Attacks on SIDH and applications

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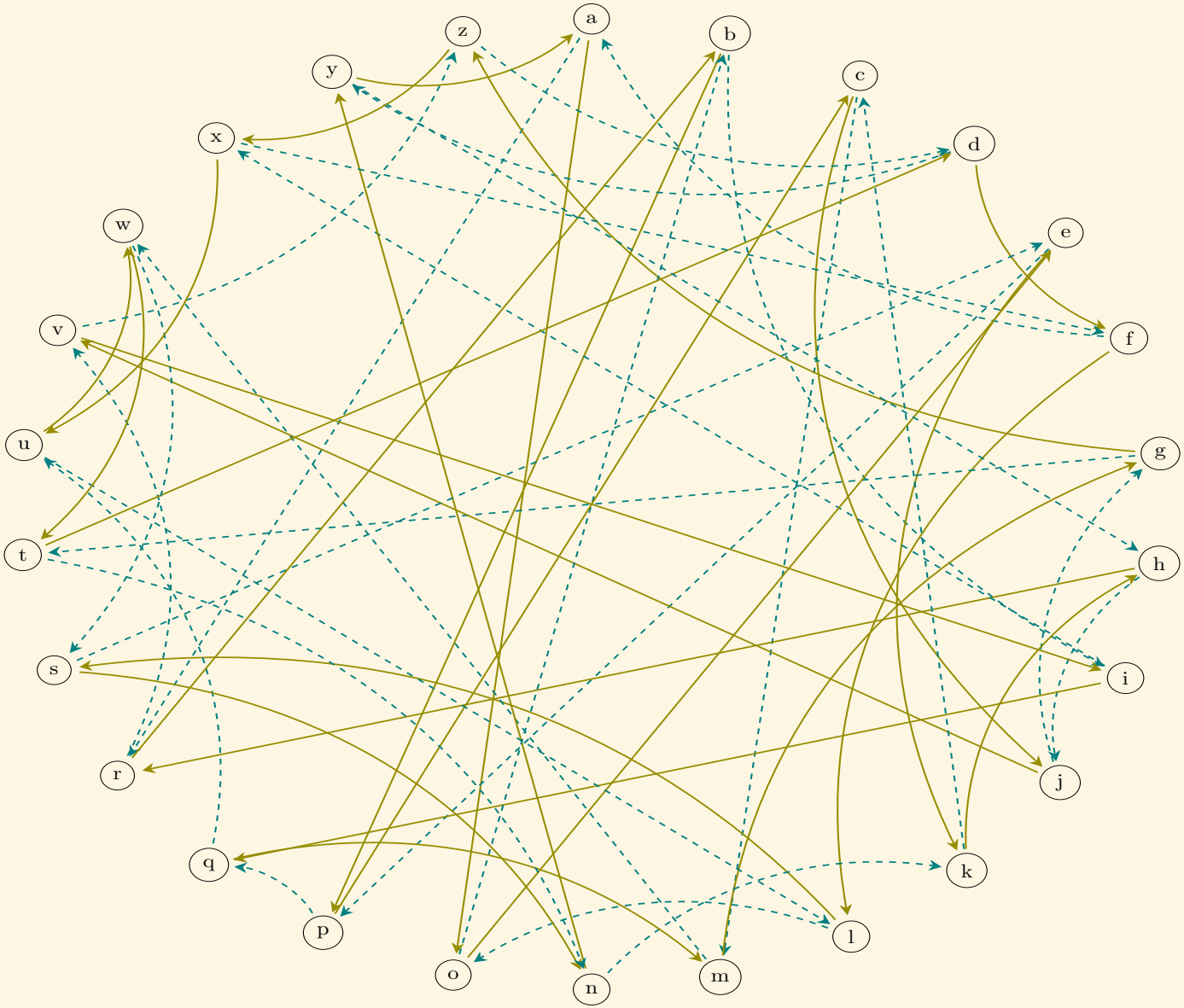
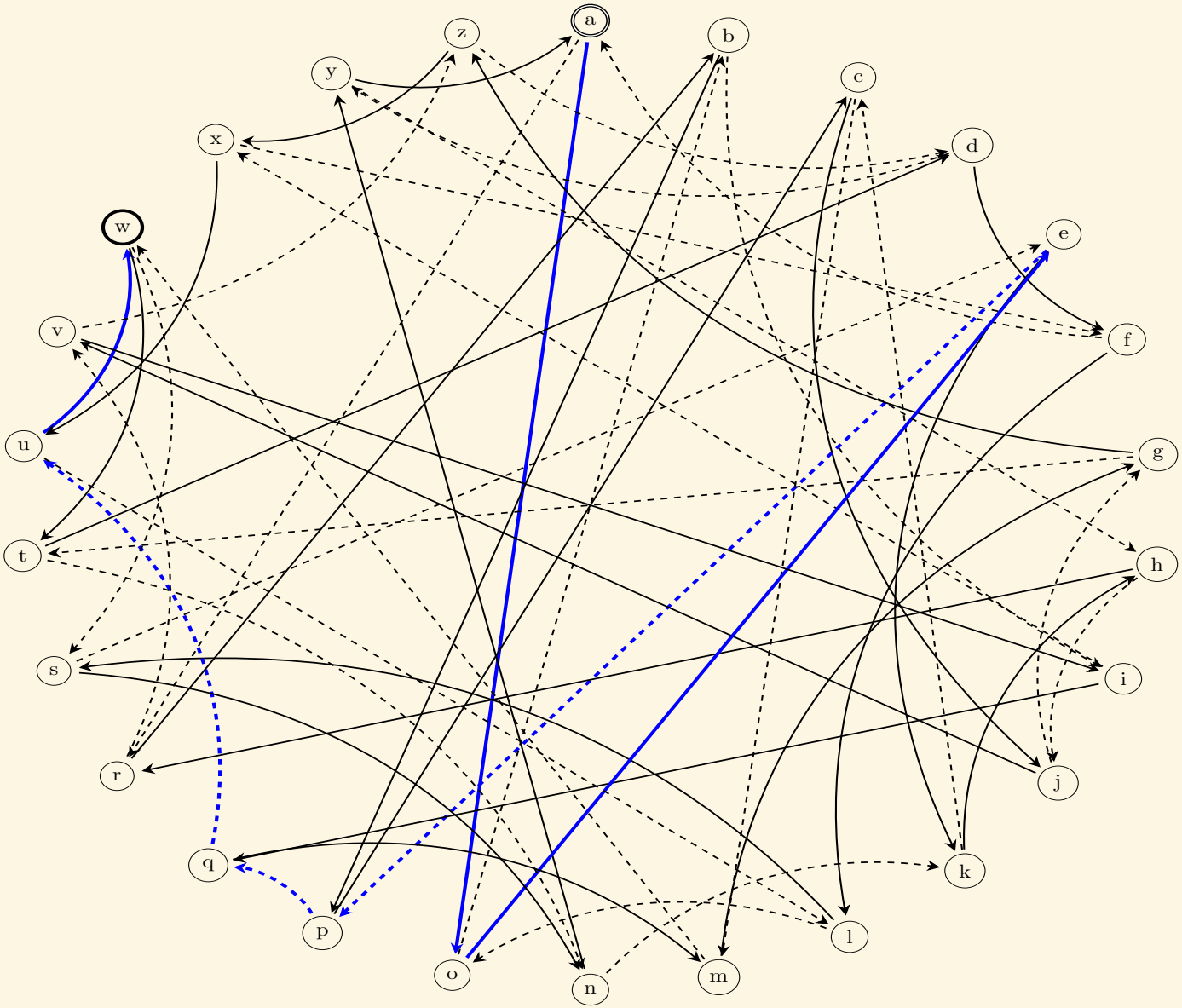
Key exchange on a (commutative) graph



