test on rotations

Monday 13th May 24

Exos résolus p. 213

Suggested correction

Wording 19.

1. Define ℓ to be the common side length of the table's five squares (the wording gives $\ell = 2\,\mathrm{m}$).

   (a) Polygon $ABCDEFGH$ has as many sides as it has vertices, *i. e.* as there are letters from A to H, that is: eight.

   (b) Since the « table is made up of [...] squares and [...] isosceles right-angled triangles side by side », each of its (eight) angles is made up of two "corners": one from a square and one being an acute angle of an isosceles right-angled triangle. Now, in any such triangle both *acute* angles are half a right angle, and any square "corner" is a right angle. Therefore, polygon $ABCDEFGH$'s angles have all the same measure[1] – that of a right angle plus that of half of a right angle – and hence are equal.

   (c) Let us now show this polygon has two sides of different lengths, which will prove it is *not* regular: to this aim, let us prove strict comparison $BC > AB$.

   Points A, B, O are three consecutive vertices of one of the table's squares: call its fourth vertex V. Triangle BCO can then be obtained by "halving" square $ABOV$ along its diagonal $[AO]$, hence length BC is that of diagonal $[AO]$, which is strictly greater than square $ABOV$'s side length AB, *q. e. d.*[2].

   Remark: one actually has $AO = \sqrt{2}AB$ thanks to Pythagoras' theorem, hence

$$BC = \ell\sqrt{2} > \ell = AB.$$

2. (a) Polygon $ABCDEFGH$'sides of are of two types: those – like segment $[AB]$ – being a side of one of the table's squares; those – like segment $[BC]$ – being the hypotenuse of one of the table's right-angled triangles. Sides of the first type have length ℓ (since all table's squares have the same side length ℓ) while sides of the second type have length $\ell\sqrt{2}$ (according to the remark above). Since each type comprises four sides, the sought-after perimeter is

$$\overbrace{\underbrace{\begin{matrix}AB+CD\\+EF+GH\end{matrix}}_{\text{each length equals }\ell}}^{\text{first type}} + \overbrace{\underbrace{\begin{matrix}BC+DE\\+FG+HA\end{matrix}}_{\text{each length equals }\ell\sqrt{2}}}^{\text{second type}} = 4\ell+4\ell\sqrt{2} = 4\ell\left(1+\sqrt{2}\right) \underset{\text{application}}{\overset{\text{numerical}}{=}} 4\left(1+\sqrt{2}\right)2\,\mathrm{m} \simeq 19,31\,\mathrm{m}\,.$$

   (b) According to the wording, octagon $ABCDEFGH$ is made up of five identical squares and four identical triangles. Now, as we remarked of above, any *two* of these triangles make up any of these squares (using suitable rotations and translations), so that the surface area of the four triangles is that of *two* squares. Therefore, octagon $ABCDEFGH$'s surface area is that of five *plus two* squares, *i. e.* that of *seven* squares, that is

$$7\ell^2 \underset{\text{application}}{\overset{\text{numerical}}{=}} 7\,(2\,\mathrm{m})^2 = 7\cdot 2^2\,\mathrm{m}^2 = 28\,\mathrm{m}^2\,.$$

3. The new table's perimeter is π times its diameter, *i. e.*

$$\pi\cdot 6\,\mathrm{m} = 6\pi\,\mathrm{m} \simeq 18,85\,\mathrm{m}\,,$$

while its surface area is π times its squared half-diameter *i. e.*

$$\pi\left(\frac{6\,\mathrm{m}}{2}\right)^2 = \pi 3^2\,\mathrm{m}^2 = 9\pi\,\mathrm{m}^2 \simeq 28,27\,\mathrm{m}^2\,.$$

Comment: the new perimeter is slightly under the old one ($\simeq 19,31\,\mathrm{m}$), while the new surface area is almost (but very slightly above) the old one ($28\,\mathrm{m}^2$).

[1] which we need not compute, though it would equal $90^\circ + 45^\circ = 135^\circ$

[2] abbreviates "*quod erat demonstrandum*", which stands for "which was to be proved"

Wording 20.

1. (on a rotation centred at O)

   (a) We have a problem here: the wording *says nothing* about point O, which is quite problematic when 1) defining something depending on it 2) asking to prove something about it! Yet, figure 1 strongly suggests it lies on segments $[AD]$ and $[BE]$, *which we will take as a definition of* O. (Figure 1 also suggests O lies on $[CF]$ but we will not need that – which, incidentally, would require an extra proof, which we will need neither.)

   Our first hitch being solved, we have yet another one: the considered rotation being meaningful *presupposes equality* $OA = OB$, since the distance from the centre to any point must equals that to the image of that point. We will therefore *assume* this equality in the sequel, which will be more than enough for the Brevet level.

   (b) The way having been cleared, let us call the considered rotation r.

   By the definition of any rotation, its angle is that at its centre between any point and its image, while its direction is that followed when turning around its centre from any point to its image. For rotation r, we will use particular point A and its image, which is point B by the very definition of r.

   When turning around r's centre from A to its image by r, one turns around O from A to B, which is a *clockwise* turn on the figure. Therefore, so is the direction of the considered rotation.

   Moreover, still using the above with particular point A, the sought-after angle is $\widehat{AOB}$, which is – *provided all six angles around* O *are equal* – one sixth of a full angle, hence has measure one sixth of $360°$, *i. e.* $\frac{360°}{6} = 60°$.

   (c) **(Very) long remark.** The previous proof would be enough for Brevet's level. However, let us now prove equality $\widehat{AOB} = 60°$ with more elementary material.

