Trigonometry test

for Thu 14 March 24

trans. from exercise notebook p. 75 (ex. 1, 3 & 4)

Suggested correction

Exercise 1. Let us define respectively

1. h Averell's $\underline{\underline{\textbf{h}}}$eight ;
2. d the $\underline{\underline{\textbf{d}}}$istance between the pistol and Averell;
3. a the pistol $\underline{\underline{\textbf{a}}}$ltitude in relation to the ground;
4. B (like "$\underline{\underline{\textbf{b}}}$ottom" or "$\underline{\underline{\textbf{b}}}$oot") the point where Averell touches the ground.

In triangle ACP, which is right-angled at A, the tangent of sought-after angle $\angle APC$ is $\frac{AC}{AP}$, hence $\angle APC = \arctan \frac{AC}{AP}$. *If we assume momentarily that* numerator AC is difference $h - a$, one can conclude:

$$\begin{aligned} \angle APC &= \arctan \frac{h-a}{d} \\ &\underset{\text{application}}{\overset{\text{numerical}}{=}} \arctan \frac{2.13\,\text{m} - 1\,\text{m}}{6\,\text{m}} = \arctan \frac{1.13}{6} \overset{\text{calculator}}{\simeq} 10.66^\circ \\ &\simeq 11^\circ \text{ (rounded to the nearest degre).} \end{aligned}$$

Let us now prove the assumed equality, *i. e.* $AC \overset{?}{=} BC - PS$ or yet $PS + AC \overset{?}{=} BC$. For this we need only establish equalities $PS \overset{?}{=} BA$ and $BA + AC \overset{?}{=} BC$.

For the 1st, we need only prove that quadrilateral $ABSP$ is a rectangle: opposite sides $[PS]$ and $[AB]$ will then have same length. Now, the wording tells us on the one hand that « [t]riangle PAC is right-angled at A », hence orthogonality $(PA) \perp (AC)$, on the other hand that « both cow-boys are standing perpendicular to the ground », hence on the one hand parallelism $(AC) = (AB)$ (Averell stands "upright"), on the other hand orthogonalities $(AB) \perp (BS) \perp (SP)$. Quadrilateral $ABSP$ has consequently three right angles (in A, B and S), therefore it is a rectangle as announced.

2d equality ($BA + AC \overset{?}{=} BC$) equates elementhood[1] $A \overset{?}{\in} [BC]$. But points P, A and C are on the same side on straight line (BS) (the pistol and Averell are above ground), hence points A and C are on the same side of B on half straight line $[BA) = [BC)$. The wanted elementhood therefore amounts to comparison $BA \overset{?}{<} BC$, *i. e.* $a \overset{?}{<} h$, *i. e.* $1\,\text{m} \overset{?}{<} 2,13\,\text{m}$, which obtains[2].

Exercise 4. The wording gives no unit whatsoever for the lengths, which is *bad*: a length cannot be a dimensionless stark naked number! We will write $\heartsuit := \frac{1}{10} MC$ for our length unit.

Since segments « $[CH]$ et $[HM]$ are perpendicular to $[HA]$ », triangles AEM and AHC are right-angled resp. at E and M. We deduce thereof on the one hand equality

$$\begin{aligned} CH &= AC \sin \angle CAH, \qquad \text{on the other hand equalities} \\ EM &= AM \sin \angle EAM = (16\heartsuit) \sin 30^\circ = 16 \frac{1}{2} \heartsuit = \underline{\underline{8\heartsuit}}. \end{aligned}$$

Now, « points H, E and A are aligned » (that is a hypothesis) *and are so in this order* (not in the hypotheses but assumed given the figure), as are points C, M and A (same reasoning), hence, on the one hand, equalities $\begin{cases} AM + MC = AC \\ AE + EH = AH \end{cases}$ and, on the other hand, equality $\angle CAH = \angle MAE$. We can thus simplify the 2d sought-after length:

$$HC = (AM + MC) \sin \angle MAE = (16\heartsuit + 10\heartsuit) \sin 30^\circ = 26\heartsuit \frac{1}{2} = \underline{\underline{13\heartsuit}}.$$

