-1} Z^2$. More precisely:

- For any partition λ, the series $H^g(\lambda)$ belongs to $\mathcal{A}$ and has Z-degree $5g' + 2\ell(\lambda)$ (Kazarian).
- For any partitions λ, μ, the series $H^g(\lambda, \mu)$ is a polynomial in the series $H^\gamma(\nu)$ with $\gamma \leq g$ and $|\nu| \leq |\lambda| + |\mu|$ (Zvonkine):

$$H^g(\lambda, \mu) = \sum_{\overrightarrow{\nu}, \overrightarrow{g}} f_{\lambda,\mu}^{\overrightarrow{\nu}} \prod_j H^{g^j}(\nu^j) = H^g(\lambda \sqcup \mu) + \cdots$$

where one sums over pairs $(\overrightarrow{\nu}, \overrightarrow{g})$ such that $g' = \frac{r(\lambda)+r(\mu)-r(\overrightarrow{\nu})}{2} + \sum_j g^{j\prime}$. The coefficients $f_{\lambda,\mu}^{\overrightarrow{\nu}}$ of these polynomials can be interpreted as numbers of factorisations satisfying certain conditions.

Examples. In genus zero and in genus one, for partitions with 0, 1 or 2 parts, one has the identities

$$\widetilde{H}^0(a-b, b) = \frac{Y^a}{a} \quad ; \quad \widetilde{H}^0(a) = \frac{1}{a}\left(\frac{Y^a}{a} - \frac{Y^{a+1}}{a+1}\right) \quad ; \quad \widetilde{H}^0(1) = Y - \frac{Y^2}{2} \quad ;$$

$$\widetilde{H}^0(\emptyset) = Y - \frac{3}{2}\left(\frac{Y^2}{2}\right) + \frac{1}{2}\left(\frac{Y^3}{3}\right) = \left(Y - \frac{Y^2}{2}\right) - \frac{1}{2}\left(\frac{Y^2}{2} - \frac{Y^3}{3}\right) = D^{-1}\widetilde{H}^0(1) \quad ;$$

$$24\,\widetilde{H}^1(\emptyset) = D^{-1}Z^2 \quad ; \quad 24\,\widetilde{H}^1(1) = Z^2 \quad ; \quad 24\,\widetilde{H}^1(2) = Z^2 \quad ;$$

$$24\,\widetilde{H}^1(d+1) = Y^d Z(Z+d) = Z^2 - Y^2 - 2Y^3 - 3Y^4 - \cdots - (d-1)Y^d.$$

Now, Kazarian's result allows one to compute the asymptotics of Hurwitz numbers *with one partition*

$$\frac{h_n^g(\lambda)}{T_n!} \sim_{n\to\infty} D_\lambda\, c_g\, e^n\, n^{\frac{5}{2}g' + \ell(\lambda) - 1},$$

while Zvonkine's result permits to *reduce the number of partitions* and hence use the latter asymptotics. Zvonkine proved his formula by merging the first two permutations of a transitive factorisation. It remains obviously a factorisation of the identity, but might be no longer transitive. Restraining oneself to the orbits yields transitive factorisations, hence a product of series $H^\gamma(\nu)$. When the merged permutations have disjoint support, the factorisation remains transitive, hence a term $H^g(\lambda \sqcup \mu)$. It so happens that the latter is the Z-leading term, hence the heuristic

$$\frac{h_n^g\left(\lambda^1, \lambda^2, \ldots, \lambda^k\right)}{T_n\left(\lambda^1, \lambda^2, \ldots, \lambda^k\right)!} \sim_{n\to\infty} \frac{h_n^g\left(\lambda^1 \sqcup \lambda^2 \sqcup \cdots \sqcup \lambda^k\right)}{T_n\left(\lambda^1 \sqcup \lambda^2 \sqcup \cdots \sqcup \lambda^k\right)!}.$$

References

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