On isogenies and polarisations

2013/11/12 — **Lfant seminar** — Bordeaux

Gaëtan Bisson, Romain Cosset, Alina Dudeanu, Dimitar Jetchev, David Lubicz, **Damien Robert**

Outline

Polarised abelian varieties over $\mathbb{C}$

Definition

A complex abelian variety A of dimension g is isomorphic to a compact Lie group V/Λ with

- A complex vector space V of dimension g;
- A $\mathbb{Z}$-lattice Λ in V (of rank $2g$);

such that there exists an Hermitian form H on V with $E(\Lambda, \Lambda) \subset \mathbb{Z}$ where $E = \operatorname{Im} H$ is symplectic.

- Such an Hermitian form H is called a polarisation on A. Conversely, any symplectic form E on V such that $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(ix, iy) = E(x, y)$ for all $x, y \in V$ gives a polarisation H with $E = \operatorname{Im} H$.
- Over a symplectic basis of Λ, E is of the form.

$$\begin{pmatrix} 0 & D_\delta \\ -D_\delta & 0 \end{pmatrix}$$

  where D_δ is a diagonal positive integer matrix $\delta = (\delta_1, \delta_2, \ldots, \delta_g)$, with $\delta_1 \mid \delta_2 | \cdots \mid \delta_g$.
- The product $\prod \delta_i$ is the degree of the polarisation; H is a principal polarisation if this degree is 1.

Principal polarisations

- Let E_0 be the canonical principal symplectic form on $\mathbb{R}^{2g}$ given by $E_0((x_1,x_2),(y_1,y_2)) = {}^t x_1 \cdot y_2 - {}^t y_1 \cdot x_2$;
- If E is a principal polarisation on $A = V/\Lambda$, there is an isomorphism $j : \mathbb{Z}^{2g} \to \Lambda$ such that $E(j(x), j(y)) = E_0(x,y)$;
- There exists a basis of V such that $j((x_1,x_2)) = \Omega x_1 + x_2$ for a matrix Ω;
- In particular $E(\Omega x_1 + x_2, \Omega y_1 + y_2) = {}^t x_1 \cdot y_2 - {}^t y_1 \cdot x_2$;
- The matrix Ω is in $\mathfrak{H}_g$, the Siegel space of symmetric matrices Ω with $\mathrm{Im}\,\Omega$ positive definite;
- In this basis, $\Lambda = \Omega\mathbb{Z}^g + \mathbb{Z}^g$ and H is given by the matrix $(\mathrm{Im}\,\Omega)^{-1}$.

Action of the symplectic group

- Every principal symplectic form (hence symplectic basis) on $\mathbb{Z}^{2g}$ comes from the action of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ on $(\mathbb{Z}^{2g}, E_0)$;
- This action gives a new equivariant bijection $j_M : \mathbb{Z}^{2g} \to \Lambda$ via $j_M((x_1, x_2)) = (A\Omega x_1 + Bx_2, C\Omega x_1 + Dx_2)$;
- Normalizing this embedding via the action of $(C\Omega + D)^{-1}$ on $\mathbb{C}^g$, we get that $j_M((x_1, x_2)) = \Omega_M x_1 + x_2$ with $\Omega_M = (A\Omega + B)(C\Omega + D)^{-1} \in \mathfrak{H}_g$;
- The moduli space of principally polarised abelian varieties is then isomorphic to $\mathfrak{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z})$.

Isogenies

Let $A = V/\Lambda$ and $B = V'/\Lambda'$.

Definition

An isogeny $f : A \to B$ is a bijective linear map $f : V \to V'$ such that $f(\Lambda) \subset \Lambda'$. The kernel of the isogeny is $f^{-1}(\Lambda')/\Lambda \subset A$ and its degree is the cardinal of the kernel.

Remark

Up to a renormalization, we can always assume that $V = V' = \mathbb{C}^g$, $f = \mathrm{Id}$ *and the isogeny is simply* $\mathbb{C}^g/\Lambda \to \mathbb{C}^g/\Lambda'$ *for* $\Lambda \subset \Lambda'$.

The dual abelian variety

Definition

If $A = V/\Lambda$ is an abelian variety, its dual is $\widehat{A} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})/\Lambda^*$. Here $\mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ is the space of anti-linear forms and $\Lambda^* = \{f \mid f(\Lambda) \subset \mathbb{Z}\}$ is the orthogonal of Λ.

- If H is a polarisation on A, its dual H^* is a polarisation on $\widehat{A}$. Moreover, there is an isogeny $\Phi_H : A \to \widehat{A}$:
$$x \mapsto H(x, \cdot)$$
of degree $\deg H$. We note $K(H)$ its kernel.
- If $f : A \to B$ is an isogeny, then its dual is an isogeny $\widehat{f} : \widehat{B} \to \widehat{A}$ of the same degree.

