SHARP EQUIVALENCY BETWEEN ρ- AND τ-MIXING COEFFICIENTS

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Abstract. For $\mathcal{A}$ and $\mathcal{B}$ two σ-algebras, the ρ-mixing coefficient $\rho(\mathcal{A},\mathcal{B})$ between $\mathcal{A}$ and $\mathcal{B}$ is the supremum correlation between two real random variables X and Y being resp. $\mathcal{A}$- and $\mathcal{B}$-measurable; the $\tau'(\mathcal{A},\mathcal{B})$ coefficient is defined similarly, but restricting to the case where X and Y are indicator functions. It has been known for long that the bound $\rho \leqslant C\tau'(1+|\log\tau'|)$ holds for some constant C; in this article, I show that $C=1$ fits and that that value cannot be improved.

1. Introduction

In this article, we consider two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$, σ-algebras whose correlation level we aim at quantifying. A classical definition for such correlation quantification is the *ρ-mixing coefficient* (a.k.a. “maximal correlation coefficient”):

$$\rho(\mathcal{A},\mathcal{B}) := \sup_{\substack{X\in L^2(\mathcal{A})\\ Y\in L^2(\mathcal{B})}} \frac{|\mathrm{Cov}(X,Y)|}{\mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}} \tag{1}$$

(where the supremum is taken only for non-constant X and Y). This coefficient is 0 if and only if $\mathcal{A}$ and $\mathcal{B}$ are independent; and we will say that $\mathcal{A}$ and $\mathcal{B}$ are as correlated (in the ρ-mixing sense) as $\rho(\mathcal{A},\mathcal{B})$ is large. Note that one always has $\rho(\mathcal{A},\mathcal{B})\leqslant 1$, because of the Cauchy–Schwarz inequality.

There are other ways to measure dependence between $\mathcal{A}$ and $\mathcal{B}$ (see for instance the review paper [2]): in particular, rather than looking at correlation between $\mathcal{A}$- and $\mathcal{B}$-measurable *random variables*, we can look at correlation between *events*. The most classical measure of dependence in this category is the *α-mixing coefficient*:

$$\alpha(\mathcal{A},\mathcal{B}) := \sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}} |\mathbb{P}(A\cap B)-\mathbb{P}(A)\,\mathbb{P}(B)|. \tag{2}$$

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Still in the same category, the τ-*mixing coefficient* is useful to catch strong correlation between small probability events:

$$\tau(\mathcal{A},\mathcal{B}) := \sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}} \frac{|\mathbb{P}(A\cap B)-\mathbb{P}(A)\,\mathbb{P}(B)|}{\mathbb{P}(A)^{1/2}\,\mathbb{P}(B)^{1/2}}. \tag{3}$$

In this article, we will rather consider a variant of τ:

$$\tau'(\mathcal{A},\mathcal{B}) := \sup_{\substack{A\in\mathcal{A}\\ B\in\mathcal{B}}} \frac{|\mathbb{P}(A\cap B)-\mathbb{P}(A)\,\mathbb{P}(B)|}{\mathbb{P}(A)^{1/2}(1-\mathbb{P}(A))^{1/2}\,\mathbb{P}(B)^{1/2}(1-\mathbb{P}(B))^{1/2}}. \tag{4}$$

The τ'-mixing coefficient is essentially the same as τ, as one has for all σ-algebras $\mathcal{A},\mathcal{B}$ that

$$\tau(\mathcal{A},\mathcal{B}) \leqslant \tau'(\mathcal{A},\mathcal{B}) \leqslant 2\tau(\mathcal{A},\mathcal{B}) : \tag{5}$$

indeed, on the one hand $\tau' \geqslant \tau$ is obvious, and on the other hand it can always be assumed that $\mathbb{P}(A),\mathbb{P}(B) \leqslant 1/2$ in (4), since the right-hand side of (4) remains unchanged when A or B is replaced by its respective complement set.

But, (4) is the same definition as (1), except that one restricts to the case when the r.v. X and Y are indicator functions; so one always has $\tau'(\mathcal{A},\mathcal{B}) \leqslant \rho(\mathcal{A},\mathcal{B})$. Then, it is a natural question whether some kind of converse link between τ' and ρ also holds, i.e., can one find some non-trivial[1] bound on ρ as a function of τ'? (or equivalently of τ). That question was answered positively by Bradley [1] in 1983.

The next question is, what is the best bound for ρ as a function of τ' that one can get? Bradley & Bryc [3, Theorem 1.1-(ii)] (and independently Bulinskiĭ [5]) showed that one always has

$$\rho \leqslant C\tau'(1+|\log\tau'|) \tag{6}$$

for some constant C; and Bradley, Bryc and Janson [4, Theorem 3.1] showed that the shape of that bound was sharp, i.e. that essentially nothing can be improved in (6) but the value of C. However, the optimal value of C remained unknown…

In this article I will show that $C = 1$ fits (Theorem 3.1) and that the corresponding bound is optimal (Theorem 4.1).

2. A first result

In this section we are going to prove a first result on bounding the ρ-mixing coefficient thanks to some condition on events. This result, in addition to having its own interest, is also interesting for its proof, which shall

[1] By "non-trivial", I mean that the bound on ρ would tend to 0 as τ' tends to 0.

involve some ideas which we will re-use in the proof of our main theorem (namely Theorem 3.1).

To state our result, we first need to define a certain Sobolev space:

Definition 2.1. For $f\colon (0,1) \to \mathbb{R}$ a $\mathcal{C}^1$ function with compact support —which we shall denote by "$f \in \mathcal{C}_c^1((0,1))$"—, one defines

$$\|f\|_{\dot{H}_0^1((0,1))} := \Big(\int_0^1 |f'(x)|^2 \, dx \Big)^{1/2}. \tag{7}$$

Equation (7) defines a norm on the set $\mathcal{C}_c^1((0,1))$; the completion of this set for that norm is denoted by $\dot{H}_0^1((0,1))$ —or merely H in the sequel, as there shall be no ambiguity.

Some non-differentiable functions can nevertheless be seen as elements of H: in particular, if f is a continuous function defined on $[0,1]$ with $f(0), f(1) = 0$ and if f is $\mathcal{C}^1$ at all points but a finite number, then Equation (7) remains valid, and f is in H if and only if $\int_0^1 |f'(x)|^2 \, dx < \infty$. Conversely, the Sobolev embedding theorem asserts that any element of H can be seen as a continuous function defined on $[0,1]$ and being zero at 0 and 1.

The main result of this section is the following one:

Theorem 2.2. *Take $f, g \in H$. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras such that, for all $A \in \mathcal{A}, B \in \mathcal{B}$:*

$$\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B) \leqslant f(\mathbb{P}(A)) g(\mathbb{P}(B)). \tag{8}$$

Then,

$$\rho(\mathcal{A}, \mathcal{B}) \leqslant \|f\|_H \|g\|_H. \tag{9}$$

Remark 2.3. Note that you need not put an absolute value in the left-hand side of (8).

Before proving that theorem, let us give a particular case of it:

Corollary 2.4. *For $p, q > 1/2$, define*

$$\alpha_{p,q}(\mathcal{A}, \mathcal{B}) := \sup_{\substack{A \in \mathcal{A} \\ B \in \mathcal{B}}} \frac{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B)|}{\mathbb{P}(A)^p\,\mathbb{P}(B)^q}. \tag{10}$$

Then,

$$\rho(\mathcal{A}, \mathcal{B}) \leqslant \frac{2^{2-p-q} pq}{(2p-1)^{1/2}(2q-1)^{1/2}} \alpha_{p,q}(\mathcal{A}, \mathcal{B}). \tag{11}$$

Proof of the corollary. Since the numerator of the right-hand side of (10) remains unchanged when A or B is replaced by its respective complement set, the hypotheses of Theorem 2.2 are satisfied with $f(x) = \alpha_{p,q}(\mathcal{A}, \mathcal{B}) \times$

$(x^p \wedge (1-x)^p)$ and $g(y) = y^q \wedge (1-y)^q$. Then the conclusion follows from the computation of $\|f\|_H$ and $\|g\|_H$. □

Proof of Theorem 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras satisfying the assumption (8). Since our goal is to bound above $\rho(\mathcal{A}, \mathcal{B})$, let us consider two L^2 real r.v. X and Y being resp. $\mathcal{A}$- and $\mathcal{B}$-measurable; and let us try to bound $|\mathrm{Cov}(X,Y)|$ by some multiple of $\mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}$. Actually we will only bound above $\mathrm{Cov}(X,Y)$, since then $-\mathrm{Cov}(X,Y)$ will also be bounded above *via* writing it as $\mathrm{Cov}(X,-Y)$.

In order to write the covariance as a function of probabilities of events, we need the following lemma, known as the *Hoeffding identity*:

Lemma 2.5 (Hoeffding [6])**.** *Let X, Y be two L^2 real r.v. defined on the same probability space. Then,*

(12)
$$\mathrm{Cov}(X,Y) = \int_{\mathbb{R}\times\mathbb{R}} \big(\mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) - \mathbb{P}(X \leqslant x)\,\mathbb{P}(Y \leqslant y)\big)\,dxdy.$$

We will give a quick proof of the Hoeffding identity here for the sake of completeness:

Proof of the Hoeffding identity. Up to using a standard approximation argument, we can assume that X and Y are bounded above. Since both members of Equation (12) remain unchanged when a constant is added to X or to Y, we can even assume that X and Y only take nonpositive values, so that the integral in the right-hand side of (12) may actually be taken over $\mathbb{R}_- \times \mathbb{R}_-$.

