

# A Vélú's like formula for computing isogenies on Abelian Varieties

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Octobre 2010, Réunion CHIC

# Vélu's formula

## Théorème

Let  $E : y^2 = f(x)$  be an elliptic curve. Let  $G \subset E(k)$  be a finite subgroup. Then  $E/G$  is given by  $Y^2 = g(X)$  where

$$X(P) = x(P) + \sum_{Q \in G \setminus \{0_E\}} x(P + Q) - x(Q)$$

$$Y(P) = y(P) + \sum_{Q \in G \setminus \{0_E\}} y(P + Q) - y(Q)$$

- Uses the fact that  $x$  and  $y$  are characterised in  $k(E)$  by

$$v_{0_E}(x) = -3 \quad v_P(x) \geq 0 \quad \text{if } P \neq 0_E$$

$$v_{0_E}(y) = -2 \quad v_P(y) \geq 0 \quad \text{if } P \neq 0_E$$

$$y^2/x^3(O_E) = 1$$

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# The modular polynomial

## Définition

- The **modular polynomial** is a polynomial  $\varphi_n(x, y) \in \mathbb{Z}[x, y]$  such that  $\varphi_n(x, y) = 0$  iff  $x = j(E)$  and  $y = j(E')$  with  $E$  and  $E'$   $n$ -isogeneous.
- If  $E : y^2 = x^3 + ax + b$  is an elliptic curve, the  **$j$ -invariant** is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

- Roots of  $\varphi_n(j(E), \cdot) \Leftrightarrow$  elliptic curves  $n$ -isogeneous to  $E$ .
  - In genus 2, modular polynomials use Igusa invariants. The height explodes :  $\varphi_2 = 50$  MB.
- $\Rightarrow$  Use the moduli space given by theta functions.
- $\Rightarrow$  Fix the form of the isogeny and look for coordinates compatible with the isogeny.

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# Complex abelian varieties

- Abelian variety over  $\mathbb{C}$  :  $A = \mathbb{C}^g / (\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ , where  $\Omega \in \mathcal{H}_g(\mathbb{C})$  the Siegel upper half space.
- The **theta functions with characteristic** give a lot of analytic (quasi periodic) functions on  $\mathbb{C}^g$ .

$$\vartheta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i n' \Omega n + 2\pi i n' z}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = e^{\pi i a' \Omega a + 2\pi i a' (z+b)} \vartheta(z + \Omega a + b, \Omega) \quad a, b \in \mathbb{Q}^g$$

- The quasi-periodicity is given by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z + m + \Omega n, \Omega) = e^{2\pi i (a' m - b' n) - \pi i n' \Omega n - 2\pi i n' z} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$$

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# Projective embeddings given by theta functions

## Théorème

- Let  $\mathcal{L}_\ell$  be the space of analytic functions  $f$  satisfying :

$$\begin{aligned} f(z+n) &= f(z) \\ f(z+n\Omega) &= \exp(-\ell \cdot \pi i n' \Omega n - \ell \cdot 2\pi i n' z) f(z) \end{aligned}$$

- A basis of  $\mathcal{L}_\ell$  is given by

$$\left\{ \vartheta \begin{bmatrix} 0 \\ b \end{bmatrix} (z, \Omega/\ell) \right\}_{b \in \frac{1}{\ell} \mathbb{Z}^g / \mathbb{Z}^g}$$

- Let  $\mathcal{Z}_\ell = \mathbb{Z}^g / \ell \mathbb{Z}^g$ . If  $i \in \mathcal{Z}_\ell$  we define  $\vartheta_i = \vartheta \begin{bmatrix} 0 \\ i/\ell \end{bmatrix} (\cdot, \Omega/\ell)$ . If  $l \geq 3$  then

$$z \mapsto (\vartheta_i(z))_{i \in \mathcal{Z}_\ell}$$

is a projective embedding  $A \rightarrow \mathbb{P}_{\mathbb{C}}^{\ell^g - 1}$ .

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# The action of the Theta group

- The point  $(a_i)_{i \in \mathcal{Z}_\ell} := (\vartheta_i(0))_{i \in \mathcal{Z}_\ell}$  is called the **theta null point** of level  $\ell$  of the Abelian Variety  $A := \mathbb{C}^g / (\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ .
- $(a_i)_{i \in \mathcal{Z}_\ell}$  determines the equations of the projective embedding of  $A$  of level  $\ell$ .
- The symplectic basis  $\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$  induce a decomposition into isotropic subgroups for the commutator pairing :

$$\begin{aligned} A[\ell] &= A[\ell]_1 \oplus A[\ell]_2 \\ &= \frac{1}{\ell} \mathbb{Z}^g / \mathbb{Z}^g \oplus \frac{1}{\ell} \Omega\mathbb{Z}^g / \Omega\mathbb{Z}^g \end{aligned}$$

This decomposition can be recovered by  $(a_i)_{i \in \mathcal{Z}_\ell}$ .