Remark

There is a canonical polarisation on $A \times \widehat{A}$ (the Poincaré bundle):
$$(x, f) \mapsto f(x).$$

Isogenies and polarisations

Definition

- An isogeny $f:(A,H_1)\to(B,H_2)$ between polarised abelian varieties is an isogeny such that
$$f^*H_2 := H_2(f(\cdot),f(\cdot)) = H_1.$$
- By abuse of notations, we say that f is an ℓ-isogeny between principally polarised abelian varieties if H_1 and H_2 are principal and $f^*H_2 = \ell H_1$.

An isogeny $f:(A,H_1)\to(B,H_2)$ respect the polarisations iff the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Phi_{H_1}"] & B \arrow[d, "\Phi_{H_2}"] \\ \widehat{A} & \widehat{B} \arrow[l, "\widehat{f}"] \end{tikzcd}$$

Isogenies and polarisations

Definition

- An isogeny $f:(A,H_1)\rightarrow(B,H_2)$ between polarised abelian varieties is an isogeny such that
$$f^*H_2 := H_2(f(\cdot),f(\cdot)) = H_1.$$
- By abuse of notations, we say that f is an ℓ-isogeny between principally polarised abelian varieties if H_1 and H_2 are principal and $f^*H_2 = \ell H_1$.

$f:(A,H_1)\rightarrow(B,H_2)$ is an ℓ-isogeny between principally polarised abelian varieties iff the following diagram commutes

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[dl, "{[\ell]}"'] \arrow[d, "\Phi_{\ell H_1}"] & B \arrow[d, "\Phi_{H_2}"] \\
A \arrow[r, "\Phi_{H_1}"'] & \widehat{A} & \widehat{B} \arrow[l, "\widehat{f}"]
\end{tikzcd}$$

Jacobians

- Let C be a curve of genus g;
- Let V be the dual of the space V^* of holomorphic differentials of the first kind on C;
- Let $\Lambda \simeq H^1(C,\mathbb{Z}) \subset V$ be the set of periods (integration of differentials on loops);
- The intersection pairing gives a symplectic form E on Λ;
- Let H be the associated hermitian form on V;

$$H^*(w_1, w_2) = \int_C w_1 \wedge w_2;$$

- Then $(V/\Lambda, H)$ is a principally polarised abelian variety: the Jacobian of C.

Theorem (Torelli)

$\operatorname{Jac} C$ with the associated principal polarisation uniquely determines C.

Remark (Howe)

There exists an hyperelliptic curve H of genus 3 and a quartic curve C such that $\operatorname{Jac} C \simeq \operatorname{Jac} H$ as non polarised abelian varieties!

Projective embeddings

Proposition

Let $\Phi : A = V/\Lambda \mapsto \mathbb{P}^{m-1}$ be a projective embedding. Then the linear functions f associated to this embedding are Λ-automorphics:

$$f(x+\lambda) = a(\lambda, x) f(x) \quad x \in V, \lambda \in \Lambda;$$

for a fixed automorphy factor a:

$$a(\lambda+\lambda', x) = a(\lambda, x+\lambda') a(\lambda', x).$$

Theorem (Appell-Humbert)

All automorphy factors are of the form

$$a(\lambda, x) = \pm e^{\pi(H(x,\lambda)+\frac{1}{2}H(\lambda,\lambda))}$$

for a polarisation H on A.

Theta functions

- Let (A, H_0) be a principally polarised abelian variety over $\mathbb{C}$: $A = \mathbb{C}^g / (\Omega\mathbb{Z}^g + \mathbb{Z}^g)$ with $\Omega \in \mathfrak{H}_g$.
- All automorphic forms corresponding to a multiple of H_0 come from the theta functions with characteristics:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i \, {}^t(n+a)\Omega(n+a) + 2\pi i \, {}^t(n+a)(z+b)} \qquad a, b \in \mathbb{Q}^g$$

- Automorphic property:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z + m_1\Omega + m_2, \Omega) = e^{2\pi i({}^t a \cdot m_2 - {}^t b \cdot m_1) - \pi i \, {}^t m_1 \Omega m_1 - 2\pi i \, {}^t m_1 \cdot z} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega).$$

Theta functions of level n

- Define $\vartheta_i = \vartheta \left[\begin{smallmatrix} 0 \\ \frac{i}{n} \end{smallmatrix}\right](., \frac{\Omega}{n})$ for $i \in Z(\overline{n}) = \mathbb{Z}^g / n\mathbb{Z}^g$ and
- This is a basis of the automorphic functions for $H = nH_0$ (theta functions of level n);
- This is the unique basis such that in the projective coordinates:

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{P}^{n^g-1}_{\mathbb{C}} \\ z & \longmapsto & (\vartheta_i(z))_{i \in Z(\overline{n})} \end{array}$$

  the translation by a point of n-torsion is normalized by

$$\vartheta_i(z + \frac{m_1}{n}\Omega + \frac{m_2}{n}) = e^{-\frac{2\pi i}{n} {}^t i \cdot m_1} \vartheta_{i+m_2}(z).$$

- $(\vartheta_i)_{i \in Z(\overline{n})} = \begin{cases} \text{coordinates system} & n \geqslant 3 \\ \text{coordinates on the Kummer variety } A/\pm 1 & n = 2 \end{cases}$
- $(\vartheta_i)_{i \in Z(\overline{n})}$: basis of the theta functions of level n $\Leftrightarrow A[n] = A_1[n] \oplus A_2[n]$: symplectic decomposition.
- Theta null point: $\vartheta_i(0)_{i \in Z(\overline{n})} =$ modular invariant.

The differential addition law ($k = \mathbb{C}$)

$$\Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{i+t}(x+y)\vartheta_{j+t}(x-y)\Big).\Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{k+t}(0)\vartheta_{l+t}(0)\Big) =$$
$$\Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{-i'+t}(y)\vartheta_{j'+t}(y)\Big).\Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{k'+t}(x)\vartheta_{l'+t}(x)\Big).$$

where $\chi \in \hat{Z}(\overline{2}), i, j, k, l \in Z(\overline{n})$

$$(i', j', k', l') = A(i, j, k, l)$$

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Cryptographic usage of isogenies

- Transfer the Discrete Logarithm Problem from one Abelian variety to another;
- Point counting algorithms (ℓ-adic or p-adic) $\Rightarrow$ Verify an abelian variety is secure;
- Compute the class field polynomials (CM-method) $\Rightarrow$ Construct a secure abelian variety;
- Compute the modular polynomials $\Rightarrow$ Compute isogenies;
- Determine $\mathrm{End}(A) \Rightarrow$ CRT method for class field polynomials;
- Speed up the arithmetic;
- Hash functions and cryptosystems based on isogeny graphs.

The isogeny theorem

Theorem

- *Let* $\varphi : Z(\overline{n}) \to Z(\overline{\ell n}), x \mapsto \ell.x$ *be the canonical embedding.*
  Let $K = A_2[\ell] \subset A_2[\ell n]$.
- *Let* $(\vartheta_i^A)_{i \in Z(\overline{\ell n})}$ *be the theta functions of level* ℓn *on* $A = \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g)$.
- *Let* $(\vartheta_i^B)_{i \in Z(\overline{n})}$ *be the theta functions of level* n *of* $B = A/K = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$.
- *We have:*

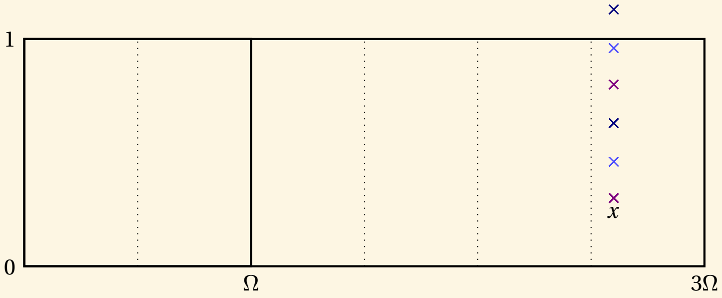
$$(\vartheta_i^B(x))_{i \in Z(\overline{n})} = (\vartheta_{\varphi(i)}^A(x))_{i \in Z(\overline{n})}$$

Example

$f : (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}) \mapsto (x_0, x_3, x_6, x_9)$ is a 3-isogeny between elliptic curves.

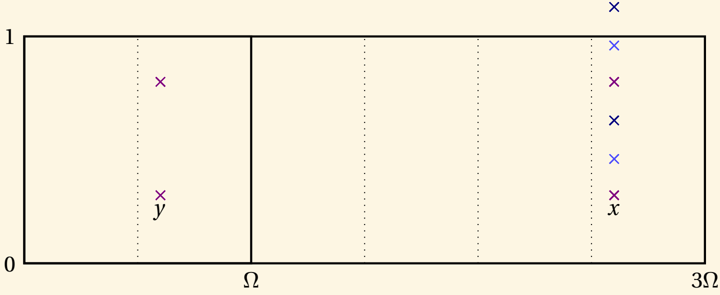
An example with $g=1$, $n=2$, $\ell=3$

$$\begin{tikzcd}
z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \arrow[rr, "{[\ell]}"] \arrow[dr, "f"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \\
& z \in \mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g), \text{ level } n \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$