Now, we start from the formula expressing covariance from expectations:

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y). \tag{13}$$

But, for a nonpositive r.v. X, one has the classical relation:

$$\mathbb{E}(X) = -\int_{\mathbb{R}_-} \mathbb{P}(X \leqslant x)\,dx, \tag{14}$$

which you prove by writing $X(\omega) = -\int_{\mathbb{R}_-} \mathbf{1}_{X(\omega)\leqslant x}\,dx$ and then applying Fubini's theorem. With a similar argument, one has that for nonpositive X and Y:

$$\mathbb{E}(XY) = \int_{\mathbb{R}_-\times\mathbb{R}_-} \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y)\,dxdy. \tag{15}$$

Then, (13) turns into:

(16) $$\operatorname{Cov}(X,Y) = \int_{\mathbb{R}_-\times\mathbb{R}_-} \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y)\,dxdy - \int_{\mathbb{R}_-} \mathbb{P}(X \leqslant x)\,dx \int_{\mathbb{R}_-} \mathbb{P}(Y \leqslant y)\,dy$$
$$= \int_{\mathbb{R}_-\times\mathbb{R}_-} \big(\mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) - \mathbb{P}(X \leqslant x)\,\mathbb{P}(Y \leqslant y)\big)\,dxdy.$$

□

Now we go back to proving Theorem 2.2. In our case, the hypothesis (8) yields:

(17) $$\operatorname{Cov}(X,Y) \leqslant \int_{\mathbb{R}\times\mathbb{R}} f\big(\mathbb{P}(X \leqslant x)\big) g\big(\mathbb{P}(Y \leqslant y)\big)\,dxdy$$
$$= \int_{\mathbb{R}} f\big(\mathbb{P}(X \leqslant x)\big)\,dx \int_{\mathbb{R}} g\big(\mathbb{P}(Y \leqslant y)\big)\,dy.$$

Thus, to prove the theorem, it suffices to show that for all r.v. $X \in L^2(\mathbb{P})$:

(18) $$\int_{\mathbb{R}} f\big(\mathbb{P}(X \leqslant x)\big)\,dx \leqslant \|f\|_H \operatorname{Var}(X)^{1/2}.$$

So, let us consider such a random variable. For $p \in (0,1)$, denote

(19) $$\chi(p) := \inf\{x \in \mathbb{R} : \ \mathbb{P}(X \leqslant x) \geqslant p\}.$$

Up to using a perturbation argument, we can assume that χ is strictly increasing, so that $\mathbb{P}(X \leqslant \chi(p)) = p$ for all $p \in (0,1)$, and also that $\chi \in \mathcal{C}^1([0,1])$.[2] Then we can perform the change of variables $x = \chi(p)$, getting:

(20) $$\int_{\mathbb{R}} f\big(\mathbb{P}(X \leqslant x)\big)\,dx = \int_0^1 f(p)\chi'(p)\,dp.$$

(This change of variables is legal here because $f(0), f(1) = 0$, so that you can actually take the integral in the left-hand side of (18) over the interval $[\chi(0), \chi(1)]$).

$\operatorname{Var}(X)$ can also be expressed as a function of χ'. Indeed, applying Equation (12) to the case $Y = X$, one finds:

(21)
$$\operatorname{Var}(X) = \int_{\mathbb{R}\times\mathbb{R}} \big(\mathbb{P}(X \leqslant x_1) \wedge \mathbb{P}(X \leqslant x_2) - \mathbb{P}(X \leqslant x_1)\,\mathbb{P}(X \leqslant x_2)\big)\,dx_1dx_2$$
$$= \int_{(0,1)^2} (p_1 \wedge p_2 - p_1p_2)\chi'(p_1)\chi'(p_2)\,dp_1dp_2.$$

[2] When we write χ as a function over the *closed* interval $[0,1]$, the values of $\chi(0)$ and $\chi(1)$ are taken by continuous continuation.

In the end, our goal has become the following one: to show that for all $f \in H$, $\varphi \in \mathcal{C}([0,1])$, one has

$$\int_0^1 f(p)\varphi(p)\,dp \leqslant \|f\|_H \Big(\int_{(0,1)^2} (p_1 \wedge p_2 - p_1 p_2) \varphi(p_1)\varphi(p_2)\,dp_1 dp_2 \Big)^{1/2}. \tag{22}$$

Note that, by a density argument, it will be enough to show (22) only for $f \in \mathcal{C}_c^2((0,1))$.

Now we have to deal a bit with bilinear forms. Denote, for $f, g \in \mathcal{C}([0,1])$,

$$\langle f, g \rangle_{L^2} := \int_{(0,1)} f(p) g(p)\,dp \tag{23}$$

and

$$\langle f, g \rangle_V := \int_{(0,1)^2} (p_1 \wedge p_2 - p_1 p_2) f(p_1) g(p_2)\,dp_1 dp_2, \tag{24}$$

so that (22) may be written as

$$\langle f, \varphi \rangle_{L^2} \leqslant \|f\|_H \langle \varphi, \varphi \rangle_V^{1/2}. \tag{25}$$

When φ is of the form χ', $\langle \varphi, \varphi \rangle_V$ is nonnegative since it corresponds to $\mathrm{Var}(X)$ in (21); and for general φ, drawing our inspiration from the formula

$$\mathrm{Var}(X) = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}} (x_1 - x_2)^2\, d\mathbb{P}(X = x_1) d\mathbb{P}(X = x_2), \tag{26}$$

we find that

$$\langle \varphi, \varphi \rangle_V = \int_{p_1 < p_2} \Big(\int_{p_1}^{p_2} \varphi(q)\,dq \Big)^2 dp_1 dp_2 \geqslant 0, \tag{27}$$

which shows that $\langle \bullet, \bullet \rangle_V$ is a scalar product indeed.

Having this scalar product property at hand suggests us to use the Cauchy–Schwarz inequality to show (25). We define

$$\begin{aligned} L \colon \mathcal{C}_c((0,1)) &\to \mathcal{C}_0([0,1]) \\ \big(Lf\big)(q) &:= \textstyle\int_0^1 (p \wedge q - pq) f(p)\,dp, \end{aligned} \tag{28}$$

so that we can write

$$\langle f, g \rangle_V = \langle Lf, g \rangle_{L^2}. \tag{29}$$

Then, if we could find a (right) inverse M for L (i.e. an operator such that $LM = \mathrm{Id}$), we would have

$$\langle f, \varphi \rangle_{L^2} = \langle L(Mf), \varphi \rangle_{L^2} = \langle Mf, \varphi \rangle_V \leqslant \|Mf\|_V \|\varphi\|_V, \tag{30}$$

which would be a good step towards our goal. But, indeed, such a right inverse is given by the operator "minus second derivative", that is, $M \colon f \in$

$\mathcal{C}_c^2((0,1)) \mapsto -f'' \in \mathcal{C}_c((0,1))$! We compute indeed that, for $q \in (0,1)$:

$$\begin{aligned}(31)\quad \big(L(-f'')\big)(q) &= -\int_0^1 (p \wedge q - pq) f''(p)\,dp \\ &= -(1-q)\int_0^q pf''(p)\,dp - q\int_q^1 (1-p)f''(p)\,dp \\ = -(1-q)\big[pf'(p)\big]_0^q + (1-q)&\int_0^q f'(p)\,dp - q\big[(1-p)f'(p)\big]_q^1 - q\int_q^1 f'(p)\,dp \\ &= -q(1-q)f'(q) + (1-q)f(q) + q(1-q)f'(q) + qf(q) = f(q)\end{aligned}$$

by integrating by parts (and using that $f'(0), f'(1), f(0), f(1) = 0$).

So, we have got that

$$(32)\qquad \langle f, \varphi\rangle_{L^2} \leqslant \|f''\|_V \|\varphi\|_V.$$

To end the proof, we finally observe that $\|f''\|_V$ is actually equal to $\|f\|_H$:

$$\begin{aligned}(33)\quad \|f''\|_V^2 = \langle f'', f''\rangle_V = \langle L(f''), f''\rangle_{L^2} &= -\langle f, f''\rangle_{L^2} \\ &= -\int_0^1 f''(p)f(p)\,dp = \int_0^1 f'(p)^2\,dp = \|f\|_H^2\end{aligned}$$

(where the penultimate equality ensues from integrating by parts). □

3. $\rho \leqslant \tau'(1 + |\log \tau'|)$

3.1. **Statement.**

The goal of this third section is to prove the statement of its title:

Theorem 3.1. *For $\mathcal{A}, \mathcal{B}$ any two σ-algebras, the coefficient $\rho(\mathcal{A}, \mathcal{B})$ can be bounded using the coefficient $\tau'(\mathcal{A}, \mathcal{B})$, according to the following formula:*

$$(34)\qquad \rho \leqslant \tau'(1 - \log \tau')$$

(where we take by continuity, for $\tau' = 0$, $\tau'(1 - \log \tau') = 0$).

Remark 3.2. Since $\tau'(\mathcal{A}, \mathcal{B})$ is always $\leqslant 1$, we can re-write the right-hand side of (34) as "$\tau'(1 + |\log \tau'|)$", which makes easier to see that that right-hand side is never less than τ' —it is obvious indeed that one always has $\rho \geqslant \tau'$.

3.2. **Comparison technique.**

Proof of Theorem 3.1 (first part). Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras on the same probability space; we denote $\tau := \tau'(\mathcal{A}, \mathcal{B})$. If $\tau = 0$ or $\tau = 1$ then Theorem 3.1 is immediate, since in the first case $\mathcal{A}$ and $\mathcal{B}$ are independent, while in the second case (34) is automatic by the Cauchy–Schwarz inequality. Therefore we will assume that $\tau \in (0,1)$.

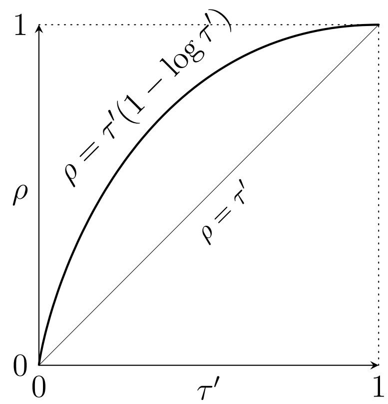

FIGURE 1. How we bound ρ as a function of τ'.