   Our one tool will be: *in any regular polygon with an even number of vertices, any diagonal joining two opposite vertices is a symmetry axis of that polygon.*

   In our regular hexagon, segment $[BE]$ is such a diagonal, hence divides angle $\widehat{ABC}$ in two equal angles $\widehat{ABE} = \widehat{CBE}$. Now, our regular hexagon has all its angles equal, so it is meaningful to define m to be their common measure, which allows us to affirm equality $\widehat{ABE} = \frac{m}{2}$. Using then diagonal $[AD]$ instead of $[BE]$, one similarly gets equality $\widehat{BAD} = \frac{m}{2}$. But, since O lies on both these diagonals, the previous equalities become $\widehat{ABO} = \frac{m}{2} = \widehat{BAO}$.

   If we managed to establish $m \overset{?}{=} 120°$, these equalities would become $\widehat{ABO} = 60° = \widehat{BAO}$, hence the sought-after angle

   $$\widehat{AOB} = 180° - \widehat{ABO} - \widehat{BAO} = 180° - 60° - 60° = 60°.$$

   Let us therefore prove equality $m = 120°$, which will conclude.

   Regardless of whether point O lies on $[FC]$ or not, it is the common vertex of six triangles dividing our regular hexagon. When summing up the $6 \cdot 3$ angles of these triangles, one gets six flat angles (since in any of these triangles the angle measures sum up to $180°$), *i. e.* $6 \cdot 180°$. On the other hand, when gathering first the angles around O (which sum up to a full angle) then all the others (who sum up to the six angles in our regular hexagon), one gets $360° + 6m$. Equalling both sums gives

   $$\begin{aligned} 360° + 6m &= 6 \cdot 180°, \qquad \text{hence (subtract } 360° = 2 \cdot 180°) \\ 6m &= 4 \cdot 180° = 4 \cdot 3 \cdot 60°, \qquad \text{hence (divide by 6)} \\ m &= 2 \cdot 60° = 120°, \qquad \textit{q. e. d.} \end{aligned}$$

2. (on triangle GAS)

   (a) The full angle around $\widehat{A}$ is constituted of five subangles: two angles from equilateral triangles ($\widehat{FAO}$ and $\widehat{BAO}$, both measuring $60°$), two square angles ($\widehat{FAS}$ and $\widehat{BAG}$, both measuring $90°$) and sought-after angle $\widehat{GAS}$. Their five measures add up to that of the full angle, *i. e.* $360°$, which writes

   $$\begin{aligned} 360° &= \underbrace{\underbrace{2 \cdot 60°}_{=120°} + \underbrace{2 \cdot 90°}_{=180°}}_{=300°} + \widehat{GAS}, \qquad \text{hence (subtract } 300°) \\ \widehat{GAS} &= 360° - 300° = 60°. \end{aligned}$$

(b) Let us prove triangle GAS is equilateral, in *two* steps. *Step 1*: we will show that triangle GAS is isosceles by proving equality $AG = AS$. *Step 2*: we will show that, when any isosceles triangle has one angle measuring $60°$, it is an equilateral triangle.

Step 1. Since base hexagon $ABCDEF$ is *regular* (says the wording: « consists of a regular hexagon »), sides $[AB]$ and $[AF]$ have equal lengths, which writes $AB = AF$. One the other hand, both quadrilaterals $ABHG$ and $AFRS$ are squares (says – again – the wording: « surrounded by six squares »), so in each of them all four sides have the same length, hence equalities $AB = AG$ (in square $ABHG$) and $AF = AS$ (in square $AFRS$). Putting these three equalities together enables one to conclude:

$$AG = AB = AF = AS, \ q.\ e.\ d.$$

Step 2. Let be an isosceles triangle whose angle measures we call r, s and s (by hypothesis, two of these measures are equal). These three measures sum up to $180°$, which writes

$$r + 2s = 180°. \qquad \text{Assume now } r \text{ or } s \text{ to be } 60°.$$

If it is r, then the previous equality becomes $60° + 2s = 180°$, hence (subtract $60°$) $2s = 180° - 60° = 120°$, hence (halve)

$$s = \frac{120°}{2} = 60° = r.$$

If[3] now s equals $60°$, the equality above then becomes $r + 2 \cdot 60° = 180°$, hence (subtract $120°$)

$$r = 180° - 120° = 60° = s.$$

In both cases, we find all measures to be equal and our given triangle is therefore equilateral.

3. Let us show answers to both questions to be "yes, it is".

(a) The sides of the resulting dodecagon are of two types: those (like $[GH]$) who are a side of one of the six surrounding squares; those (like $[GS]$) who link two vertices of two such squares. As proved in the second step above: on the one hand, all surrounding squares have the same side length; one the other hand, since any of the six triangles is equilateral, the side length "furthest" from centre O is the same as that of the six squares. Therefore, whether of the first or second type, all sides of the dodecagon have the same length.

To conclude this polygon is regular, we now need only prove all its angles have the same measure. But any of these is made up of the corner of a square combined with the corner of an equilateral triangle, hence they all have same measure[4], *q. e. d.*

(b) The dodecagon is made up of twelve sides, each of which has the same length as $[AB]$ (as established just above), hence its perimeter is $12AB$. Now, since the base hexagon is regular, its perimeter equals six times its side length, *i. e.* $6AB$. Remarking of equality $12 = 2 \cdot 6$ enables one to conclude.

[3] This case is actually pointless for our application in triangle GAS (which is isosceles *precisely* at the vertex whose angle measures $60°$) but the result is still good to know regardless of which angles are equal. It is incidentally an opportunity to introduce *case discussion* in a proof.

[4] once again, we need *not* compute this measure (which, incidentally, equals $90° + 60° = 120°$)