# Breaking SIDH in polynomial time

2022/09/13 — LFANT seminar, Bordeaux

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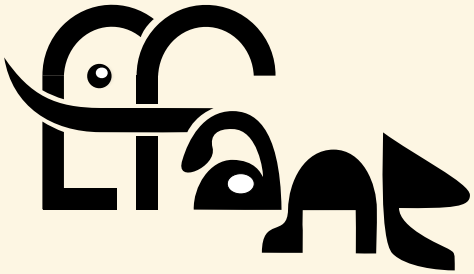
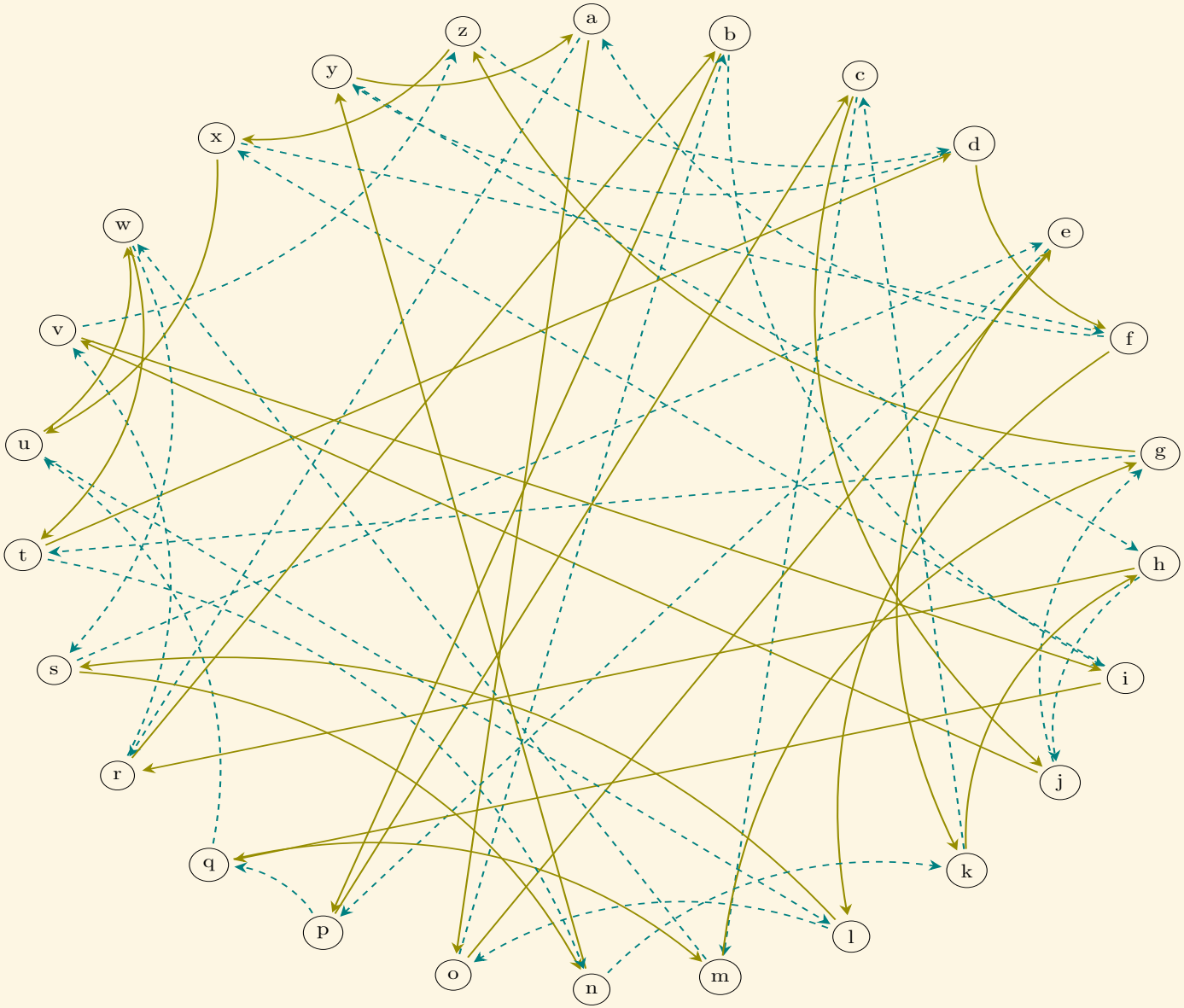
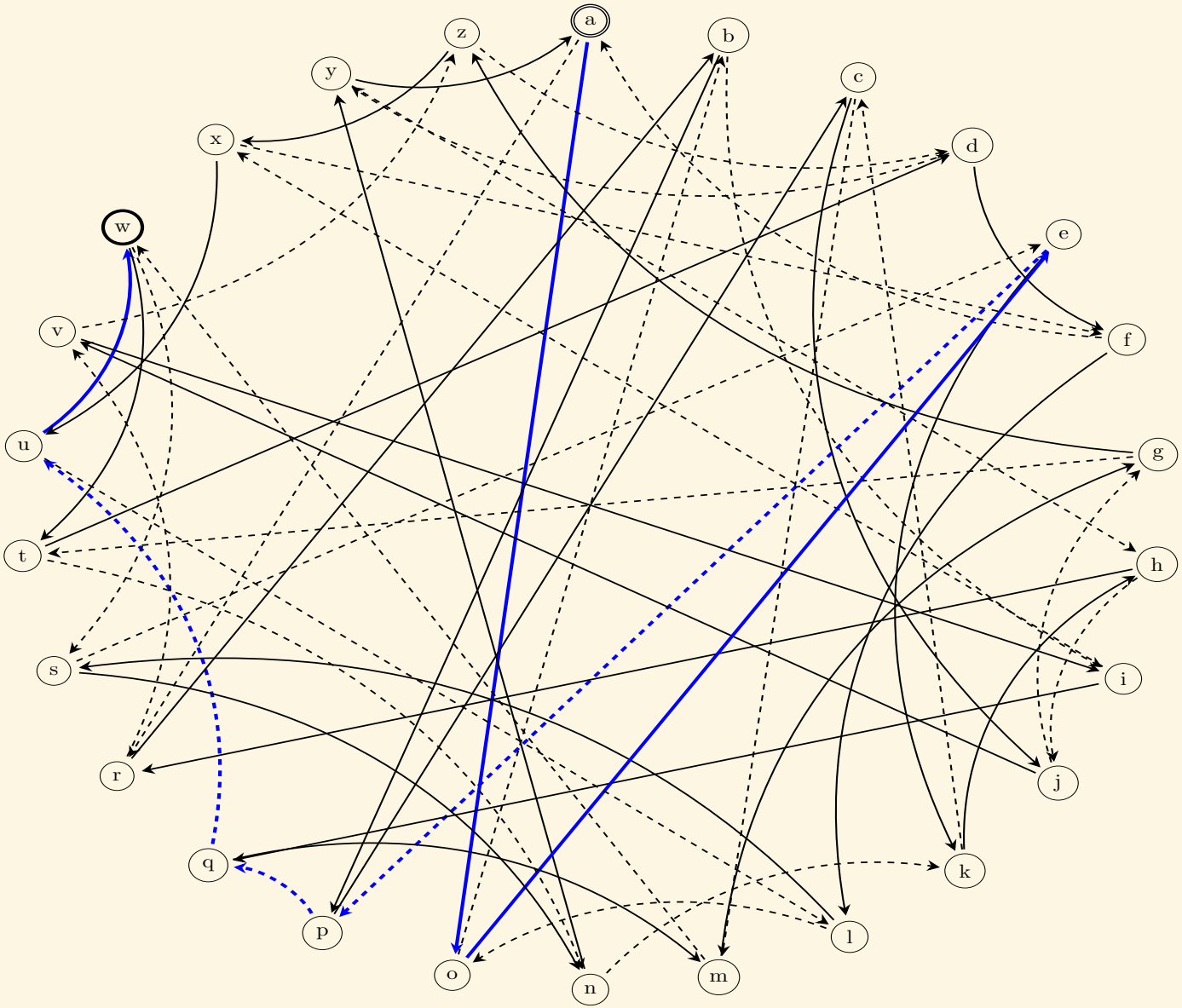