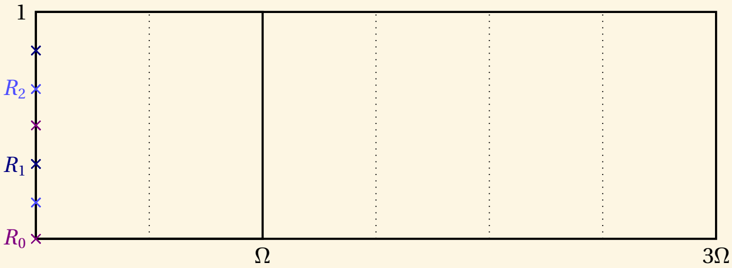
An example with $g=1$, $n=2$, $\ell=3$

$$\begin{tikzcd} z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \arrow[rr, "{[\ell]}"] \arrow[dr, "f"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \\ & z \in \mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g), \text{ level } n \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$



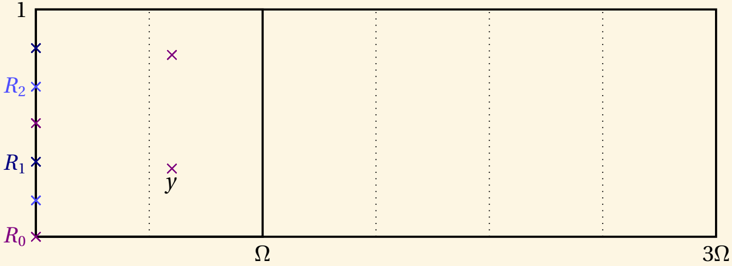
An example with $g=1$, $n=2$, $\ell=3$

$$\begin{tikzcd}
z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \arrow[rr, "{[\ell]}"] \arrow[dr, "f"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \\
& z \in \mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g), \text{ level } n \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$



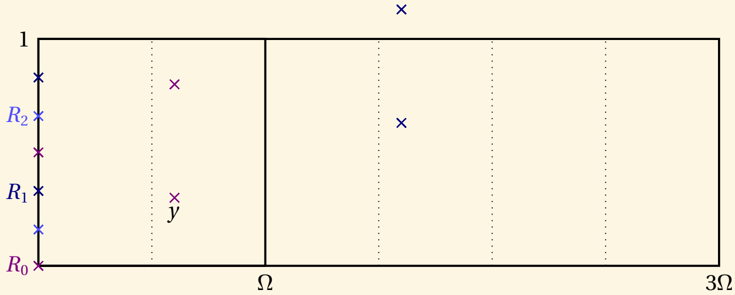
An example with $g=1$, $n=2$, $\ell=3$

$$\begin{tikzcd}
z \in \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g),\ \text{level } \ell n \arrow[rr, "{[\ell]}"] \arrow[dr, "f"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g),\ \text{level } \ell n \\
& z \in \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g),\ \text{level } n \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$



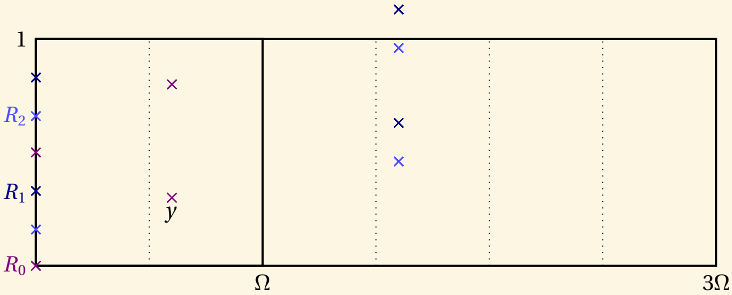
An example with $g=1$, $n=2$, $\ell=3$

$$\begin{tikzcd}
z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \arrow[rr, "{[\ell]}"] \arrow[dr, "f"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \\
& z \in \mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g), \text{ level } n \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$



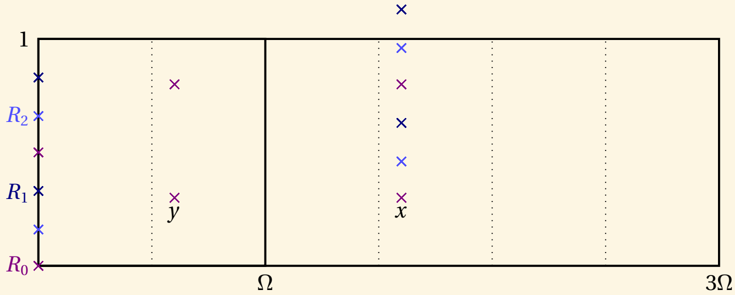
An example with $g=1$, $n=2$, $\ell=3$

$$\begin{tikzcd}
z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \arrow[rr, "{[\ell]}"] \arrow[dr, "f"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \\
& z \in \mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g), \text{ level } n \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}$$



