# Comparison between $W_2$ distance and $\dot{H}^{-1}$ norm, and Localisation of Wasserstein distance

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**Abstract**

It is well known that the quadratic Wasserstein distance $W_2(\cdot,\cdot)$ is formally equivalent, for infinitesimally small perturbations, to some weighted $H^{-1}$ homogeneous Sobolev norm. In this article I show that this equivalence can be integrated to get non-asymptotic comparison results between these distances. Then I give an application of these results to prove that the $W_2$ distance exhibits some localisation phenomenon: if $\mu$ and $\nu$ are measures on $\mathbf{R}^n$ and $\varphi\colon \mathbf{R}^n \to \mathbf{R}_+$ is some bump function with compact support, then under mild hypotheses, you can bound above the Wasserstein distance between $\varphi \cdot \mu$ and $\varphi \cdot \nu$ by an explicit multiple of $W_2(\mu,\nu)$.

**Keywords:** Wasserstein distance; homogeneous Sobolev norm; localisation

## Foreword

This article is divided into two sections, each of which having its own introduction. § 1 deals with general results of comparison between Wasserstein distance and homogeneous Sobolev norm, while § 2 handles an application to localisation of $W_2$ distance.

## 1 Non-asymptotic equivalence between $W_2$ distance and $\dot{H}^{-1}$ norm

### 1.1 Introduction

In all this section, $M$ denotes a connected Riemannian manifold endowed with its distance $dist(\cdot,\cdot)$ and its Lebesgue measure $\lambda$. Let us give a few standard definitions which will be at the core of our work:

- For $\mu,\nu$ two positive measures on $M$, denoting by $\Pi(\mu,\nu)$ the set of (positive) measures on $M \times M$ whose respective marginals are $\mu$ and $\nu$, for $\pi \in \Pi(\mu,\nu)$ one defines

$$I(\pi) := \int_{M\times M} dist(x,y)^2 \pi(dx,dy) \tag{1}$$

and then

$$W_2(\mu,\nu) := \inf\{I(\pi) \mid \pi \in \Pi(\mu,\nu)\}^{1/2}. \tag{2}$$

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$W_2$ is a (possibly infinite) distance, called the *quadratic Wasserstein distance* [Villani, 2003, § 7.1]. Note that this distance is finite only between measures having the same total mass.

- On the other hand, for $\mu$ a (positive) measure on $M$, if $f$ is a $\mathcal{C}^1$ real function on $M$, one denotes

$$\|f\|_{\dot{H}^1(\mu)} \coloneqq \Big(\int_M |\nabla f(x)|^2 d\mu(x)\Big)^{1/2}, \tag{3}$$

which defines a semi-norm; for $\nu$ a signed measure on $M$, one then denotes

$$\|\nu\|_{\dot{H}^{-1}(\mu)} \coloneqq \sup\{|\langle f, \nu\rangle| \mid \|f\|_{\dot{H}^1(\mu)} \leqslant 1\}, \tag{4}$$

which defines a (possibly infinite) norm, which we will call the $\dot{H}^{-1}(\mu)$ *weighted homogeneous Sobolev norm*. Note that this norm is finite only for measures having zero total mass. In the case $\mu$ is the Lebesgue measure, we will merely write "$\dot{H}^{-1}$" for "$\dot{H}^{-1}(\lambda)$".

The $W_2$ Wasserstein distance is an important object in analysis; but it is non-linear, which makes it harder to study. For infinitesimal perturbations however, the linearised behaviour of $W_2$ is well known: if $\mu$ is a positive measure on $M$ and $d\mu$ is an infinitesimally small perturbation of this measure,[*] one has formally (see [Villani, 2003, § 7.6] or [Otto and Villani, 2000, § 7])

$$W_2(\mu, \mu + d\mu) = \|d\mu\|_{\dot{H}^{-1}(\mu)} + o(d\mu). \tag{5}$$

More precisely, one has the following equality, known as the *Benamou–Brenier formula* [Benamou and Brenier, 2000, Prop. 1.1]: for two positive measures $\mu, \nu$ on $M$,

$$W_2(\mu, \nu) = \inf\Big\{\int_0^1 \|d\mu_t\|_{\dot{H}^{-1}(\mu_t)} \;\Big|\; \mu_0 = \mu,\ \mu_1 = \nu\Big\}. \tag{6}$$

Then, a natural question is the following: are there *non-asymptotic* comparisons between the $W_2$ distance and the $\dot{H}^{-1}$ norm? Concretely, we are looking for inequalities like

$$C_{\mathrm{a}}\|\mu - \nu\|_{\dot{H}^{-1}(\mu)} \leqslant W_2(\mu, \nu) \leqslant C_{\mathrm{b}}\|\mu - \nu\|_{\dot{H}^{-1}(\mu)} \tag{7}$$

for constants $0 < C_{\mathrm{a}} \leqslant C_{\mathrm{b}} < \infty$, under mild assumptions on $\mu$ and $\nu$.

## 1.2 Controlling $W_2$ by $\dot{H}^{-1}$

**Theorem 1.** *For any positive measures $\mu, \nu$ on $M$,*

$$W_2(\mu, \nu) \leqslant 2\|\mu - \nu\|_{\dot{H}^{-1}(\mu)}. \tag{8}$$

*Proof.* We suppose that $\|\mu - \nu\|_{\dot{H}^{-1}(\mu)} < \infty$, otherwise there is nothing to prove. For $t \in [0, 1]$, let

$$\mu_t \coloneqq (1 - t)\mu + t\nu, \tag{9}$$

so that $\mu_0 = \mu$, $\mu_1 = \nu$ and $d\mu_t = (\mu - \nu)dt$. Then, by the Benamou–Brenier formula (6):

$$W_2(\mu, \nu) \leqslant \int_0^1 \|\mu - \nu\|_{\dot{H}^{-1}(\mu_t)} dt. \tag{10}$$

Now, we use the following key lemma, whose proof is postponed:

[*]Beware that here $d\mu$ denotes a small measure on $M$, not the value of $\mu$ on a small area.