# Key exchange on a (commutative) graph



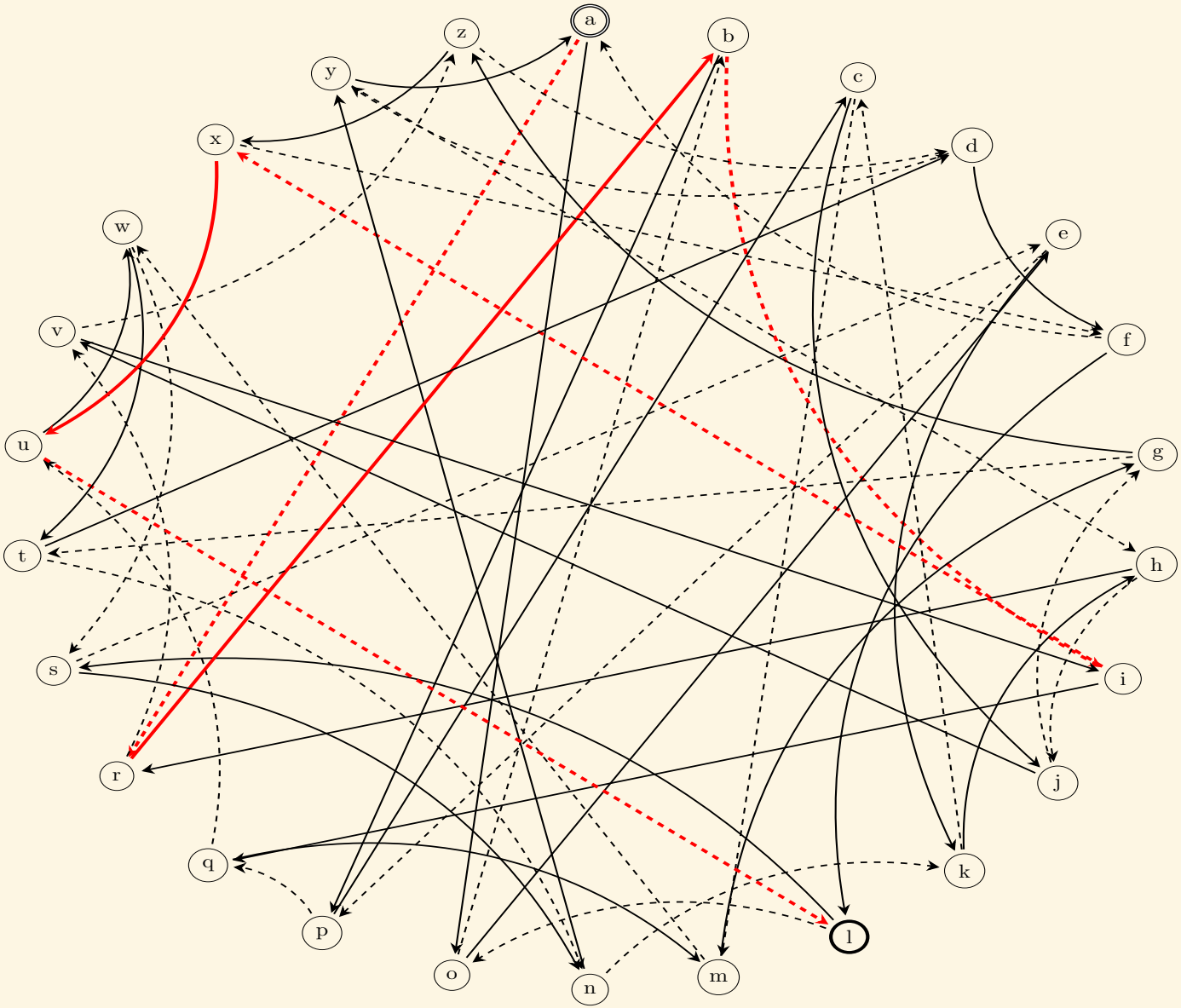
# Key exchange on a (commutative) graph

Alice starts from ‘a’, follows the path 001110, and get ‘w’.



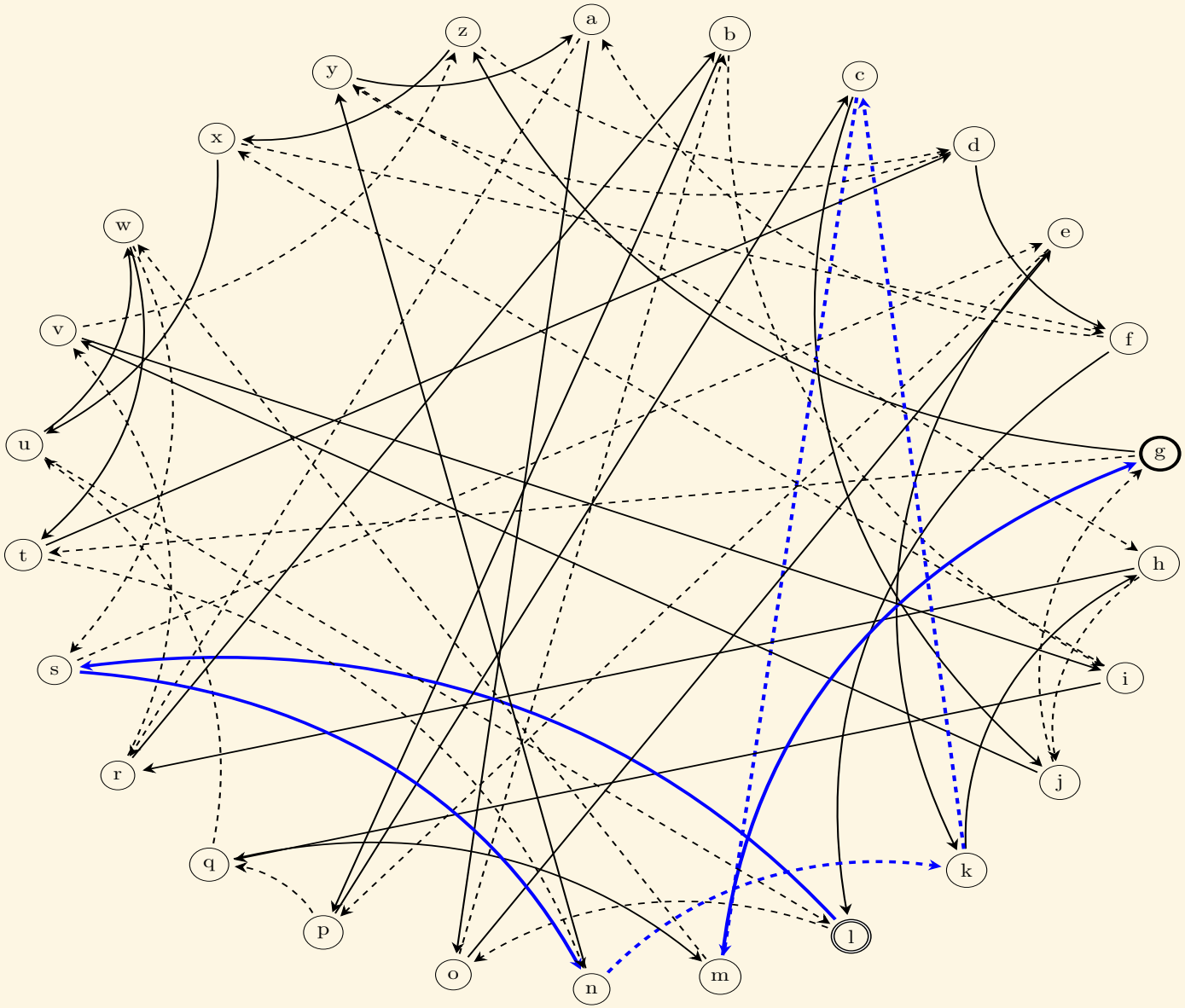
# Key exchange on a (commutative) graph

Bob starts from 'a', follows the path 101101, and get 'l'.