An example with $g=1$, $n=2$, $\ell=3$

$$\begin{tikzcd}
z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \arrow[rr, "{[\ell]}"] \arrow[dr, "f"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g+\ell\Omega\mathbb{Z}^g), \text{ level } \ell n \\
& z \in \mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g), \text{ level } n \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$



Changing level

Theorem (Koizumi–Kempf)

Let F be a matrix of rank r such that ${}^tFF = \ell\,\mathrm{Id}_r$. Let $X \in (\mathbb{C}^g)^r$ and $Y = F(X) \in (\mathbb{C}^g)^r$. Let $j \in (\mathbb{Q}^g)^r$ and $i = F(j)$. Then we have

$$\vartheta\begin{bmatrix} 0 \\ i_1 \end{bmatrix}(Y_1, \frac{\Omega}{n}) \ldots \vartheta\begin{bmatrix} 0 \\ i_r \end{bmatrix}(Y_r, \frac{\Omega}{n}) = \sum_{\substack{t_1,\ldots,t_r \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g \\ F(t_1,\ldots,t_r)=(0,\ldots,0)}} \vartheta\begin{bmatrix} 0 \\ j_1 \end{bmatrix}(X_1 + t_1, \frac{\Omega}{\ell n}) \ldots \vartheta\begin{bmatrix} 0 \\ j_r \end{bmatrix}(X_r + t_r, \frac{\Omega}{\ell n}),$$

(This is the isogeny theorem applied to $F_A : A^r \to A^r$.)

- If $\ell = a^2 + b^2$, we take $F = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, so $r = 2$.
- In general, $\ell = a^2 + b^2 + c^2 + d^2$, we take F to be the matrix of multiplication by $a + bi + cj + dk$ in the quaternions, so $r = 4$.

The isogeny formula

$$\ell \wedge n = 1, \quad B = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g), \quad A = \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g)$$

$$\vartheta_b^B := \vartheta\begin{bmatrix} 0 \\ \frac{b}{n} \end{bmatrix}\left(\cdot, \frac{\Omega}{n}\right), \quad \vartheta_b^A := \vartheta\begin{bmatrix} 0 \\ \frac{b}{n} \end{bmatrix}\left(\cdot, \frac{\ell\Omega}{n}\right)$$

Proposition

Let F be a matrix of rank r such that ${}^tFF = \ell\,\mathrm{Id}_r$. Let X in $(\mathbb{C}^g)^r$ and $Y = XF^{-1} \in (\mathbb{C}^g)^r$. Let $i \in (Z(\overline{n}))^r$ and $j = iF^{-1}$. Then we have

$$\vartheta_{i_1}^A(Y_1)\ldots\vartheta_{i_r}^A(Y_r) = \sum_{\substack{t_1,\ldots,t_r \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g \\ (t_1,\ldots,t_r)F=(0,\ldots,0)}} \vartheta_{j_1}^B(X_1+t_1)\ldots\vartheta_{j_r}^B(X_r+t_r),$$

Corollary

$$\vartheta_k^A(0)\vartheta_0^A(0)\ldots\vartheta_0^A(0) = \sum_{\substack{t_1,\ldots,t_r \in K \\ (t_1,\ldots,t_r)F=(0,\ldots,0)}} \vartheta_{j_1}^B(t_1)\ldots\vartheta_{j_r}^B(t_r), \quad (j = (k,0,\ldots,0)F^{-1} \in Z(\overline{n}))$$

The Algorithm [Cosset, R.]

$$
\begin{tikzcd}
 & x \in (A, \ell H_1) \arrow[r, dashed] \arrow[dl, "f"] \arrow[dd, "{[\ell]}"] & (x, 0, \dots, 0) \in (A^r, \ell H_1 \star \dots \star \ell H_1) \arrow[d, "{}^t F"] \\
y \in (B, H_2) \arrow[dr, "\widetilde{f}"'] & & {}^t F(x, 0, \dots, 0) \in (A^r, \ell H_1 \star \dots \star \ell H_1) \arrow[d, "F"] \\
 & \widetilde{f}(y) \in (A, H_1) & F \circ {}^t F(x, 0, \dots, 0) \in (A^r, H_1 \star \dots \star H_1) \arrow[l, dashed]
\end{tikzcd}
$$