So, let X and Y be L^2 real r.v. being resp. $\mathcal{A}$- and $\mathcal{B}$- measurable; our goal is to bound above $\mathrm{Cov}(X,Y)$ (as in the proof of Theorem 2.2, bounding above $\mathrm{Cov}(X,Y)$ will actually yield a bound for $|\mathrm{Cov}(X,Y)|$). We start from the formula (12):

$$\mathrm{Cov}(X,Y) = \int_{\mathbb{R}\times\mathbb{R}} \big(\mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) - \mathbb{P}(X \leqslant x)\,\mathbb{P}(Y \leqslant y)\big)\, dxdy. \tag{35}$$

Now we use the τ'-mixing property. τ'-mixing means that for $\mathcal{A}$-, resp. $\mathcal{B}$-measurables events A, B,

$$\mathbb{P}(A\cap B) \leqslant \mathbb{P}(A)\,\mathbb{P}(B) + \tau\,\mathbb{P}(A)^{1/2}(1-\mathbb{P}(A))^{1/2}\,\mathbb{P}(B)^{1/2}(1-\mathbb{P}(B))^{1/2}. \tag{36}$$

Yet, if we use that formula naively, we shall not get anything better than Theorem 2.2 —which in the present case would yield an infinite bound, that is to say, nothing. The new idea consists in noticing that (36) can be automatically improved into:

$$\mathbb{P}(A\cap B) \leqslant \big(\mathbb{P}(A)\,\mathbb{P}(B) + \tau\,\mathbb{P}(A)^{1/2}(1-\mathbb{P}(A))^{1/2}\,\mathbb{P}(B)^{1/2}(1-\mathbb{P}(B))^{1/2}\big) \wedge \mathbb{P}(A) \wedge \mathbb{P}(B). \tag{37}$$

To alleviate notation, we pose

$$Z(p,q) := \big(pq + \tau p^{1/2}(1-p)^{1/2}q^{1/2}(1-q)^{1/2}\big) \wedge p \wedge q \tag{38}$$

(actually Z is also a function of τ, but in all the sequel of the proof τ will be fixed), so that the right-hand side of (37) becomes "$Z(\mathbb{P}(A),\mathbb{P}(B))$". So, we get

$$\mathrm{Cov}(X,Y) \leqslant \int_{\mathbb{R}\times\mathbb{R}} \Big(Z\big(\mathbb{P}(X \leqslant x), \mathbb{P}(Y \leqslant y)\big) - \mathbb{P}(X \leqslant x)\,\mathbb{P}(Y \leqslant y)\Big)\, dxdy. \tag{39}$$

As in the proof of Theorem 2.2, we now define

$$\chi(p) := \inf\{x \in \mathbb{R} : \ \mathbb{P}(X \leqslant x) \geqslant p\} \tag{40}$$

and likewise

$$\upsilon(q) := \inf\{y \in \mathbb{R} : \ \mathbb{P}(Y \leqslant y) \geqslant q\}. \tag{41}$$

Then, the change of variables $(x, y) = (\chi(p), \upsilon(q))$ in (39) yields:

$$\mathrm{Cov}(X, Y) \leqslant \int_{(0,1)^2} (Z(p, q) - pq)\chi'(p)\upsilon'(q)\, dpdq \tag{42}$$

(using if needed an approximation argument to make as if χ and υ were $\mathcal{C}^1$ and strictly increasing). As in the proof of 2.2 again, one also has

$$\mathrm{Var}(X) = \int_{(0,1)^2} (p_1 \wedge p_2 - p_1 p_2)\chi'(p_1)\chi'(p_2)\, dp_1 dp_2, \tag{43}$$

$$\text{resp.} \qquad \mathrm{Var}(Y) = \int_{(0,1)^2} (q_1 \wedge q_2 - q_1 q_2)\upsilon'(q_1)\upsilon'(q_2)\, dq_1 dq_2. \tag{44}$$

Then, it turns out that the right-hand sides of (42), (43) and (44) can be seen as the covariance and variances of two random variables which we will now introduce. But first, we need define a probability law which shall play a central role in the sequel:

Definition 3.3. The *Chogosov law*[3], denoted by Γ, is the probability law on $(0, 1)^2$ characterized by

$$\forall p, q \in (0, 1)^2 \qquad \Gamma\big((0, p) \times (0, q)\big) = Z(p, q). \tag{45}$$

(It shall be proved in Subsection 3.3 that that law actually exists).

Now, on the space $\{(p, q) \in (0, 1)^2\}$ equipped with the Chogosov law, we define the following random variables:

$$X^* := \chi(p); \tag{46}$$

$$Y^* := \upsilon(q). \tag{47}$$

I claim that X^* and Y^* have the same distributions as resp. X and Y. Under the Chogosov law indeed, both p and q have a *Uniform*$(0, 1)$ distribution (this follows from taking $q = 1$, resp. $p = 1$ in (45)), so that the function "χ_{X^*}" got by replacing X by X^* in (40) coincides with the actual function χ (which proves that X and X^* have the same law), and likewise $\upsilon_{Y^*} = \upsilon$. So, the respective right-hand sides of (43) and (44) are equal to $\mathrm{Var}(X^*)$ and $\mathrm{Var}(Y^*)$. Furthermore, applying (35) to X^* and Y^*, the very definition of

[3] So called in honour of my dear friend M. K. Chogosov.

the Chogosov law shows that $\mathrm{Cov}(X^*, Y^*)$ is *exactly* equal to the right-hand side of (42). In the end, proving the theorem is tantamount to proving that

$$\mathrm{Cov}(X^*, Y^*) \leqslant \tau(1 - \log\tau)\,\mathrm{Var}(X^*)^{1/2}\,\mathrm{Var}(Y^*)^{1/2}. \tag{48}$$

Now, denoting by $\mathcal{A}^*$ the σ-algebra on $(0,1)^2$ spanned by p, resp. by $\mathcal{B}^*$ the σ-algebra spanned by q, we observe that X^* and Y^* are resp. $\mathcal{A}^*$- and $\mathcal{B}^*$-measurable; thus, to show (48), it will be enough to show that

$$\rho(\mathcal{A}^*, \mathcal{B}^*) \leqslant \tau(1 - \log\tau). \tag{49}$$

END OF THE FIRST PART OF THE PROOF

3.3. The Chogosov law. ☛ **To alleviate notation, from now on we will denote $\bar{p} := 1 - p$ and $\hat{p} := p - \frac{1}{2}$ (with similar notation for q).**

In this subsection, we make a pause in the proof of Theorem (3.1) to prove the existence of the Chogosov law and to describe its structure. We recall that the Chogosov law Γ is the probability law on $\{(p,q) \in (0,1)^2\}$ defined by

$$\Gamma\big((0,p) \times (0,q)\big) = \big(pq + \tau(p\bar{p})^{1/2}(q\bar{q})^{1/2}\big) \wedge p \wedge q =: Z(p,q). \tag{50}$$

First we notice that, due to the presence of minimum symbols in the definition of $Z(p,q)$, its analytic expression shall depend on the zone of $(0,1)^2$ in which (p,q) lies (see Figure 2):

1. If $q\bar{p} \,/\, p\bar{q} < \tau^2$, then $Z(p,q) = q$; in this case we will say that we are in Zone 1;
2. If $\tau^2 < q\bar{p} \,/\, p\bar{q} < \tau^{-2}$, then $Z(p,q) = pq + \tau(p\bar{p})^{1/2}(q\bar{q})^{1/2}$; in this case we will say that we are in Zone 2;
3. If $q\bar{p} \,/\, p\bar{q} > \tau^{-2}$, then $Z(p,q) = p$; in this case we will say that we are in Zone 3.

It will be convenient too to give a name to the boundaries between the different zones: the boundary between Zones 1 and 2 (corresponding to $q\bar{p}/p\bar{q} = \tau^2$) will be denoted by $\mathfrak{d}$, and the boundary between Zones 2 and 3 (corresponding to $q\bar{p}/p\bar{q} = \tau^{-2}$) will be denoted by $\mathfrak{u}$. One can parametrize these boundaries by p: $\mathfrak{d}$ is the graph of the function "$q = q_d(p)$" and $\mathfrak{u}$ is the graph of "$q = q_u(p)$", where we define $q_d(p) := \tau^2 p \,/\, (\bar{p} + \tau^2 p)$ and $q_u(p) := p \,/\, (\tau^2\bar{p} + p)$.

First, we have to check that the Chogosov law actually exists. In fact, (50) automatically describes a measure on $(0,1)^2$ whose density is $\partial^2 Z/\partial q\partial p$ (in the sense of distributions), but we have to make sure that this measure is nonnegative!

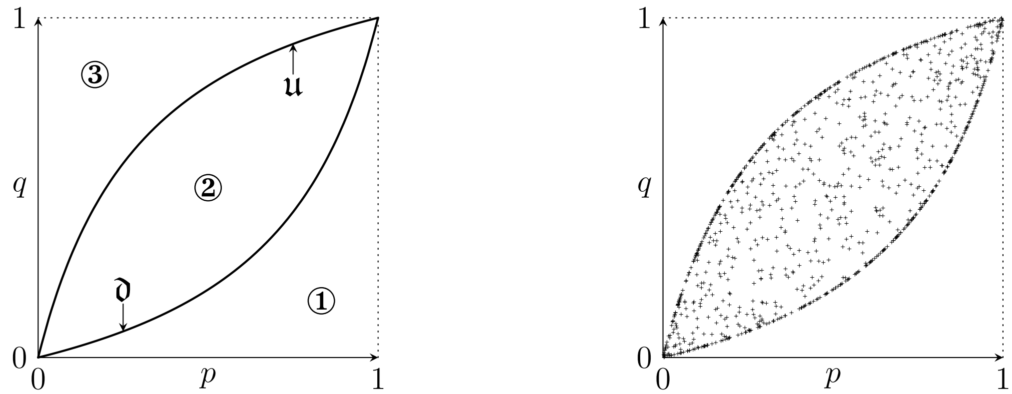

FIGURE 2. The Chogosov law Γ. On the left are drawn the different zones relative to the support of the measure; on the right is a cloud of 1,024 independent points with law Γ. (The drawings are made for $\tau = 1/2$).

So, we have to compute $\partial^2 Z/\partial q \partial p$ to know its sign. We start with computing $\partial Z/\partial p$:

- In Zone 1, $\partial Z(p,q)/\partial p = 0$;
- In Zone 2, $\partial Z(p,q)/\partial p = q - \tau(q\bar{q})^{1/2}\hat{p} \,/\, (p\bar{p})^{1/2}$;
- In Zone 3, $\partial Z(p,q)/\partial p = 1$.

(As Z is continuous at the boundaries $\mathfrak{d}$ and $\mathfrak{u}$, it is not important to know what happens there). Next we compute $\partial^2 Z/\partial q \partial p$:

- In Zone 1, $\partial^2 Z/\partial q \partial p = 0$;
- At $q = q_d(p)$, $\partial Z(p,\boldsymbol{\cdot})/\partial p$ makes a jump of amplitude $q_d(p) - \tau \times (q_d(p)\bar{q}_d(p))^{1/2}\hat{p}/(p\bar{p})^{1/2}$; since $\bar{q}_d(p)/\bar{p} = \tau^{-2} q_d(p)/p$, that amplitude can be simplified into $q_d(p) - \hat{p} q_d(p) \,/\, p = q_d(p) \,/\, 2p$;
- In Zone 2, $\partial^2 Z/\partial q \partial p = 1 + \tau \hat{p}\hat{q} \,/\, (p\bar{p})^{1/2}(q\bar{q})^{1/2}$;
- At $q = q_u(p)$, $\partial Z(p,\boldsymbol{\cdot})/\partial p$ makes a jump of amplitude $\bar{q}_u(p) + \tau \times (q_u(p)\bar{q}_u(p))^{1/2}\hat{p}/(p\bar{p})^{1/2}$; since $q_u(p)/p = \tau^{-2}\bar{q}_u(p)/\bar{p}$, that amplitude can be simplified into $\bar{q}_u(p) + \hat{p}\bar{q}_u(p) \,/\, \bar{p} = \bar{q}_u(p) \,/\, 2\bar{p}$;
- Finally, in Zone 3, $\partial^2 Z/\partial q \partial p = 0$.