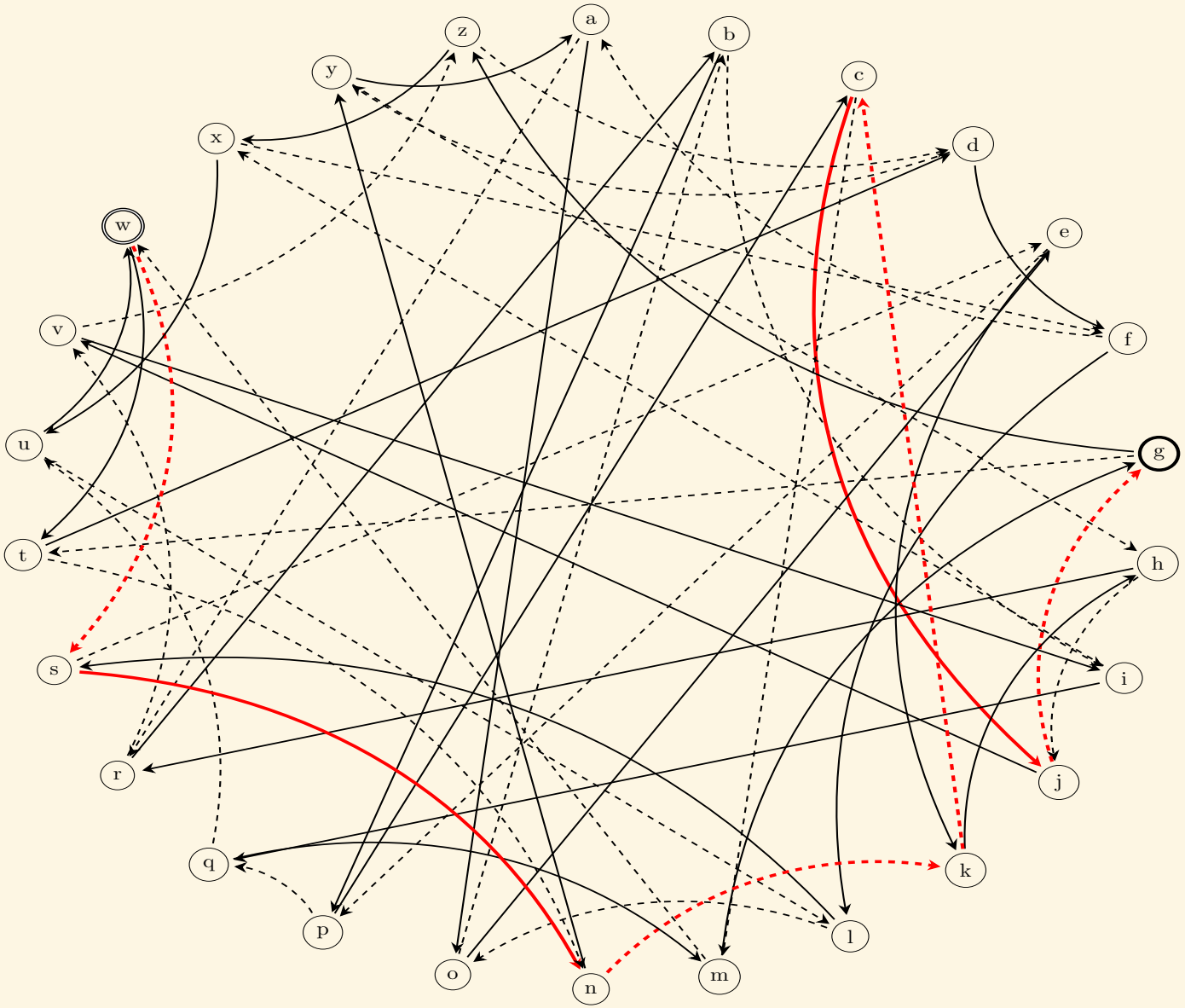
# Key exchange on a (commutative) graph

Alice starts from 'l', follows the path 001110, and get 'g'.



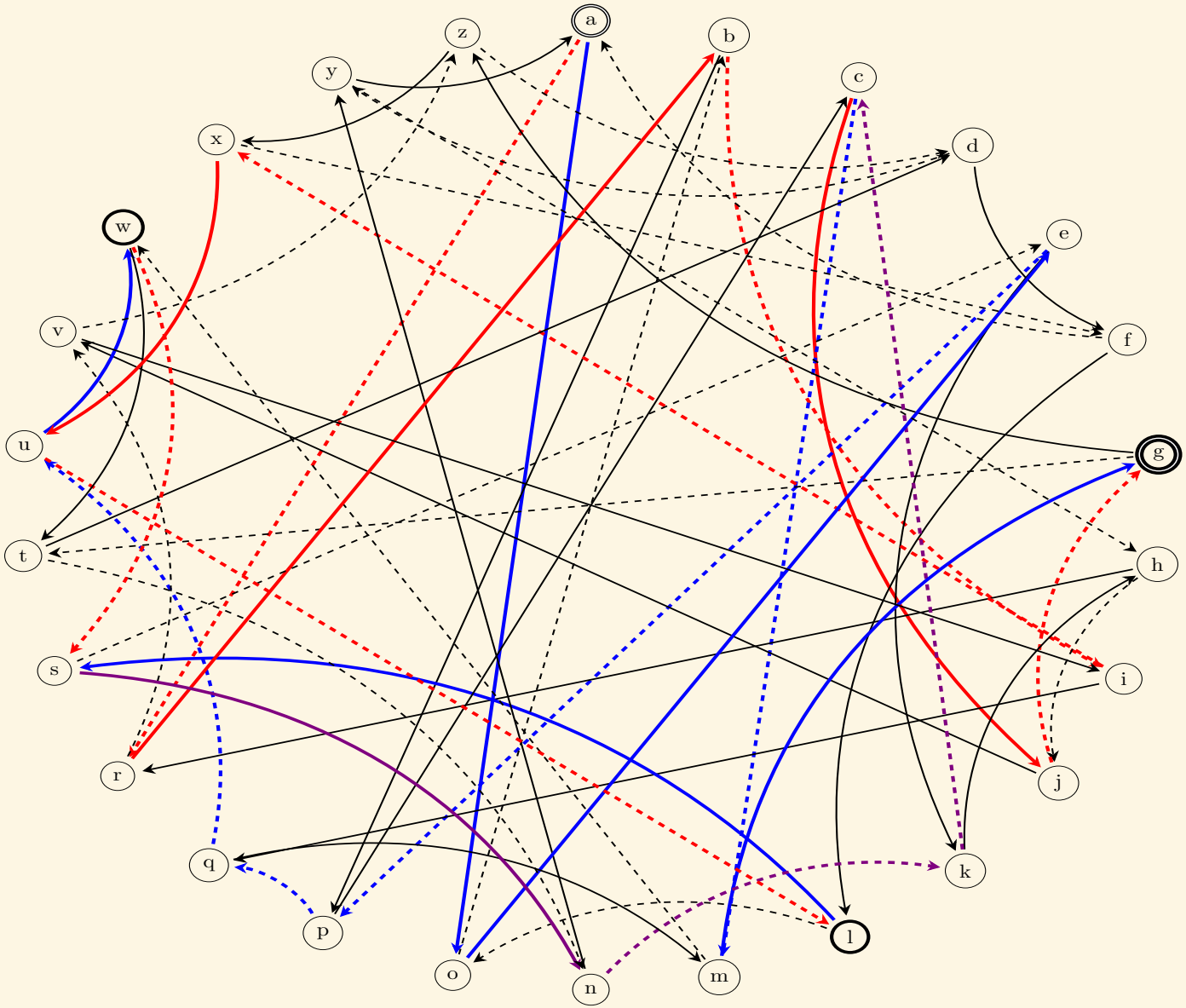
# Key exchange on a (commutative) graph

Bob starts from ‘w’, follows the path 101101, and get ‘g’.



