Computing optimal pairings on abelian varieties with theta functions
06/06/2013 — AGCT

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Outline

1. Pairings on curves
2. Abelian varieties
3. Theta functions
4. Pairings with theta functions
5. Performance
The Weil pairing on elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve over $k$ (char $k \neq 2,3$).
Let $P, Q \in E[\ell]$ be points of $\ell$-torsion.
Let $f_P$ be a function associated to the principal divisor $\ell(P) - \ell(0)$, and $f_Q$ to $\ell(Q) - \ell(0)$. We define:

$$e_{W,\ell}(P,Q) = \frac{f_P((Q) - (0))}{f_Q((P) - (0))}.$$

The application $e_{W,\ell} : E[\ell] \times E[\ell] \rightarrow \mu_\ell(k)$ is a non degenerate pairing: the Weil pairing.

**Definition (Embedding degree)**

The embedding degree $d$ is the smallest number thus that $\ell \mid q^d - 1$; $\mathbb{F}_{q^d}$ is then the smallest extension containing $\mu_\ell(k)$.
The Tate pairing on elliptic curves over $\mathbb{F}_q$

**Definition**

The Tate pairing is a non degenerate (on the right) bilinear application given by

$$e_T: E_0[\ell] \times E(\mathbb{F}_q)/\ell E(\mathbb{F}_q) \rightarrow \mathbb{F}_q^*/\ell^d \mathbb{F}_q^*$$

$$(P,Q) \mapsto f_P((Q) - (0))$$

where

$$E_0[\ell] = \{P \in E[\ell](\mathbb{F}_q^d) \mid \pi(P) = [q]P\}.$$

- On $\mathbb{F}_q^d$, the Tate pairing is a non degenerate pairing

$$e_T: E[\ell](\mathbb{F}_q^d) \times E(\mathbb{F}_q^d)/\ell E(\mathbb{F}_q^d) \rightarrow \mathbb{F}_q^*/\ell^d \mathbb{F}_q^* \simeq \mu_\ell;$$

- We normalise the Tate pairing by going to the power of $(q^d - 1)/\ell$. 
We need to compute the functions $f_P$ and $f_Q$. More generally, we define the Miller’s functions:

**Definition**

Let $\lambda \in \mathbb{N}$ and $X \in E[\ell]$, we define $f_{\lambda,X} \in k(E)$ to be a function thus that:

$$(f_{\lambda,X}) = \lambda(X) - ([\lambda]X) - (\lambda - 1)(0).$$

We want to compute (for instance) $f_{\ell,P}((Q) - (0))$. 
Miller’s algorithm

- The key idea in Miller’s algorithm is that
  \[ f_{\lambda+\mu,X} = f_{\lambda,X} f_{\mu,X} \hat{f}_{\lambda,\mu,X} \]
  where \( \hat{f}_{\lambda,\mu,X} \) is a function associated to the divisor
  \[ ([\lambda + \mu]X) - ([\lambda]X) - ([\mu]X) + (0). \]

- We can compute \( \hat{f}_{\lambda,\mu,X} \) using the addition law in \( E \): if \( [\lambda]X = (x_1, y_1) \) and \( [\mu]X = (x_2, y_2) \) and \( \alpha = (y_1 - y_2)/(x_1 - x_2) \), we have
  \[ \hat{f}_{\lambda,\mu,X} = \frac{y - \alpha(x - x_1) - y_1}{x + (x_1 + x_2) - \alpha^2}. \]
Let $C$ be a curve of genus $g$;

- Let $P \in \text{Jac}(C)[\ell]$ and $D_P$ a divisor of degree 0 on $C$ representing $P$;
- By definition of $\text{Jac}(C)$, $\ell D_P$ corresponds to a principal divisor $(f_P)$ on $C$;
- The same formulas as for elliptic curve define the Weil and Tate pairings:

$$e_W(P,Q) = f_P(D_Q)/f_Q(D_P)$$
$$e_T(P,Q) = f_P(D_Q).$$
Let $C$ be a curve of genus $g$;

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\[
e_W(P,Q) = f_P(D_Q)/f_Q(D_P)
\]
\[
e_T(P,Q) = f_P(D_Q).
\]

A key ingredient for evaluating $f_P(D_Q)$ comes from Weil reciprocity theorem.