Then, checking the nonnegativity of Γ is equivalent to verifying that both $\partial^2 Z/\partial q \partial p$ (wherever it is defined) and the jumps of $\partial Z/\partial p$ are nonnegative. Obviously the only non-trivial case is Zone 2. To show that $1 + \tau \times \hat{p}\hat{q} \,/\, (p\bar{p})^{1/2}(q\bar{q})^{1/2}$ is nonnegative on the whole Zone 2, we consider four cases separately:

- If $p \leqslant 1/2$ and $q \leqslant 1/2$, then $\hat{p}\hat{q} \geqslant 0$, so that the nonnegativity of the density is trivial.
- Likewise, nonnegativity is trivial if $p \geqslant 1/2$ and $q \geqslant 1/2$.

- If $p \leqslant 1/2$ and $q \geqslant 1/2$, we use that $q\bar{p} \,/\, p\bar{q} \leqslant \tau^{-2}$ (since we are in Zone 2) to get that $|\tau\hat{p}\hat{q} \,/\, (p\bar{p})^{1/2}(q\bar{q})^{1/2}| \leqslant |\hat{p}\hat{q}| \,/\, q\bar{p} = |\hat{p} \,/\, \bar{p}||\hat{q} \,/\, q|$; and since $p \leqslant 1/2$ and $q \geqslant 1/2$, one has $|\hat{p} \,/\, \bar{p}||\hat{q} \,/\, q| \leqslant 1/2 \times 1/2 \leqslant 1$, which shows that the density is nonnegative.
- Likewise, if $p \geqslant 1/2$ and $q \leqslant 1/2$, we use that $q\bar{p} \,/\, p\bar{q} \geqslant \tau^2$ to get that $|\tau\hat{p}\hat{q} \,/\, (p\bar{p})^{1/2}(q\bar{q})^{1/2}| \leqslant |\hat{p}\hat{q}| \,/\, p\bar{q} = |\hat{p} \,/\, p||\hat{q} \,/\, \bar{q}| \leqslant 1/2 \times 1/2 \leqslant 1$.

So we have proved that the Chogosov law actually exists (see also Figure 2). Moreover, the computations we have been making permit us to get a more detailed description of this law:

Definition 3.4. For $p \in (0,1)$, we define the law Γ^p on $(0,1)$ in the following way:

- On $(q_d(p), q_u(p))$, Γ^p has density $1 + \tau\hat{p}\hat{q} \,/\, (p\bar{p})^{1/2}(q\bar{q})^{1/2}$ w.r.t. the Lebesgue measure;
- At $q_d(p)$, Γ^p has an atom of mass $q_d(p) \,/\, 2p$; and at $q_u(p)$, it has an atom of mass $\bar{q}_u(p) \,/\, 2\bar{p}$;
- Outside $[q_d(p), q_u(p)]$, Γ^p is zero.

Then we may describe the Chogosov law in the following way:

Proposition 3.5. *(p,q) is distributed according to the Chogosov law Γ if and only if p is uniformly distributed on $(0,1)$ and that, conditionally to p, q is distributed according to the law Γ^p. In other words, for all $A, B \subset (0,1)$,*

$$\Gamma(A \times B) = \int_A \Gamma^p(B)\, dp. \tag{51}$$

3.4. **ρ-mixing for the Chogosov law.**

Proof of Theorem 3.1 (second part). The second and last part of the proof is to show (49). Remember that we are working on the space $\{(p,q) \in (0,1)^2\}$ equipped with the Chogosov law (whose law was defined by (50)), and that $\mathcal{A}^*$ is the σ-algebra spanned by p, resp. $\mathcal{B}^*$ the σ-algebra spanned by q.

Let us consider random variables X and Y being resp. $\mathcal{A}^*$ and $\mathcal{B}^*$-measurable, that is to say, X and Y are resp. of the form $X = f(p)$ and $Y = g(q)$. Our goal will be to bound $|\mathrm{Cov}(X,Y)|$ by some multiple of $\mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}$. Up to substracting their respective expectations from X and Y (which will not change any of the sides of the inequality to be proved), it will be convenient to assume that X and Y are centered: then, indeed, one will have $\mathrm{Var}(X) = \mathbb{E}(X^2) = \|f\|^2_{L^2((0,1))}$ and likewise $\mathrm{Var}(Y) = \|g\|^2_{L^2((0,1))}$, since both p and q are uniformly distributed on $(0,1)$.

Moreover, centering X and Y means that f and g lie in the (closed) subspace of the zero mean functions of $L^2((0,1))$: in the sequel of the proof, this Hilbert (sub)space will be denoted by H.

Since X and Y are centered, one has $\mathrm{Cov}(X,Y) = \mathbb{E}(XY)$. Thus, we can write $\mathrm{Cov}(X,Y)$ in terms of some linear operator:

Definition 3.6. We define $L\colon H \to H$ by

$$\big(Lg\big)(p) := \mathbb{E}_{\Gamma^p}(g), \tag{52}$$

where we recall that Γ^p is the Chogosov law conditioned to the value of p (cf. Definition 3.4 and Proposition 3.5). In other words, $\big(Lg\big)(p)$ is the expectation of $g(q)$ conditionally to p when (p,q) is distributed according to the Chogosov law. (That interpretation ensures that L actually maps H into itself).

Thus, conditioning w.r.t. p, one gets that

$$\mathrm{Cov}(X,Y) = \langle f, Lg\rangle_H. \tag{53}$$

Therefore, to show (49), it is (necessary and) sufficient to prove that the operator norm $\|L\|_{H\to H}$ is not greater than $\tau(1-\log\tau)$.

But, it turns out that L has the nice property of being self-adjoint on H. Indeed, we have defined L so that $\langle f, Lg\rangle_H = \mathbb{E}_\Gamma\big(f(p)g(q)\big)$; but the law Γ is invariant under the permutation $(p,q)\mapsto(q,p)$ (since $Z(p,q)$ is), so that $\langle f, Lg\rangle_H = \mathbb{E}_\Gamma\big(f(p)g(q)\big) = \mathbb{E}_\Gamma\big(f(q)g(p)\big) = \langle Lf, g\rangle_H$, that is, L is self-adjoint.

In the sequel we will use the following lemma on self-adjoint operators, whose proof you can find in § 3.5:

Lemma 3.7. *Let L be a self-adjoint operator (possibly unbounded) on a Hilbert space H. Assume there exists a dense subset $D\subset H$ such that, for some $C<\infty$,*

$$\forall h\in D \qquad \limsup_{k\to\infty} |\langle L^k h, h\rangle_H|^{1/k} \leqslant C. \tag{54}$$

Then $\|L\|_{H\to H}\leqslant C$.

Thanks to Lemma 3.7, we can focus on some dense subset of H on which the work is easier:

Definition 3.8. Let $\varepsilon>0$ be a parameter that we fix for the time being (though in the sequel we will make it tend to 0). We define formally, for $f\in H$,

$$\|f\|_{Lip} := \sup_{p\in(0,1)} \frac{|f'(p)|}{(p\bar{p})^{-3/2+\varepsilon}}, \tag{55}$$

or, in rigorous terms,

$$\|f\|_{Lip} := \sup_{p_1<p_2} \frac{|f(p_2)-f(p_1)|}{\int_{p_1}^{p_2} (p\bar{p})^{-3/2+\varepsilon}\,dp}. \tag{56}$$

We denote by *Lip* the space of the functions of H such that $\|f\|_{Lip} < \infty$, which we equip with the norm $\|\bullet\|_{Lip}$.

Obviously *Lip* is a dense subset of H. Moreover the canonical injection $Lip \hookrightarrow H$ is continuous: indeed, for $f \in Lip$, we write that $\int_{1/2}^{p} f'(p_1)\,dp_1 = f(p) - f(1/2)$ (here we do as if $f \in \mathcal{C}^1((0,1))$ to alleviate notations, but it would actually work for all $f \in Lip$), and since f is orthogonal in $L^2((0,1))$ to the constant functions (for it has zero mean),

$$\Big\| p \mapsto \int_{1/2}^{p} f'(p_1)\,dp_1 \Big\|_{L^2((0,1))}^2 = \|f\|_{L^2((0,1))}^2 + \|f(1/2)\|_{L^2((0,1))}^2, \tag{57}$$

whence

$$\begin{aligned} \|f\|_H = \|f\|_{L^2((0,1))} &\leqslant \Big\| p \mapsto \int_{1/2}^{p} f'(p_1)\,dp_1 \Big\|_{L^2((0,1))} \\ &= \Big(\int_0^1 \Big(\int_{1/2}^{p} f'(p_1)\,dp_1 \Big)^2 dp \Big)^{1/2} \leqslant \Big(\int_0^1 \Big(\int_{1/2}^{p} |f'(p_1)|\,dp_1 \Big)^2 dp \Big)^{1/2} \\ &\leqslant \|f\|_{Lip} \times \Big(\int_0^1 \Big(\int_{1/2}^{p} (p_1\bar{p_1})^{-3/2+\varepsilon}\,dp_1 \Big)^2 dp \Big)^{1/2}. \end{aligned} \tag{58}$$

The factor in the very-right-hand side of (58) being finite because $\varepsilon > 0$, this proves that the injection $Lip \hookrightarrow H$ is continuous. Denoting by C the continuity constant of this injection, it ensues that for all $f \in Lip$ and $k \in \mathbb{N}$, one has $|\langle L^k f, f\rangle_H| \leqslant \|f\|_H \|L^k f\|_H \leqslant \|f\|_H C \|L^k f\|_{Lip} \leqslant \|f\|_H C \|L\|_{Lip\to Lip}^k \|f\|_{Lip}$ (note that $C\|f\|_{Lip}\|f\|_H < \infty$), whence

$$\limsup_{k\to\infty} |\langle L^k f, f\rangle_H|^{1/k} \leqslant \|L\|_{Lip\to Lip}. \tag{59}$$

Thus, by Lemma 3.7,

$$\|L\|_{H\to H} \leqslant \|L\|_{Lip\to Lip}. \tag{60}$$

As we will see, $\|L\|_{Lip\to Lip}$ is easier to bound than $\|L\|_{H\to H}$.