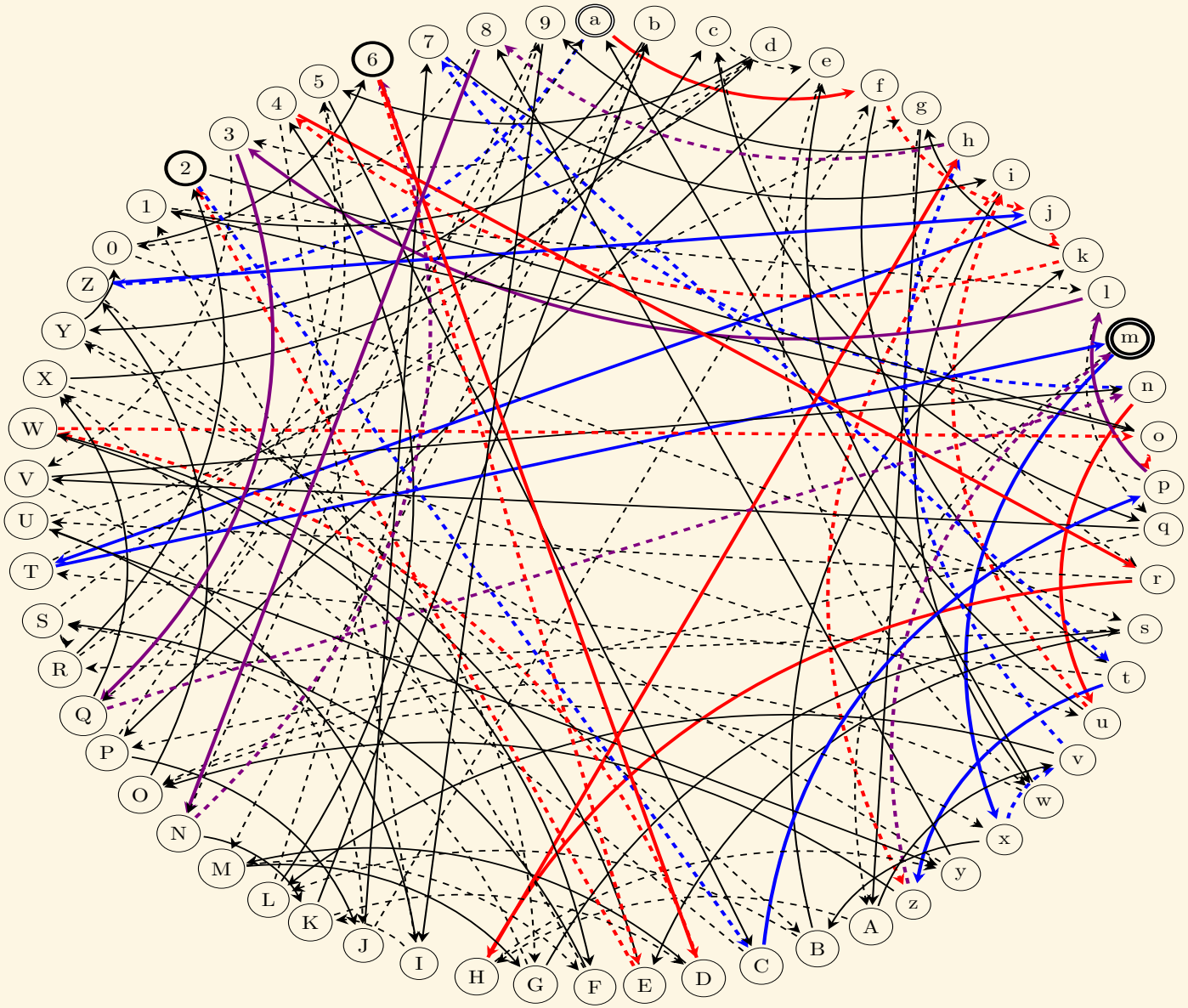
# Key exchange on a (commutative) graph

The full exchange:



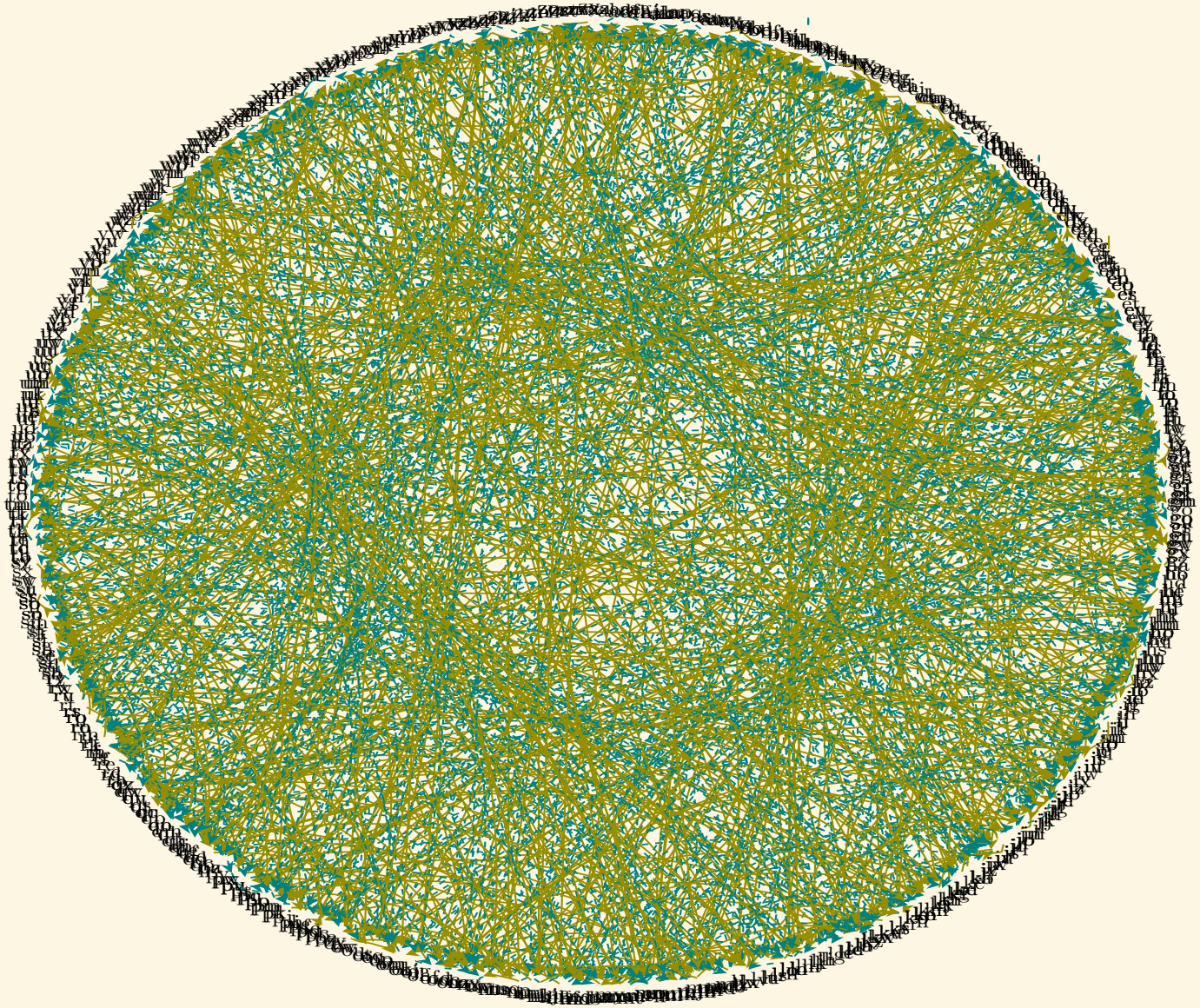
# Key exchange on a (commutative) graph

## Bigger graph (62 nodes)



# Key exchange on a (commutative) graph

## Even bigger graph (676 nodes)



# Graph size for 128 bits of security (arbitrary graph)

- Need a graph with good mixing properties:
  A path of length $O(\log N)$ gives a uniform node $\Rightarrow$ Ramanujan/expander graph.

- Finding a path by exhaustive search: $O(N)$.
- Grover (quantum): $O(\sqrt{N})$.
- Meet in the middle: $\widetilde{O}(\sqrt{N})$ and $O(\sqrt{N})$ memory.
- Time/Memory trade off: van Oorschot-Wiener Golden Collision Claw Finding.
  $\widetilde{O}\left(\frac{N^{3/4}}{\sqrt{M}}\right)$ with $O(M)$-memory ($M \leq \sqrt{N}$).
- Quantum claw finding: $\widetilde{O}(N^{1/3})$ but needs $\widetilde{O}(N^{1/3})$ memory.
- Tani's quantum claw finding algorithm:
  find $g_1(x) = g_2(y)$ in $O(\#X^{2/3})$ where $g_1 : X \to Z, g_2 : Y \to Z, \#Z \gg \#X \approx \#Y$
  (generalized Grover).
- Needs $N \approx 2^{256}$ or $N \approx 2^{384}$.

- The graph does not fit in memory.
- Need an algorithm taking a node as input and giving the neighbour nodes as output.

# Isogeny graph of ordinary elliptic curves [Couveignes (1997)], [Rostovtsev–Stolbunov (2006)]

- Isogeny graph of an ordinary elliptic curve $E/\mathbb{F}_p$.
- Graph of size $\approx \sqrt{p}$.
- Torsor (principal homogeneous space) under the class group $\mathbf{Cl}(\mathbf{End}(E_0))$.
- Hard to find good parameters.
- CSIDH: supersingular elliptic curves over $\mathbb{F}_p$, torsor under $\mathbf{Cl}(\mathbb{Z}[\sqrt{-p}])$.