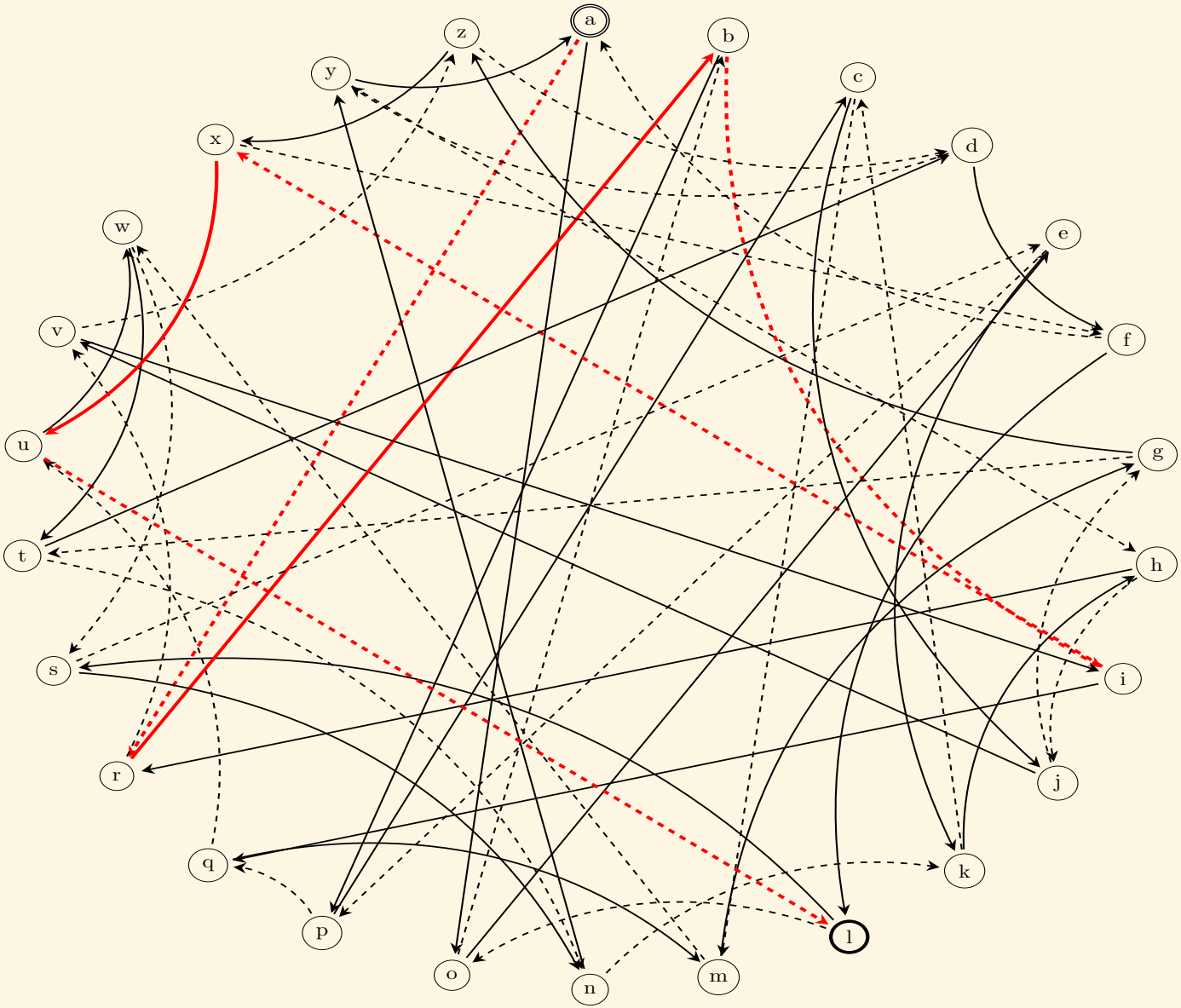
Key exchange on a (commutative) graph

Alice starts from ‘a’, follows the path oo1110, and get ‘w’.



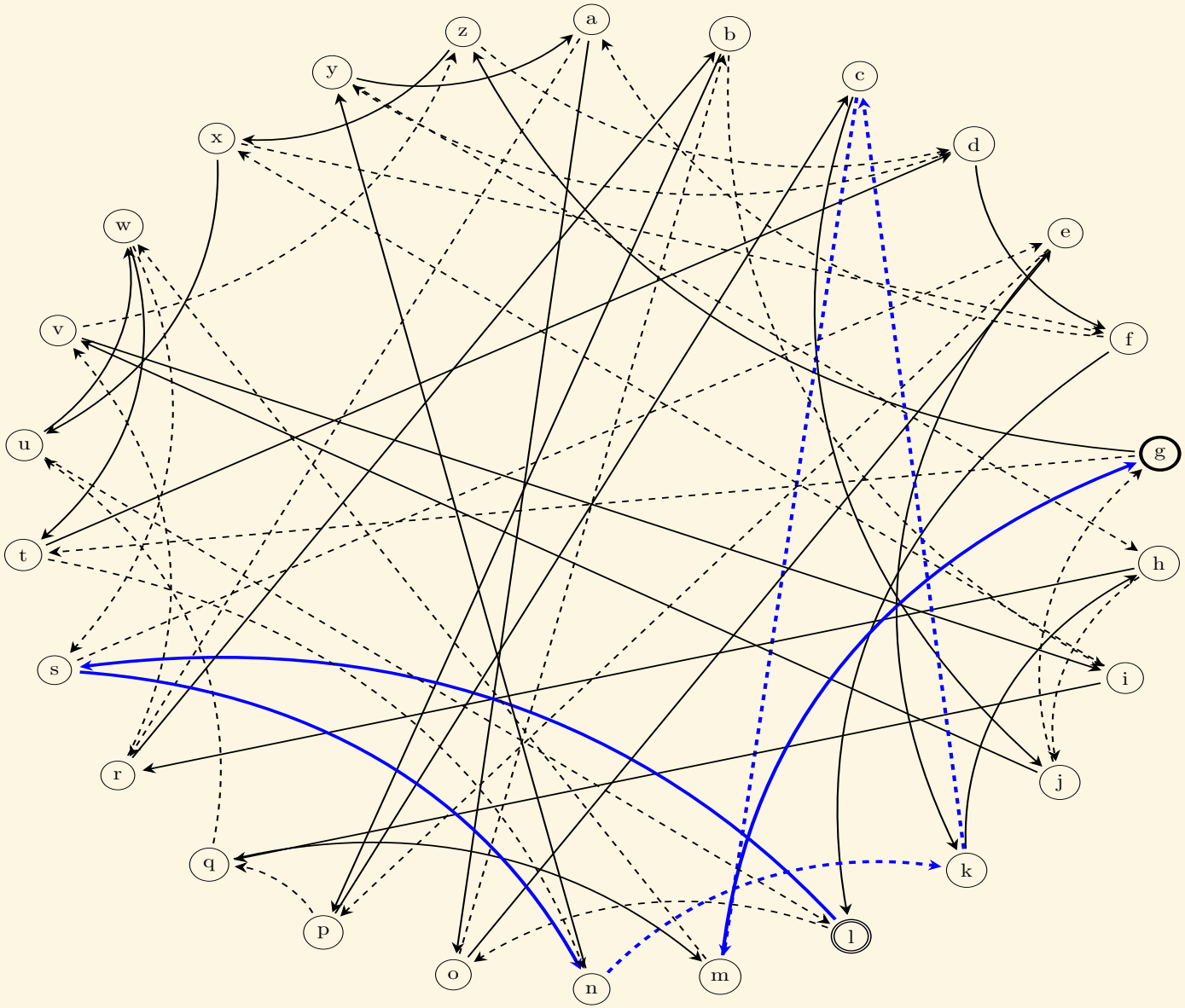
Key exchange on a (commutative) graph

Bob starts from 'a', follows the path 101101, and get 'l'.



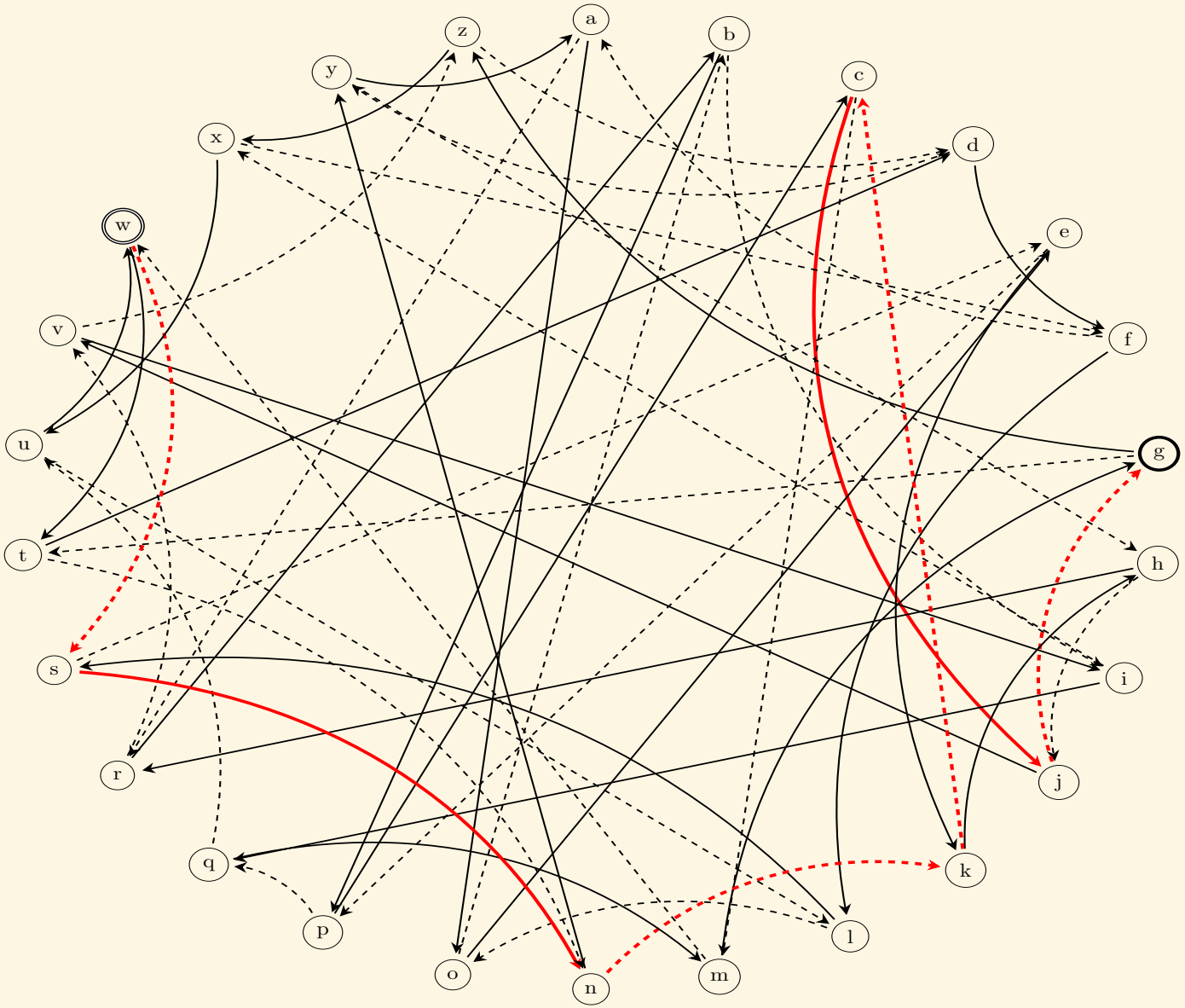
Key exchange on a (commutative) graph

Alice starts from 'l', follows the path 001110, and get 'g'.