**Lemma 2.** *If $\mu, \mu'$ are two measures such that $\mu' \geqslant \rho\mu$ for some $\rho > 0$, then* $\|\cdot\|_{\dot{H}^{-1}(\mu')} \leqslant \rho^{-1/2}\|\cdot\|_{\dot{H}^{-1}(\mu)}$.[†]

Here obviously $\mu_t \geqslant (1-t)\mu$, so

$$W_2(\mu,\nu) \leqslant \int_0^1 (1-t)^{-1/2}\|\mu-\nu\|_{\dot{H}^{-1}(\mu)}dt = 2\|\mu-\nu\|_{\dot{H}^{-1}(\mu)}, \tag{11}$$

QED. □

**Corollary 3.** *If $\mu \geqslant \rho\lambda$ for some $\rho > 0$, then*

$$W_2(\mu,\nu) \leqslant 2\rho^{-1/2}\|\mu-\nu\|_{\dot{H}^{-1}}. \tag{12}$$

*Proof.* Just use that $\|\cdot\|_{\dot{H}^{-1}(\mu)} \leqslant \rho^{-1/2}\|\cdot\|_{\dot{H}^{-1}}$ by Lemma 2. □

*Proof of Lemma 2.* Take $\mu' \geqslant \rho\mu$ and let $\nu$ be a signed measure on $M$ such that $\mu+\nu$ is positive; then $\mu'+\rho\nu$ is also positive. For $m$ a measure on $M$, we denote by $diag(m)$ the measure on $M\times M$ supported by the diagonal whose marginals (which are equal) are $m$, i.e.:

$$\big(diag(m)\big)(A\times B) \coloneqq m(A\cap B); \tag{13}$$

with that notation,

$$\pi \in \Pi(\mu,\mu+\nu) \Rightarrow \rho\pi + diag(\mu'-\rho\mu) \in \Pi(\mu',\mu'+\rho\nu), \tag{14}$$

and

$$I\big(\rho\pi + diag(\mu'-\rho\mu)\big) = \rho I(\pi). \tag{15}$$

Therefore, taking infima,

$$\begin{aligned} W_2(\mu',\mu'+\rho\nu)^2 &= \inf\{I(\pi') \mid \pi' \in \Gamma(\mu',\mu'+\rho\nu)\} \\ &\leqslant \inf\big\{I\big(\rho\pi + diag(\mu'-\rho\mu)\big) \;\big|\; \pi\in\Gamma(\mu,\mu+\nu)\big\} \\ &= \rho\inf\{I(\pi) \mid \pi\in\Gamma(\mu,\mu+\nu)\} = \rho W_2(\mu,\nu)^2. \end{aligned} \tag{16}$$

For infinitesimally small $\nu$, it follows by Equation (5) that $\|\rho\nu\|^2_{\dot{H}^{-1}(\mu')} \leqslant \rho\|\nu\|^2_{\dot{H}^{-1}(\mu)}$, hence $\|\nu\|_{\dot{H}^{-1}(\mu')} \leqslant \rho^{-1/2}\|\nu\|_{\dot{H}^{-1}(\mu)}$. This relation remains true even for non-infinitesimal $\nu$ by linearity, which ends the proof. □

*Remark* 4. Lemma 2 could also be proved very quickly by using the definition (3)-(4) of the $\dot{H}^{-1}(\mu)$ norm. The proof above, however, has the advantage that it does not need the precise expression of $\|\cdot\|_{\dot{H}^{-1}(\mu)}$, but only the fact that it is the linearised $W_2$ distance.

[†]Beware that here '$\cdot$' stands for a *measure*, not for a function: otherwise the formula would be false.—When $f$ is a function, $\|f\|_{\dot{H}^{-1}(\mu)}$ stands for the $\dot{H}^{-1}(\mu)$ norm of the measure having density $f$ w.r.t. $\mu$.

### 1.3 Controlling $\dot{H}^{-1}$ by $W_2$

**Theorem 5.** *Assume $M$ has nonnegative Ricci curvature. Then for any positive measures $\mu, \nu$ on $M$ such that $\mu \leqslant \rho_0 \lambda$ and $\nu \leqslant \rho_1 \lambda$,*

$$\|\mu - \nu\|_{\dot{H}^{-1}} \leqslant \frac{2(\rho_0^{1/2} - \rho_1^{1/2})}{\ln(\rho_0 \,/\, \rho_1)} W_2(\mu, \nu). \tag{17}$$

*(For $\rho_1 = \rho_0$, the right-hand side of* (17) *is to be taken as $\rho_0^{1/2} W_2(\mu, \nu)$ by continuity).*

*Remark* 6. For $M = \mathbf{R}^n$ a similar result was already stated in [Loeper, 2006, Proposition 2.8], with a different proof.

*Proof.* Let $(\mu_t)_{0 \leqslant t \leqslant 1}$ be the displacement interpolation between $\mu$ and $\nu$ (cf. [Villani, 2009, chap. 7]), which is such that $\mu_0 = \mu$, $\mu_1 = \nu$ and the infimum in (6) is attained with $\|d\mu_t\|_{\dot{H}^{-1}(\mu_t)} = W_2(\mu, \nu) dt \ \forall t$. Since Ricci curvature is nonnegative, the Lott–Sturm–Villani theory tells us that, denoting by $\|\mu\|_\infty$ the essential supremum of the density of $\mu$ w.r.t. $\lambda$, one has $\|\mu_t\|_\infty \leqslant \|\mu_0\|_\infty^{1-t} \|\mu_1\|_\infty^t = \rho_0^{1-t} \rho_1^t$ (see [Villani, 2009, Corollary 17.19] or [Cordero-Erausquin et al., 2001, Lemma 6.1]); so that $\|\cdot\|_{\dot{H}^{-1}} \leqslant \rho_0^{(1-t)/2} \rho_1^{t/2} \|\cdot\|_{\dot{H}^{-1}(\mu_t)}$ by Lemma 2.

Then, by the integral triangle inequality for normed vector spaces,

$$\begin{aligned}
\|\mu - \nu\|_{\dot{H}^{-1}} &= \Big\| \int_0^1 d\mu_t \Big\|_{\dot{H}^{-1}} \leqslant \int_0^1 \|d\mu_t\|_{\dot{H}^{-1}} \\
&\leqslant \int_0^1 \rho_0^{(1-t)/2} \rho_1^{t/2} \|d\mu_t\|_{\dot{H}^{-1}(\mu_t)} = \Big( \int_0^1 \rho_0^{(1-t)/2} \rho_1^{t/2} dt \Big) W_2(\mu, \nu) \\
&= \frac{2(\rho_0^{1/2} - \rho_1^{1/2})}{\ln(\rho_0 \,/\, \rho_1)} W_2(\mu, \nu),
\end{aligned} \tag{18}$$

QED. □

*Remark* 7. Taking into account the dimension $n$ of the manifold $M$, the bound on $\|\mu_t\|_\infty$ could be refined into

$$\|\mu_t\|_\infty \leqslant \big( (1-t) \|\mu_0\|_\infty^{-1/n} + t \|\mu_1\|_\infty^{-1/n} \big)^{-n}, \tag{19}$$

which would yield a slightly sharper bound in Equation (17), namely:

$$\begin{aligned}
\|\mu - \nu\|_{\dot{H}^{-1}} &\leqslant \Big( \int_0^1 \big( (1-t) \rho_0^{-1/n} + t \rho_1^{-1/n} \big)^{-n/2} dt \Big) W_2(\mu, \nu) \\
&= \begin{cases} \dfrac{\rho_0^{1/2-1/n} - \rho_1^{1/2-1/n}}{(n/2-1)(\rho_1^{-1/n} - \rho_0^{-1/n})} W_2(\mu, \nu) & n \geqslant 2; \\[2ex] \dfrac{\ln(\rho_1/\rho_0)}{2(\rho_0^{-1/2} - \rho_1^{-1/2})} W_2(\mu, \nu) & n = 2. \end{cases}
\end{aligned} \tag{20}$$