☺ Commutative graph!

☹ Hidden shift problem solvable in quantum subexponential $L(1/2)$ time for an abelian group action via Kuperberg's algorithm.

# SIDH: supersingular elliptic curve Diffie-Helmann [De Feo, Jao (2011)][De Feo, Jao, Plût (2014)]

- Use the isogeny graph of a supersingular elliptic curve $E$ over $\mathbb{F}_{p^2}$.
- There are $\approx p/12$ nodes and the graph is an expander graph.
- The endomorphism ring is a quaternion algebra (ramified at $p$ and infinity), which is non commutative.
- The isogeny graph is a Cayley graph for the class groupoid.
- Non commutative graph.

- Practical parameters: $p+1 = 2^a 3^b$ with $2^a \approx 3^b$
- $E(\mathbb{F}_{p^2}) \simeq \mathbb{Z}/(p+1)\mathbb{Z} \times \mathbb{Z}/(p+1)\mathbb{Z}$ has rational $2^a$ and $3^b$-torsion.
- Paths given by $2^a$ and $3^b$ isogenies: $O(\sqrt{p})$ possible paths.
- Attacks: $p^{1/4}$ (classical) or $p^{1/6}$ (Tani's quantum claw algorithm).

$\Rightarrow$ For $128$ bits of security, needs $p$ of $512$ bits or $768$ bits.

- SIKE: supersingular isogeny key encapsulation (KEM).
- Short key size: $3.5 \log p$ via key compression.
  Total key size: $1792$ or $2688$ bits

# SIDH: supersingular elliptic curve Diffie-Hellmann [De Feo, Jao (2011)][De Feo, Jao, Plût (2014)]

# SIDH in practice

- $p = 2^a 3^b - 1$. $N_A = 2^a, N_B = 3^b, N_A$ prime to $N_B$.
- $E_0 : y^2 = x^3 + x$ (supersingular when $a \geq 2$) or $E_0 : y^2 = x^3 + 6x^2 + x$.
- $E_0[N_A] = \langle P_A, Q_A \rangle, E_0[N_B] = \langle P_B, Q_B \rangle$.
- Alice's secret isogeny: $\phi_A$ of kernel $\langle P_A + s_A Q_A \rangle$.
- Bob's secret isogeny: $\phi_B$ of kernel $\langle P_B + s_B Q_B \rangle$.
- Key exchange:

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_B} & E_B \\ \downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi'_A} \\ E_A & \xrightarrow{\phi'_B} & E_{AB} \end{array}$$

- $E_{AB}$ is the shared secret.
- $\phi'_A \circ \phi_B = \phi'_B \circ \phi_A : E_0 \to E_{AB}$ has kernel $\operatorname{Ker} \phi_A + \operatorname{Ker} \phi_B$.
- $\phi'_A$ has kernel $\phi_B(P_A + s_A Q_A)$, $\phi'_B$ has kernel $\phi_A(P_B + s_B Q_B)$.
- Alice publishes: $P'_B = \phi_A(P_B), Q'_B = \phi_A(Q_B)$.
  Bob publishes: $P'_A = \phi_B(P_A), Q'_A = \phi_B(Q_A)$. ("Torsion points".)
- $\operatorname{Ker} \phi'_A = \langle P'_A + s_A Q'_A \rangle$, $\operatorname{Ker} \phi'_B = \langle P'_B + s_B Q'_B \rangle$.
- Key exchange in $\widetilde{O}(\log N_A \ell_A^{1/2} + \log N_B \ell_B^{1/2})$
  (Via fast smooth isogeny computation [De Feo, Jao, Plût (2014)] and Velusqrt [Bernstein, De Feo, Leroux, Smith (2020)]).

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# NIST PQC competition (July 5th 2022)

- Standardized KEM: Kyber (structured lattice)
- Standardized signatures: Dilithium, Falcon (structured lattices), SPHINCS (hash function)
- Fourth round KEM: BIKE, HQC (structured codes), Classic McEliece (code), SIKE (SIDH).

# Torsion points attacks

- Eve knows $E_0, E_B, P'_A = \phi_B(P_A), Q'_A = \phi_B(Q_A), P_A, Q_A$ basis of the $N_A$-torsion.
- Goal: recover $\phi_B : E_0 \to E_B$, an $N_B$-isogeny.

- [Petit (2017)]: build an isogeny $\alpha$ on $E_0$ and combine it with $\phi_B$ (and/or $\widetilde{\phi_B}$) to get an $N_A$-isogeny $F$.
- Recover $\mathrm{Ker}\, F \subset E_0[N_A]$: we know the action of $\phi_B$ on the $N_A$-torsion!
- We know the action of $\phi_B$ on $E_0[N_A]$ so we also know the action of $\widetilde{\phi_B}$ on $E_B[N_A]$, so we can also use $\widetilde{\phi_B}$ to build $F$.
- Compute $F$ via an isogeny algorithm.
- Extract $\phi_B$ from $F$.

# $N$-isogenies

- Polarised abelian variety $(A, \lambda_A)$: where $\lambda_A : A \to \widehat{A}$ is an isogeny
  Technicality: the morphism $\lambda_A$ needs to be induced from a divisor, $\Leftrightarrow \widehat{\lambda_A} : \widehat{\widehat{A}} \simeq A \to \widehat{A} = \lambda_A$.
- Principal polarisation: $\lambda_A$ is an isomorphism $\Rightarrow$ principally polarized abelian variety (ppav)
- $f : (A, \lambda_A) \to (B, \lambda_B)$ is an $N$-isogeny between ppav if $f^* \lambda_B = N \lambda_A$.
- Dual isogeny: $\hat{f} : \hat{B} \to \widehat{A}$
- Contragredient isogeny / Dual with respect to the principal polarisations: $\tilde{f} = \lambda_A^{-1} \hat{f} \lambda_B : B \to A$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \lambda_A^{-1}}\uparrow & & \downarrow{\scriptstyle \lambda_B} \\ \widehat{A} & \xleftarrow[\hat{f}]{} & \hat{B} \end{array}$$

- $f$ is an $N$-isogeny $\Leftrightarrow \tilde{f} f = N \Leftrightarrow f \tilde{f} = N$.
- $\mathrm{Ker} f = \mathrm{Im} \tilde{f} \mid B[N]$.

# Algorithms for $N$-isogenies

- [Cosset-R. (2014), Lubicz-R. (2012–2022)]: An $N$-isogeny in dimension $g$ can be evaluated in linear time $O(N^g)$ arithmetic operations in the theta model given generators of its kernel.
- Warning: exponential dependency $2^g$ or $4^g$ in the dimension $g$.

- [Couveignes-Ezome (2015)]: Algorithm in $O(N^g)$ in the Jacobian model.
- Not hard to extend to product of Jacobians.
- Restricted to $g \leq 3$.

# $N$-isogenies and isotropic kernels

- $f : (A, \lambda_A) \to (B, \lambda_B)$ $N$-isogeny $\Rightarrow \operatorname{Ker} f$ is maximal isotropic in $A[N]$ for the Weil pairing
- Conversely, if $K \subset A[N]$ maximal isotropic, $N\lambda_A$ descends to a principal polarisation on $B = A/K$.

- An elliptic curve only has one principal polarisation ($NS(E) = \mathbb{Z}$).
- So $f : E_1 \to E_2$ is an $N$-isogeny $\Leftrightarrow \#\operatorname{Ker} f = N$.
- But in higher dimension there may be many non equivalent principal polarisations.

**Example (Superspecial abelian surfaces)**

$A = E^2, E/\mathbb{F}_{p^2}$ supersingular. It admits $\approx p^2/288$ product polarisations $(E_1 \times E_2, \lambda_{E_1} \times \lambda_{E_2})$ where $E_1, E_2$ are supersingular and $\approx p^3/2880$ indecomposable polarisations $(\operatorname{Jac} C, \Theta_C)$ where $C$ is an hyperelliptic curve of genus 2.

- If $f : (A, \lambda_A) \to (B, \lambda_B)$ has maximal isotropic kernel in $A[N]$, $N\lambda_A$ descends to a principal polarisation $\lambda'_B$ on $B$.
- But we may have $\lambda'_B \neq \lambda_B$.
- $\tilde{f} \circ f = N$ is a stronger condition that ensures compatibility of $f$ with $\lambda_B$.