Complexity over $\mathbb{F}_q$

- The geometric points of the kernel live in a extension k' of degree at most $\ell^g - 1$ over $k = \mathbb{F}_q$;
- The isogeny formula assumes that the points are in affine coordinates. In practice, given $A/\mathbb{F}_q$ we only have projective coordinates $\Rightarrow$ we use differential additions to normalize the coordinates;
- Computing the normalization factors takes $O(\log \ell)$ operations in k';
- Computing the points of the kernel via differential additions take $O(\ell^g)$ operations in k';
- If $\ell \equiv 1 \pmod 4$, applying the isogeny formula take $O(\ell^g)$ operations in k';
- If $\ell \equiv 3 \pmod 4$, applying the isogeny formula take $O(\ell^{2g})$ operations in k';

$\Rightarrow$ The total cost is $\widetilde{O}(\ell^{2g})$ or $\widetilde{O}(\ell^{3g})$ operations in $\mathbb{F}_q$.

Remark

The complexity is much worse over a number field because we need to work with extensions of much higher degree.

Complexity over $\mathbb{F}_q$

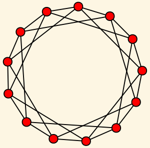
- The geometric points of the kernel live in a extension k' of degree at most $\ell^g - 1$ over $k = \mathbb{F}_q$;
- The isogeny formula assumes that the points are in affine coordinates. In practice, given $A/\mathbb{F}_q$ we only have projective coordinates $\Rightarrow$ we use differential additions to normalize the coordinates;
- Computing the normalization factors takes $O(\log \ell)$ operations in k';
- Computing the points of the kernel via differential additions take $O(\ell^g)$ operations in k';
- If $\ell \equiv 1 \pmod 4$, applying the isogeny formula take $O(\ell^g)$ operations in k';
- If $\ell \equiv 3 \pmod 4$, applying the isogeny formula take $O(\ell^{2g})$ operations in k';

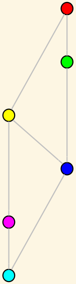
$\Rightarrow$ The total cost is $\widetilde{O}(\ell^{2g})$ or $\widetilde{O}(\ell^{3g})$ operations in $\mathbb{F}_q$.

Theorem ([Lubicz, R.])

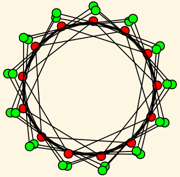
We can compute the isogeny directly given the equations (in a suitable form) of the kernel K of the isogeny. When K is rational, this gives a complexity of $\widetilde{O}(\ell^g)$ or $\widetilde{O}(\ell^{2g})$ operations in $\mathbb{F}_q$.

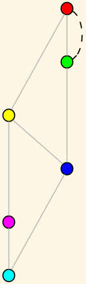
An (ℓ,ℓ)-isogeny graph in dimension 2 [Bisson, Cosset, R.]



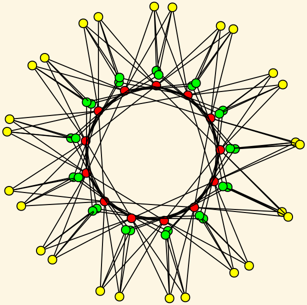


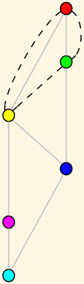
An (ℓ,ℓ)-isogeny graph in dimension 2 [Bisson, Cosset, R.]



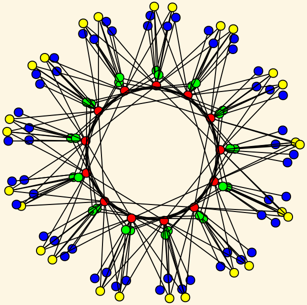


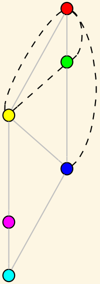
An (ℓ,ℓ)-isogeny graph in dimension 2 [Bisson, Cosset, R.]



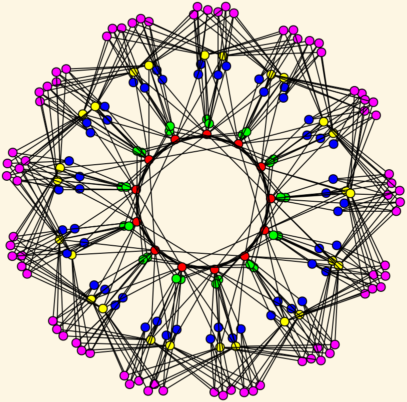


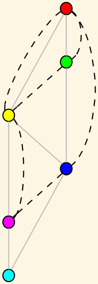
An (ℓ,ℓ)-isogeny graph in dimension 2 [Bisson, Cosset, R.]