For $n = 1$ it turns out that one can let tend $\rho_1 \to \infty$ in (20) without making the integral diverge; which leads to a much more powerful result:

*Theorem* 8. *When $M$ is an interval of $\mathbf{R}$, then under the sole assumption that $\mu \leqslant \rho_0 \lambda$, one has for all positive measures $\nu$ on $M$:*

$$\|\mu - \nu\|_{\dot{H}^{-1}} \leqslant 2 \rho_0^{1/2} W_2(\mu, \nu). \tag{21}$$

*Remark* 9. For $n \geqslant 2$ there is no hope to get a bound valid for all $\nu$, because then it can occur that $W_2(\mu, \nu) < \infty$ but $\|\mu - \nu\|_{\dot{H}^{-1}} = \infty$: for instance, take $\mu$ to be the uniform measure on the 2-dimensional sphere and $\nu$ a Dirac mass.

# 2 Application to localisation of Wasserstein distance

## 2.1 Introduction

In all this section, we work in the Euclidian space $\mathbf{R}^n$, whose norm is denoted by $|\cdot|$. $dist(x, A) \coloneqq \inf\{|x-y| \mid y \in A\}$ denotes the distance between a point $x$ and a set $A$; $A^{\mathsf{c}}$ denotes the complement of $A$; $\lambda$ denotes the Lebesgue measure. We will use the following notation to handle measures:

- For $\mu$ a measure on $\mathbf{R}^n$ and $f\colon \mathbf{R}^n \to \mathbf{R}^n$ a measurable map, $f_*\mu$ denotes the pushforward of $\mu$ by $f$, that is, $\big(f_*\mu\big)(A) \coloneqq \mu(f^{-1}(A))$.
- For $\mu$ a measure on $\mathbf{R}^n$ and $\varphi\colon \mathbf{R}^n \to \mathbf{R}_+$ a nonnegative measurable function, $\varphi \cdot \mu$ denotes the measure such that $d\big(\varphi\cdot\mu\big)(x) \coloneqq \varphi(x)d\mu(x)$.

We will also use the following norms on measures:

- $\|\mu\|_{\dot{H}^{-1}(\nu)}$ has the same definition as in § 1;
- $\|\mu\|_1 \coloneqq \int_{\mathbf{R}^n} |d\mu(x)|$ is the total variation norm of $\mu$;[‡]
- For $\nu$ a positive measure with $\mu \ll \nu$, we define

$$\|\mu\|_{L^2(\nu)} \coloneqq \left(\int_{\operatorname{supp}\nu} \left(\frac{d\mu}{d\nu}(x)\right)^2 d\nu(x)\right)^{1/2}. \tag{22}$$

For $A \subset \mathbf{R}^n$, we also denote $\|\cdot\|_{L^2(A)}$ for $\|\cdot\|_{L^2(\mathbf{1}_A\cdot\lambda)}$.

The goal of this section is to give an application of Theorem 1 to the problem of *localisation* of the quadratic Wasserstein distance. Morally, the question is the following: take two measures $\mu, \nu$ on $\mathbf{R}^n$ being close to each other in the sense of $W_2$ distance; is it true that $\mu$ and $\nu$ remain close when you consider their restrictions to a subset of $\mathbf{R}^n$? Concretely, if $\varphi$ is a non-negative real function on $\mathbf{R}^n$ with compact support (plus some technical assumptions to be specified later), we want to bound above $W_2(a\varphi\cdot\mu, \varphi\cdot\nu)$ by some multiple of $W_2(\mu,\nu)$—where, in the former expression, $a$ is a constant factor ensuring that $a\varphi\cdot\mu$ and $\varphi\cdot\nu$ have the same mass (for otherwise the distance between $\varphi\cdot\mu$ and $\varphi\cdot\nu$ is generically infinite).

This question, which was my initial motivation for the results of § 1, was asked to me by Xavier Tolsa, who needed such a result for his paper [Tolsa, 2012] on characterizing uniform rectifiability in terms of mass transport. Actually Xavier managed to devise a proof of his own [Tolsa, 2012, Theorem 1.1], but it was quite long (about thirty pages) and involved arguments of multi-scale analysis. With Theorem 1 at hand, however, the reasoning becomes far more direct; moreover we will be able to relax some of the assumptions of Xavier's theorem.

## 2.2 Statement of the theorem

**Theorem 10.** *Let $\mu, \nu$ be (positive) measures on $\mathbf{R}^n$ having the same total mass; let $B$ be a ball of $\mathbf{R}^n$ (whose radius will be denoted by $R$ when needed). Assume that on $B$, the density of $\mu$ w.r.t. the Lebesgue measure is bounded above and below:*

$$\exists\, 0 < m_1 \leqslant m_2 < \infty \quad \forall x \in B \qquad m_1\lambda(dx) \leqslant d\mu(x) \leqslant m_2\lambda(dx). \tag{23}$$

*Let $\varphi\colon \mathbf{R}^n \to \mathbf{R}_+$ be a function such that:*

[‡]Note that in the case $\mu$ is a positive measure on $\mathbf{R}^n$, then $\|\mu\|_1$ is noting but $\mu(\mathbf{R}^n)$.

*(i)* $\varphi$ *is zero outside* $B$*;*

*(ii) There exist* $0 < c_1 \leqslant c_2 < \infty$ *such that for all* $x \in B$*,* $c_1\, dist(x, B^{\mathsf{c}})^2 \leqslant \varphi(x) \leqslant c_2\, dist(x, B^{\mathsf{c}})^2$*.*

*(iii)* $\varphi$ *is* $k$*-Lipschitz for some* $k < \infty$*.*

*Then, denoting* $a \coloneqq \|\varphi \cdot \nu\|_1 \,/\, \|\varphi \cdot \mu\|_1$*,*

$$W_2(a\varphi \cdot \mu, \varphi \cdot \nu) \leqslant C(n) \frac{c_2^{3/2} m_2^{3/2}}{c_1^{3/2} m_1^{3/2}} k c_1^{-1/2} W_2(\mu, \nu), \tag{24}$$

*for* $C(n) < \infty$ *some absolute constant only depending on* $n$*. Moreover, one can bound explicitly* $C(n)$ *in such a way that* $C(n) = O(n^{1/2})$ *when* $n \to \infty$*.*[§]

*Remark* 11. Actually the constraint that the support of $\varphi$ is a ball is of little importance: we could assume as well that it would be a cube, a simplex, or many other shapes, as the corollary below shows:

*Corollary* 12. *Make the same assumptions as in Theorem 10, except that* $B$ *need not be a ball: instead, we only assume that, denoting by* $B_\circ$ *the (true) ball having the same volume as* $B$*, there exists a bijection* $\Phi\colon B \leftrightarrow B_\circ$ *mapping the uniform measure on* $B$ *onto the uniform measure on* $B_\circ$ *(i.e. such that* $\Phi_*(\mathbf{1}_B \cdot \lambda) = \mathbf{1}_{B_\circ} \cdot \lambda$*) such that* $\Phi$ *is bi-Lipschitz (i.e. such that both* $\Phi$ *and* $\Phi^{-1}$ *are Lipschitz). Denote by* $\|\Phi\|_{Lip}$ *and* $\|\Phi^{-1}\|_{Lip}$ *the optimal Lipschitz constants for resp.* $\Phi$ *and* $\Phi^{-1}$*. Then, the conclusion of Theorem 10 remains true, except that now you have to replace the factor* $C(n)$ *by*

$$(\|\Phi\|_{Lip} \|\Phi^{-1}\|_{Lip})^5 C(n). \tag{25}$$

*Proof.* Consider the measures $\mu_\circ \coloneqq \Phi_* \mu$ and $\nu_\circ \coloneqq \Phi_* \nu$, and the bump function $\varphi_\circ \coloneqq \varphi \circ \Phi^{-1}$; then, $\mu_\circ$, $\nu_\circ$ and $\varphi_\circ$ satisfy the original assumptions of Theorem 10, the roles of '$m_1$' and '$m_2$' (in the ball situation) being held by $m_1$ and $m_2$ (in the general situation) themselves, the role of '$k$' being held by $\|\Phi^{-1}\|_{Lip} k$, and the roles of '$c_1$' and '$c_2$' being held by $c_1 \,/\, \|\Phi\|_{Lip}^2$ and $c_2 \|\Phi^{-1}\|_{Lip}^2$. Therefore, applying (24):

$$W_2(a\varphi_\circ \cdot \mu_\circ, \varphi_\circ \cdot \nu_\circ) \leqslant C(n) \|\Phi\|_{Lip}^4 \|\Phi^{-1}\|_{Lip}^4 \frac{c_2^{3/2} m_2^{3/2}}{c_1^{3/2} m_1^{3/2}} W_2(a\mu_\circ, \nu_\circ). \tag{26}$$

But the optimal transportation plan from $a\mu$ to $\nu$, with cost $W_2(\mu, \nu)^2$, can be pushed forward by $\Phi$ into a (not optimal in general) transportation plan from $a\mu_\circ$ to $\nu_\circ$, whose cost will then be $\leqslant \|\Phi\|_{Lip}^2 W_2(\mu, \nu)^2$; so $W_2(a\mu_\circ, \nu_\circ) \leqslant \|\Phi\|_{Lip} W_2(a\mu, \nu)$. Similarly $W_2(a\varphi \cdot \mu, \varphi \cdot \nu) \leqslant \|\Phi^{-1}\|_{Lip} W_2(a\varphi_\circ \cdot \mu_\circ, \varphi_\circ \cdot \nu_\circ)$. The announced result follows. □

### 2.3 Proof of the main theorem

In the sequel we will shorthand $W_2(\mu, \nu) =: w$, and also $\varphi \cdot \mu =: \hat{\mu}$, resp. $\varphi \cdot \nu =: \hat{\nu}$. Let $g =: Id + S$ be a map achieving optimal transportation from $\nu$ to $\mu$, i.e. such that $\mu = g_* \nu$ with $\int_{\mathbf{R}^n} |S(y)|^2 d\nu(y) = w^2$.[¶]

[§] For instance, with the estimates of this article, one finds that $C(n) \coloneqq 47 n^{1/2}$ fits—though this may be strongly suboptimal.

[¶] Actually such an $g$ does not always exist, as it can occur that the optimal transportation plan from $\nu$ to $\mu$ "splits points" if $\nu$ is not regular enough. However it would suffice to use the general

Our strategy will consist in transforming $\hat{\nu}$ into $a\hat{\mu}$ according to the following procedure:

① We apply the transportation plan $g$ to $\hat{\nu}$; this transforms $\hat{\nu}$ into some measure $\hat{\mu}^*$. The measure $\hat{\mu}^*$ is not supported by $B$ *a priori*, so we split it into $\hat{\mu}^*_B + \hat{\mu}^*_{\mathsf{c}} := \mathbf{1}_B \cdot \hat{\mu}^* + \mathbf{1}_{B^{\mathsf{c}}} \cdot \hat{\mu}^*$.

② Denoting $a_{\mathsf{c}} := \|\hat{\mu}^*_{\mathsf{c}}\|_1 \,/\, \|\hat{\mu}\|_1$, we then transform $\hat{\mu}^*_{\mathsf{c}}$ into $a_{\mathsf{c}}\hat{\mu}$ according to an arbitrary transference plan.

③ Finally, denoting $a_B := \|\hat{\mu}^*_B\|_1 \,/\, \|\hat{\mu}\|_1$,[iii] we transform $\hat{\mu}^*_B$ into $a_B\hat{\mu}$ according to the optimal transference plan: the cost of this operation is $W_2(\hat{\mu}^*_B, a_B\hat{\mu})$, which we bound above by $2\|\hat{\mu}^*_B - a_B\hat{\mu}\|_{\dot{H}^{-1}(a_B\hat{\mu})}$ thanks to Theorem 1.

Then, denoting by $W_2(①), W_2(②), W_2(③)$ the respective Wasserstein distances of these steps, we shall have $W_2(\hat{\nu}, a\hat{\mu}) \leqslant W_2(①) + (W_2(②)^2 + W_2(③)^2)^{1/2}$.

Let us begin with bounding the cost of Step ①. The squared cost of this step is

$$W_2(①)^2 = \int |S(y)|^2 d\hat{\nu}(y) = \int |S(y)|^2 \varphi(y) d\nu(y)$$
$$\leqslant \sup \varphi \times \int |S(y)|^2 d\nu(y) = \sup \varphi \times w^2 \leqslant c_2 R^2 w^2, \quad (27)$$

whence $W_2(①) \leqslant c_2^{1/2} R w$.

Now consider Step ②. As $a_{\mathsf{c}}\hat{\mu}$ is supported by $B$, one has obviously

$$W_2(②)^2 \leqslant \int \big(dist(x,B) + 2R\big)^2 d\hat{\mu}^*_{\mathsf{c}}(x) = \int_{B^{\mathsf{c}}} \big(dist(x,B) + 2R\big)^2 d\hat{\mu}^*(x). \quad (28)$$

From that we deduce that $W_2(②) \leqslant 2c_2^{1/2} R w$ by the following computation:

$$\int_{B^{\mathsf{c}}} \big(dist(x,B) + 2R\big)^2 d\hat{\mu}^*(x) = \int_{g(y)\notin B} \big(dist(g(y),B) + 2R\big)^2 \varphi(y) d\nu(y)$$
$$\leqslant c_2 \int_{\substack{y\in B\\ g(y)\notin B}} \big(dist(g(y),B) + 2R\big)^2 \, dist(y, B^{\mathsf{c}})^2 d\nu(y)$$
$$\leqslant c_2 \int_{\substack{y\in B\\ g(y)\notin B}} \big(R\, dist(g(y),B) + 2R\, dist(y, B^{\mathsf{c}})\big)^2 d\nu(y)$$
$$\leqslant 4c_2R^2 \int_{\substack{y\in B\\ g(y)\notin B}} \big(dist(g(y),B) + dist(y, B^{\mathsf{c}})\big)^2 d\nu(y)$$
$$\leqslant 4c_2R^2 \int |y - g(y)|^2 d\nu(y) = 4c_2 R^2 w^2. \quad (29)$$

formalism of transportation plans to handle that case: we do not do it here to keep notation light, but this is straightforward. Also note that it is not obvious that the infimum in (2) is attained: again, that is not a real problem as our proof still works by considering a sequence of transportation plans approaching optimality.