# Composition and product polarisations

- Composition: $f : A \to B$ a $N$-isogeny, $g : B \to C$ a $M$-isogeny, $g \circ f : A \to C$.
- $\widehat{g \circ f} = \hat{f} \circ \hat{g} : \hat{C} \to \hat{A}$;
- $\widetilde{g \circ f} = \tilde{f} \circ \tilde{g} : C \to A$;
- $(\widetilde{g \circ f}) \circ (g \circ f) = \tilde{f} \circ \tilde{g} \circ g \circ f = NM$.
- The composition $g \circ f$ is an $NM$-isogeny.
- Conversely, if $g \circ f$ is an $N$-isogeny and $f$ (resp. $g$) is an $M$-isogeny, then $g$ (resp. $f$) is an $N/M$-isogeny.

- Product polarisation: $(A, \lambda_A) \times (B, \lambda_B) = (A \times B, \lambda_A \times \lambda_B)$ where $\lambda_A \times \lambda_B : A \times B \to \widehat{A} \times \hat{B}$ is the product.

- $F = \begin{pmatrix} a & c \\ b & d \end{pmatrix} : (A \times B, \lambda_A \times \lambda_B) \to (C \times D, \lambda_C \times \lambda_D)$.
- $\hat{F} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} : \hat{C} \times \hat{D} \to \hat{A} \times \hat{B}$.
- $\tilde{F} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} : C \times D \to A \times B$.

# Dimension 1 torsion points attacks

Recall: construct $\alpha$ and combine it with $\phi_B$ to get an $N_A$-isogeny $F$.

- [Petit (2017)]: $F = \phi_B \circ \gamma \circ \widetilde{\phi_B} + [d]$.

$$E_B \underset{\phi_B}{\overset{\widetilde{\phi_B}}{\rightleftarrows}} E_0 \circlearrowright \gamma$$

- $\gamma$ a $c$-endomorphism of $E_0$ of "trace 0": $\tilde{\gamma} = -\gamma$.
- $\tilde{F} = -\phi_B \circ \gamma \circ \widetilde{\phi_B} + [d]$.
- $\tilde{F}F = N_B^2 c + d^2$.

- Find parameters $b, d$ such that $N_A = b^2 c N_B^2 + d^2, F = b\phi_B \gamma \widetilde{\phi_B} + [d]$ is then an $N_A$-endomorphism on $E_B$.
- Extract $\phi_B$ from $F$.

$\Rightarrow$ Needs a non trivial endomorphism $\gamma$ on $E_0$.

$\Rightarrow$ Needs unbalanced parameters $N_A > N_B^2$.

# Non trivial endomorphisms

- On $E_0 : y^2 = x^3 + x$, endomorphism $\gamma = [i]$ over $\mathbb{F}_{p^2}$,

$$[i](x, y) = (-x, iy).$$

- $\widetilde{[i]} = [-i]$: the Rosati involution is the complex conjugation

- $\mathrm{End}(E_0) \supset \mathbb{Z}[i]$.
- If $\alpha = a_1 + a_2 i, \tilde{\alpha} = a_1 - a_2 i, \tilde{\alpha} \circ \alpha = a_1^2 + a_2^2$.
- Can construct $a$-isogenies whenever $a = a_1^2 + a_2^2$.

- [QKLMPPS (2021)]: dimension 1 attack when $N_A^2 = b^2 N_B^2 + a^2$ (and other variants).

# Non trivial endomorphisms

# Building $N$-isogenies in higher dimension?

- $f \mapsto \tilde{f}$ behaves like the complex conjugation $z \mapsto \overline{z}$,
  (or like the $^*$ operator in a $\mathbb{C}^*$-algebra)
- $f \mapsto \tilde{f} \circ f$ behaves like the complex norm $z \mapsto \|z\| = \overline{z}z = |z|^2$.

- Matrix representation of the Gaussian integers $z = \alpha + \beta i, \alpha, \beta \in \mathbb{Z}$:

$$M_z = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

- $\widetilde{M_z} := M_Z^* = \overline{M_z}^T = M_{\overline{z}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\widetilde{M_z} M_z = \alpha^2 + \beta^2 = a + b$, $a = \|\alpha\|, b = \|\beta\|$.
- $M_z$ combines an "$a$-endomorphism" $\alpha$ and a "$b$-endomorphism" $\beta$ into a "$a + b$-endomorphism" $M_z$ provided $\alpha, \beta$ are "symmetric" ($\tilde{\alpha} = \alpha, \tilde{\beta} = \beta$).

- Generalisation: $\alpha = \alpha_1 + \alpha_2 i \in \mathbb{Z}[i]$ of norm $a = \|\alpha\|$, $\beta = \beta_1 + \beta_2 i \in \mathbb{Z}[i]$ of norm $b = \|\beta\|$
- $F = \begin{pmatrix} \alpha & \overline{\beta} \\ -\beta & \overline{\alpha} \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}[i])$.
- $\tilde{F} := F^* = \overline{F}^T = \begin{pmatrix} \overline{\alpha} & -\overline{\beta} \\ \beta & \alpha \end{pmatrix}$, $\tilde{F}F = \overline{\alpha}\alpha + \overline{\beta}\beta = \|\alpha\| + \|\beta\| = a + b = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2$.
- Matrix representation of Hamilton's quaternions $\mathbb{Z}[i, j, k]$.
- $F$ combines an "$a$-endomorphism" and a "$b$-endomorphism" into a "$a + b$-endomorphism".

# Kani's lemma [Kani (1997)], [R. (2022-08-10)]

- $\alpha : A \to B$ a $a$-isogeny, $\beta : A \to C$ a $b$-isogeny.
- $\alpha' : C \to D$ a $a$-isogeny, $\beta' : C \to D$ a $b$-isogeny with $\beta'\alpha = \alpha'\beta$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ C & \xrightarrow{\alpha'} & D \end{array}$$

- NB: If $a$ prime to $b$, the pushforward $\alpha', \beta'$ of $\alpha, \beta$ by $\beta, \alpha$ satisfy these conditions.

- $F = \begin{pmatrix} \alpha & \widetilde{\beta'} \\ -\beta & \widetilde{\alpha'} \end{pmatrix} : A \times D \to B \times C.$
- $\tilde{F} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \beta' & \alpha' \end{pmatrix} : B \times C \to A \times D, \quad \tilde{F}F = a + b.$
- $F$ is an $a + b$-isogeny with respect to the product polarisations.
- $\ker F = \{\tilde{\alpha}(P), \beta'(P) \mid P \in B[N_A]\}$ (if $a$ is prime to $b$)

# Dimension 2 attacks [Castryck-Decru (2022-07-30)], [Maino-Martindale (2022-08-08)]

- if $\alpha : E_0 \to E_0'$ is an $a$-isogeny, combine $\alpha$ with $\phi_B : E_0 \to E_B$ an $N_B$-isogeny to build an $N_B + a$-isogeny: $F = E_0' \times E_B \to E_0 \times E_X$.

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_B} & E_B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ E_0' & \xrightarrow{\phi_B'} & E_X \end{array}$$

- $F = \begin{pmatrix} \tilde{\alpha} & \widetilde{\phi_B} \\ -\phi_B' & \alpha' \end{pmatrix}$.
- $\ker F = \{\alpha(P), \phi_B(P) \mid P \in E_0[N_A]\}$.
- $\phi_B$ can be directly extracted from $F$.
- Needs a $a := N_A - N_B$ isogeny on $E_0$.
- Breaking SIDH then reduces to evaluating the isogeny $F$ in dimension 2.
- Cost $\widetilde{O}(\log N_A \ell_A^2)$ arithmetic operations.

# Dimension 2 attacks [Castryck-Decru (2022-07-30)], [Maino-Martindale (2022-08-08)]

# Dimension 2 attacks: general case

- Easy to construct smooth isogenies from $E_0$.
- Look for parameters $N_A = bN_B + a$ with $a, b$ smooth.
- Parameter tweaks: $eN_A = bN_B/D_B + a$, $e$ small integer, $D_B$ small divisor of $N_B$.
- [De Feo (2022-08-25)]: (heuristic) subexponential $L(1/2)$ attack (tweaks of subexponential size).