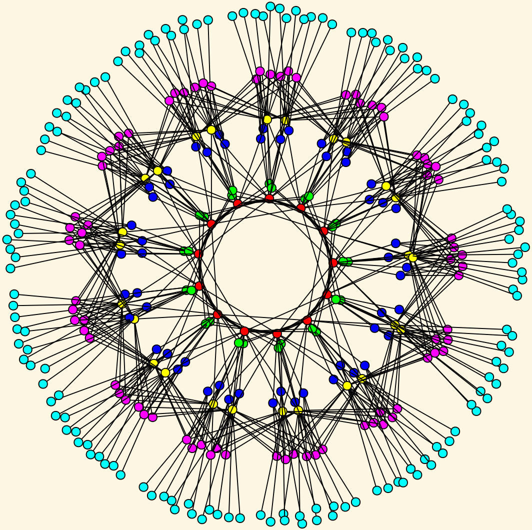


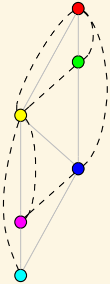
An (ℓ,ℓ)-isogeny graph in dimension 2 [Bisson, Cosset, R.]





An (ℓ,ℓ)-isogeny graph in dimension 2 [Bisson, Cosset, R.]





Non principal polarisations

- Let $f:(A,H_1) \rightarrow (B,H_2)$ be an isogeny between principally polarised abelian varieties;
- When $\operatorname{Ker} f$ is not maximal isotropic in $A[\ell]$ then $f^* H_2$ is not of the form ℓH_1;
- How can we go from the principal polarisation H_1 to $f^* H_1$?

Non principal polarisations

Theorem (Birkenhake-Lange, Th. 5.2.4)

Let A be an abelian variety with a principal polarisation $\mathscr{L}_1$;

- *Let $O_0 = \mathrm{End}(A)^s$ be the real algebra of endomorphisms symmetric under the Rosati involution;*
- *Let $\mathrm{NS}(A)$ be the Néron-Severi group of line bundles modulo algebraic equivalence.*

Then

- *$\mathrm{NS}(A)$ is a torsor under the action of O_0;*
- *This induces a bijection between polarisations of degree d in $\mathrm{NS}(A)$ and totally positive symmetric endomorphisms of norm d in O_0;*
- *The isomorphic class of a polarisation $\mathscr{L}_f \in \mathrm{NS}(A)$ for $f \in O_0^+$ correspond to the action $\varphi \mapsto \varphi^* f \varphi$ of the automorphisms of A.*

Cyclic isogeny

- Let $f:(A,H_1)\to(B,H_2)$ be an isogeny between principally polarised abelian varieties with cyclic kernel of degree ℓ;
- There exists φ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[dl, "\varphi"'] \arrow[d, "\Phi_{f^*H_2}"] & B \arrow[d, "\Phi_{H_2}"] \\ A \arrow[r, "\Phi_{H_1}"'] & \widehat{A} & \widehat{B} \arrow[l, "\widehat{f}"] \end{tikzcd}$$

- φ is an $(\ell,0,\ldots,\ell,0,\ldots)$-isogeny whose kernel is not isotropic for the H_1-Weil pairing on $A[\ell]$!
- φ commutes with the Rosatti involution so is a real endomorphism (φ is H_1-symmetric);
- φ is totally positive.

Descending a polarisation via φ

- The isogeny f induces a compatible isogeny between $\varphi H_1 = f^* H_2$ and H_2 where φH_1 is given by the following diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\Phi_{\varphi H_1}"'] & A \arrow[d, "\Phi_{H_1}"] \\ & \widehat{A} \end{tikzcd}$$

- φ plays the same role as $[\ell]$ for ℓ-isogenies;
- We then define the φ-contragredient isogeny $\widetilde{f}$ as the isogeny making the following diagram commute

$$\begin{tikzcd} & x \in (A, \varphi^* H_1) \arrow[dl, "f"] \arrow[dd, "\varphi"] \\ y \in (B, \varphi H_2) \arrow[dr, "\widetilde{f}"] & \\ & \widetilde{f}(y) \in (A, H_1) \end{tikzcd}$$

φ-change of level

- We can use the isogeny theorem to compute f from $(A, \varphi H_1)$ down to (B, H_2) or $\widetilde{f}$ from (B, H_2) up to $(A, \varphi H_1)$ as before;
- What about changing level between $(A, \varphi H_1)$ and (A, H_1)?
- φH_1 fits in the following diagram:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\Phi_{\varphi^* H_1}"'] \arrow[dr, "\Phi_{\varphi H_1}"] & A \arrow[d, "\Phi_{H_1}"] \\ \widehat{A} & \widehat{A} \arrow[l, "\hat{\varphi}"] \end{tikzcd}$$

- Applying the isogeny theorem on φ allows to find relations between $\varphi^* H_1$ and H_1 but we want φH_1.