Theorem (Weil)

Let $D_1$ and $D_2$ be two divisors with disjoint support linearly equivalent to (0) on a smooth curve $C$. Then

\[
f_{D_1}(D_2) = f_{D_2}(D_1).
\]
Let $C$ be a curve of genus $g$;
Let $P \in \text{Jac}(C)[\ell]$ and $D_P$ a divisor of degree 0 on $C$ representing $P$;
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\]
\[
e_T(P,Q) = f_P(D_Q).
\]

The extension of Miller's algorithm to Jacobians is “straightforward”;
For instance if $g = 2$, the function $f_{\lambda,\mu,P}$ is of the form
\[
\frac{y - l(x)}{(x - x_1)(x - x_2)}
\]
where $l$ is of degree 3.
**Definition**

An **Abelian variety** is a complete connected group variety over a base field $k$.

**Example**

- **Elliptic curves** = Abelian varieties of dimension 1;
- If $C$ is a (projective smooth absolutely irreducible) curve of genus $g$, its Jacobian is an abelian variety of dimension $g$;
- In dimension $g \geq 4$, not every abelian variety is a Jacobian.
Let \( f : A \rightarrow B \) be a separable isogeny with kernel \( K \) between two abelian varieties defined over \( k \):

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
0 & \leftarrow & \hat{A} & \leftarrow & \hat{f} & \leftarrow & \hat{B} & \leftarrow & \hat{K} & \leftarrow & 0
\end{array}
\]

- \( \hat{K} \) is the Cartier dual of \( K \), and we have a non degenerate pairing \( e_f : K \times \hat{K} \rightarrow \overline{k}^* \):
  1. If \( Q \in \hat{K}(\overline{k}) \), \( Q \) defines a divisor \( D_Q \) on \( B \);
  2. \( \hat{f}(Q) = 0 \) means that \( f^*D_Q \) is equal to a principal divisor \( (g_Q) \) on \( A \);
  3. \( e_f(P,Q) = g_Q(x)/g_Q(x+P) \). (This last function being constant in its definition domain).

- The Weil pairing \( e_{W,\ell} \) is the pairing associated to the isogeny \( [\ell] : A \rightarrow A \).
If $\mathcal{L}$ is an ample line bundle, the polarisation $\varphi_\mathcal{L}$ is a morphism $A \to \hat{A}$, $x \mapsto t^*_\mathcal{L} x \otimes \mathcal{L}^{-1}$.

**Definition**

Let $\mathcal{L}$ be a principal polarization on $A$. The (polarized) Weil pairing $e_{W,\mathcal{L},\ell}$ is the pairing

$$e_{W,\mathcal{L},\ell} : A[\ell] \times A[\ell] \to \mu_\ell(k),$$

$$(P, Q) \mapsto e_{W,\ell}(P, \varphi_{\mathcal{L}}(Q))$$

associated to the polarization $\mathcal{L}^\ell$:

$$A \xrightarrow{[\ell]} A \xrightarrow{\mathcal{L}} \hat{A}$$
The Tate pairings on abelian varieties over finite fields

- From the exact sequence
  \[ 0 \rightarrow A[\ell](\overline{F}_{q^d}) \rightarrow A(\overline{F}_{q^d}) \rightarrow A[\ell](\overline{F}_{q^d}) \rightarrow 0 \]
  we get from Galois cohomology a connecting morphism
  \[ \delta : A(\overline{F}_{q^d})/\ell A(\overline{F}_{q^d}) \rightarrow H^1(\text{Gal}(\overline{F}_{q^d}/F_{q^d}), A[\ell]); \]