To bound $\|L\|_{Lip\to Lip}$, we will use the idea of *monotone coupling* between the Γ^p's. For $\omega \in (0,1)$, let

$$Q(p,\omega) := \inf\{q \in (0,1) \ :\ \Gamma^p((0,q]) \geqslant \omega\} \tag{61}$$

be the inverse repartition function of Γ^p. Then, Γ^p is the pushforward of the *Uniform*$(0,1)$ distribution by the map $Q(p,\bullet)$, so that

$$\big(Lf\big)(p) = \int_0^1 f\big(Q(p,\omega)\big)\,d\omega. \tag{62}$$

To alleviate notation, we will do as if f were of class $\mathcal{C}^1$ (treating the general case $f \in Lip$ would raise absolutely no more difficulty but heavier formalism). Then, differentiating (62), one finds

$$(Lf)'(p) = \int_0^1 Q'(p,\omega) f'\big(Q(p,\omega)\big)\, d\omega, \tag{63}$$

where Q' is the derivative of $Q(p,\omega)$ *with respect to* p. (Justification for having differentiated under the integral sign will follow from the upcoming computations on Q').

Consequently,

$$\big|(Lf)'(p)\big| \leqslant \|f\|_{Lip} \int_0^1 |Q'(p,\omega)| \big(Q(p,\omega)\bar{Q}(p,\omega)\big)^{-3/2+\varepsilon}\, d\omega. \tag{64}$$

As that formula is valid for all p and f, it follows that

$$\|L\|_{Lip\to Lip} \leqslant \sup_{p\in(0,1)} \Big\{ (p\bar{p})^{3/2-\varepsilon} \int_0^1 |Q'(p,\omega)| \big(Q(p,\omega)\bar{Q}(p,\omega)\big)^{-3/2+\varepsilon}\, d\omega \Big\}, \tag{65}$$

hence

$$\|L\|_{H\to H} \leqslant \textit{Right-hand side of } (65). \tag{66}$$

Before starting with explicit computations, we prove that it is licit to take directly $\varepsilon = 0$ in Equation (66) (recall that ε was *a priori* defined to be any *strictly* positive parameter). To prove that point, we first notice that, denoting by $S_p := [q_d(p), q_u(p)]$ the support of Γ^p, one always has $Q(p,\omega) \in S_p$. But for all $q \in S_p$, one has $q\bar{q}/p\bar{p} \leqslant \tau^{-2}$: in the case $q \leqslant p$ indeed, having $q \in S_p$ implies that $q\bar{p}/p\bar{q} \geqslant \tau^2$, thus $q\bar{q}/p\bar{p} = (q/p)^2/(q\bar{p}/p\bar{q}) \leqslant 1/\tau^2 = \tau^{-2}$; and there is a similar argument for the case $q \geqslant p$. Then, for all p:

$$(p\bar{p})^{3/2-\varepsilon} \int_0^1 |Q'(p,\omega)| \big(Q(p,\omega)\bar{Q}(p,\omega)\big)^{-3/2+\varepsilon} d\omega \leqslant \\ \tau^{-2\varepsilon} \times (p\bar{p})^{3/2} \int_0^1 |Q'(p,\omega)| \big(Q(p,\omega)\bar{Q}(p,\omega)\big)^{-3/2} d\omega. \tag{67}$$

In that formula, the factor $\tau^{-2\varepsilon}$ does not depend on p and tends to 1 when $\varepsilon \to 0$; therefore (66) remains valid for $\varepsilon = 0$.

So, we have to compute the right-hand side of (66) for $\varepsilon = 0$. The first step is to compute Q and Q'. Because of the strcture of Γ^p (see Definition 3.4), there are three cases for the analytic expression of $Q(p,\omega)$:

- If $0 < \omega \leqslant q_d(p) \,/\, 2p$, then $Q(p,\omega) = q_d(p)$;
- Likewise, if $1 - \bar{q}_u(p) \,/\, 2\bar{p} \leqslant \omega < 1$, then $Q(p,\omega) = q_u(p)$;
- The case $q_d(p) \,/\, 2p < \omega < 1 - \bar{q}_u(p) \,/\, 2\bar{p}$ is more complicated... As Γ^p is the p-conditional law of Γ, the definition (50) of Γ implies that

$\Gamma^p((0,q]) = \partial_p Z(p,q)$; thus $Q(p,\omega)$ is the number Q such that

$$Q - \tau\hat{p}(Q\bar{Q})^{1/2} \,/\, (p\bar{p})^{1/2} = \omega. \tag{68}$$

(Indeed, remember that in that case one has $Q(p,\omega) \in (q_d(p), q_u(p))$, so that $Z(p,Q) = pQ + \tau(p\bar{p})^{1/2}(Q\bar{Q})^{1/2}$).

From that we get the formula for $Q'(p,\omega)$ —recall that Q' is the derivative of Q *w.r.t.* p—:

- For $\omega < q_d(p) \,/\, 2p$, one has $Q'(p,\omega) = dq_d/dp$. Since q_d is characterized by "$q_d(p)\bar{p} = \tau^2 p\bar{q}_d(p)$", differentiating the latter formula w.r.t. p yields that $Q'(p,\omega) = (q_d(p) + \tau^2\bar{q}_d(p)) \,/\, (\bar{p} + \tau^2 p)$. Using again that $q_d\bar{p} = \tau^2 p\bar{q}_d$, that expression then simplifies into "$Q'(p,\omega) = q_d(p)\bar{q}_d(p) \,/\, p\bar{p}$".
- Likewise, for $\omega > 1 - \bar{q}_u(p) \,/\, 2\bar{p}$, one has $Q'(p,\omega) = q_u(p)\bar{q}_u(p) \,/\, p\bar{p}$.
- Finally for $q_d(p) \,/\, 2p < \omega < 1 - \bar{q}_u(p) \,/\, 2\bar{p}$, we differentiate (68), getting

$$Q'(p,\omega) = \frac{\tau(Q\bar{Q})^{1/2}}{4(p\bar{p})^{3/2}\big(1 + \tau\hat{p}\hat{Q} \,/\, (p\bar{p}Q\bar{Q})^{1/2}\big)} \tag{69}$$

—where "Q" is a shorthand for "$Q(p,\omega)$".

(Note by the way that these computations ensure that $Q'(p,\omega)$ actually exists [for all p, for almost-all ω] and that $|Q'|$ is bounded by τ^{-2} [that point, which is unsubstantial and tedious, is left to the reader to verify], which gives *a posteriori* justification to (63)).

Then we can compute the right-hand side of (66) (for $\varepsilon = 0$):

$$(p\bar{p})^{3/2}\int_0^1 |Q'(p,\omega)|\big(Q(p,\omega)\bar{Q}(p,\omega)\big)^{-3/2}\,d\omega =$$

$$(p\bar{p})^{3/2}\frac{q_d(p)}{2p}\,\frac{q_d(p)\bar{q}_d(p)}{p\bar{p}}\big(q_d(p)\bar{q}_d(p)\big)^{-3/2} \tag{70}$$

$$+(p\bar{p})^{3/2}\frac{\bar{q}_u(p)}{2\bar{p}}\,\frac{q_u(p)\bar{q}_u(p)}{p\bar{p}}\big(q_u(p)\bar{q}_u(p)\big)^{-3/2} \tag{71}$$

$$+(p\bar{p})^{3/2}\int_{q_d/2p}^{1-\bar{q}_u/2\bar{p}} \frac{\tau(Q(\omega)\bar{Q}(\omega))^{1/2}}{4(p\bar{p})^{3/2}\big(1 + \tau\hat{p}\hat{Q}(\omega) \,/\, (p\bar{p}Q(\omega)\bar{Q}(\omega))^{1/2}\big)}\big(Q(\omega)\bar{Q}(\omega)\big)^{-3/2}d\omega \tag{72}$$

—where, in (72), "q_d", "q_u" and "$Q(\omega)$" are shortcuts for resp. "$q_d(p)$", "$q_u(p)$" and "$Q(p,\omega)$".

Using the formula characterizing $q_d(p)$, Term (70) simplifies into $\big(q_d(p)\bar{p}/p\bar{q}_d(p)\big)^{1/2}/2 = \tau/2$. Similarly, Term (71) simplifies into $\tau/2$ too. To compute the value of Term (72), we make the change of variables "$q = Q(\omega)$". Differentiating (68) *with respect to* ω, we get that for that change of variables,

$$(73)\qquad \Big(1 + \frac{\tau\hat{p}\hat{Q}(\omega)}{\big(p\bar{p}Q(\omega)\bar{Q}(\omega)\big)^{1/2}}\Big)dq = d\omega,$$

whence

(74)

$$(p\bar{p})^{3/2}\int_{q_d/2p}^{1-\bar{q}_u/2\bar{p}} \frac{\tau(Q(\omega)\bar{Q}(\omega))^{1/2}}{4(p\bar{p})^{3/2}\big(1+\tau\hat{p}\hat{Q}(\omega)\ /\ (p\bar{p}Q(\omega)\bar{Q}(\omega))^{1/2}\big)}\big(Q(\omega)\bar{Q}(\omega)\big)^{-3/2}d\omega$$
$$= \frac{\tau}{4}\int_{q_d(p)}^{q_u(p)}\frac{1}{q\bar{q}}dq = \frac{\tau}{4}\Big[\log\frac{q}{\bar{q}}\Big]_{q_d(p)}^{q_u(p)} = \frac{\tau}{4}\Big(\log\frac{q_u(p)}{\bar{q}_u(p)} - \log\frac{q_d(p)}{\bar{q}_d(p)}\Big)$$

which, using the formulas characterizing $q_d(p)$ and $q_u(p)$, is finally equal to

$$(75)\qquad \frac{\tau}{4}\Big(\log\frac{p}{\tau^2\bar{p}} - \log\frac{p\tau^2}{\bar{p}}\Big) = \frac{\tau}{4}\log\frac{1}{\tau^4} = -\tau\log\tau.$$

So, the sum (70)–(72) is equal to $\tau(1-\log\tau)$ for all p, and thus the rigt-hand side of (66) (for $\varepsilon = 0$) is $\tau(1-\log\tau)$, which proves the theorem. □

3.5. **Appendix: On the norm of self-adjoint operators.** This appendix aims at proving Lemma 3.7.