φ-change of level

- φ is a totally positive element of a totally positive order O_0;
- A theorem of Siegel show that φ is a sum of m squares in $K_0 = O_0 \otimes \mathbb{Q}$;
- Clifford's algebras give a matrix $F \in \mathrm{Mat}_r(K_0)$ such that $\mathrm{diag}(\varphi) = F^* F$;
- We can use this matrix F to change level as before: If $X \in (\mathbb{C}^g)^r$ and $Y = F(X) \in (\mathbb{C}^g)^r$, $j \in (\mathbb{Q}^g)^r$ and $i = F(j)$, we have

$$\vartheta\begin{bmatrix} 0 \\ i_1 \end{bmatrix}(Y_1, \frac{\Omega}{n}) \ldots \vartheta\begin{bmatrix} 0 \\ i_r \end{bmatrix}(Y_r, \frac{\Omega}{n}) = \sum_{\substack{t_1,\ldots,t_r \in K(\varphi H_1) \\ F(t_1,\ldots,t_r)=(0,\ldots,0)}} \vartheta\begin{bmatrix} 0 \\ j_1 \end{bmatrix}(X_1 + t_1, \frac{\varphi^{-1}\Omega}{n}) \ldots \vartheta\begin{bmatrix} 0 \\ j_r \end{bmatrix}(X_r + t_r, \frac{\varphi^{-1}\Omega}{n}),$$

Remark

- *In general r can be larger than m;*
- *The matrix F acts by real endomorphism rather than by integer multiplication;*
- *There may be denominators in the coefficients of F.*

The Algorithm for cyclic isogenies [Dudeanu, Jetchev, R.]

$$B = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g), \quad A = \mathbb{C}^g/(\mathbb{Z}^g + \varphi\Omega\mathbb{Z}^n)$$

$$\vartheta_b^B := \vartheta\begin{bmatrix} 0 \\ \frac{b}{n} \end{bmatrix}\left(\cdot, \frac{\Omega}{n}\right), \quad \vartheta_b^A := \vartheta\begin{bmatrix} 0 \\ \frac{b}{n} \end{bmatrix}\left(\cdot, \frac{\varphi\Omega}{n}\right)$$

Theorem

Let X in $(\mathbb{C}^g)^r$ and $Y = XF^{-1} \in (\mathbb{C}^g)^r$. Let $i \in (Z(\overline{n}))^r$ and $j = iF^{-1}$.

$$\vartheta_{i_1}^A(Y_1)\dots\vartheta_{i_r}^A(Y_r) = \sum_{\substack{t_1,\dots,t_r \in K(\varphi H_2) \\ (t_1,\dots,t_r)F = (0,\dots,0)}} \vartheta_{j_1}^B(X_1+t_1)\dots\vartheta_{j_r}^B(X_r+t_r),$$

$$\begin{tikzcd}
 & x \in (A, \varphi H_1) \arrow[dl, "f"'] \arrow[dd, "\varphi"] \arrow[r, dashed] & (x,0,\dots,0) \in (A^r, \varphi H_1 \star \cdots \star \varphi H_1) \arrow[d, "{}^tF"] \\
y \in (B, H_2) \arrow[dr, "\tilde{f}"'] & & {}^tF(x,0,\dots,0) \in (A^r, \varphi H_1 \star \cdots \star \varphi H_1) \arrow[d, "F"] \\
 & \tilde{f}(y) \in (A, H_1) & F \circ {}^tF(x,0,\dots,0) \in (A^r, H_1 \star \cdots \star H_1) \arrow[l, dashed]
\end{tikzcd}$$

Hidden details

- We normalize the coordinates by using multi-way additions;
- The real endomorphisms are codiagonalisables (in the ordinary case), this is important to apply the isogeny theorem;
- If $g = 2$, $K_0 = \mathbb{Q}(\sqrt{d})$, the action of $\sqrt{d}$ is given by a standard (d, d)-isogeny, so we can compute it using the previous algorithm for d-isogenies!
- The important point is that this algorithm is such that we can keep track of the projective factors when computing the action of $\sqrt{d}$.

Remark

Computing the action of $\sqrt{d}$ directly may be expensive if d is big.

AVIsogenies [Bisson, Cosset, R.]

- AVIsogenies: Magma code written by Bisson, Cosset and R.
  http://avisogenies.gforge.inria.fr
- Released under LGPL 2+.
- Implement isogeny computation (and applications thereof) for abelian varieties using theta functions.
- Current release 0.6.
- Cyclic isogenies coming “soon”!