[1] "$e \in E$" reads "e **is an element of** E" or "e **belongs to** E"

[2] in this context, *to obtain* means *to be the case*

Last, one has in right-angled triangle AEM equalities $\frac{AE}{AM}=\cos\angle EAM=\cos 30^\circ=\frac{\sqrt{3}}{2}$, hence $AE=\frac{\sqrt{3}}{2}AM$; we would similarly get $AH=\frac{\sqrt{3}}{2}AC$, hence

$$HE=AH-AE=\frac{\sqrt{3}}{2}AC-\frac{\sqrt{3}}{2}AM=\frac{\sqrt{3}}{2}(AC-AM)=\frac{\sqrt{3}}{2}CM=\frac{\sqrt{3}}{2}10\heartsuit=\underline{\underline{5\sqrt{3}\heartsuit}}.$$

Conclusion: taking as length unit the tenth of length MC (where the chimney will come out of the roof), the "tallest" side of the chimney will measure 13 units, the "smallest" 8, and its "roof" $5\sqrt{3}\simeq 8.66$.

Alternative solution (without trigonometry): given the 30° angle, triangle EAM is "half" of the equilateral triangle whose $[AE]$ is one height, so lengths EM and AE equal resp. half and $\frac{\sqrt{3}}{2}$ times that of corresponding side $[AM]$, hence equalities $\begin{cases} EM=\frac{1}{2}AM \\ AE=\frac{\sqrt{3}}{2}AM \end{cases}$. We would get in the same way $\begin{cases} CH=\frac{1}{2}AC \\ AH=\frac{\sqrt{3}}{2}AC \end{cases}$ and conclude as in the 1st solution.

Exercise 3. According to the wording, the slope of a road is merely the tangent of the angle it makes with the horizontal.

The 1st can be read straight on the sign: 24%.

The 3rd equals $\tan 12,4^\circ \overset{\text{calculator}}{\simeq} 0.21986 \simeq 22.0\%$ (rounded to the thousandth): needless here to compute lengths since we are directly given the angle 12.4° with the horizontal.

For the 2d slope, one can compute either the missing length by way of Pythagoras' theorem (then compute the tangent as a length ratio) or the angle with the horizontal thanks to its sine (then compute its tangent directly). Let us present these two ways, followed by a third one that encompasses them both – and a fourth way out of elegance.

Some notations. Define g, h, r and α to be resp. the elevation **g**ain (here: 280 m), the sought-after length (= **h**orizontal displacement), the **r**oad length (here: 1.5 km) and the **a**ngle with the horizontal (whose tangent is the sought-after slope). Thus, lengths g, h and r are resp. the opposite side, the adjacent side and the hypotenuse relative to angle α.

1. *Sine way.* Angle α has a sine equalling $\frac{g}{r}=\frac{280\,\text{m}}{1500\,\text{m}}=\frac{14\cdot\cancel{20}}{75\cdot\cancel{20}}=\frac{14}{75}$, hence equals $\arcsin\frac{14}{75}\overset{\text{calculator}}{\simeq}10.75^\circ$, so its tangent is[3] about $\tan 10.75^\circ \overset{\text{calculator}}{\simeq} 0.19001\simeq 19.0\%$ (rounded to the thousandth).

2. *Pythagoras' way.* Pythagoras' theorem yields equality $g^2+h^2=r^2$, hence equalities[4]

$$\begin{aligned} h^2 &= r^2-g^2=(1.5\,\text{km})^2-(0.28\,\text{km})^2=\left(1.5^2-0.28^2\right)\text{km}^2 \\ &= (2.2500-0.0784)\,\text{km}^2=2.1716\ \text{km}^2. \end{aligned}$$