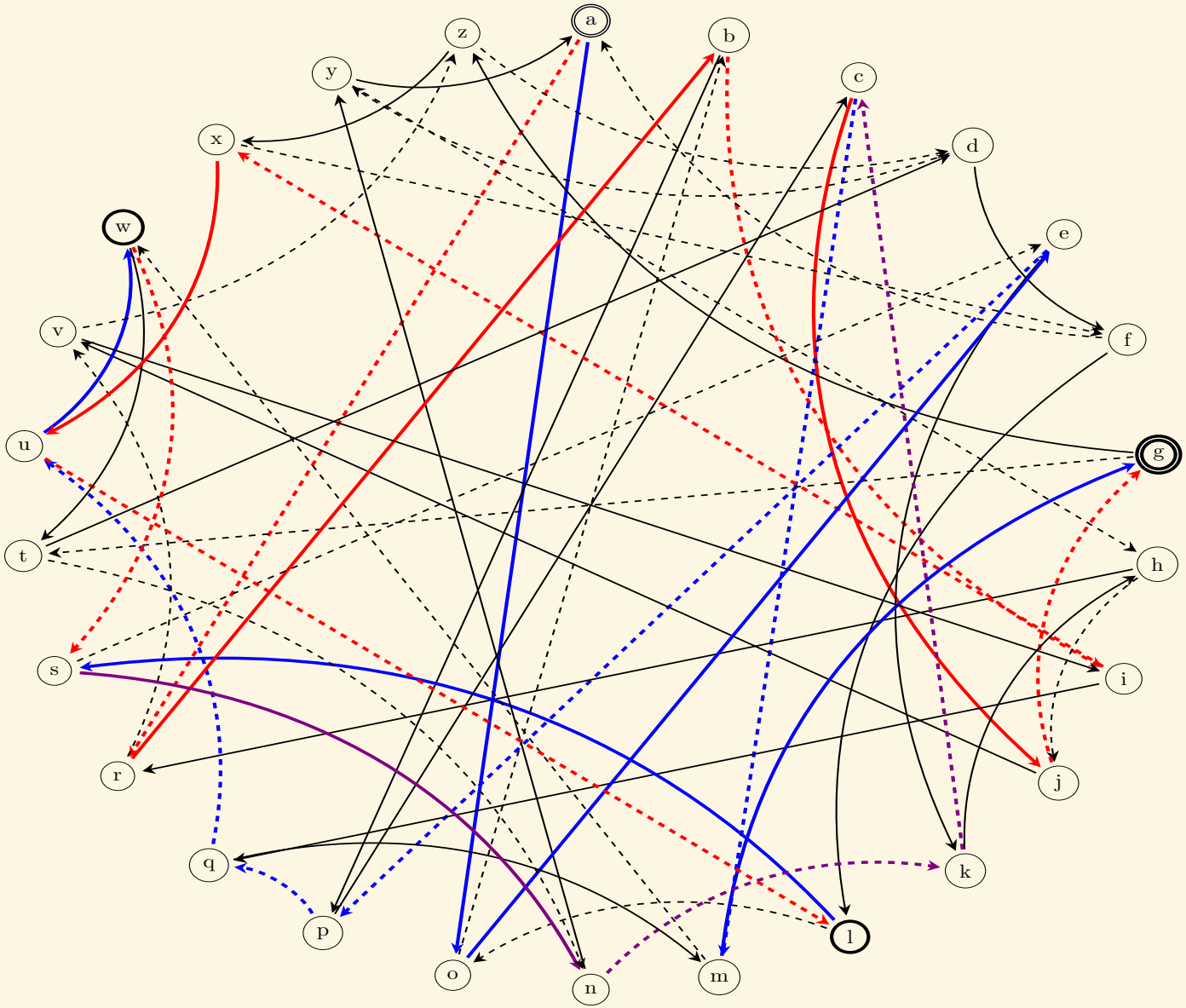
Key exchange on a (commutative) graph

Bob starts from ‘w’, follows the path 101101, and get ‘g’.



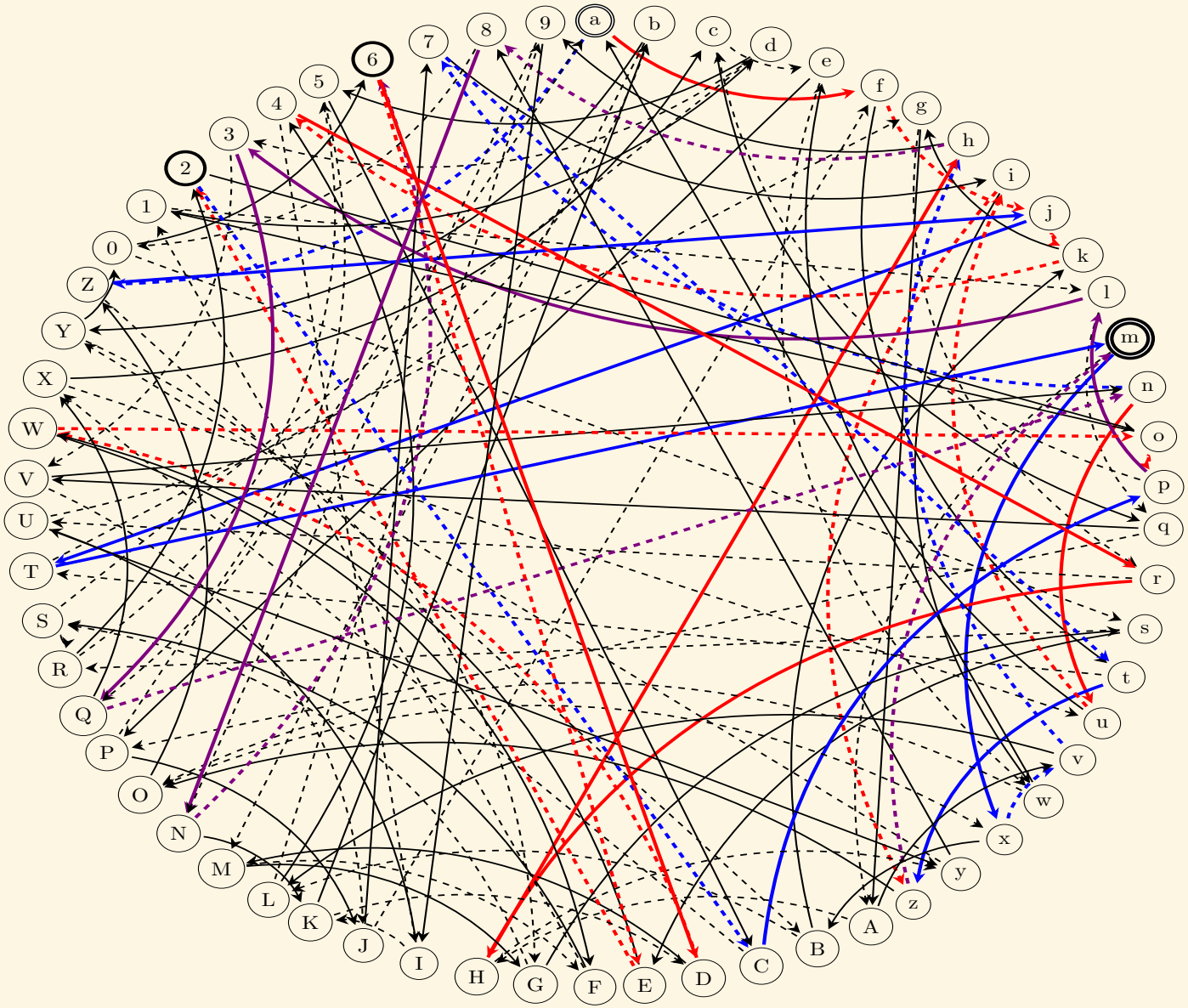
Key exchange on a (commutative) graph

The full exchange:



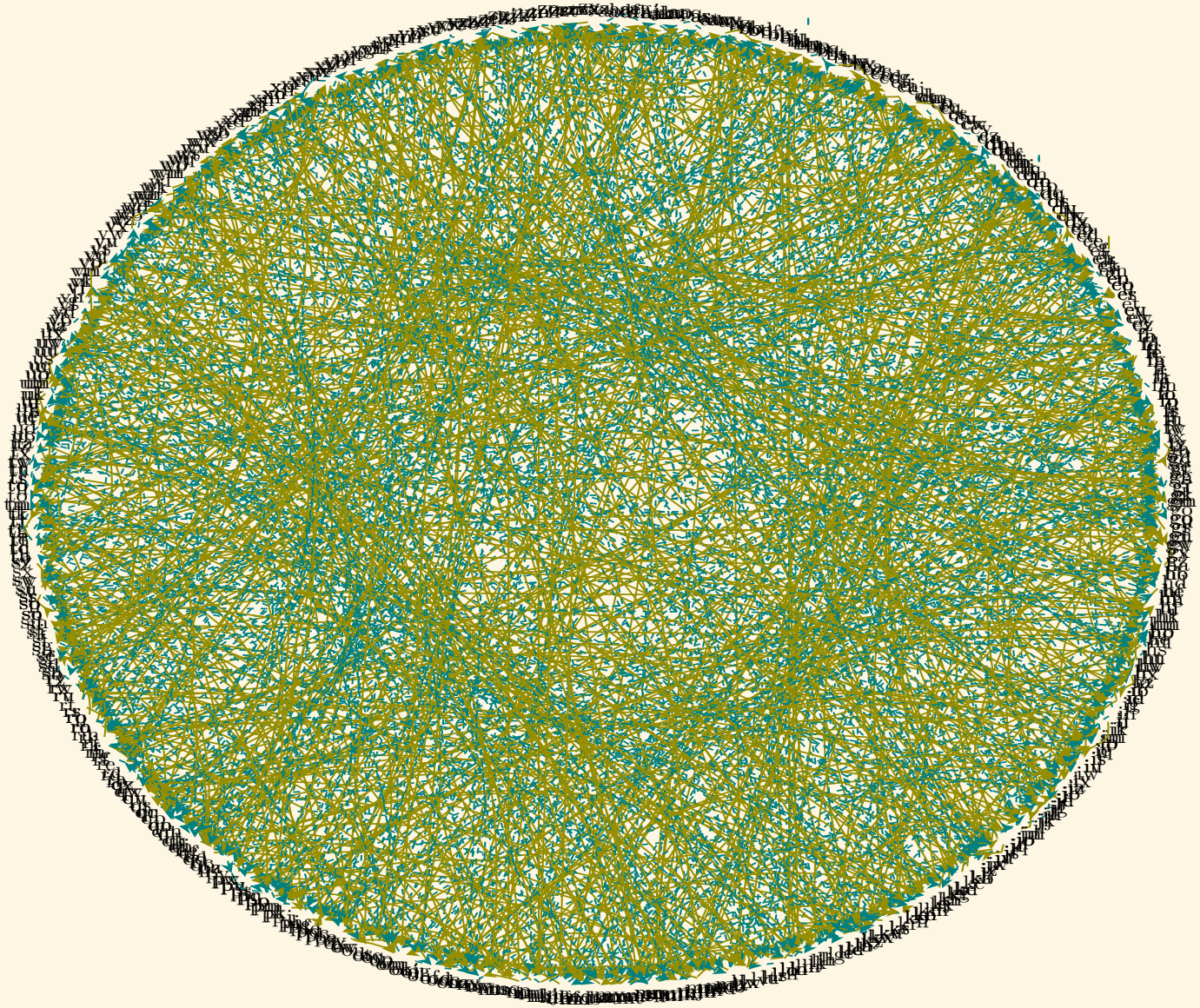
Key exchange on a (commutative) graph

Bigger graph (62 nodes)



Key exchange on a (commutative) graph

Even bigger graph (676 nodes)



Graphs for key exchange

- Needs a graph with good mixing properties:
  A path of length $O(\log N)$ gives a uniform node $\Rightarrow$ Ramanujan/expander graph.
- The graph does not fit in memory $(N = 2^{256})$.
- Needs an algorithm taking a node as input and giving the neighbour nodes as output.

- Isogeny graphs of elliptic curves

Isogenies

- Elliptic curve: $E/k : y^2 = x^3 + ax + b$. Algebraic group law!

- Isogeny: $\phi : E_1 \to E_2$ with $\phi(0_{E_1}) = 0_{E_2}$
- $\phi(P+Q) = \phi(P) + \phi(Q)$

$$\phi(x,y) = \left(\frac{g(x)}{h(x)}, cy \left(\frac{g(x)}{h(x)} \right)' \right)$$

- Isogeny $\phi \Leftrightarrow$ Kernels $K = \mathrm{Ker}\,\phi$

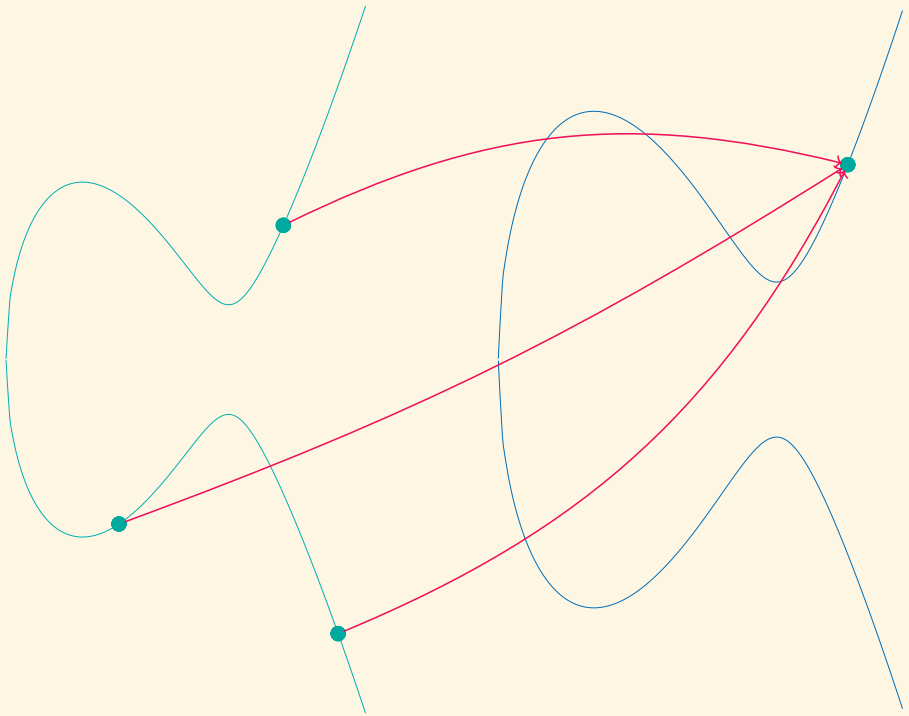
Isogeny based cryptography:

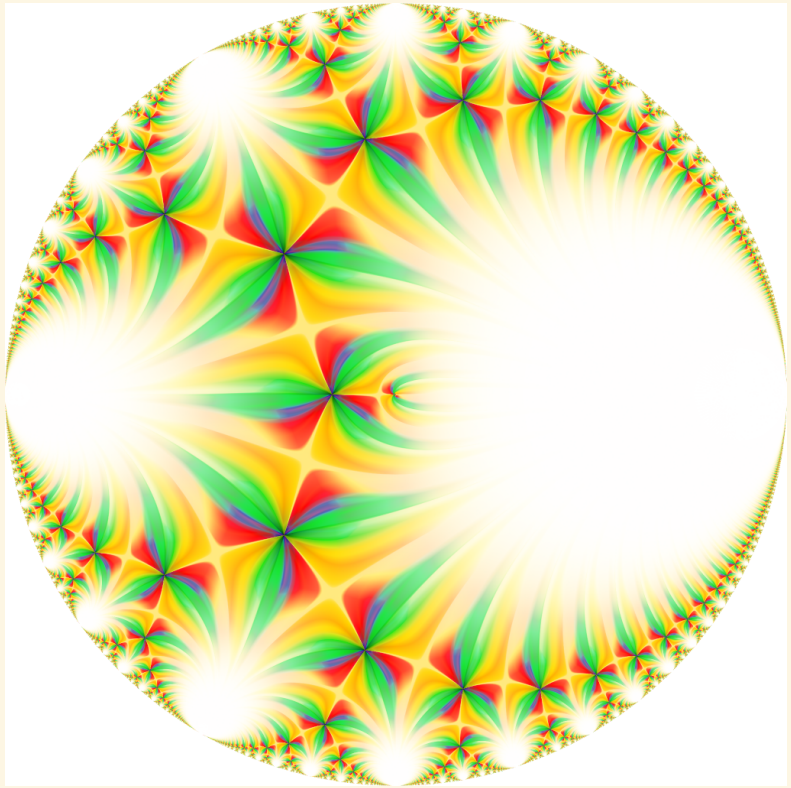
- Computing an isogeny $\phi : E_1 \to E_2$: Easy!
- Given (E_1, E_2), find an isogeny path $\phi : E_1 \to E_2$: Hard! (Even for quantum computers!)

$\Rightarrow$ Post quantum cryptosystems

Isogenies

$E_1 : y^2 = x^3 + a_1 x + b_1$ $E_2 : y^2 = x^3 + a_2 x + b_2$





Isogeny graphs for key exchange

- Isogeny graph of ordinary elliptic curves $E/\mathbb{F}_p$
  [Couveignes (1997)], [Rostovtsev–Stolbunov (2006)]
- Graph of size $N \approx \sqrt{p}$.

☺ Commutative graph!