- Composing with the Weil pairing, we get a bilinear application
  \[ A[\ell](\overline{F}_{q^d}) \times A(\overline{F}_{q^d})/\ell A(\overline{F}_{q^d}) \rightarrow H^1(\text{Gal}(\overline{F}_{q^d}/F_{q^d}), \mu_\ell) \simeq \overline{F}_{q^d}^* / F_{q^d}^\ell \simeq \mu_\ell \]
  where the last isomorphism comes from the Kummer sequence
  \[ 1 \rightarrow \mu_\ell \rightarrow \overline{F}_{q^d}^* \rightarrow \overline{F}_{q^d}^* \rightarrow 1 \]
  and Hilbert 90;

- Explicitly, if \( P \in A[\ell](\overline{F}_{q^d}) \) and \( Q \in A(\overline{F}_{q^d}) \) then the (reduced) Tate pairing is given by
  \[ e_T(P, Q) = e_W(\overline{P}, \pi(\overline{Q_0}) - \overline{Q_0}) \]
  where \( Q_0 \) is any point such that \( Q = [\ell]Q_0 \) and \( \pi \) is the Frobenius of \( \overline{F}_{q^d} \).
Let \((A, \mathcal{L})\) be a principally polarized abelian variety;

- To a degree 0 cycle \(\sum(P_i)\) on \(A\), we can associate the line bundle \(\otimes t_{P_i}^* \mathcal{L}\) on \(A\);
- The cycle \(\sum(P_i)\) corresponds to a trivial line bundle iff \(\sum P_i = 0\) in \(A\);
- If \(f\) is a function on \(A\) and \(D = \sum(P_i)\) a cycle whose support does not contain a zero or pole of \(f\), we let

\[
    f(D) = \prod f(P_i).
\]

(In the following, when we write \(f(D)\) we will always assume that we are in this situation.)

**Theorem ([Lan58])**

Let \(D_1\) and \(D_2\) be two cycles equivalent to 0, and \(f_{D_1}\) and \(f_{D_2}\) be the corresponding functions on \(A\). Then

\[
    f_{D_1}(D_2) = f_{D_2}(D_1)
\]
The Weil and Tate pairings on abelian varieties

**Theorem**

Let $P, Q \in A[\ell]$. Let $D_P$ and $D_Q$ be two cycles equivalent to $(P) - (0)$ and $(Q) - (0)$. The Weil pairing is given by

$$e_W(P, Q) = \frac{f_{\ell D_P}(D_Q)}{f_{\ell D_Q}(D_P)}.$$

**Theorem**

Let $P \in A[\ell](\mathbb{F}_{q^d})$ and $Q \in A(\mathbb{F}_{q^d})$, and let $D_P$ and $D_Q$ be two cycles equivalent to $(P) - (0)$ and $(Q) - (0)$. The (non reduced) Tate pairing is given by

$$e_T(P, Q) = f_{\ell D_P}(D_Q).$$
Cryptographic usage of pairings on abelian varieties

- The moduli space of abelian varieties of dimension $g$ is a space of dimension $g(g + 1)/2$. We have more liberty to find optimal abelian varieties in function of the security parameters.

- If $A$ is an abelian variety of dimension $g$, $A[\ell]$ is a \((\mathbb{Z}/\ell\mathbb{Z})\)-module of dimension $2g \Rightarrow$ the structure of pairings on abelian varieties is richer.

- Supersingular abelian varieties can have larger embedding degree than supersingular elliptic curves.