We deduce thereof[5]

$$h=\sqrt{2.1716\ \text{km}^2}=\sqrt{2.1716}\,\text{km}\overset{\text{calculator}}{\simeq}1.4736\,\text{km},$$

hence the sought-after slope

$$\tan\alpha=\frac{g}{h}\simeq\frac{0.28\,\text{km}}{1.4736\,\text{km}}\overset{\text{calculator}}{\simeq}0.19001 \text{ (we get the same approximate value).}$$

These two ways share a problem – the successive use of approximate values: *how can one make sure of the final result precision if each computation increasingly modifies the previous approximation?* Here is indeed how the sought-after slope varies with lesser approximations of h:

$h\simeq$	1.4736 km	1.474 km	1.48 km	1.5 km
$\tan\alpha\simeq$	0.19001	0.18996	0.18919	0.18667
slope to the thousandth	19.0%	19.0%	18.9%	18.7%

[3] To compare with the 1st and 3rd slopes, *i. e.* 24% (a supposedly exact value) and ≃ 22.0% (which we computed), we need keep a precision of the same order, we chose *one digit after the percentage point*, *i. e.* a thousandth precision.

[4] mental arithmetic training:

$$\begin{aligned} 15^2 &= (3\cdot5)^2=3^2 5^2=9\cdot25=(10-1)\,25=250-25=\underline{\underline{225}} \quad \text{et} \\ 28^2 &= (2\cdot14)^2=2^2 14^2=4\cdot(15-1)^2=4\left(15^2-2\cdot15+1^2\right)=4\,(225-30+1)=900-120+4=\underline{\underline{784}} \end{aligned}$$

[5] *reminder*: a length being always *non-negative*, the negative solution $h=-\sqrt{\cdots}$ is to be ruled out

The result varies of some thousandths indeed – and this variation could escape us... One way out of this problem is to handle all computations *literally* and, *only at the end*, replace the letters with their corresponding numerical values. This is the...

3. ... *literal (or formal) way.* As above, Pythagoras' theorem gives us equality $h = \sqrt{r^2 - g^2}$, hence the sought-after slope:

$$\frac{g}{h} \overset{g \geq 0}{=} \frac{\sqrt{g^2}}{\sqrt{r^2 - g^2}} = \sqrt{\frac{g^2}{r^2 - g^2}} \underset{\text{under the root}}{\overset{\text{divide by } g^2}{=}} \sqrt{\frac{1}{\frac{r^2}{g^2} - 1}} = \sqrt{\frac{1}{\left(\frac{r}{g}\right)^2 - 1}}$$

$$= \quad f(\lambda) \text{ where we defined } \lambda := \frac{r}{g} \text{ and } f := \begin{cases}]1, \infty[& \longrightarrow & \mathbf{R} \\ t & \longmapsto & \sqrt{\frac{1}{t^2-1}} \end{cases}.$$

Our slope, therefore, depends (by way of map f) only on ratio $\lambda = \frac{1500}{280} = \frac{75}{14}$, hence equality

$$\tan \alpha = f\left(\frac{75}{14}\right) \overset{\text{calculator}}{\simeq} 0.19001 \text{ (the same approximate value again).}$$

The only approximation we do not have control over is that given by the calculator (when applying f on $\frac{75}{14}$): we have thus solved our approximations issue.

4. *Elegant way.* Let us remind (!) we wanted to put in order the three slopes: since the 1st and 3rd equal resp. 24% and $\simeq$ 22% $\geq$ 21.5% $>$ 20%, we will be able to prove the ordering

$$\text{1st slope} > \text{3rd slope} \, (> 20\%) \overset{?}{>} \text{2d slope}$$

if we manage to prove that the 2d slope $\frac{g}{h}$ is smaller than $20\% = \frac{1}{5}$. Now, here is a (subtle) minoration:

$$\frac{h}{\text{km}} = \sqrt{2.1716} > \underbrace{\sqrt{2}}_{\simeq 1.414} > 1.4 = 5 \cdot 0.28 = 5\frac{g}{\text{km}}, \text{ hence } \frac{g}{h} < \frac{1}{5}, \textit{ QED.}$$