# Dimension 2 attacks: NIST's starting curve [Castryck-Decru]

- When $E_0 : y^2 = x^3 + x$, $\mathrm{End}(E_0) \supset \mathbb{Z}[i]$.
- Can efficiently build $a$-isogenies if $a := N_A - N_B = a_1^2 + a_2^2$.
- Probability: $\Omega(1/\sqrt{\log N_A})$ (heuristic).
- [Castryck-Decru]: (heuristic) polynomial time attack.
- Require factorisation oracles to decompose $a$ as a sum of two squares.
- [Wesolowski (2022-08-12)]: if $\mathrm{End}(E_0)$ is known, can always build an $a$-isogeny in proven polynomial time.
- Polynomial time precomputation (depending on $E_0$), then attack in $\widetilde{O}(\log N_A \ell_A^2)$ arithmetic operations.
- Sage implementation: `https://github.com/jack4818/Castryck-Decru-SageMath`
- [R.], [Oudompheng]: Look for $N_A = (b_1^2 + b_2^2)N_B/D_B + (a_1^2 + a_2^2)$.
- Heuristic precomputation of $O(\log^3 N_A)$ then attack (heuristically) in $\widetilde{O}(\log^2 N_A \ell_A^2)$ arithmetic operations.
- Can be reduced to $\widetilde{O}(\log^{3/2} N_A \ell_A^2)$ with $O(\log^{1/2} N_A)$ factorisation calls.

# Revisiting the general case

Recall: we need to construct an $a := N_A - N_B$ isogeny on $E_0$.

- If $X$ is generic: can build smooth isogenies or the $d^2$-endomorphism $[d]$.
- But on $X^2$, for $a_1, a_2 \in \mathbb{Z}$, $\alpha = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}$ is an $a_1^2 + a_2^2$-endomorphism.
- More generally, if $\alpha_1$ is an $a_1$-endomorphism, $\alpha_2$ an $a_2$-endomorphism of $X$ and $\alpha_1\alpha_2 = \alpha_2\alpha_1$, $\alpha = \begin{pmatrix} \alpha_1 & \tilde{\alpha}_2 \\ -\alpha_2 & \widetilde{\alpha_1} \end{pmatrix}$ is an $a_1 + a_2$-endomorphism on $X^2$.
- So if $a = a_1^2 + a_2^2 + a_3^2 + a_4^2$, we can build an $a_1^2 + a_2^2$ endomorphism on $X^2$, an $a_3^2 + a_4^2$ endomorphism on $X^2$, and an $a$-endomorphism on $X^4$.
- Decomposition of $a$ as a sum of four squares: randomized polynomial time $\widetilde{O}(\log^2 a)$.

$\Rightarrow$ We can always build an $a$-endomorphism on $E_0^4$!

# Revisiting the general case

# Dimension $8$ attack in the general case [R. (2022-08-10)]

- Assume $N_A > N_B$.
- $a := N_A - N_B$. Decompose $a = a_1^2 + a_2^2 + a_3^2 + a_4^2$ to build $\alpha$ an $a$-endomorphism on $E_0^4$.

$$\begin{array}{ccc} E_0^4 & \xrightarrow{\phi_B} & E_B^4 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ E_0^4 & \xrightarrow{\phi_B} & E_B^4 \end{array}$$

- $F = \begin{pmatrix} \alpha & \widetilde{\phi_B} \\ -\phi_B & \tilde{\alpha} \end{pmatrix} \in \mathrm{End}(E_0^4 \times E_B^4)$.
- $\mathrm{Ker}\, F = \{\tilde{\alpha}(P), \phi_B(P) \mid P \in E_0[N_A]\}$.
- Breaking SIDH in the general case reduces to evaluating the isogeny $F$ in dimension $8$.

- Precomputation: randomized $O(\log^2 N_A)$.
- Attack: $\widetilde{O}(\log N_A \ell_A^8)$ arithmetic operations.
- Quasi-linear if $\ell_A = O(1)$ (or $\ell_A = O(\log\log N_A)$).

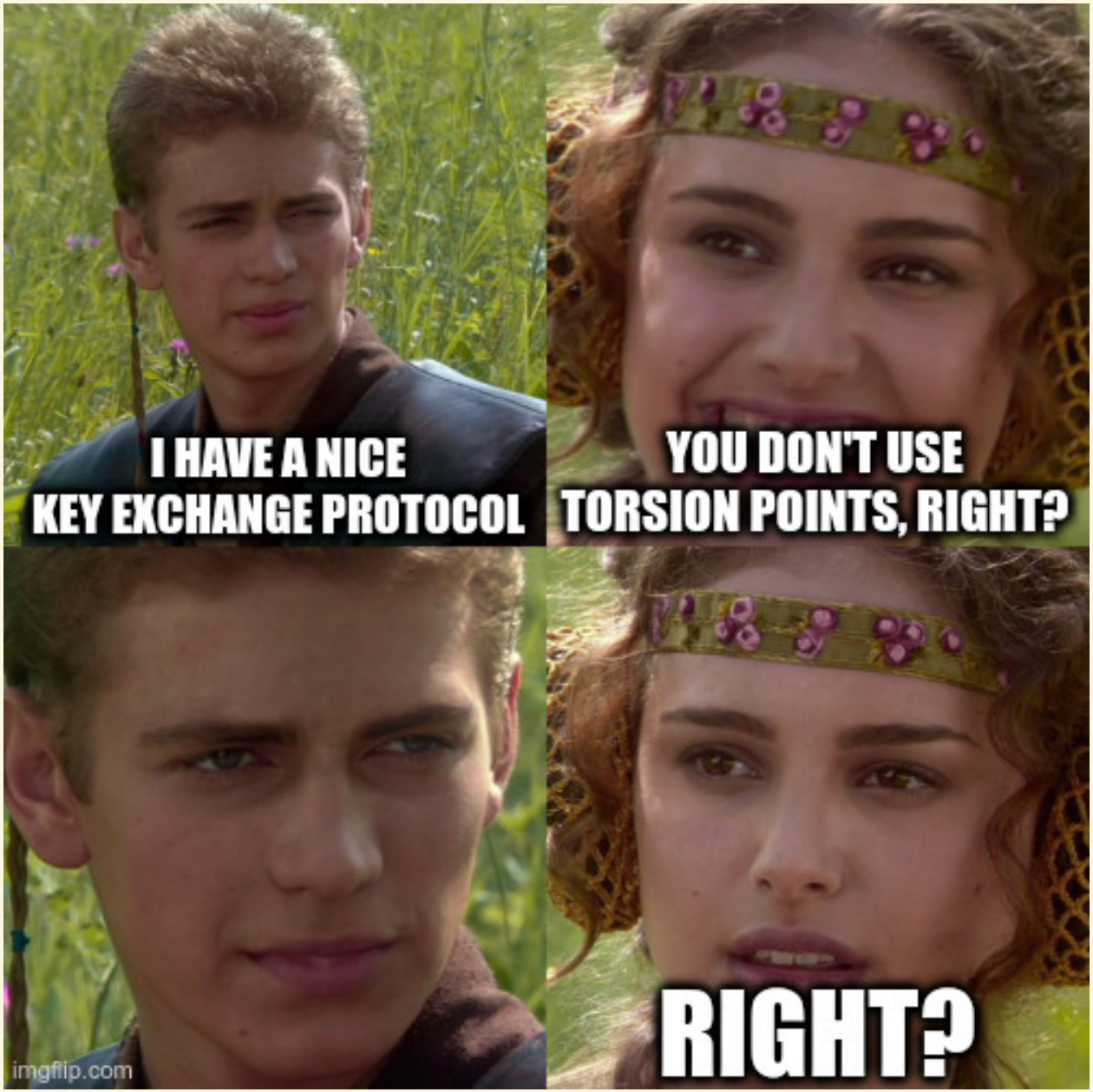
- Actually only need $N_A^2 > N_B$: we can reconstruct the $N_A^2$-isogeny $F$ from its action on the $N_A$-torsion.

# Dimension $4$ attack in the general case [R. (2022-08-10)]

- If $N_A^2 = (b_1^2 + b_2^2)N_B + (a_1^2 + a_2^2)$, we can construct $\beta$ a $b_1^2 + b_2^2$-endomorphism, $\alpha$ a $a_1^2 + a_2^2$-endomorphism on $E_0^2$.
- $F$ is a $N_A^2$-endomorphism on $E_0^2 \times E_B^2$.
- Attack: $\widetilde{O}(\log N_A \ell_A^4)$ arithmetic operations.
- Precomputation in (randomized heuristic) $\widetilde{O}(\log^3 N_A)$.