- Over a Jacobian, we can use twists even if they are not coming from twists of the underlying curve.
A complex abelian variety is of the form $A = V/\Lambda$ where $V$ is a $\mathbb{C}$-vector space and $\Lambda$ a lattice, with a polarization (actually an ample line bundle) $\mathcal{L}$ on it;

The Chern class of $\mathcal{L}$ corresponds to a symplectic real form $E$ on $V$ such that $E(i x, i y) = E(x, y)$ and $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$;

The commutator pairing $e_\mathcal{L}$ is then given by $\exp(2i \pi E(\cdot, \cdot))$;

A principal polarization on $A$ corresponds to a decomposition $\Lambda = \Omega \mathbb{Z}^g + \mathbb{Z}^g$ with $\Omega \in \mathfrak{H}_g$ the Siegel space;

The associated Riemann form on $A$ is then given by $E(\Omega x_1 + x_2, \Omega y_1 + y_2) = ^t x_1 \cdot y_2 - ^t y_1 \cdot x_2$. 
The theta functions of level \( n \) give a system of projective coordinates:

\[
\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] (z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (n+a)\Omega(n+a)+2\pi i (n+a)(z+b)} \quad a, b \in \mathbb{Q}^g
\]

If \( n = 2 \), we get (in the generic case) an embedding of the Kummer variety \( A/\pm 1 \).

**Remark**

*Working on level \( n \) mean we take a \( n \)-th power of the principal polarisation. So in the following we will compute the \( n \)-th power of the usual Weil and Tate pairings.*
The differential addition law \((k = \mathbb{C})\)

\[
\left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{i+t}(x+y) \vartheta_{j+t}(x-y) \right) \cdot \left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{k+t}(0) \vartheta_{l+t}(0) \right) = \\
\left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{-i+t}(y) \vartheta_{j+t}(y) \right) \cdot \left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{k'+t}(x) \vartheta_{l'+t}(x) \right).
\]

where \(\chi \in \hat{\mathbb{Z}}(2), \ i, j, k, l \in \mathbb{Z}(n)\)

\((i', j', k', l') = A(i, j, k, l)\)

\[
A = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\]
Example: differential addition in dimension 1 and in level 2

Algorithm

Input \( z_P = (x_0, x_1), z_Q = (y_0, y_1) \) and \( z_{P-Q} = (z_0, z_1) \) with \( z_0 z_1 \neq 0 \);
\( z_0 = (a, b) \) and \( A = 2(a^2 + b^2), B = 2(a^2 - b^2) \).

Output \( z_{P+Q} = (t_0, t_1) \).

1. \( t_0' = (x_0^2 + x_1^2)(y_0^2 + y_2^2)/A \)
2. \( t_1' = (x_0^2 - x_1^2)(y_0^2 - y_1^2)/B \)
3. \( t_0 = (t_0' + t_1')/z_0 \)
4. \( t_1 = (t_0' - t_1')/z_1 \)

Return \( (t_0, t_1) \)
Miller functions with theta coordinates

Proposition (Lubicz-R. [LR13])

- For $P \in A$ we note $z_P$ a lift to $\mathbb{C}^g$. We call $P$ a projective point and $z_P$ an affine point (because we describe them via their projective, resp affine, theta coordinates);
- We have (up to a constant)

$$f_{\lambda, P}(z) = \frac{\vartheta(z)}{\vartheta(z + \lambda z_P)} \left( \frac{\vartheta(z + z_P)}{\vartheta(z)} \right)^{\lambda};$$

- So (up to a constant)

$$f_{\lambda, \mu, P}(z) = \frac{\vartheta(z + \lambda z_P)\vartheta(z + \mu z_P)}{\vartheta(z)\vartheta(z + (\lambda + \mu)z_P)}.$$
Three way addition

Proposition (Lubicz-R. [LR13])

From the affine points \( z_P, z_Q, z_R, z_{P+Q}, z_{P+R} \) and \( z_{Q+R} \) one can compute the affine point \( z_{P+Q+R} \).

(In level 2, the proposition is only valid for “generic” points).

Proof.