Proof of Lemma 3.7. Let L be a self-adjoint operator on H. Then by the spectral theorem for self-adjoint operators, we have that, up to some isomorphism, we may assume that H is the space $L^2(\mu)$ corresponding to some measured Radon space (X,μ) and that L is a real multiplication operator on that space —i.e. that there exists $\lambda \in L^\infty(\mu,\mathbb{R})$ such that

$$(76)\qquad \forall x\in X \qquad \big(Lf\big)(x) = \lambda(x)f(x)$$

(where "$\forall x$" actually means "for μ-almost all x").

Once L is written under that form, we have that for all $f\in L^2(\mu)$,

$$(77)\qquad \limsup_{k\to\infty}|\langle L^k f, f\rangle_H|^{1/k} = \sup\big\{c>0\ :\ \mu(\{f\neq 0 \text{ and } |\lambda|>c\})>0\big\}.$$

(To prove the "$\geqslant$" sense, use that for even k one has $\lambda(x)^k \geqslant 0\ \forall x$). But if one had $\mu(\{|\lambda| > C\}) > 0$, the set of the $f\in H$ such that $\mu(\{f\neq 0 \text{ and } |\lambda|>C\})>0$ would be a non-empty open subset of H, and then (77) would contradict the assumption of the lemma. Therefore one necessarily has $\mu(\{|\lambda|>C\}) = 0$, thus $\|L\|_{H\to H}\leqslant C$, QED. □

4. Optimality of our bound

☛ **In all this section, all the sets considered will be tacitly understood to be Borel.**

4.1. **Statement of the theorem and outline of the proof.** In this section we will prove that our bound (34) cannot be improved. More precisely, we are going to prove the following theorem:

Theorem 4.1. *Let $\tau \in [0,1]$; let $\rho < \tau(1-\log\tau)$. Then, there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and two σ-algebras $\mathcal{A},\mathcal{B}$ of this space such that $\tau'(\mathcal{A},\mathcal{B}) \leqslant \tau$ and $\rho(\mathcal{A},\mathcal{B}) \geqslant \rho$.*

Since the map $\tau \mapsto \tau(1-\log\tau)$ is continuous, that theorem will occur as an immediate corollary of the following one:

Theorem 4.2. *Let $\tau \in [0,1]$; let $\tau_1 > \tau$. Then, there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and two σ-algebras $\mathcal{A},\mathcal{B}$ of this space such that $\rho(\mathcal{A},\mathcal{B}) \geqslant \tau(1-\log\tau)$ and $\tau'(\mathcal{A},\mathcal{B}) \leqslant \tau_1$.*

Note that Theorem 4.2 is immediate for $\tau = 0$ and for $\tau = 1$, so it is enough to prove it for $\tau \in (0,1)$.

In order to prove Theorem 4.2, we will have to find a sharp bound for some τ'-mixing coefficient, which is not an easy challenge in general. For that reason, we are first going to focus on some particular measure for which finding this kind of bound is easier. However this measure will not be a probability measure (it shall have infinite total mass), so that we will have to use a truncation argument in a second step to get a genuine probability measure.

☛ **In all the sequel of this section, we are considering some fixed $\tau \in (0,1)$, and our goal is to prove 4.2 for that value of τ.**

4.2. **The measure Γ_∞.**

Definition 4.3.

(1) We define, for $(p,q) \in (0,\infty)^2$,

$$Z_\infty(p,q) := \tau p^{1/2} q^{1/2} \wedge p \wedge q. \tag{78}$$

(2) We define the measure Γ_∞ on $(0,\infty)^2$ by

$$\forall p,q \in (0,\infty)^2 \qquad \Gamma_\infty\big((0,p)\times(0,q)\big) = Z_\infty(p,q). \tag{79}$$

(This actually defines a nonnegative measure: the reasoning to prove that point is similar —and easier— to the one we followed for the Chogosov law. See also Figure 3).

(3) For $p \in (0,\infty)$, we define the measure Γ_∞^p on $(0,\infty)$ in the following way:

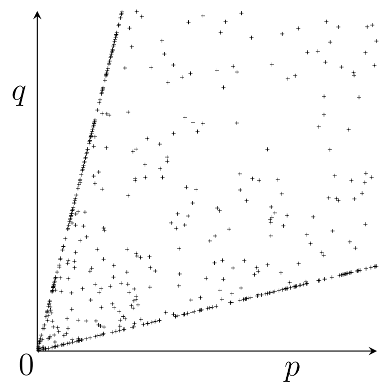

FIGURE 3. The measure Γ_∞: this is a Poisson cloud of points with density Γ_∞. (The scale and density of the cloud are consistent with Figure 2).

- On $(\tau^2 p, \tau^{-2} p)$, Γ^p_∞ has the density $\tau/4p^{1/2}q^{1/2}$ w.r.t. the Lebesgue measure;
- At $\tau^2 p$, Γ^p_∞ has an atom of mass $\tau^2/2$; and at $\tau^{-2}p$, it has an atom of mass $1/2$;
- Outside $[\tau^2 p, \tau^{-2} p]$, Γ^p_∞ is zero.

(Note that Γ^p_∞ is a probability measure).

Like for the Chogosov law, we can prove the following properties of the measure Γ_∞:

Proposition 4.4.

(1) *Both marginals of* Γ_∞ *(i.e. its marginals on* p *and on* q*) are equal to the Lebesgue measure on* $(0, \infty)$.

(2) Γ^p_∞ *is the "*p*-conditional law" of* Γ_∞*, in the sense that for all* $A, B \subset (0, \infty)$*, one has* $\Gamma_\infty(A \times B) = \int_A \Gamma^p_\infty(B) dp$.

So, like for the Chogosov law, the measure Γ_∞ is made of several components: first, a component with density $\tau \,/\, 4p^{1/2}q^{1/2}$ w.r.t. the Lebesgue measure on the cone $\{(p, q) \in (0, \infty)^2 : \ \tau^2 p < q < \tau^{-2} p\}$; then, components with a *lineic* density on the half-lines $\{(p, \tau^2 p)\}$ and $\{(p, \tau^{-2} p)\}$. We will give a name to the (surfacic) density component:

Definition 4.5.

(1) We denote by $\tilde{\Gamma}_\infty$ the absolutely continuous part of the measure Γ_∞ w.r.t. the Lebesgue measure; in other words, $\tilde{\Gamma}_\infty$ is the measure on $(0, \infty)^2$ defined by

$$d\tilde{\Gamma}_\infty(p, q) = \mathbf{1}_{\tau^2 p < q < \tau^{-2} p} \frac{\tau}{4p^{1/2}q^{1/2}} dp dq. \tag{80}$$

(2) We also denote by $\tilde\Gamma^p_\infty$ the "p-conditional measure" of $\tilde\Gamma_\infty$, that is, the measure on $(0,\infty)$ defined by

$$d\tilde\Gamma^p_\infty(q) = \mathbf{1}_{\tau^2 p<q<\tau^{-2}p}\frac{\tau}{4p^{1/2}q^{1/2}}dq, \tag{81}$$

which is such that $\tilde\Gamma_\infty(A\times B) = \int_A \tilde\Gamma^p_\infty(B)dp$. (Beware, $\tilde\Gamma^p_\infty$ is not a probability measure).

Now we are going to state and prove two lemmas essential for the proof of Theorem 4.2:

Lemma 4.6. *For all $A,B\subset(0,\infty)$ such that $\tau^2A\subset B$ and $\tau^2|B|\leqslant|A|<\infty$ ($|A|$ denotes the Lebesgue measure of A):*

$$\tilde\Gamma_\infty(A\times B)\leqslant \tau|A|^{1/2}|B|^{1/2} - \tau^2(|A|+|B|)/2. \tag{82}$$

Proof. We start from the "p-conditional" decomposition of $\tilde\Gamma_\infty$:

$$\tilde\Gamma_\infty(A\times B) = \int_A \tilde\Gamma^p_\infty(B)dp. \tag{83}$$

For $x\in(0,|A|)$, we pose

$$\pi(x) := \inf\{p\in(0,\infty)\ :\ |(0,p]\cap A|\geqslant x\}, \tag{84}$$

so that the restriction of the Lebesgue measure to A is the pushforward by π of the Lebesgue measure on $(0,|A|)$. Then, changing variables in the right-hand side of (83), one has

$$\tilde\Gamma_\infty(A\times B) = \int_0^{|A|}\tilde\Gamma^{\pi(x)}_\infty(B)\,dx. \tag{85}$$

Now we are going to bound $\tilde\Gamma^{\pi(x)}_\infty(B)$. (In the next computations I will shorthand "$\pi(x)$" into "π", not to be confused with Archimedes' constant [which is nowhere involved in this article]). There are three steps:

(1) First, observing that $\tilde\Gamma^\pi_\infty$ does not give any mass to $(0,\tau^2\pi]$, we have that $\tilde\Gamma^\pi_\infty(B) = \tilde\Gamma^\pi_\infty\big(B\smallsetminus(0,\tau^2\pi]\big)$. Let us denote $B\smallsetminus(0,\tau^2\pi] =: B^\times$.
(2) For $y\in(0,|B^\times|)$, let us define

$$\kappa(y) := \inf\{q\in(\tau^2\pi,\infty)\ :\ |(\tau^2\pi,q]\cap B^\times|\geqslant y\}, \tag{86}$$

so that the restriction of the Lebesgue measure to $B^\times$ is the pushforward by κ of the Lebesgue measure on $(0,|B^\times|)$. Then, changing variables,

$$\tilde\Gamma^\pi_\infty(B) = \int_{B^\times}\mathbf{1}_{q<\tau^{-2}\pi}\frac{\tau}{4\pi^{1/2}q^{1/2}}dq = \int_0^{|B^\times|}\mathbf{1}_{\kappa(y)<\tau^{-2}\pi}\frac{\tau}{4\pi^{1/2}\kappa(y)^{1/2}}dy. \tag{87}$$

But, for all y, one has $\kappa(y) \geqslant \tau^2\pi + y$ (that is obvious from (86)), so that (87) yields:

$$\tilde{\Gamma}^\pi_\infty(B) \leqslant \int_0^{|B^\times|} \mathbf{1}_{\tau^2\pi+y<\tau^{-2}\pi} \frac{\tau}{4\pi^{1/2}\big(\tau^2\pi+y\big)^{1/2}} dy \tag{88}$$
$$= \frac{\tau(\tau^2\pi+|B^\times|)^{1/2}}{2\pi^{1/2}} \wedge 1/2 - \tau^2/2 \leqslant \frac{\tau(\tau^2 x+|B^\times|)^{1/2}}{2x^{1/2}} \wedge 1/2 - \tau^2/2,$$

where the last inequality comes from the fact that $\pi \geqslant x$ (because of (84)).