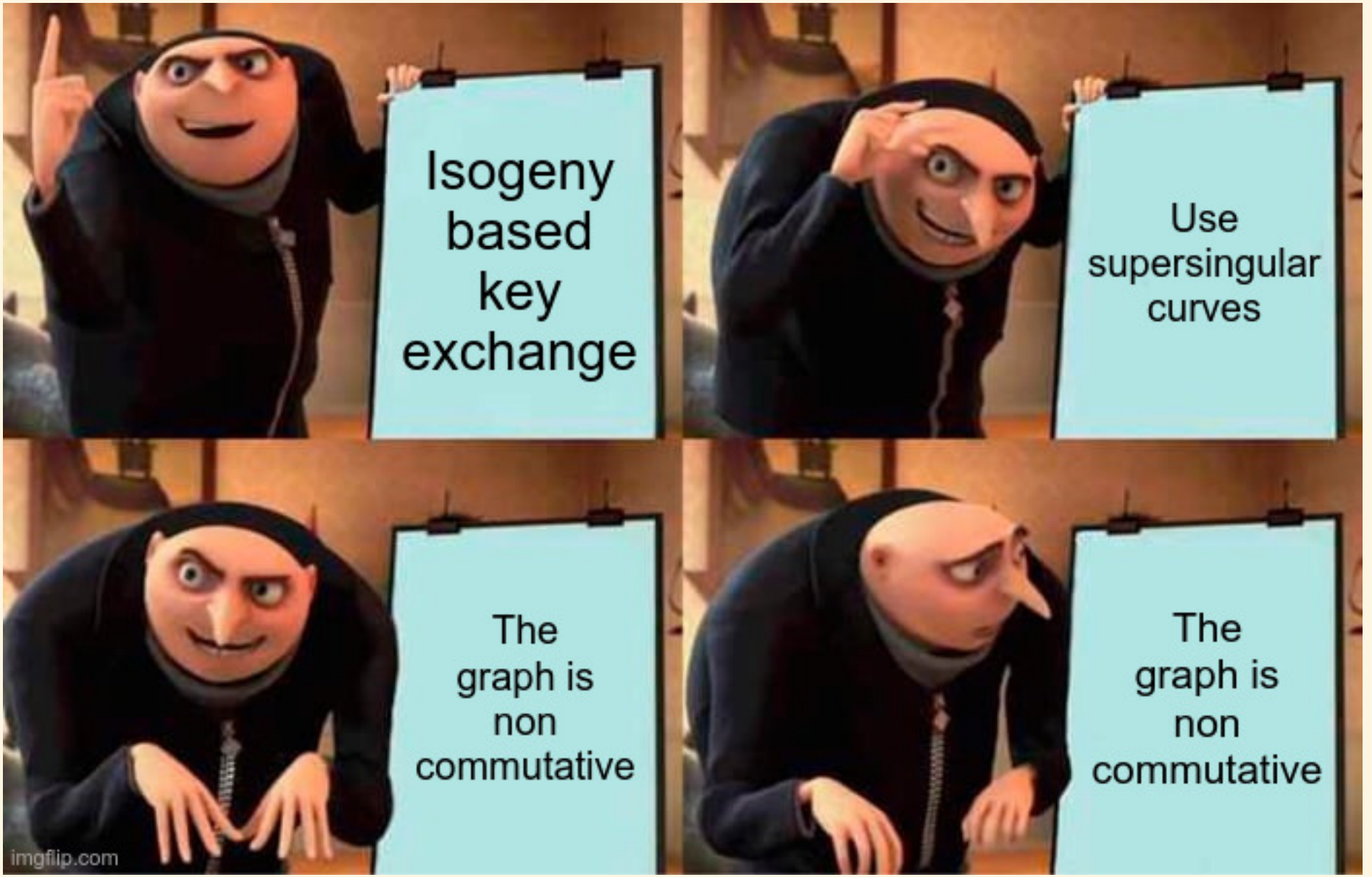
☺ Group action framework! (By $\mathbf{Cl}(\mathbf{End}(E_0))$)

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_{\mathfrak{a}}} & E_a \\ \downarrow{\scriptstyle \phi_{\mathfrak{b}}} & & \downarrow{\scriptstyle \phi_{\mathfrak{b}}} \\ E_b & \xrightarrow{\phi_{\mathfrak{a}}} & E_{ab} \end{array}$$

☹ Hidden shift problem solvable in quantum subexponential $L(1/2)$ time for an abelian group action via Kuperberg's algorithm.

- SIDH: supersingular elliptic curve Diffie-Hellmann [De Feo, Jao (2011)],[De Feo, Jao, Plût (2014)]
- Use the isogeny graph of a supersingular elliptic curve E over $\mathbb{F}_{p^2}$ ($N \approx p$).

Isogeny graphs for key exchange



SIDH in practice

- $p = 2^a 3^b - 1$. $N_A = 2^a, N_B = 3^b$
- $E_0 : y^2 = x^3 + x$ (supersingular when $a \geq 2$)
- $E_0[N_A] = \langle P_A, Q_A \rangle, E_0[N_B] = \langle P_B, Q_B \rangle$.
- Alice's secret isogeny: ϕ_A of kernel $\langle P_A + s_A Q_A \rangle$.
- Bob's secret isogeny: ϕ_B of kernel $\langle P_B + s_B Q_B \rangle$.
- Key exchange:

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_B} & E_B \\ \downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi'_A} \\ E_A & \xrightarrow{\phi'_B} & E_{AB} \end{array}$$

- E_{AB} is the shared secret.
- $\phi'_A \circ \phi_B = \phi'_B \circ \phi_A : E_0 \to E_{AB}$ has kernel $\operatorname{Ker} \phi_A + \operatorname{Ker} \phi_B$.

- ϕ'_A has kernel $\langle \phi_B(P_A + s_A Q_A) \rangle$, ϕ'_B has kernel $\langle \phi_A(P_B + s_B Q_B) \rangle$.
- Alice publishes: $P'_B = \phi_A(P_B), Q'_B = \phi_A(Q_B)$.
  Bob publishes: $P'_A = \phi_B(P_A), Q'_A = \phi_B(Q_A)$. ("Torsion points".)
- $\operatorname{Ker} \phi'_A = \langle P'_A + s_A Q'_A \rangle$, $\operatorname{Ker} \phi'_B = \langle P'_B + s_B Q'_B \rangle$.

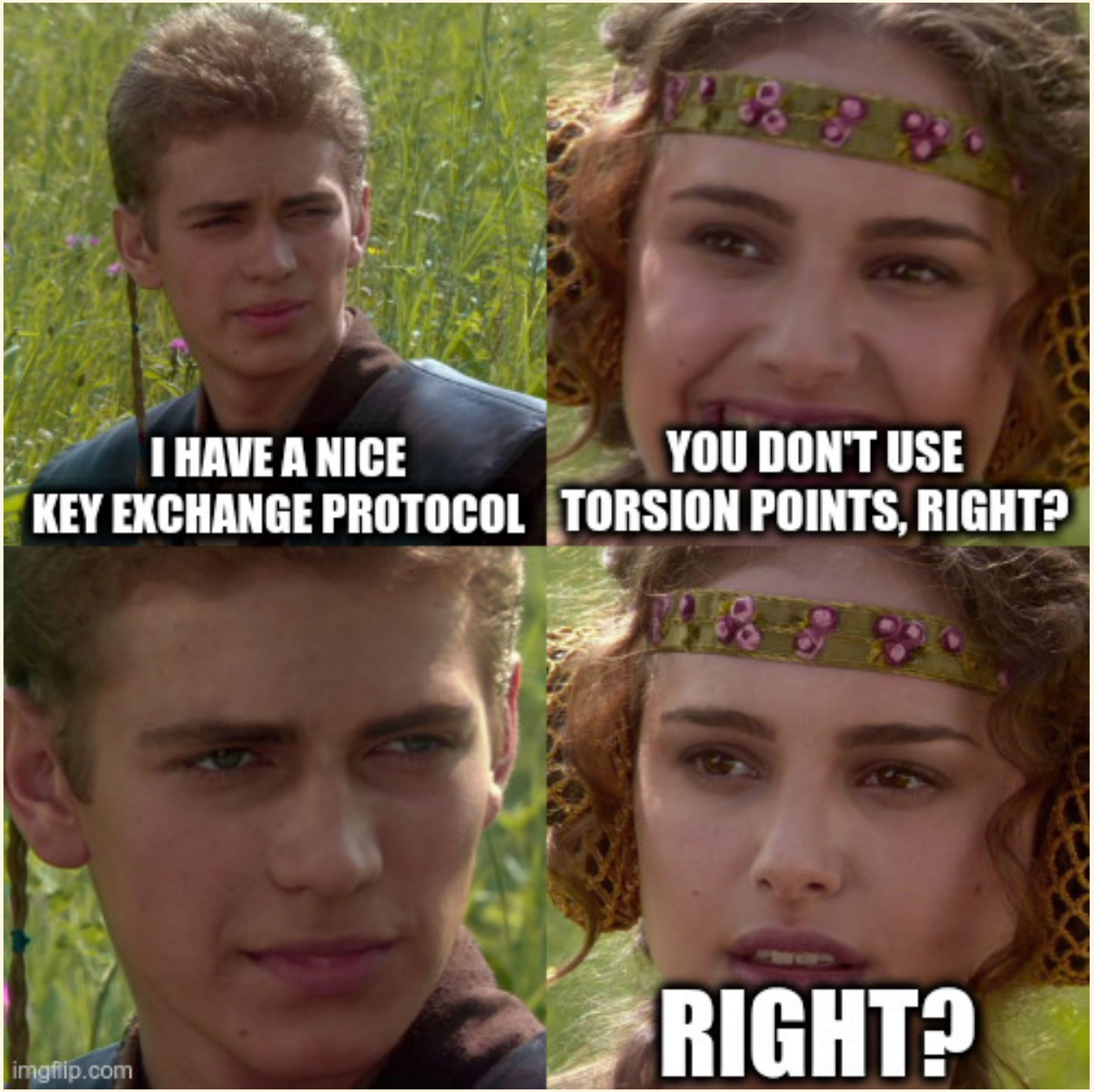
Isogeny evaluation and interpolation

- Evaluation: given an n-isogeny ϕ and a point $Q \in E(\mathbb{F}_q)$, evaluate $\phi(Q)$.
- n-evaluation problem: ϕ is an n-isogeny = Ker ϕ is of degree n.

- Interpolation: given a tuple $(P, \phi(P))$, recover ϕ.
- (n, N)-interpolation problem: given ϕ an n-isogeny and P a point of N-torsion, from $(P, \phi(P))$ and $Q \in E(\mathbb{F}_q)$, evaluate $\phi(Q)$
- Weak interpolation: we are given $(P_1, \phi(P_1))$, $(P_2, \phi(P_2))$ for (P_1, P_2) a basis of $E[N]$.