We can compute the three way addition using a generalised version of Riemann’s relations:

\[
\left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{i+t}(z_{P+Q+R}) \vartheta_{j+t}(z_P) \right) \cdot \left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{k+t}(z_Q) \vartheta_{l+t}(z_R) \right) =
\]

\[
\left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{-i'+t}(z_0) \vartheta_{j'+t}(z_{Q+R}) \right) \cdot \left( \sum_{t \in \mathbb{Z}(2)} \chi(t) \vartheta_{k'+t}(z_{P+R}) \vartheta_{l'+t}(z_{P+Q}) \right).
\]
Computing the Miller function $f_{\lambda, \mu, P}((Q) - (0))$

**Algorithm**

**Input** $\lambda P, \mu P$ and $Q$;

**Output** $f_{\lambda, \mu, P}((Q) - (0))$

1. **Compute** $(\lambda + \mu)P, Q + \lambda P, Q + \mu P$ using normal additions and take any affine lifts $z_{(\lambda+\mu)P}, z_{Q+\lambda P}$ and $z_{Q+\mu P}$;

2. **Use a three way addition to compute** $z_{Q+(\lambda+\mu)P}$;

**Return**

$$f_{\lambda, \mu, P}((Q) - (0)) = \frac{\vartheta(z_Q + \lambda z_P) \vartheta(z_Q + \mu z_P) \vartheta((\lambda + \mu)z_P) \vartheta(z_P)}{\vartheta(z_Q) \vartheta(z_Q + (\lambda + \mu)z_P) \vartheta(\lambda z_P) \vartheta(\mu z_P)}.$$  

**Lemma**

*The result does not depend on the choice of affine lifts in Step 2.*

- This allow us to evaluate the Weil and Tate pairings and derived pairings;
- Not possible *a priori* to apply this algorithm in level 2.
The Tate pairing with Miller’s functions and theta coordinates

Let $P \in A[\ell](\mathbb{F}_{q^d})$ and $Q \in A(\mathbb{F}_{q^d})$; choose any lift $z_P$, $z_Q$ and $z_{P+Q}$.

- The algorithm loop over the binary expansion of $\ell$, and at each step does a doubling step, and if necessary an addition step.

Given $z_{\lambda P}$, $z_{\lambda P+Q}$;

Doubling Compute $z_{2\lambda P}$, $z_{2\lambda P+Q}$ using two differential additions;

Addition Compute $(2\lambda + 1)P$ and take an arbitrary lift $z_{(2\lambda+1)P}$. Use a three way addition to compute $z_{(2\lambda+1)P+Q}$.

- At the end we have computed affine points $z_{\ell P}$ and $z_{\ell P+Q}$. Evaluating the Miller function then gives exactly the quotient of the projective factors between $z_{\ell P}$, $z_0$ and $z_{\ell P+Q}$, $z_Q$.

- Described this way can be extended to level 2 by using compatible additions;

- Can we get rid of three way additions?
The Weil and Tate pairing with theta coordinates (Lubicz-R. [LR10])

$P$ and $Q$ points of $\ell$-torsion.

\[
\begin{align*}
&z_0 & z_P & 2z_P & \ldots & \ell z_P = \lambda^0_P z_0 \\
&z_Q & z_P \oplus z_Q & 2z_P + z_Q & \ldots & \ell z_P + z_Q = \lambda^1_P z_Q \\
&2z_Q & z_P + 2z_Q & & & \\
&\ldots & \ldots & & & \\
&\ell Q = \lambda^0_Q 0_A & z_P + \ell z_Q = \lambda^1_Q z_P & & & \\
\end{align*}
\]

\[
e_{W,\ell}(P,Q) = \frac{\lambda^1_P \lambda^0_Q}{\lambda^0_P \lambda^1_Q}.
\]

\[
e_{T,\ell}(P,Q) = \frac{\lambda^1_P}{\lambda^0_P}.
\]
Ate pairing

- Let \( P \in G_2 = A[\ell] \cap \text{Ker}(\pi_q - [q]) \) and \( Q \in G_1 = A[\ell] \cap \text{Ker}(\pi_q - 1); \) \( \lambda \equiv q \mod \ell. \)
- In projective coordinates, we have \( \pi^d_q(P + Q) = \lambda^d P + Q = P + Q; \)
- Of course, in affine coordinates, \( \pi^d_q(z_{P+Q}) \neq \lambda^d z_P + z_Q. \)
- But if \( \pi_q(z_{P+Q}) = C \ast (\lambda z_P + z_Q), \) then \( C \) is exactly the (non reduced) ate pairing (up to a renormalisation)!