[iii]Observe that $a_B + a_{\mathsf{c}} = a$.

Step ③ is the difficult one. We begin with observing that it is easy to bound the $L^2(B)$ distance between $\hat{\mu}^*_B$ and $\hat{\mu}$: indeed, denoting by $f =: Id + T$ the inverse map of $g$[**],

$$\begin{aligned}\|\hat{\mu}^*_B - \hat{\mu}\|^2_{L^2(\mathbf{1}_B\cdot\mu)} &= \int_B \Big(\frac{d\hat{\mu}^*(x) - \varphi(x)d\mu(x)}{d\mu(x)}\Big)^2 d\mu(x) \\ &= \int_B \big(\varphi(f(x)) - \varphi(x)\big)^2 d\mu(x) \\ &\leqslant k^2 \int_{\mathbf{R}^n} |x - f(x)|^2 d\mu(x) = k^2 \int |T(x)|^2 d\mu(x) = k^2 w^2, \quad (30)\end{aligned}$$

(where we used that $d\hat{\mu}^*(x) = d\hat{\nu}(f(x)) = \varphi(f(x))d\nu(f(x)) = \varphi(f(x))d\mu(x)$), so that

$$\|\hat{\mu}^*_B - \hat{\mu}\|^2_{L^2(B)} \leqslant k^2 m_2 w^2. \quad (31)$$

Now we have to link $\|\cdot\|_{L^2(B)}$ with $\|\cdot\|_{\dot{H}^{-1}(\mu)}$. This is achieved by the following lemma, whose proof is postponed:

**Lemma 13.** *Define $\hat{\lambda}$ to be the measure on $B$ such that $\hat{\lambda}(dx) := dist(x, B^{\mathsf{c}})^2 \lambda(dx)$. Then, for any signed measure $m$ on $B$ having total mass zero:*

$$\|m\|_{\dot{H}^{-1}(\hat{\lambda})} \leqslant C_1(n)^{1/2} \|m\|_{L^2(B)}, \quad (32)$$

*where $C_1(n)$ is some absolute constant only depending on $n$. Moreover, taking $C_1(n) := \big((2e+1)n - 1\big) \vee 8e$ fits.*

Thanks to Theorem 1 and Lemma 13, we have that

$$\begin{aligned}W_2(③) \leqslant 2\|a_B\hat{\mu} - \hat{\mu}^*_B\|_{\dot{H}^{-1}(a_B\hat{\mu})} &\leqslant 2(a_B c_1 m_1)^{-1/2} \|a_B\hat{\mu} - \hat{\mu}^*_B\|_{\dot{H}^{-1}(\hat{\lambda})} \\ &\leqslant 2C_1(n)^{1/2}(a_B c_1 m_1)^{-1/2} \|a_B\hat{\mu} - \hat{\mu}^*_B\|_{L^2(B)}. \quad (33)\end{aligned}$$

Next, we compute

$$\begin{aligned}\|a_B\hat{\mu} - \hat{\mu}^*_B\|_{L^2(B)} &= \Big\| \tfrac{\|\hat{\mu}^*_B\|_1}{\|\hat{\mu}\|_1}\hat{\mu} - \hat{\mu}^*_B \Big\|_{L^2(B)} \leqslant \tfrac{|\|\hat{\mu}^*_B\|_1 - \|\hat{\mu}\|_1|}{\|\hat{\mu}\|_1} \|\hat{\mu}\|_{L^2(B)} + \|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \\ &\leqslant \tfrac{\|\hat{\mu}\|_{L^2(B)}}{\|\hat{\mu}\|_1} \|\hat{\mu}^*_B - \hat{\mu}\|_1 + \|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \leqslant \Big(\tfrac{\|\hat{\mu}\|_{L^2(B)}}{\|\hat{\mu}\|_1}\lambda(B)^{1/2} + 1\Big)\|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \\ &\leqslant \Big(\tfrac{c_2 m_2}{c_1 m_1}\frac{\lambda(B)^{1/2}\|\hat{\lambda}\|_{L^2(B)}}{\|\hat{\lambda}\|_1} + 1\Big)\|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \underset{[\dagger\dagger]}{\leqslant} \big(\sqrt{6}\tfrac{c_2 m_2}{c_1 m_1} + 1\big)\|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \\ &\underset{(31)}{\leqslant} (\sqrt{6} + 1)\tfrac{c_2 m_2}{c_1 m_1} k m_2^{1/2} w, \quad (34)\end{aligned}$$

so that, combining (33) and (34), we have got:

$$W_2(③) \leqslant (2\sqrt{6} + 2)C_1(n)^{1/2} a_B^{-1/2} \frac{c_2 m_2^{3/2}}{c_1 m_1^{3/2}} \frac{k}{c_1^{1/2}} w. \quad (35)$$

[**]For $f$ to exist, $g$ should be bijective, which is not always true *stricto sensu*; but we can safely carry out the reasoning with pretending so, by the same argument as in Footnote [¶] on page 6.

[††]This step comes from the computation $\lambda(B)^{1/2}\|\hat{\lambda}\|_{L^2(B)} \,/\, \|\hat{\lambda}\|_1 = (\int_0^1 r^{n-1}dr)^{1/2} \times \big(\int_0^1 (1-r)^4 r^{n-1}dr\big)^{1/2} \,/\, \big(\int_0^1 (1-r)^2 r^{n-1}dr\big) = \big(6(1+n)(2+n) \,/\, (3+n)(4+n)\big)^{1/2} \leqslant \sqrt{6}\ \forall n.$

Equation (35) is the kind of bound we were looking for, *provided* $a_B \lesssim 1$. Though this will be the case in practice (since we are mainly interested in cases where $\nu$ is close to $\mu$ and thus $\hat{\mu}^*$ is close to $\hat{\mu}$), this is not quite satisfactory yet. So, what can we do when $a_B \ll 1$, that is, when $\|\hat{\mu}^*_B\|_1 \ll \|\hat{\mu}\|_1$? In fact that case is easier, because transportation between small measures has low cost, while $w$ has to be large to make $\hat{\mu}^*_B$ very different from $\hat{\mu}$.

