Some More Functions That Are Not APN Infinitely Often

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Outline

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Boolean functions.

- Vectorial Boolean functions are useful in private key cryptography for designing block ciphers.

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- Two main attacks on these ciphers are differential attacks and linear attacks.
  An important criterion on boolean functions is a high resistance to the differential cryptanalysis.

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- Vectorial Boolean functions are useful in private key cryptography for designing block ciphers.
- Two main attacks on these ciphers are differential attacks and linear attacks.
  An important criterion on boolean functions is a high resistance to the differential cryptanalysis.
- Kaisa Nyberg has introduced in 1993 the notion of almost perfect nonlinearity (APN) to characterize those functions which have the best resistance to differential attacks.

APN functions

Let us consider a (vectorial) Boolean function $f : \mathbb{F}_2^m \longrightarrow \mathbb{F}_2^m$.

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Definition

The function f is said to be APN (almost perfect nonlinear) if for every $a \neq 0$ in $\mathbb{F}_2^m$ and $b \in \mathbb{F}_2^m$, there exists at most 2 elements x of $\mathbb{F}_2^m$ such that

$$f(x+a) + f(x) = b.$$

APN power functions

Up to now, the study of APN functions was especially devoted to the power functions.

The following functions $f(x) = x^d$ are APN on $\mathbb{F}_{2^m}$, where d is given by:

- $d = 2^h + 1$ where $\gcd(h, m) = 1$ (Gold functions).

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F. Hernando and G. McGuire proved recently the following :

Theorem

The Gold and Kasami functions are the only monomials where d is odd and which give APN functions for an infinity of values of m.

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In 2005, Edel, Kyureghyan and Pott proved that the function

$$\begin{array}{ccc} \mathbb{F}_{2^{10}} & \longrightarrow & \mathbb{F}_{2^{10}} \\ x & \longmapsto & x^3 + \omega x^{36} \end{array}$$

where ω is a primitive cube root of unity in $\mathbb{F}_{2^{10}}^*$ was APN and not CCZ equivalent to a power function.

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A number of people (Budaghyan, Carlet, Felke, Leander, Bracken, Byrne, Markin, McGuire, Dillon. . .) showed that some polynomials were APN and not CCZ equivalent to known power functions.

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Dillon found an APN polynomial on $\mathbb{F}_{2^6}$ which was a permutation.
First APN permutation on an even number of variables.
Not CCZ equivalent to a power function.

New Conjecture

G. McGuire proposed the following conjecture.

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We will give some results toward this conjecture.

Outline

A bound for the degree of an APN polynomial

Let $q = 2^m$ and let f be a polynomial mapping of $\mathbb{F}_q$ in itself.

- which has no term of degree a power of 2
- and with no constant term.

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We have the following result:

Theorem (FR)

Let f be a polynomial mapping from $\mathbb{F}_q$ to $\mathbb{F}_q$, d its degree.

Suppose that the surface X with affine equation

$$\frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(z + y)(x + z)} = 0$$

is absolutely irreducible.

Then f is not APN over $\mathbb{F}_{q^n}$ for all n sufficiently large.

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  has all of its rational points contained in the surface

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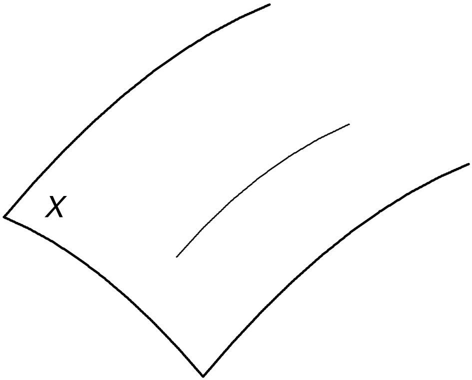
- The surface X has a number of rational points bounded by Lang-Weil bound.
- If f is APN and q too big, then the surface X has too many rational points to be contained in the surface $(x + y)(z + y)(x + z) = 0$.

Irreducibility of X

$$X : \frac{f(x)+f(y)+f(z)+f(x+y+z)}{(x+y)(z+y)(x+z)} = 0$$



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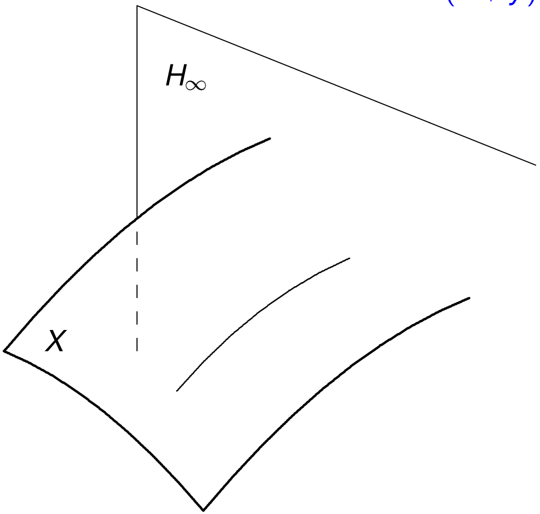