- SIDH: the key exchange uses the N_A and N_B evaluation problems
- If we can solve the weak interpolation problem when $n = N_A, N = N_B$ are smooth in polylogarithmic time, we can break SIDH.

Isogeny evaluation and interpolation



Evaluation

- $\phi : E_1 \to E_2$ an n-isogeny, $\phi(x,y) = \left(\frac{g(x)}{h(x)}, cy\left(\frac{g(x)}{h(x)}\right)'\right)$, $\deg g, \deg h \leq n$
- $K : h(x) = 0, h(x) = \prod_{P \in K - 0_E}(x - x(P))$. If $E : y^2 = f(x)$, [Kohel 1996]:

$$\phi(x,y) = \left(\frac{g(x)}{h(x)}, y\left(\frac{g(x)}{h(x)}\right)'\right)$$

$$\frac{g(x)}{h(x)} = \#K.x - \sigma - f'(x)\frac{h'(x)}{h(x)} - 2f(x)\left(\frac{h'(x)}{h(x)}\right)'$$

- Kernel representation: Linear time and linear space.

- $\ker f = \langle T \rangle, T \in \mathbb{F}_{q^d}$, evaluate $\phi(Q)$ in $O(n)$ operations in $\mathbb{F}_{q^d}$ [Vélu 1971] :

$$x(f(P)) = x(P) + \sum_{i=1}^{n-1}(x(P+iT) - x(iT))$$

$$y(f(P)) = y(P) + \sum_{i=1}^{n-1}(y(P+iT) - y(iT))$$

- $\sqrt{\text{élu}}$: $\widetilde{O}(\sqrt{n})$ (time/memory trade off)
- Generator representation: Compact representation if d small.

Decomposing a smooth degree isogeny

- $\phi : E_1 \to E_2, K = \mathrm{Ker}\,\phi = \langle T \rangle$ of degree $n = 2^a, T \in \mathbb{F}_{q^d}$
- $\phi = \phi_1' \circ \phi_1$
- $\phi_1 : E_1 \to E_1'$ of degree 2 with kernel $K_1 = \langle 2^{a-1} T \rangle$
- $\phi_1' : E_1' \to E_2$ of degree 2^{a-1} with kernel $K = \langle \phi_1(T) \rangle$

- Complexity: $O(a^2)$ arithmetic operations in $\mathbb{F}_{q^d}$
- [De Feo, Jao, Plût 2011]: $\widetilde{O}(a)$ operations in $\mathbb{F}_{q^d}$

d can be large, $d = \Theta(n)$ in the worst case $\Rightarrow$ quasi-linear time

- In SIDH: $N_A = 2^a$ and $N_B = 3^b$ and the N_A, N_B-torsion points are rational, so the decomposition is fast!

- Can decompose isogenies of smooth degree N (if the N-torsion is accessible)

Interpolation

- Given $(P, \phi(P))$, P a point of order $N > 4n$, recover the rational function $\frac{g(x)}{h(x)}$ in $\widetilde{O}(N)$ by interpolating the points $(x(mP), x(m\phi(P))), m = 1, \dots, N-1$.
- Can evaluate on Q directly.
- Quasi-linear time.

- Faster algorithm when N is smooth?
- Yes if $\phi(P) = 0$. Then $n = N$ and $\mathrm{Ker}\,\phi = \langle P \rangle$.
- If $n = N$, the weak interpolation problem reduces via the DLP to the N-evaluation problem.
- This is why the SIDH key exchange is fast: Bob uses the torsion point information published by Alice to find the kernel of his pushforward isogeny.
- No reason to expect a fast algorithm when N is prime to n.

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Revisiting isogeny evaluation

- Can an n-isogeny be evaluated faster than linear time when n has a large prime factor?
- If $\phi = [\ell]$ (so $n = \ell^2$): double and add in $O(\log \ell)$ to evaluate ℓQ.

- $\Phi : E^2 \to E^2, (P_1, P_2) \mapsto (P_1 + P_2, P_1 - P_2)$ is a 2-isogeny in dimension 2.
- $\Phi = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$
- Double: $\Phi(T, T) = (2T, 0)$.
- Add: $\Phi(T, Q) = (T + Q, T - Q)$.

- We can evaluate ℓQ as a composition of $O(\log \ell)$ evaluations of Φ, projections $E^2 \to E$ and embeddings $E \to E^2$.
- Double and add on E = 2-isogenies in dimension 2

Kani's lemma [Kani 1997] ($g = 1$), [R. 2022] ($g > 1$)

- $\alpha : A \to B$ a a-isogeny, $\beta : A \to C$ a b-isogeny.
- $\alpha' : C \to D$ a a-isogeny, $\beta' : C \to D$ a b-isogeny with $\beta'\alpha = \alpha'\beta$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ C & \xrightarrow{\alpha'} & D \end{array}$$

- If a prime to b, the pushforward α', β' of α by β satisfy these conditions.

- $\Phi = \begin{pmatrix} \alpha & \widetilde{\beta'} \\ -\beta & \widetilde{\alpha'} \end{pmatrix} : A \times D \to B \times C.$
- $\widetilde{\Phi} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \beta' & \alpha' \end{pmatrix} : B \times C \to A \times D, \quad \widetilde{\Phi}\Phi = a + b.$
- Φ is an $a + b$-isogeny with respect to the product polarisations.
- $\operatorname{Ker} \Phi = \{\tilde{\alpha}(P), \beta'(P) \mid P \in B[a + b]\}$ (if a is prime to b)

Using Kani's lemma for the interpolation problem

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ E_1' & \xrightarrow{\phi'} & E_2' \end{array}$$

- $\phi : E_1 \to E_2$ an n-isogeny.
- Goal: replace ϕ by Φ an N-isogeny.
- Find $\alpha : E_1 \to E_1'$ an m-isogeny, with $N = n + m$.
- Kani's lemma: $\Phi = \begin{pmatrix} \alpha & \widetilde{\phi'} \\ -\phi & \widetilde{\alpha'} \end{pmatrix} : E_1 \times E_2' \to E_1' \times E_2$ is an N-isogeny.
- We know $\phi(E[N])$ and we can evaluate α on $E[N] \Rightarrow$ recover $\mathrm{Ker}\ \Phi$ (or $\mathrm{Ker}\ \widetilde{\Phi}$)
- Evaluate Φ, hence ϕ at any point: $\Phi(P, 0) = (\alpha(P), -\phi(P))$.
- Evaluation is fast if N is (power) smooth.

Examples:

- m smooth [Castryck–Decru; Maino–Martindale (2022)]
- $m = \ell^2$: take $\alpha = [\ell]$
- $\mathrm{End}(E)$ has an efficient endomorphism α of norm m [Castryck–Decru; Wesolowski (2022)].

Using Kani's lemma for the interpolation problem

The general case: Zahrin's trick

- $\alpha = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}$ is always an endomorphism of norm $m = a_1^2 + a_2^2$ on E^2
- Gaussian integers $\mathbb{Z}[i]$

- $\alpha = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix}$ is always an endomorphism of norm $m = a_1^2 + a_2^2 + a_3^2 + a_4^2$ on E^4
- Hamilton's quaternion algebra
- Evaluating α: $O(\log m)$ arithmetic operations
- Every integer is a sum of four squares.

$$\begin{array}{ccc} E_1^4 & \xrightarrow{\phi} & E_2^4 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ E_1^4 & \xrightarrow{\phi} & E_2^4 \end{array}$$

- $\Phi : E_1^4 \times E_2^4 \to E_1^4 \times E_2^4$ is an N-isogeny.

Kani's lemma + Zahrin's trick = the embedding lemma [R. 2022]

- A n-isogeny $\phi : A \to B$ in dimension g can always be efficiently embedded into a N isogeny $\Phi : A' \to B'$ in dimension $8g$ (and sometimes $4g, 2g$) for any $N \geq n$.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \twoheadrightarrow \\ A' & \xrightarrow{\Phi} & B' \end{array}$$

- Considerable flexibility (at the cost of going up in dimension).

- Reduces the weak (n, N)-interpolation problem to the N-evaluation problem in higher dimension
- Actually only need the image of ϕ on a subgroup of size $N, N > 4n$ (via further tricks by Castryck, De Feo, R., Wesolowski…)