Algorithm (Computing the ate pairing)

**Input**  \( P \in G_2, Q \in G_1; \)

1. **Compute** \( z_Q + \lambda z_P, \lambda z_P \) using differential additions;
2. **Find the projective factors** \( C_1 \) and \( C_0 \) such that \( z_Q + \lambda z_P = C_1 \ast \pi(z_{P+Q}) \) and \( \lambda z_P = C_0 \ast \pi(z_P) \) respectively;

**Return** \( (C_1/C_0)^{q^d-1}/\ell. \)
Let $\lambda = m\ell = \sum c_i q^i$ be a multiple of $\ell$ with small coefficients $c_i$. ($\ell \nmid m$)

The pairing

$$a_\lambda : G_2 \times G_1 \rightarrow \mu_\ell$$

$$(P, Q) \mapsto \left( \prod_i f_{c_i, P}(Q)^q^i \prod_i \sum_{j>i} c_j q^j, c_i q^i, P(Q) \right)^{(q^d-1)/\ell}$$

is non-degenerate when $md q^{d-1} \neq (q^d - 1)/r \sum_i i c_i q^{i-1} \mod \ell$.

Since $\varphi_d(q) = 0 \mod \ell$ we look at powers $q, q^2, \ldots, q^{\varphi(d)-1}$.

We can expect to find $\lambda$ such that $c_i \approx \ell^{1/\varphi(d)}$. 

Optimal ate pairing
Optimal ate pairing with theta functions

Algorithm (Computing the optimal ate pairing)

Input \( \pi_q(P) = [q]P, \pi_q(Q) = Q, \lambda = m\ell = \sum c_i q^i; \)

1. Compute the \( z_Q + c_i z_P \) and \( c_i z_P; \)
2. Apply Frobeniuses to obtain the \( z_Q + c_i q^i z_P, c_i q^i z_P; \)
3. Compute \( c_i q^i z_P \oplus \sum_j c_j q^j z_P \) (up to a constant) and then do a three way addition to compute \( z_Q + c_i q^i z_P + \sum_j c_j q^j z_P \) (up to the same constant);
4. Recurse until we get \( \lambda z_P = C_0 * z_P \) and \( z_Q + \lambda z_P = C_1 * z_Q; \)

Return \( (C_1/C_0)^{q^d-1\over \ell}. \)
One step of the pairing computation

Algorithm (A step of the Miller loop with differential additions)

Input  $nP = (x_n, z_n); (n+1)P = (x_{n+1}, z_{n+1}), (n+1)P + Q = (x'_{n+1}, z'_{n+1})$.

Output $2nP = (x_{2n}, z_{2n}); (2n+1)P = (x_{2n+1}, z_{2n+1});$

$(2n+1)P + Q = (x'_{2n+1}, z'_{2n+1})$.