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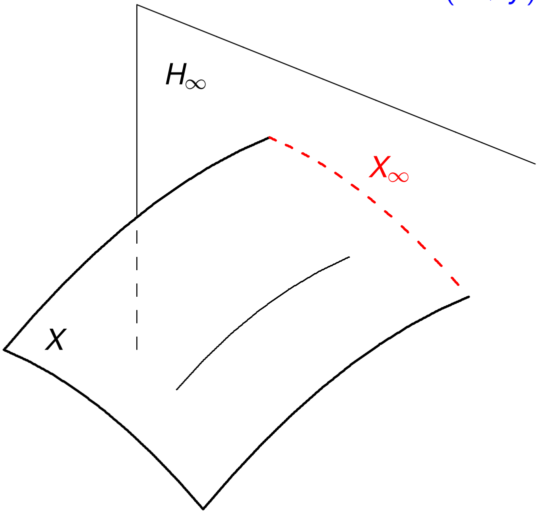

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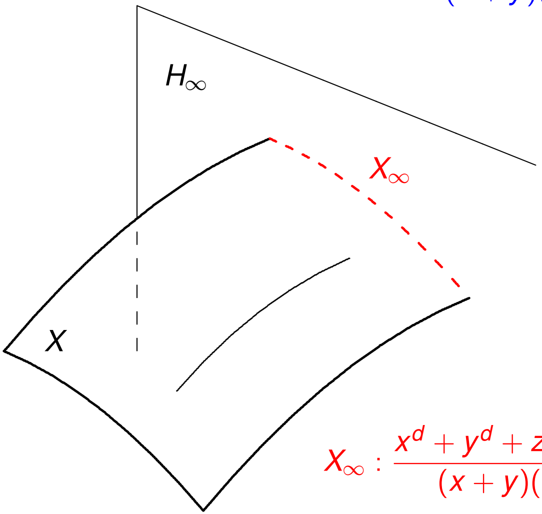

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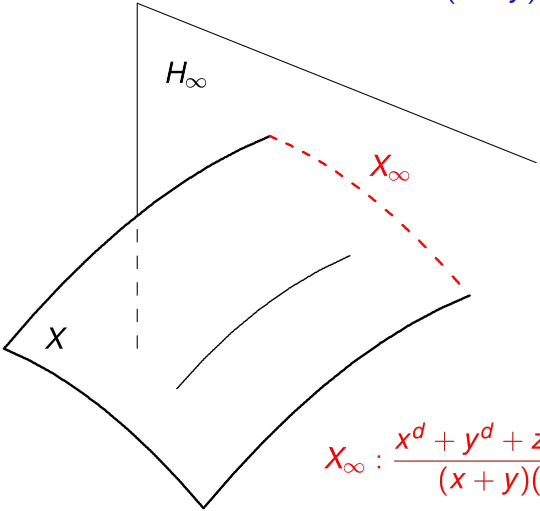

$$X_\infty : \frac{x^d+y^d+z^d+(x+y+z)^d}{(x+y)(z+y)(x+z)} = 0$$

Proposition

X_∞ *absolutely irreducible* $\Rightarrow X$ *absolutely irreducible*

Irreducibility of X_∞

Janwa, McGuire and Wilson have studied the curve X_∞ and have deduced a certain number of cases where it is absolutely irreducible.

Proposition

The curve X_∞ is absolutely irreducible for

$$d \equiv 3 \pmod 4$$

or

$$d \equiv 5 \pmod 8 \quad \textit{and} \quad d > 13.$$

Outline

The conjecture on APN functions

Irreducibility of X_∞

Let $\mathbb{F}_q$ the field of definition of f.
If the surface X is absolutely irreducible,
then the polynomial function f can be APN only for a finite number of extensions.

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Proposition (Hernando, McGuire)

The curve X_∞ of degree d has an irreducible component defined over $\mathbb{F}_2$ for d odd, not equal to Gold or Kasami exponent.

The conjecture on APN functions

Polynomials of odd degree d

Theorem (Aubry, McGuire, Rodier)

If the degree of the polynomial function f is d with d odd, not equal to Gold or Kasami exponent, then f is not APN over $\mathbb{F}_{q^n}$ for all n sufficiently large.

The conjecture on APN functions

Polynomials of degree $d = 2e$

Theorem (Aubry, McGuire, Rodier)

If the degree of the polynomial function f is $2e$ with e odd, and if f contains a term of odd degree, then f is not APN over $\mathbb{F}_{q^n}$ for all n sufficiently large.