The computations are the following. First, it is obvious that

$$W_2(③) = W_2(\hat{\mu}^*_B, a_B\hat{\mu}) \leqslant 2R\|\hat{\mu}^*_B\|_1^{1/2}. \tag{36}$$

Next, observing that $\varphi(f(x)) \geqslant \frac{c_1}{c_2}\varphi(x) - 2c_1\, \mathit{dist}(x, B^{\mathsf{c}})|T(x)|$,[‡‡] we compute that

$$\begin{aligned}\|\hat{\mu}^*_B\|_1 = \int_B \varphi(f(x))d\mu(x) &\geqslant \int_B \Big(\tfrac{c_1}{c_2}\varphi(x) - 2c_1\, \mathit{dist}(x, B^{\mathsf{c}})|T(x)|\Big)d\mu(x) \\ &\geqslant \tfrac{c_1}{c_2}\|\hat{\mu}\|_1 - 2c_1\Big(\int_B \mathit{dist}(x,B^{\mathsf{c}})^2 d\mu(x)\Big)^{1/2}\Big(\int_B |T(x)|^2 d\mu(x)\Big)^{1/2} \\ &= \tfrac{c_1}{c_2}\|\hat{\mu}\|_1 - 2c_1\|\mathit{dist}(\cdot, B^{\mathsf{c}})^2\cdot\mu\|_1^{1/2} w \geqslant \tfrac{c_1}{c_2}\|\hat{\mu}\|_1 - 2c_1 m_2^{1/2}\|\hat{\lambda}\|_1^{1/2} w,\end{aligned} \tag{38}$$

whence

$$w \geqslant \frac{\big(\frac{c_1}{c_2}\|\hat{\mu}\|_1 - \|\hat{\mu}^*_B\|_1\big)_+}{2c_1 m_2^{1/2}\|\hat{\lambda}\|_1^{1/2}} = \frac{\big(\frac{c_1}{c_2} - a_B\big)_+\|\hat{\mu}\|_1}{2c_1 m_2^{1/2}\|\hat{\lambda}\|_1^{1/2}} \geqslant \frac{m_1^{1/2}}{2c_1 m_2^{1/2}}\big(\tfrac{c_1}{c_2} - a_B\big)_+\|\hat{\mu}\|_1^{1/2}. \tag{39}$$

So,

$$W_2(③) \leqslant 2R\|\hat{\mu}^*_B\|_1^{1/2} = 2Ra_B^{1/2}\|\hat{\mu}\|_1^{1/2} \leqslant 4Rc_1^{1/2}\frac{m_2^{1/2}}{m_1^{1/2}}\frac{a_B^{1/2}}{(\frac{c_1}{c_2} - a_B)_+}w. \tag{40}$$

In the end, choosing either (35) if $a_B \geqslant c_1 \,/\, 2c_2$ or (40) if $c_1 \,/\, 2c_2$, and observing that $c_1 \leqslant kR^{-1}$, one has always:

$$W_2(③) \leqslant \big((4\sqrt{3} + 2\sqrt{2})C_1(n)^{1/2} \vee 4\sqrt{2}\big)\frac{c_2^{3/2}m_2^{3/2}}{c_1^{3/2}m_1^{3/2}}\frac{k}{c_1^{1/2}}w. \quad \square \tag{41}$$

*Remark* 14. To bound $W_2(③)$ in the situation where $a_B \ll 1$, we could also have started from "$\varphi(f(x)) \geqslant \varphi(x) - k|T(x)|$" (instead of "$\varphi(f(x)) \geqslant \frac{c_1}{c_2}\varphi(x) - 2c_1\, \mathit{dist}(x, B^{\mathsf{c}})|T(x)|$") to get another bound analogous to (38). Following such an approach, the factor $(c_2 \,/\, c_1)^{3/2}$ in (40) would be improved into $(c_2 \,/\, c_1)$ in the analogous formula; however the dimensional factor would behave in $O(n)$ rather than in $O(n^{1/2})$.

## 2.4 Proof of Lemma 13

It still remains to prove Lemma 13, whose statement we recall to be:

[‡‡]This follows from the computation:

$$\begin{aligned}\varphi(f(x)) \geqslant c_1\, \mathit{dist}(f(x), B^{\mathsf{c}})^2 &\geqslant c_1\big(\mathit{dist}(x, B^{\mathsf{c}}) - |T(x)|\big)_+^2 \\ &\geqslant c_1\, \mathit{dist}(x, B^{\mathsf{c}})^2 - 2c_1\, \mathit{dist}(x, B^{\mathsf{c}})|T(x)| \geqslant \frac{c_1}{c_2}\varphi(x) - 2c_1\, \mathit{dist}(x, B^{\mathsf{c}})|T(x)|.\end{aligned} \tag{37}$$

**Lemma.** *Denoting $\hat{\lambda} \coloneqq dist(\cdot, B^c)^2 \cdot \lambda$, one has, for any signed measure $m$ on $B$ having total mass zero:*

$$\|m\|_{\dot{H}^{-1}(\hat{\lambda})} \leqslant \big(\big((2e+1)n-1\big) \vee 8e\big)^{1/2} \|m\|_{L^2(B)}. \tag{42}$$

*—In the sequel, "$\big((2e+1)n-1\big) \vee 8e$" will be shorthanded into "$C_1(n)$".*

*Remark* 15. The bound (42) is within a constant factor of being optimal, uniformly in $n$, as one sees by $f$ in (45) to be linear.

*Proof of the lemma.* We begin with translating the lemma into a functional analysis statement by a duality argument. Recall the duality definition of $\|m\|_{\dot{H}^{-1}(\hat{\lambda})}$ from § 1:

$$\|m\|_{\dot{H}^{-1}(\hat{\lambda})} \coloneqq \sup\{|\langle f, m\rangle| \mid \|f\|_{\dot{H}^1(\hat{\lambda})} \leqslant 1\}. \tag{43}$$

There is a similar duality formula for $\|m\|_{L^2(B)}$:

$$\|m\|_{L^2(B)} = \sup\{|\langle f, m\rangle| \mid \|f\|_{L^2(B)} \leqslant 1\}, \tag{44}$$

where, for $f$ a *function*, $\|f\|_{L^2(B)}$ has its usual meaning, namely $\|f\|_{L^2(B)} \coloneqq \big(\int_B f(x)^2 d\lambda(x)\big)^{1/2}$. Since $m$ is assumed to have total mass zero, $|\langle f, m\rangle|$ does not change when one adds a constant to $f$. On the other hand, when $f$ describes the set $\{\|f_0 + a\| \mid a \in \mathbf{R}\}$, $\|f\|_{L^2(B)}$ is minimal when $a$ is such that $f$ has zero mean on $B$, while the value of $\|f\|_{\dot{H}^1(\hat{\lambda})}$ remains constant.[*] As a consequence, we can restrict the supremum in (43) and (44) to those $f$ *having zero mean on* $B$. Thus, the lemma will be implied[†] by proving that

$$\langle f, \mathbf{1}_B \cdot \lambda\rangle = 0 \quad \Rightarrow \quad \|f\|_{L^2(B)} \leqslant C_1(n)^{1/2} \|f\|_{\dot{H}^1(\hat{\lambda})}. \tag{45}$$

Going back to the definitions of $\|\cdot\|_{\dot{H}^{-1}(\hat{\lambda})}$ and $\|\cdot\|_{L^2(B)}$, relaxing the condition on $f$ to be centred by projecting it orthogonally in $L^2(B)$ onto the subspace of centred functions, and denoting by $P$ the uniform probability measure on $B$, Equation (45) turns into:

$$\forall f \qquad \mathrm{Var}_P(f) \leqslant C_1(n) \int dist(x, B^c)^2 |\nabla f(x)|^2 dP(x), \tag{46}$$

which we recognize to be a weighted Poincaré inequality.