1. $\alpha = (x^2_n + z^2_n); \beta = \frac{A}{B}(x^2_n - z^2_n)$.
2. $X_n = \alpha^2; X_{n+1} = \alpha(x^2_{n+1} + z^2_{n+1}); X'_{n+1} = \alpha(x'^2_{n+1} + z'^2_{n+1});$
3. $Z_n = \beta(x^2_n - z^2_n); Z_{n+1} = \beta(x^2_{n+1} - z^2_{n+1}); Z'_{n+1} = \beta(x'^2_{n+1} + z'^2_{n+1});$
4. $x_{2n} = X_n + Z_n; x_{2n+1} = (X_{n+1} + Z_{n+1})/x_P; x'_{2n+1} = (X'_{n+1} + Z'_{n+1})/x_Q;$
5. $z_{2n} = \frac{a}{b}(X_n - Z_n); z_{2n+1} = (X_{n+1} - Z_{n+1})/z_P; z'_{2n+1} = (X'_{n+1} - Z'_{n+1})/z_Q;$

Return $(x_{2n}, z_{2n}); (x_{2n+1}, z_{2n+1}); (x'_{2n+1}, z'_{2n+1})$.
Weil and Tate pairing over $\mathbb{F}_{q^d}$

\[
\begin{align*}
g = 1 & \quad 4M + 2m + 8S + 3m_0 \\
g = 2 & \quad 8M + 6m + 16S + 9m_0
\end{align*}
\]

Tate pairing with theta coordinates, $P, Q \in A[\ell](\mathbb{F}_{q^d})$ (one step)

Operations in $\mathbb{F}_q$: $M$: multiplication, $S$: square, $m$ multiplication by a coordinate of $P$ or $Q$, $m_0$ multiplication by a theta constant;

Mixed operations in $\mathbb{F}_q$ and $\mathbb{F}_{q^d}$: $M$, $m$ and $m_0$;

Operations in $\mathbb{F}_{q^d}$: $M$, $m$ and $S$.

Remark

- **Doubling step for a Miller loop with Edwards coordinates**: $9M + 7S + 2m_0$;
- **Just doubling a point in Mumford projective coordinates using the fastest algorithm** [Lan05]: $33M + 7S + 1m_0$;
- **Asymptotically the final exponentiation is more expensive than Miller’s loop, so the Weil’s pairing is faster than the Tate’s pairing!**
Tate pairing

\[
g = 1 \quad 1m + 2s + 2M + 2M + 1m + 6s + 3m_0 \\
g = 2 \quad 3m + 4s + 4M + 4M + 3m + 12s + 9m_0
\]

Tate pairing with theta coordinates, \( P \in A[\ell](\F_q), Q \in A[\ell](\F_{q^d}) \) (one step)

<table>
<thead>
<tr>
<th>( g = 1 )</th>
<th>Miller</th>
<th>Theta coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d ) even</td>
<td>( 1M + 1S + 1M )</td>
<td>( 1M + 1M )</td>
</tr>
<tr>
<td>( d ) odd</td>
<td>( 2M + 2S + 1M )</td>
<td>( 2M + 1M )</td>
</tr>
<tr>
<td>( g = 2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q ) degenerate + ( d ) even</td>
<td>( 1M + 1S + 3M )</td>
<td>( 1M + 3M )</td>
</tr>
<tr>
<td>General case</td>
<td>( 2M + 2S + 18M )</td>
<td>( 2M + 18M )</td>
</tr>
</tbody>
</table>

\( P \in A[\ell](\F_q), Q \in A[\ell](\F_{q^d}) \) (counting only operations in \( \F_{q^d} \)).
### Ate and optimal ate pairings

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<thead>
<tr>
<th>$g$</th>
<th>Equation</th>
<th>( 4M + 1m + 8S + 1m + 3m_0 )</th>
<th>( 8M + 3m + 16S + 3m + 9m_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
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</tr>
</tbody>
</table>

- **Ate pairing with theta coordinates**, \( P \in G_2, Q \in G_1 \) (one step)

**Remark**

Using affine Mumford coordinates in dimension 2, the hyperelliptic ate pairing costs \([Gra+07]\):

- **Doubling** \( 1I + 29M + 9S + 7M \)
- **Addition** \( 1I + 29M + 5S + 7M \)

*(where \( I \) denotes the cost of an affine inversion in \( \mathbb{F}_{q^d} \)).*
Bibliography