# Lessons learned

- Publishing the image of the torsion points was a key weakness.



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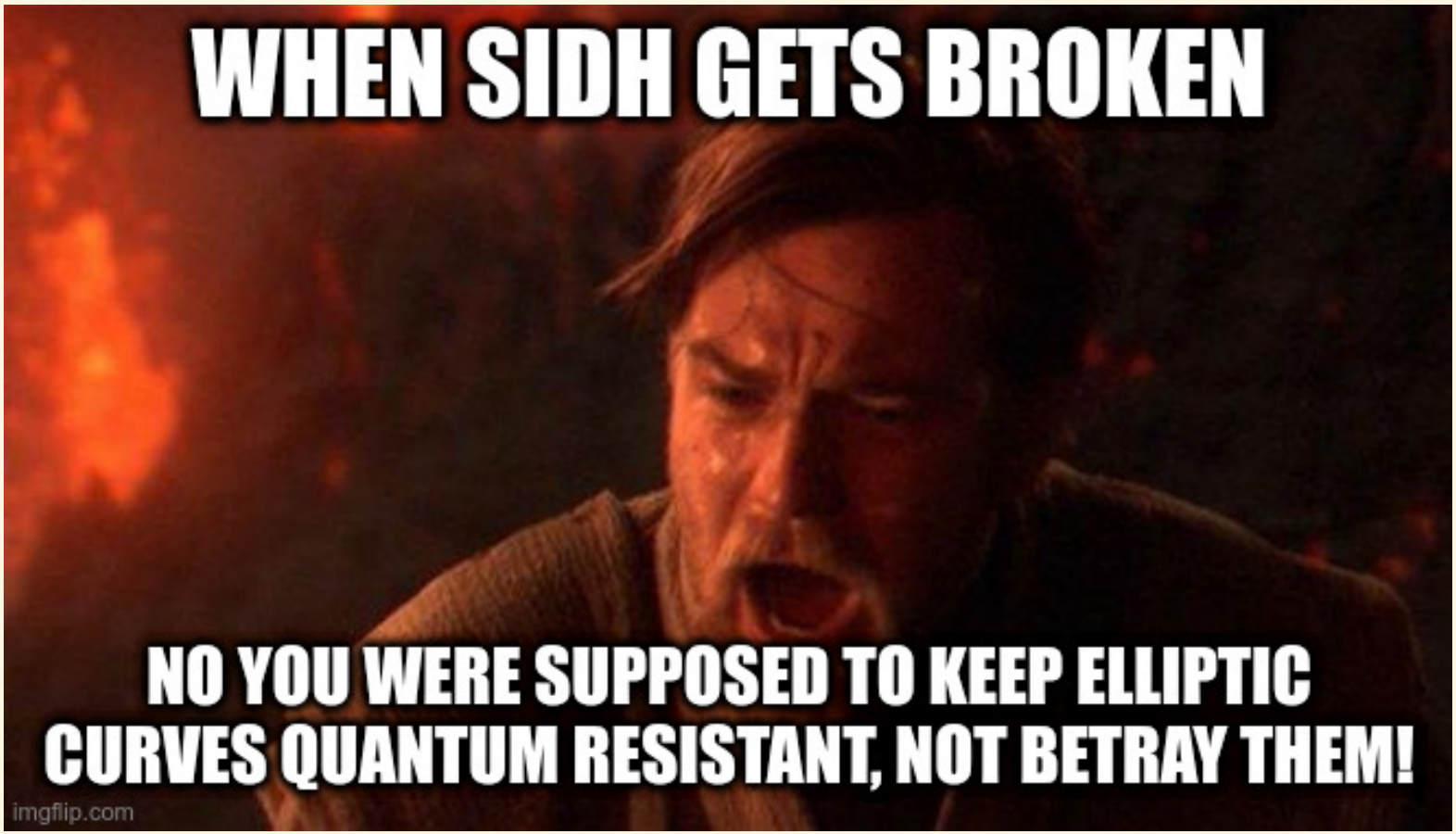
- CSIDH, SQISign still secure.


# Lessons learned

- Publishing the image of the torsion points was a key weakness.
- CSIDH, SQISign still secure.
- No post-quantum isogeny based KEM with exponential quantum security anymore. (OSIDH, masking torsion points [Moriya, Fouotsa]?)



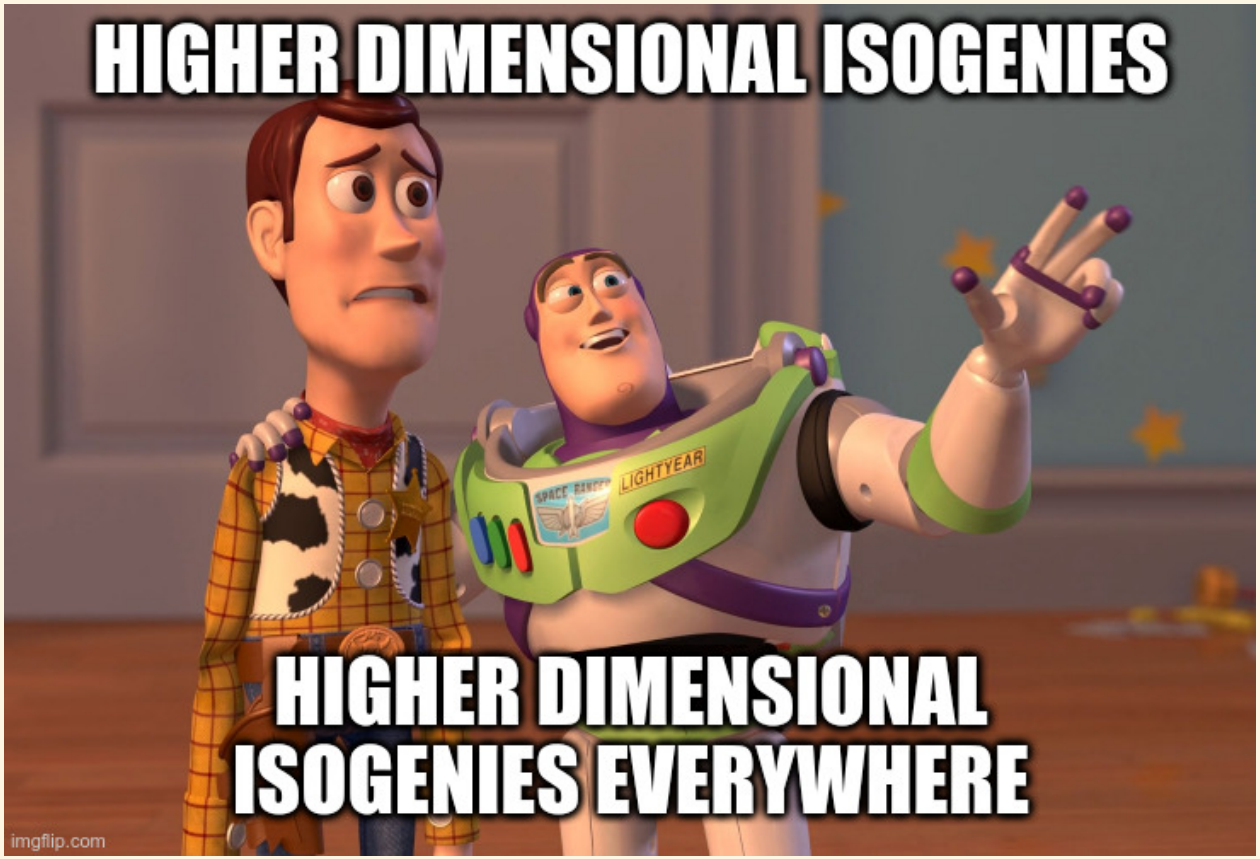
# New tools

- Embedding lemma: for any $N' > N$, a $N$-isogeny $f : A \to B$ in dimension $g$ can always be efficiently embedded into a $N'$-isogeny in dimension $8g$ (and sometimes $4g$ or $2g$).

$\Rightarrow$ An $N$-isogeny $f$ over a finite field always admit an efficient representation which allows for evaluation in polylogarithmic time [R. (2022-08-17)]. (Take $N'$ power smooth.)

# New tools

- Can we use this new toolbox to build new cryptosystems or break other existing ones?



# Conclusion