To prove (46), the first key idea (inspired by [Bobkov, 2003]) is to separate radial and spherical coordinates. This is, considering the bijection

$$\begin{aligned} \varphi \colon (0, R) \times \mathbb{S}^{n-1} &\to B \smallsetminus \{0\} \\ (r, \theta) &\mapsto r\theta \end{aligned} \tag{47}$$

(the origin of space being set at the center of $B$), we introduce the measure $\tilde{P} \coloneqq {\varphi^{-1}}_* P$, which is obviously the product measure $\tilde{P}_r \otimes \tilde{P}_\theta$, where $\tilde{P}_r$ is the probability measure on $(0, R)$ such that $d\tilde{P}_r(r) \coloneqq nR^{-n} r^{n-1} dr$, resp. $\tilde{P}_\theta$ is the uniform measure

[*] Here we implicitly assume that $\int_B |f(x)| d\lambda(x) < \infty$, which is legit since an approximation argument allows to restrict the suprema in (43) and (44) to those $f$ having a $\mathcal{C}^\infty$ continuation on $\mathrm{cl}(B)$.

[†] Actually there is even equivalence.

on the sphere $\mathbb{S}^{n-1}$. With this notation, we perform can a change of variables to see that (46) is equivalent to proving that, for all $g \in L^2(\tilde{P})$:

$$C_1(n)^{-1} \operatorname{Var}_{\tilde{P}}(g) \leqslant \int_0^R \int_{\mathbb{S}^{n-1}} (R-r)^2 \big(|\nabla_r g(r,\theta)|^2 + r^{-2}|\nabla_\theta g(r,\theta)|^2\big) d\tilde{P}_r(r) d\tilde{P}_\theta(\theta), \tag{48}$$

where $\nabla_r$ and $\nabla_\theta$ denote the gradient along resp. the $r$ coordinate and the $\theta$ coordinate.[‡] We will denote the right-hand side of (48) by $\mathcal{E}(g,g)$.

Because $\tilde{P} = \tilde{P}_r \otimes \tilde{P}_\theta$, we know that $L^2(\tilde{P})$ can be seen as (the closure of) the tensor product of $L^2(\tilde{P}_r)$ and $L^2(\tilde{P}_\theta)$:

$$L^2(\tilde{P}) = \operatorname{cl}(L^2(\tilde{P}_r) \overset{\perp}{\otimes} L^2(\tilde{P}_\theta)), \tag{49}$$

where the symbol '$\overset{\perp}{\otimes}$' means that the Hilbertian structure of $L^2(\tilde{P})$ is compatible with the Hilbertian structures of $L^2(\tilde{P}_r)$ and $L^2(\tilde{P}_\theta)$—i.e., that $\langle h_{\mathrm{a}} \otimes u_{\mathrm{a}}, h_{\mathrm{b}} \otimes u_{\mathrm{b}} \rangle_{L^2(\tilde{P})} = \langle h_{\mathrm{a}}, h_{\mathrm{b}} \rangle_{L^2(\tilde{P}_r)} \times \langle u_{\mathrm{a}}, u_{\mathrm{b}} \rangle_{L^2(\tilde{P}_\theta)}$. Now consider the spherical harmonics $Y_0, Y_1, \ldots$, which by definition are an orthonormal basis, in $L^2(\tilde{P}_\theta)$, of eigenfunctions of the Laplace–Beltrami operator $\Delta$ on $\mathbb{S}^{n-1}$; and call $\ell_0, \ell_1, \ldots$ the associated eigenvalues, which are known to be such that (up to permuting indices) $Y_0 \equiv 1$ with $\ell_0 = 0$, and $\ell_i \leqslant -(n-1)$ $\forall i \neq 0$ (see for instance [Seeley, 1966]). By construction, $L^2(\tilde{P}_\theta) = \operatorname{cl}\big(\overset{\perp}{\bigoplus}_{i\in\mathbf{N}} (\mathbf{R} \cdot Y_i)\big)$; therefore, one has that

$$L^2(\tilde{P}) = \operatorname{cl}\Big(\overset{\perp}{\bigoplus_{i\in\mathbf{N}}} L^2(\tilde{P}_r) \cdot Y_i\Big) : \tag{50}$$

in other words, the functions of $L^2(\tilde{P})$ are those of the form

$$g(r,\theta) = \sum_{i\in\mathbf{N}} h_i(r) Y_i(\theta), \tag{51}$$

with $\sum_i \|h_i\|^2_{L^2(\tilde{P}_r)} < \infty$, and the correspondence is bijective. An interesting point is that, then, one has:

$$\operatorname{Var}_{\tilde{P}}(g) = \operatorname{Var}_{\tilde{P}_r}(h_0) + \sum_{i\neq 0} \|h_i\|^2_{L^2(\tilde{P}_r)}. \tag{52}$$

On the other hand, one has

$$\mathcal{E}(g,g) = -\langle Lg, g \rangle_{L^2(\tilde{P})}, \tag{53}$$

where

$$\big(Lg\big)(r,\theta) := (R-r)^2 \Delta_r g + \bigg((n-1)\frac{(R-r)^2}{r} - 2(R-r)\bigg) \mathbf{e}_r \cdot \nabla_r g + \frac{(R-r)^2}{r^2} \Delta_\theta g. \tag{54}$$

From (54) we see that, since the $Y_i$ are eigenfunctions of $\Delta_\theta$, all the $L^2(\tilde{P}_r) \cdot Y_i$ are invariant by $L$, and that one has:

$$\mathcal{E}(g,g) = \sum_{i\in\mathbf{N}} \int_0^R \bigg((R-r)^2 |\nabla h_i(r)|^2 - \ell_i \frac{(R-r)^2}{r^2} h_i(r)^2\bigg) \tilde{P}_r(dr). \tag{55}$$

[‡]In the latter case, we have to use the Riemannian definition of the gradient on $\mathbb{S}^{n-1}$.