$\Rightarrow$ Solves the interpolation problem when N is (power) smooth

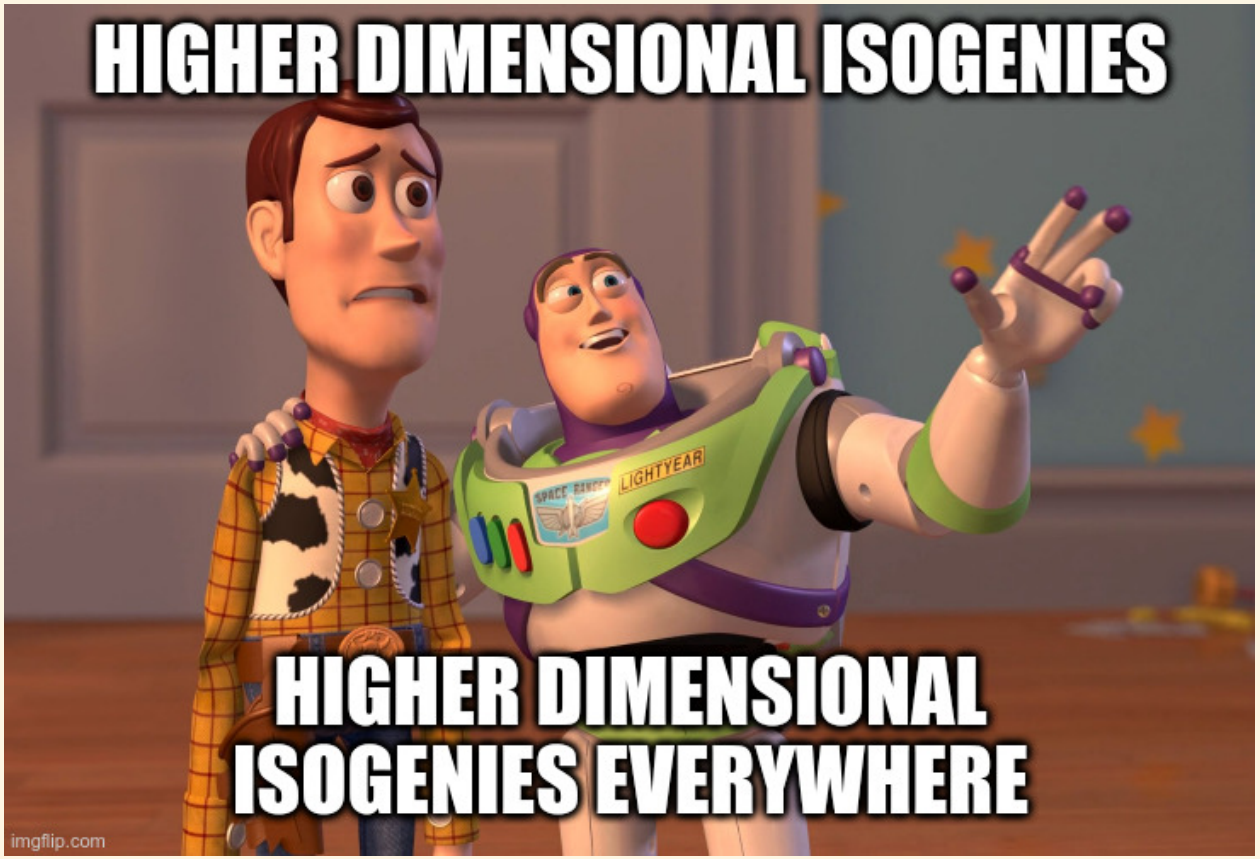
- Amazing fact: does not requires $\ker \phi$, works even if n is prime

- Breaks SIDH: [Castryck–Decru], [Maino–Martindale] in dimension 2, [R.] in dimension 4 or 8

Kani's lemma + Zahrin's trick = the embedding lemma [R. 2022]

Cryptographic applications

Castryck's invited talk at Eurocrypt 2024: "An attack became a tool: Isogeny based cryptography 2.0"



Isogeny representations

- Before 2022: could only compute smooth degree isogenies $\phi : E_1 \to E_2$ (with accessible kernel points)

- Isogeny based cryptography: correspondance between ideals $I \subset R$ and certain isogenies $\phi_I : E_1 \to E_2$
- Supersingular isogeny graph over $\mathbb{F}_{p^2}$: R is a non commutative quaternionic order. Every isogeny comes from an ideal
  Deuring's correspondance
- Ordinary isogeny graph or supersingular isogeny graph over $\mathbb{F}_p$: R is a (commutative) quadratic imaginary order.
  ☺Class group action!

Translating an ideal I to an isogeny ϕ_I:

- Needs to find a smooth equivalent ideal $J \sim I$
- KLPT: heuristic polynomial smoothening algorithm for R quaternion algebra

☹ J has very large norm $\approx p^{4.5}$

☹ If R quadratic order, only subexponential time smoothening algorithms known

☹ Restricted group action

The HD representation

- Embed $\phi : E_1 \to E_2$ into an N-isogeny Φ in dimension g ($N \geq n$)
- Represent Φ by its kernel $\mathrm{Ker}\,\Phi$:
  $\mathrm{Ker}\,\Phi$ is completely determined by n and the action of ϕ on $E_1[N]$
- CRT basis: $N = \prod_{i=1}^{m} N_i = \prod_{i=1}^{m} \ell_i^{e_i}$,

$$(P_i, Q_i, \phi(P_i), \phi(Q_i)), \quad \text{for } (P_i, Q_i) \text{ a basis of } E[\ell_i^{e_i}]$$

- Naive algorithm: reconstruct ϕ in $\widetilde{O}(n)$ via rational function interpolation
- HD approach: exploit the N-torsion structure by going to Φ in higher dimension

- Can take any $N \geq n$ (Example: N powersmooth)
- Ideal scenario: E_1 has rational $N = 2^m$-torsion and Φ in dimension 2
- Compact and efficient isogeny representation

- Universal: can be efficiently recovered from any other efficient isogeny representation of ϕ
- Philosophy: if we know how ϕ act on sufficiently many nice points, we can efficiently compute $\phi(P)$ for any point P

Algorithms for N-isogenies in higher dimension

- Analogues of Vélu's formula: [Cosset, R. (2014); Lubicz, R. (2012–2022)]
  An N-isogeny in dimension g can be evaluated in linear time $O(N^g)$ arithmetic operations in the theta model given generators of its kernel.

☺ Work in any dimension

☹ Exponential dependency 2^g in the dimension g.

☹ Need a rational level $\Gamma(2,4)$-structure (automatic for supersingular curves over $\mathbb{F}_{p^2}$)

- Algorithm in $O(N^g)$ in the Jacobian model: [Couveignes, Ezome (2015)]

☺ Rational model

☹ Restricted to $g \leq 3$

Cost of a 2^m-isogeny in dimension g:

g	1	2	4	8
Relative cost	×1	×4	×32	×1024

Dedicated fast formulas in higher dimension

Dimension 2:

- Fast 2^m-isogenies in the Mumford Jacobian or Kummer model [Kunzweiler 2022] and in the theta model [Dartois, Maino, Pope, R. 2023]

		Codomain			Evaluation		
$\log p$	m	Theta Rust	Theta SageMath	Richelot SageMath	Theta Rust	Theta SageMath	Richelot SageMath
254	126	2.13 ms	108 ms	1028 ms	161 µs	5.43 ms	114 ms
381	208	9.05 ms	201 ms	1998 ms	411 µs	8.68 ms	208 ms
1293	632	463 ms	1225 ms	12840 ms	17.8 ms	40.8 ms	1203 ms

- Fast 3^m-isogenies in the Mumford Jacobian model [Decru, Kunzweiler 2023] and in the theta model [Corte-Real Santos, Costello, Smith 2024]

Dimension 4:

- Fast 2^m-isogenies in the theta model [Dartois 2024]

Cryptographic applications

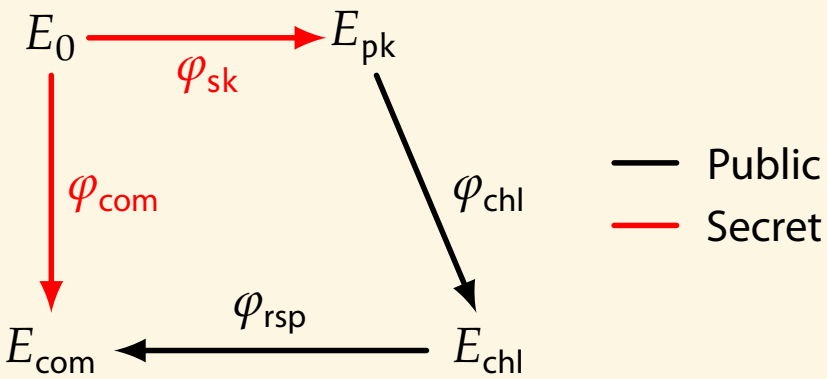
- New protocols in isogeny based cryptography: SQIsignHD [DLRW24], FESTA [BMP23] and QFESTA [NO23], the Deuring VRF [Ler23b], SCALLOP-HD [CL23] (efficient representation of orientations), IS-CUBE [Mor23], LIT-SiGamal [Mor24], SILBE [DFV24], POKE [Bas24], SQIsign2d (West and East) [BDD+24; NO24], SQIPrime [DF24]…
- New or improved security reductions in isogeny based cryptography, [MW23; ACD+23; PW24; ES24] and in classical elliptic curve cryptography [Gal24]
- New methods to convert ideals into isogenies [Ler23a; NO23; PR23; ON24; BDD+24]

Examples:

- Clapoti(s) [Page, R. 2023]: computing the class group action for an arbitrary orientation R in polynomial time
- No smoothening needed
- Unrestricted effective group action!
- SQIsignHD, SQIsign2d-West: bypass KLPT's smoothening algorithm for supersingular curves too
- KLPT: $\phi_J : E_1 \to E_2$, smoothened isogeny of degree $O(p^{4.5})$ (or $O(p^3)$ if E_1 is nice)
- HD representation: can use the smallest isogeny $\phi_J : E_1 \to E_2$ of degree $O(\sqrt{p})$ even if it is not smooth!

SQISign

- Proves knowledge of a supersingular endomorphism ring
- Most compact PK+signature out of all PQ signature schemes
- NIST submission



SQISign2d (West) and SQISignHD

	SQIsign	SQIsign2d
Public key	66B	66B
Signatures	177B	148B
Clean security proof	☹	☺
Keygen (Mcycles)	400	60
Sign (Mcycles)	1880	160
Verify (Mcycles)	29	9

- SQIsign2D: signature and verification in dimension 2
- SQIsignHD: signature in dimension 1, verification in dimension 4

  New faster variant compared to the Eurocrypt 2024 version using techniques from SQIsign2d: signatures now use dimension 2 too. Bonus: same public key as in SQIsign2d!
- Signature size: 109B
- Signature $\approx 5\times$ faster than SQIsign2d
- Verification expected $\approx 8\times$ slower

Number theoretic applications

- $E/\mathbb{F}_q$ ordinary elliptic curve, $K = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Given the factorisation of $[O_K : \mathbb{Z}[\pi]]$, compute $\mathrm{End}(E)$ in polynomial time [R. 2022].
  Factorisation: quantum polynomial time, classical subexponential time
- Previously: no quantum polynomial time algorithm known.
  Classical algorithm in $L(1/2)$ under GRH [Bisson–Sutherland 2009].