(3) Finally, I claim that $|B^\times| \leqslant |B| - \tau^2 x$: indeed, we have assumed that $B \supset \tau^2 A$, so $|B \cap (0,\tau^2\pi]| \geqslant |\tau^2 A \cap (0,\tau^2\pi]| = \tau^2|A\cap(0,\pi]| = \tau^2 x$. Therefore (88) yields:

$$\tilde{\Gamma}^\pi_\infty(B) \leqslant \frac{\tau|B|^{1/2}}{2x^{1/2}} \wedge 1/2 - \tau^2/2. \tag{89}$$

To conclude, we just have to put (89) inside (85) (it is here that the assumption that $\tau^2|B| \leqslant |A|$ is used):

$$\tilde{\Gamma}_\infty(A\times B) \leqslant \int_0^{|A|} \Big(\frac{\tau|B|^{1/2}}{2x^{1/2}} \wedge 1/2 - \tau^2/2\Big) dx \tag{90}$$
$$= \tau^2|B|/2 + \int_{\tau^2|B|}^{|A|} \frac{\tau|B|^{1/2}}{2x^{1/2}} dx - \tau^2|A|/2 = \tau|A|^{1/2}|B|^{1/2} - \tau^2(|A|+|B|)/2.$$

□

Lemma 4.7. *Let $A, B \subset (0,\infty)$ be such that $|A| \geqslant \tau^2|B|$. Then there exists $A' \subset (0,\infty)$ such that $|A'| = |A|$, $\Gamma_\infty(A'\times B) \geqslant \Gamma_\infty(A\times B)$ and $A' \supset \tau^2 B$.*

Proof. Denote $A_1 := A \cap \tau^2 B$. Since we have assumed that $|A| \geqslant \tau^2|B|$, we have $|A \setminus A_1| = |A| - |A_1| \geqslant \tau^2|B| - |A_1| = |\tau^2 B \setminus A_1|$, so that we can find $A_2 \subset |A \setminus A_1|$ such that $|A_2| = |\tau^2 B \setminus A_1|$. Now let us denote $A_3 := A \setminus A_1 \setminus A_2$ and $A_2' := \tau^2 B \setminus A_1$, and define $A' := A_1 \cup A_2' \cup A_3$. It is clear by construction that $|A'| = |A|$ and $A' \supset \tau^2 B$; and still by construction,

$$\Gamma_\infty(A'\times B) = \Gamma_\infty(A\times B) - \Gamma_\infty(A_2\times B) + \Gamma_\infty(A_2'\times B), \tag{91}$$

so to prove the lemma it only remains to show that $\Gamma_\infty(A_2\times B) \leqslant \Gamma_\infty(A_2'\times B)$.

Using the "p-conditional" decomposition of Γ_∞, one has

$$\Gamma_\infty(A_2\times B) = \int_{A_2} \Gamma^p_\infty(B)dp, \tag{92}$$

$$\text{resp.} \qquad \Gamma_\infty(A_2'\times B) = \int_{A_2'} \Gamma^p_\infty(B)dp. \tag{93}$$

But, recalling the structure of Γ^p_∞ (cf. Definition 4.3-3), we have that $\Gamma^p_\infty(B) \geqslant 1/2$ as soon as $\tau^{-2}p \in B$, and thus also that $\Gamma^p_\infty(B) \leqslant 1-1/2 = 1/2$ as soon as $\tau^{-2}p \notin B$. Since, by construction, $\tau^{-2}A_2 \cap B = \varnothing$, resp. $\tau^{-2}A'_2 \subset B$, one thus has

$$\Gamma_\infty(A_2 \times B) = \int_{A_2} \Gamma^p_\infty(B)dp \leqslant |A_2|/2, \tag{94}$$

$$\text{resp.} \qquad \Gamma_\infty(A'_2 \times B) = \int_{A'_2} \Gamma^p_\infty(B)dp \geqslant |A'_2|/2 = |A_2|/2, \tag{95}$$

so that $\Gamma_\infty(A_2 \times B) \leqslant \Gamma_\infty(A'_2 \times B)$, which is the desired result. □

As the function $Z_\infty(p,q)$ used to define Γ_∞ is invariant by switching p and q, we have for all A, B that $\Gamma_\infty(A \times B) = \Gamma_\infty(B \times A)$, so that Lemma 4.7 yields the following corollary:

Lemma 4.8. *Let $A, B \subset (0,\infty)$ be such that $|B| \geqslant \tau^2|A|$. Then there exists $B' \subset (0,\infty)$ such that $|B'| = |B|$, $\Gamma_\infty(A \times B') \geqslant \Gamma_\infty(A \times B)$ and $B' \supset \tau^2 A$.*

Thanks to Lemmas 4.6 and 4.8, we can prove the main result of this subsection:

Lemma 4.9. *For all $A, B \subset (0,\infty)$,*

$$\Gamma_\infty(A \times B) \leqslant \tau|A|^{1/2}|B|^{1/2}. \tag{96}$$

Proof. First observe that one automatically has $\Gamma_\infty(A \times B) \leqslant |A| \wedge |B|$ (since both marginals of Γ_∞ are equal to the Lebesgue measure), so that the lemma is immediate if $|A| \leqslant \tau^2|B|$ or $|B| \leqslant \tau^2|A|$; in the sequel we will therefore assume that $\tau^2|A| \leqslant |B| \leqslant \tau^{-2}|A|$. Then the assumptions of Lemma 4.8 are satisfied, so that up to replacing B by B' we can assume that $\tau^2 A \subset B$, and then apply Lemma 4.6, getting:

$$\tilde{\Gamma}_\infty(A \times B) \leqslant \tau|A|^{1/2}|B|^{1/2} - \tau^2(|A| + |B|)/2. \tag{97}$$

So we have bounded the absolutely continuous component of $\Gamma_\infty(A \times B)$. Now we have to bound the lineic density components. The first of these components is

$$\Gamma_\infty\big(\{(p,q) \in A \times B \ : \ q = \tau^2 p\}\big) \leqslant \Gamma_\infty\big(\{(p,q) \ : \ p \in A \text{ and } q = \tau^2 p\}\big), \tag{98}$$

which, using the "p-conditional" decomposition of Γ_∞ and the structure of Γ^p_∞, is equal to $|A| \times \tau^2/2$. Likewise, the second lineic density component is

$$\begin{aligned} \Gamma_\infty\big(\{(p,q) \in A \times B \ : \ q = \tau^{-2} p\}\big) \\ \leqslant \Gamma_\infty\big(\{(p,q) \ : \ p \in \tau^2 B \text{ and } q = \tau^{-2} p\}\big) = \tau^2|B| \times 1/2. \end{aligned} \tag{99}$$

Summing (97)–(99), we finally get the wanted result. □

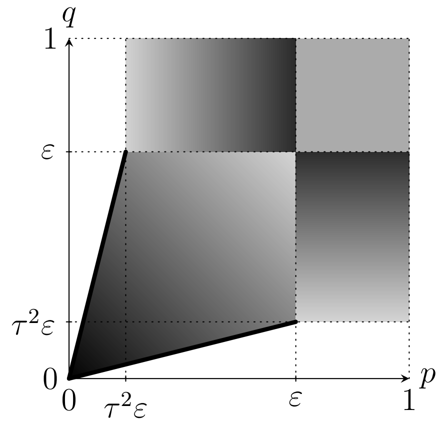

FIGURE 4. A schematic representation of the measure Γ_b.

4.3. **Proof of optimality.** Now that we are equipped with Lemma 4.9, we can at last prove Theorem 4.2. The measurable space we are going to consider is the square $\{(p,q) \in (0,1)^2\}$, on which the σ-algebras $\mathcal{A}$ and $\mathcal{B}$ will be the ones spanned by resp. p and q, so that the $\mathcal{A}$-measurable events are the events of the form $A \times (0,1)$ (resp. the $\mathcal{B}$-measurable events are the events of the form $(0,1) \times B$) and the $\mathcal{A}$-measurable functions are the functions of the form $f(p)$ (resp. the $\mathcal{B}$-measurable functions are the functions of the form $g(q)$).

The probability measure $\mathbb{P}$ we are going to build on $(0,1)^2$ will be devised so that both its p- and q-marginals are equal to the Lebesgue measure on $(0,1)$ in order to simplify computations. The principle of the definition we are now giving is that the probability $\mathbb{P}$ —which we will call Γ_b— coincides with Γ_∞ in some neighborhood of $(0,0)$:

Definition 4.10. Take a parameter $\varepsilon \in (0,1)$. On $(0,1)^2$, we define the probability measure Γ_b by

$$
\begin{aligned}
&\Gamma_b(A \times B) = \Gamma_\infty(A \times B) && \text{for } A \times B \subset (0,\varepsilon] \times (0,\varepsilon];\\
&\Gamma_b(A \times B) = \frac{|B|}{1-\varepsilon}\Big(|A| - \Gamma_\infty\big(A \times (0,\varepsilon]\big)\Big) && \text{for } A \times B \subset (0,\varepsilon] \times (\varepsilon,1);\\
&\Gamma_b(A \times B) = \frac{|A|}{1-\varepsilon}\Big(|B| - \Gamma_\infty\big((0,\varepsilon] \times B\big)\Big) && \text{for } A \times B \subset (\varepsilon,1) \times (0,\varepsilon];\\
&\Gamma_b(A \times B) = \frac{|A||B|}{(1-\varepsilon)^2}\Big(1 - 2\varepsilon + \Gamma_\infty\big((0,\varepsilon)^2\big)\Big) && \text{for } A \times B \subset (\varepsilon,1) \times (\varepsilon,1).
\end{aligned}
\tag{100}
$$

(See Figure 4).