So, proving (48) becomes equivalent to proving that both following formulas hold for all $h \in L^2(\tilde{P}_r)$:

$$\mathrm{Var}_{\tilde{P}_r}(h) \leqslant C_1(n) \int_0^R (R-r)^2 |\nabla h(r)|^2 \tilde{P}_r(dr); \tag{56}$$

$$\|h\|^2_{L^2(\tilde{P}_r)} \leqslant C_1(n) \int_0^R \Big( (R-r)^2 |\nabla h(r)|^2 + (n-1)\frac{(R-r)^2}{r^2} h(r)^2 \Big) \tilde{P}_r(dr). \tag{57}$$

Let us start with (56). In all the sequel of the proof, we introduce

$$b := 1 - n^{-1}. \tag{58}$$

By the Cauchy–Schwarz inequality, one has, for all $r \in (bR, R)$:

$$\begin{aligned}\big(h(r) - h(bR)\big)^2 = \Big(\int_{bR}^r h'(s)ds\Big)^2 &\leqslant \Big(\int_{bR}^r (R-s)^{-3/2} ds\Big) \times \int_{bR}^r (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &\leqslant 2\big((R-r)^{-1/2} - (R-bR)^{-1/2}\big) \int_{bR}^r (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &\leqslant 2(R-r)^{-1/2} \int_{bR}^r (R-s)^{3/2} |\nabla h(s)|^2 ds. \end{aligned} \tag{59}$$

Integrating and using Fubini's formula, it follows that

$$\begin{aligned} &\int_{bR}^R \big(h(r) - h(bR)\big)^2 d\tilde{P}_r(r) \leqslant \\ &\qquad 2 \int_{s=bR}^R \Big( \int_{r=s}^R nR^{-n} (R-r)^{-1/2} r^{n-1} dr \Big) (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &\qquad \leqslant 2 \int_{s=bR}^R \Big( \int_{r=s}^R nR^{-n} (b^{-1}s)^{n-1} (R-r)^{-1/2} dr \Big) (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &\qquad = 2b^{-(n-1)} \int_{s=bR}^R \Big( \int_{r=s}^R (R-r)^{-1/2} dr \Big) (R-s)^{3/2} |\nabla h(s)|^2 d\tilde{P}_r(s) \\ &\qquad = 4b^{-(n-1)} \int_{s=bR}^R (R-s)^2 |\nabla h(s)|^2 ds. \end{aligned} \tag{60}$$

One can apply the same line of reasoning for $r \in (0, bR)$: the (unweighted this time) Cauchy–Schwarz inequality then yields $\big(h(r) - h(bR)\big)^2 \leqslant (bR - r) \times \int_r^{bR} |\nabla h(s)|^2 ds$, whence:

$$\begin{aligned} \int_0^{bR} \big(h(r) - h(bR)\big)^2 d\tilde{P}_r(r) &\leqslant \int_{s=0}^{bR} \Big( \int_{r=0}^s nR^{-n} (bR - r) r^{n-1} dr \Big) |\nabla h(s)|^2 ds \\ &\leqslant bR^{-(n-1)} \int_{s=0}^{bR} \Big( \int_{r=0}^s n r^{n-1} dr \Big) |\nabla h(s)|^2 ds = bR \int_0^{bR} |\nabla h(s)|^2 s^n ds \\ \leqslant bn^{-1} R^2 \int_0^{bR} |\nabla h(s)|^2 d\tilde{P}_r(s) &\leqslant b(1-b)^{-2} n^{-1} \int_0^{bR} (R-s)^2 |\nabla h(s)|^2 d\tilde{P}_r(s). \end{aligned} \tag{61}$$

Summing (60) and (61), we get that

$$\int_0^R \big(h(r) - h(bR)\big)^2 d\tilde{P}_r(r) \leqslant \big(4b^{-(n-1)} \vee b(1-b)^{-2} n^{-1}\big) \int_0^s (R-s)^2 |\nabla h(s)|^2 d\tilde{P}_r(s), \tag{62}$$

where $\big(4b^{-(n-1)} \vee b(1-b)^{-2}n^{-1}\big)$ can itself be bounded by $\big((n-1) \vee 4e\big)$. The left-hand-side of (62) being an upper bound for $\mathrm{Var}_{\tilde{P}_r}(h)$, this proves (56).

Now we turn to (57). For $r \in (bR, R)$ we have, similarly to (59), that

$$\big(h(r) - h(br)\big)^2 \leqslant 2(R-r)^{-1/2} \int_{br}^{r} (R-s)^{3/2} |\nabla h(s)|^2 ds, \tag{63}$$

so that

$$h(r)^2 \leqslant 2h(br)^2 + 4(R-r)^{-1/2} \int_{br}^{r} (R-s)^{3/2} |\nabla h(s)|^2 ds. \tag{64}$$

Then, integrating and applying Fubini's formula:

$$\int_{bR}^{R} h(r)^2 d\tilde{P}_r(r) \leqslant 2 \int_{bR}^{R} h(br)^2 d\tilde{P}_r(r) + \\ 4 \int_{s=b^2R}^{R} \Big( \int_{r=s\vee bR}^{b^{-1}s \wedge R} nR^{-n} r^{n-1} (R-r)^{-1/2} dr \Big) (R-s)^{3/2} |\nabla h(s)|^2 ds. \tag{65}$$

By change of variables, the first term of the right-hand side of (65) is equal to $2b^{-n} \int_{b^2R}^{bR} h(s)^2 d\tilde{P}_r(s)$, which we can bound by

$$2b^{-(n-2)} \frac{(1-b)^{-2}}{n-1} \int_{b^2R}^{bR} (n-1) \frac{(R-r)^2}{r^2} h(s)^2 d\tilde{P}_r(s) \\ \leqslant 2ne \int_0^R (n-1) \frac{(R-r)^2}{r^2} h(s)^2 d\tilde{P}_r(s). \tag{66}$$

The second term of the right-hand side of (65) is itself bounded by

$$4b^{-(n-1)} \int_{s=b^2R}^{R} \Big( \int_{r=s}^{R} (R-r)^{-1/2} dr \Big) (R-s)^{3/2} |\nabla h(s)|^2 d\tilde{P}_r(s) \\ \leqslant 8e \int_0^R (R-s)^2 |\nabla h(s)|^2 d\tilde{P}_r(s). \tag{67}$$

This way, we have bounded $\int_{bR}^{R} h(r)^2 d\tilde{P}_r(r)$.

On the other hand, it is trivial that, for $r \leqslant bR$,

$$h(r)^2 \leqslant \frac{b^2}{(n-1)(1-b)^2} \times (n-1) \frac{(R-r)^2}{r^2} h(r)^2, \tag{68}$$

whence:

$$\int_0^{bR} h(r)^2 d\tilde{P}_r(r) \leqslant (n-1) \int_0^R (n-1) \frac{(R-r)^2}{r^2} h(r)^2 d\tilde{P}_r(r). \tag{69}$$

Combining (66), (67) and (69), we finally get the wanted bound (57). □

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