- Compute the canonical lift $\hat{E}/\mathbb{Z}_q$ of an ordinary elliptic curve in polynomial time [R. 2022]
  Previously: $L(1/2)$ under GRH [Couveignes–Henocq 2002]
- Compute the modular polynomial Φ_ℓ by deformation [Kunzweiler, R. 2024]

Point counting for $E/\mathbb{F}_q, q = p^n$

- [Schoof 1985]: $\widetilde{O}(n^5 \log^5 p)$ (Étale cohomology)
- [SEA 1992]: $\widetilde{O}(n^4 \log^4 p)$ (Heuristic)

- [Kedlaya 2001]: $\widetilde{O}(n^3 p)$ (Rigid cohomology)
- [Harvey 2007]: $\widetilde{O}(n^{3.5} p^{1/2} + n^5 \log p)$

- [Satoh 2000] (canonical lifts of ordinary curves): $\widetilde{O}(n^2 p^2)$ (Crystalline cohomology)
- [Maiga – R. 2021]: $\widetilde{O}(n^2 p)$
- [R. 2022]: $\widetilde{O}(n^2 \log^8 p + n \log^{11} p)$
  Use an HD representation of the Verschiebung $\hat{\pi}_p$ and canonical lifts

Example: divisions [R. 2022]

- Is an isogeny $\phi : E_1 \to E_2$ divisible by $[\ell]$?
- Prior art: test if $\phi(E[\ell]) = 0$
- Division polynomial ψ_ℓ: degree $O(\ell^2)$ $\Rightarrow$ exponential time

- HD division algorithm [R. 2022]:
- Given an HD representation $(P_i, Q_i, \phi(P_i), \phi(Q_i))$ with $N_i \wedge \ell = 1$,

$$(P_i, Q_i, \frac{\phi(P_i)}{\ell}, \frac{\phi(Q_i)}{\ell})$$

  is an HD representation of ϕ/ℓ if it exists

$\Rightarrow$ polynomial time (in $\log \ell$) division algorithm

Corollary (Computing the endomorphism ring of ordinary elliptic curves)

If $E/\mathbb{F}_q$ is an ordinary elliptic curve; point counting gives χ_π, hence $K := \mathbb{Q}(\pi_q)$, and we know $\mathbb{Z}[\pi] \subset \mathrm{End}(E) \subset O_K$. Given the factorisation of the conductor $[O_K : \mathbb{Z}[\pi]]$ of $\mathbb{Z}[\pi]$, we can determine $\mathrm{End}(E)$ in polynomial time, via efficient divisions.

Algorithms for the HD representation

$\phi/\mathbb{F}_q : E_1 \to E_2$ an n-isogeny with an efficient representation

- Equality testing, Validity
- Composition and addition: $\phi_2 \circ \phi_1, \phi_1 + \phi_2$
- Dual isogeny: $\widetilde{\phi} : E_2 \to E_1$
- Divisions: Test if $\phi \stackrel{?}{=} \psi' \circ \psi$ is divisible by ψ, and if so return the HD representation of ψ'
- Lifts and deformations: deform ϕ to $\widetilde{\phi}/R : \widetilde{E_1} \to \widetilde{E_2}$ over $R = \mathbb{F}_q[\varepsilon]/\varepsilon^m$ or $R = \mathbb{Z}_q/p^m\mathbb{Z}_q$
- Splittings: If $n = n_1 n_2, n_1 \wedge n_2 = 1$, split ϕ as $\phi = \phi_2 \circ \phi_1$

$$\phi : E_1 \xrightarrow{\phi_1} E_{12} \xrightarrow{\phi_2} E_2$$

- Pushforwards: compute the pushfoward of ϕ_1 and ϕ_2 if they are of coprime degrees

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_1} & E_1 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_2'} \\ E_1 & \xrightarrow{\phi_1'} & E_{12} \end{array}$$

- Kernel: return an equation for $\mathsf{Ker}\,\phi$ in $\widetilde{O}(n)$

Efficient representation of isogenies

Past:

- Restricted to smooth degree isogenies
- Vélu's / $\sqrt{\text{élu}}$ formulas
- Ideal smoothening

Present:

- The HD representation: recent powerful tool with many applications in isogeny based cryptography and algorithmic number theory
- Use abelian varieties to speed up algorithms on elliptic curves
- Survey paper: [Rob24]

Future?

- Switch from ideals equivalences of categories to modules equivalences of categories
  - Handles the higher dimensional isogeny graphs of E^g
  - Handles level structures
  - Go beyond Kani's lemma
- Use cyclic isogenies?

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Polarisations and isogenies on an abelian variety

- Polarisation on A = a (symmetric) isogeny $\lambda_A : A \to \widehat{A}$
- Principal polarisation: λ_A is an isomorphism.
- Warning: A may have several non equivalent principal polarisations if $g > 1$.

Example (Superspecial abelian surfaces)

$A = E^2, E/\mathbb{F}_{p^2}$ supersingular. It admits $\approx p^2/288$ product polarisations $(E_1 \times E_2, \lambda_{E_1} \times \lambda_{E_2})$ where E_1, E_2 are supersingular and $\approx p^3/2880$ indecomposable polarisations $(\mathrm{Jac}\, C, \Theta_C)$ where C is an hyperelliptic curve of genus 2.

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- Polarisation on A = a (symmetric) isogeny $\lambda_A : A \to \widehat{A}$
- Principal polarisation: λ_A is an isomorphism.
- Warning: A may have several non equivalent principal polarisations if $g > 1$.

- $\phi : (A, \lambda_A) \to (B, \lambda_B)$ N-isogeny between ppav: $\phi^* \lambda_B = N\lambda_A$.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \lambda_A^{-1}}\uparrow & & \downarrow{\scriptstyle \lambda_B} \\ \widehat{A} & \xleftarrow[\widehat{\phi}]{} & \widehat{B} \end{array}$$

- Dual isogeny: $\widehat{\phi} : \widehat{B} \to \widehat{A}$
- Contragredient isogeny: $\widetilde{\phi} = \lambda_A^{-1} \widehat{\phi} \lambda_B : B \to A$
- ϕ N-isogeny $\Leftrightarrow \widetilde{\phi} \circ \phi = N \Leftrightarrow \phi\widetilde{\phi} = N$.

- $\mathrm{Ker}\, \phi = \mathrm{Im}\left(\widetilde{\phi} \mid B[N]\right)$.

N-isogenies and isotropic kernels

- $\phi : (A, \lambda_A) \to (B, \lambda_B)$ N-isogeny $\Rightarrow \operatorname{Ker}\phi$ is maximal isotropic in $A[N]$ for the Weil pairing
- Conversely, if $K \subset A[N]$ maximal isotropic, $N\lambda_A$ descends to a principal polarisation on $B = A/K$.

- An elliptic curve only has one principal polarisation ($NS(E) = \mathbb{Z}$).
- So $\phi : E_1 \to E_2$ is an N-isogeny $\Leftrightarrow \#\operatorname{Ker}\phi = N$.
- But in higher dimension there may be many non equivalent principal polarisations.

Example (Superspecial abelian surfaces)

$A = E^2, E/\mathbb{F}_{p^2}$ supersingular. It admits $\approx p^2/288$ product polarisations $(E_1 \times E_2, \lambda_{E_1} \times \lambda_{E_2})$ where E_1, E_2 are supersingular and $\approx p^3/2880$ indecomposable polarisations $(\operatorname{Jac} C, \Theta_C)$ where C is an hyperelliptic curve of genus 2.

- If $\phi : (A, \lambda_A) \to (B, \lambda_B)$ has maximal isotropic kernel in $A[N]$, $N\lambda_A$ descends to a principal polarisation λ'_B on B.
- But we may have $\lambda'_B \neq \lambda_B$.
- $\widetilde{\phi} \circ \phi = N$ is a stronger condition that ensures compatibility of ϕ with λ_B.

Composition and product polarisations

- Composition: $f : A \to B$ a N-isogeny, $g : B \to C$ a M-isogeny, $g \circ f : A \to C$.
- $\widehat{g \circ f} = \hat{f} \circ \hat{g} : \hat{C} \to \hat{A}$;
- $\widetilde{g \circ f} = \tilde{f} \circ \tilde{g} : C \to A$;
- $(\widetilde{g \circ f}) \circ (g \circ f) = \tilde{f} \circ \tilde{g} \circ g \circ f = NM$.
- The composition $g \circ f$ is an NM-isogeny.
- Conversely, if $g \circ f$ is an N-isogeny and f (resp. g) is an M-isogeny, then g (resp. f) is an N/M-isogeny.

- Product polarisation: $(A, \lambda_A) \times (B, \lambda_B) = (A \times B, \lambda_A \times \lambda_B)$ where $\lambda_A \times \lambda_B : A \times B \to \widehat{A} \times \widehat{B}$ is the product.

- $F = \begin{pmatrix} a & c \\ b & d \end{pmatrix} : (A \times B, \lambda_A \times \lambda_B) \to (C \times D, \lambda_C \times \lambda_D)$.
- $\hat{F} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} : \hat{C} \times \hat{D} \to \hat{A} \times \hat{B}$.
- $\tilde{F} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} : C \times D \to A \times B$.