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Let

$$\phi_r = \frac{x^r + y^r + z^r + (x+y+z)^r}{(x+y)(z+y)(x+z)}.$$

The equation of X_∞ is

$$\phi_{2e}(x,y,z) = \phi_e^2(x,y,z)(x+y)(z+y)(x+z) = 0$$

The component of X_∞ which contains the line $x + y = 0$ in the plane at infinity is defined over $\mathbb{F}_2$.
It corresponds to a component of X defined over $\mathbb{F}_q$.

The conjecture on APN functions

Polynomials of degree $d = 4e$

Theorem (FR)

If the degree of the polynomial function f is even such that $\deg(f) = 4e$ *with*

- $e \equiv 3 \pmod 4$,
- $e \not\equiv 1 \pmod 7$,
- *and* $e \geq 7$.

then f is not APN over $\mathbb{F}_{q^n}$ *for n large.*

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This case is far more intricate than the previous cases, because there are some polynomials which are CCZ equivalent to monomials for $e = 3$.

Sketch of proof. We have in this case:

$$\phi_d(x,y,z) = \phi_e(x,y,z)^4((x+y)(x+z)(y+z))^3$$

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  $$P(x,y,z) = c_1(x^2+y^2+z^2) + c_2(xy+xz+zy) + b(x+y+z) + d$$

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Investigating these polynomials, one proves that

- they may divide ϕ only if $c_1 = c_2$, $b = 0$, $d = c_1^3$
- in this case there is an irreducible subcomponent defined over $\mathbb{F}_q$

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Investigating these polynomials, one proves that

- they may divide ϕ only if $c_1 = c_2$, $b = 0$, $d = c_1^3$
- in this case there is an irreducible subcomponent defined over $\mathbb{F}_q$ among the components which contain the curves $\phi_e(x, y, z)^4$.

The conjecture on APN functions

Polynomials of degree $d = 4 \times 3$

Theorem (FR)

If the degree of the polynomial f is 12, then

- *either f is not APN over $\mathbb{F}_{q^n}$ for n large*
- *or f is CCZ equivalent to the Gold function x^3.*

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- *or f is CCZ equivalent to the Gold function x^3.*

Let $d \in \mathbb{F}_{q^3}$ such that $d + d^q + d^{q^2} = 0$. The polynomials CCZ equivalent to the Gold function x^3 are the polynomial functions

$$f(x) = L(x^3) \quad \text{or} \quad f(x) = L(x)^3$$

with

$$L(x) = x^4 + (d^{1+q} + d^{1+q^2} + d^{q+q^2})x^2 + d^{1+q+q^2}x$$

and the polynomials composed with f and an affine permutation.

The conjecture on APN functions

Gold degree

Theorem (Aubry, McGuire, Rodier)

Suppose $f(x) = x^d + g(x)$ *where*
the degree of f *is* $d = 2^k + 1$ *and* $\deg(g) \leq 2^{k-1} + 1$.
Suppose moreover that there exists a nonzero coefficient of x^r *in* g *such that*

$$\frac{x^r + y^r + z^r + (x + y + z)^r}{(x + y)(z + y)(x + z)}$$

is irreducible.
Then X *is absolutely irreducible.*

So f *is not APN over* $\mathbb{F}_{q^n}$ *for* n *large.*

The conjecture on APN functions

Kasami degree

Theorem

Suppose $f(x) = x^d + g(x)$ *where*
the degree of f is $d = 2^{2k} - 2^k + 1$ *and* $\deg(g) \leq 2^{2k-1} - 2^{k-1} + 1$.
Suppose moreover that there exists a nonzero coefficient of x^r *in g such that*

$$\frac{x^r + y^r + z^r + (x+y+z)^r}{(x+y)(z+y)(x+z)}$$

is irreducible.
Then X is absolutely irreducible.

So f is not APN over $\mathbb{F}_{q^n}$ *for n large.*

Outline

Conclusion

Criteria for Boolean functions

We have shown that many polynomials cannot be APN

if their degrees are too large with respect to the number of variables.

It is a consequence of Lang-Weil bound on some surfaces on finite fields.

To prove the conjecture on APN functions we have

- to prove the bound for several classes of degrees not Gold or Kasami;
  I mean $d = 2^i(2^i\ell + 1)$ with $\ell \neq 1$ and $\ell \neq 2^i - 1$ and $i \geq 2$.

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- to prove the bound for several classes of degrees not Gold or Kasami;
  I mean $d = 2^i(2^i\ell + 1)$ with $\ell \neq 1$ and $\ell \neq 2^i - 1$ and $i \geq 2$.
- to study polynomials of Gold or Kasami degree.

THANK YOU