We see that outside $(0,\varepsilon]^2$, the measure Γ_b is absolutely continuous w.r.t. the Lebesgue measure. For $(p,q) \in (0,\varepsilon] \times (\varepsilon,1)$, we can compute the density of Γ_b to be

$$\frac{d\Gamma_b(p,q)}{dpdq} = \begin{cases} 0 & \text{if } p \leqslant \tau^2\varepsilon; \\ \big(1-\tau\varepsilon^{1/2} \,/\, 2p^{1/2}\big)/(1-\varepsilon) & \text{if } p > \tau^2\varepsilon. \end{cases} \tag{101}$$

So, provided ε was chosen so that $\varepsilon \leqslant \tau/2$ —which we will assume from now on—, that density is bounded by 1 on the whole set $(0,\varepsilon]\times(\varepsilon,1)$, and thus, using the symmetry of Γ_b under switching p and q, also on $(\varepsilon,1)\times(0,\varepsilon]$. Besides, for $(p,q)\in(\varepsilon,1)\times(\varepsilon,1)$ we compute that the density of Γ_b at (p,q) is

$$\frac{d\Gamma_b(p,q)}{dpdq} = \frac{1-2\varepsilon+\tau\varepsilon}{(1-\varepsilon)^2}. \tag{102}$$

Now let us consider $A,B\subset(0,1)$. We denote $A_1 := A\cap(0,\varepsilon]$ and $A_2 := A\cap(\varepsilon,1)$, resp. $B_1 := B\cap(0,\varepsilon], B_2 := B\cap(\varepsilon,1)$. Using the density computations we have been doing, we then have $\Gamma_b(A_1\times B_2)\leqslant |A_1||B_2|$, resp. $\Gamma_b(A_2\times B_1)\leqslant|A_2||B_1|$. Thus, splitting $A\times B$ into $A_1\times B_1\cup A_1\times B_2\cup A_2\times B_1\cup A_2\times B_2$,

$$\begin{aligned} \Gamma_b(A\times B)-|A||B| &\leqslant \Gamma_b(A_1\times B_1)-|A_1||B_1|+\Gamma_b(A_2\times B_2)-|A_2||B_2| \\ &\leqslant \tau|A_1|^{1/2}|B_1|^{1/2}+\frac{\tau\varepsilon-\varepsilon^2}{(1-\varepsilon)^2}|A_2||B_2|, \end{aligned} \tag{103}$$

where the second inequality comes from using simultaneously Lemma 4.9, nonnegativity of $|A_1||B_1|$, and the value of the density of Γ_b on $(\varepsilon,1)^2$.

Our goal is to prove that for $\mathbb{P}=\Gamma_b$, one has $\tau'(\mathcal{A},\mathcal{B})\leqslant\tau_1$ (where $\tau_1>\tau$ is the arbitrary number which was fixed in the statement of Theorem 4.2). In other words, we want to get that

$$\forall A,B\subset(0,1) \qquad \big|\Gamma_b(A\times B)-|A||B|\big| \leqslant \tau_1|A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2}. \tag{104}$$

First, we notice that is suffices to prove (104) with no absolute value in the left-hand side:

$$\forall A,B\subset(0,1) \qquad \Gamma_b(A\times B)-|A||B| \leqslant \tau_1|A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2}; \tag{105}$$

indeed, if one replaces B by its complement set $B^c := (0,1)\smallsetminus B$, the left-hand side of (105) just change signs while the right-hand side remains unchanged. It is even sufficient to prove (105) only for $|A|\leqslant 1/2$, since none of the sides of (105) changes when one replaces simultaneously A by A^c and B by B^c. Therefore, from now on we will assume that $|A|\leqslant 1/2$. But then,

(105) is automatic for $|B| \geqslant 1/(1+\tau_1^2)$, since in that case

$$\begin{aligned}(106)\quad \Gamma_b(A\times B) - |A||B| \leqslant \Gamma_b(A\times(0,1)) - |A||B| = |A|(1-|B|) \\ \leqslant |A|^{1/2}(1-|A|)^{1/2}\times\tau_1|B|^{1/2}(1-|B|)^{1/2}.\end{aligned}$$

So, it will be enough to prove (105) for $|A| \leqslant 1/2$ and $|B| \leqslant 1/(1+\tau_1^2)$.

We start from Equation (103):

$$(107)\qquad \Gamma_b(A\times B) - |A||B| \leqslant \tau|A_1|^{1/2}|B_1|^{1/2} + \frac{\tau\varepsilon-\varepsilon^2}{(1-\varepsilon)^2}|A_2||B_2|.$$

Our goal is to bound above the right-hand side of (107) by some multiple of $|A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2}$. Recall that $A_1 := A\cap(0,\varepsilon]$, resp. $B_1 := B\cap(0,\varepsilon]$, so that $|A_1|, |B_1| \leqslant \varepsilon$. First we have

$$(108)\qquad |A_1|^{1/2} \leqslant |A_1|^{1/2}\frac{(1-|A_1|)^{1/2}}{(1-\varepsilon)^{1/2}} \leqslant (1-\varepsilon)^{-1/2}|A|^{1/2}(1-|A|)^{1/2},$$

where the second inequality comes from the fact that $|A_1| \leqslant |A|$ and that $p\mapsto p^{1/2}(1-p)^{1/2}$ is increasing on $[0,1/2]$. Similarly, provided ε was chosen small enough,

(109)
$$\begin{aligned}|B_1|^{1/2} \leqslant |B_1|^{1/2}\frac{(1-|B_1|)^{1/2}}{(1-\varepsilon)^{1/2}} \leqslant (1-\varepsilon)^{-1/2}(|B|\wedge\varepsilon)^{1/2}(1-|B|\wedge\varepsilon)^{1/2} \\ \leqslant (1-\varepsilon)^{-1/2}|B|^{1/2}(1-|B|)^{1/2},\end{aligned}$$

where the last inequality comes from the fact that $\varepsilon^{1/2}(1-\varepsilon)^{1/2} \leqslant q^{1/2}(1-q)^{1/2}$ for all $q\in[\varepsilon,1-\varepsilon]$, hence for all $q\in[\varepsilon,1/(1+\tau_1^2)]$ provided $\varepsilon \leqslant \tau_1^2/(1+\tau_1^2)$ (which we will assume from now on). Next,

$$(110)\qquad |A_2| \leqslant |A| \leqslant |A|^{1/2}(1-|A|)^{1/2}$$

(using again that $|A| \leqslant 1/2$), and similarly

$$(111)\qquad |B_2| \leqslant |B| \leqslant \tau_1^{-1}|B|^{1/2}(1-|B|)^{1/2}.$$

Putting the previous bounds into (107), we get that for all A, B such that $|A| \leqslant 1/2, |B| \leqslant 1/(1+\tau_1^2)$ (and provided ε was chosen small enough):

(112)
$$\Gamma_b(A\times B) - |A||B| \leqslant \left(\frac{\tau}{1-\varepsilon} + \frac{\varepsilon\tau-\varepsilon^2}{\tau_1(1-\varepsilon)^2}\right)|A|^{1/2}|B|^{1/2}(1-|A|)^{1/2}(1-|B|)^{1/2}.$$

The numerical factor in front of the right-hand side of (112) tends to τ when $\varepsilon \searrow 0$, so it is actually $\leqslant \tau_1$ provided ε was chosen small enough. In the end we have proved that for $\mathbb{P} = \Gamma_b$, one has $\tau'(\mathcal{A},\mathcal{B}) \leqslant \tau_1$.

To end the proof of Theorem 4.2, it remains for us to prove that $\rho(\mathcal{A},\mathcal{B}) \geqslant \tau(1-\log\tau)$. That will be easier, as it suffices to find $\mathcal{A}$- and $\mathcal{B}$-measurable

L^2 r.v. X and Y such that $|\mathrm{Cov}(X,Y)| \,/\, \mathrm{Var}(X)^{1/2}\,\mathrm{Var}(Y)^{1/2}$ is arbitrarily close to $\tau(1-\log\tau)$. To do that, we take $l\in(0,\varepsilon)$ and we pose

$$X := \mathbf{1}_{l\leqslant p\leqslant\varepsilon}\, p^{-1/2}, \tag{113}$$

resp.

$$Y := \mathbf{1}_{l\leqslant q\leqslant\varepsilon}\, q^{-1/2}. \tag{114}$$

Since both p- and q-marginals of Γ_b are equal to the Lebesgue measure, we have $\mathbb{E}(X),\mathbb{E}(Y) = 2\varepsilon^{1/2} - 2l^{1/2}$ and $\mathbb{E}(X^2),\mathbb{E}(Y^2) = \log\varepsilon - \log l$, whence $\mathrm{Var}(X),\mathrm{Var}(Y)\sim|\log l|$ when $l\searrow 0$. On the other hand, XY is zero outside $(0,\varepsilon]^2$, so according to the structure of Γ_b we have $\mathbb{E}(XY) = \int \mathbf{1}_{l\leqslant p,q\leqslant\varepsilon}\times p^{-1/2}q^{-1/2}d\Gamma_\infty(p,q)$. According to the structure of Γ_∞, we compute that that quantity is equal to

$$\begin{aligned}\mathbb{E}(XY) = \tau\big(\log\tau\log l - \log\tau\log\varepsilon - \log^2\tau - \log l + 2\log\tau + \log\varepsilon\big)\\ \sim_{l\searrow 0} \tau(1-\log\tau)|\log l|.\end{aligned} \tag{115}$$

(In our computation we assumed that $l\leqslant\tau^4\varepsilon$). So, when $l\searrow 0$, the Pearson correlation between X and Y tends to $\tau(1-\log\tau)$. This shows that $\rho(\mathcal{A},\mathcal{B})\geqslant\tau(1-\log\tau)$, thus ending the proof of Theorem 4.2.

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References

[1] Richard C. Bradley. Equivalent measures of dependence. *J. Multivariate Anal.*, 13(1):167–176, 1983.

[2] Richard C. Bradley. Basic properties of strong mixing conditions. A survey and some open questions. *Probab. Surv.*, 2:107–144 (electronic), 2005. Update of the 1986 original.

[3] Richard C. Bradley and Włodzimierz Bryc. Multilinear forms and measures of dependence between random variables. *J. Multivariate Anal.*, 16(3):335–367, 1985.

[4] Richard C. Bradley, Włodzimierz Bryc, and Svante Janson. On dominations between measures of dependence. *J. Multivariate Anal.*, 23(2):312–329, 1987.

[5] A. V. Bulinskiĭ. On mixing conditions of random fields. *Theor. Probab. Appl.*, 30(1):219–220, 1986.

[6] Wassilij Höffding. Maszstabinvariante Korrelationstheorie. *Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin*, 5:181–233, 1940.

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