- The action by translation is given by

$$\vartheta_k \left( z - \frac{i}{\ell} - \Omega \frac{j}{\ell} \right) = e_{\mathcal{L}_\ell}(i+k, j) \vartheta_{i+k}$$

where  $e_{\mathcal{L}_\ell}(x, y) = e^{2\pi i/\ell \cdot x'y}$  is the commutator pairing.

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# The isogeny theorem

## Théorème

- Let  $\ell = n.m$ , and  $\varphi : \mathcal{Z}_n \rightarrow \mathcal{Z}_\ell, x \mapsto m.x$  be the canonical embedding.  
Let  $K = A[m]_2 \subset A[\ell]_2$ .
- Let  $(\vartheta_i^A)_{i \in \mathcal{Z}_\ell}$  be the theta functions of level  $\ell$  on  $A = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ .
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- We have :

$$(\vartheta_i^B(x))_{i \in \mathcal{Z}_n} = (\vartheta_{\varphi(i)}^A(x))_{i \in \mathcal{Z}_n}$$

## Démonstration.

$$\vartheta_i^B(z) = \vartheta \begin{bmatrix} 0 \\ i/n \end{bmatrix} \left( z, \frac{\Omega}{m/n} \right) = \vartheta \begin{bmatrix} 0 \\ mi/\ell \end{bmatrix} (z, \Omega/\ell) = \vartheta_{m \cdot i}^A(z)$$





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# Mumford : On equations defining Abelian varieties

## Théorème (car $k \neq \ell$ )

- The theta null point of level  $\ell$   $(a_i)_{i \in \mathbb{Z}_\ell}$  satisfy the Riemann Relations :

$$\sum_{t \in \mathbb{Z}_2} a_{x+t} a_{y+t} \sum_{t \in \mathbb{Z}_2} a_{u+t} a_{v+t} = \sum_{t \in \mathbb{Z}_2} a_{z-u+t} a_{z-y+t} \sum_{t \in \mathbb{Z}_2} a_{z-x+t} a_{z-v+t} \quad (1)$$

We note  $\mathcal{M}_\ell$  the *moduli space* given by these relations together with the relations of symmetry :

$$a_x = a_{-x}$$

- $\mathcal{M}_\ell(k)$  is the modular space of  $k$ -Abelian variety with a theta structure of level  $\ell$ . The locus of theta null points of level  $\ell$  is an open subset  $\mathcal{M}_\ell^0(k)$  of  $\mathcal{M}_\ell(k)$ .

## Remark

- Analytic action* :  $\mathrm{Sp}_{2g}(\mathbb{Z})$  acts on  $\mathcal{H}_g$  (and preserve the isomorphic classes).
- Algebraic action* :  $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$  acts on  $\mathcal{M}_\ell$ .

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# Summary

$$\begin{array}{ccc}
 A_k, A_k[\ell] = A_k[\ell]_1 \oplus A_k[\ell]_2 & \xleftarrow{\text{determines}} & (a_i)_{i \in \mathcal{Z}_\ell} \in \mathcal{M}_\ell(k) \\
 \hat{\pi} \updownarrow \pi & & \downarrow \\
 B_k, B_k[n] = B_k[n]_1 \oplus B_k[n]_2 & \xleftarrow{\dots\dots\dots} & (b_i)_{i \in \mathcal{Z}_n} \in \mathcal{M}_n(k)
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- The kernel of  $\pi$  is  $A_k[m]_2 \subset A_k[\ell]_2$ .
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- Every isogeny comes from a modular solution.

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 A_k, A_k[\ell] = A_k[\ell]_1 \oplus A_k[\ell]_2 \leftarrow \dots \dots \dots \text{determines} & & (a_i)_{i \in \mathcal{Z}_\ell} \in \mathcal{M}_\ell(k) \\
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# The addition law

## Théorème

$$\left( \sum_{t \in \mathbb{Z}_2} \chi(t) \vartheta_{i+t}(x+y) \vartheta_{j+t}(x-y) \right) \cdot \left( \sum_{t \in \mathbb{Z}_2} \chi(t) \vartheta_{k+t}(0) \vartheta_{l+t}(0) \right) =$$

$$\left( \sum_{t \in \mathbb{Z}_2} \chi(t) \vartheta_{-i'+t}(y) \vartheta_{j'+t}(y) \right) \cdot \left( \sum_{t \in \mathbb{Z}_2} \chi(t) \vartheta_{k'+t}(x) \vartheta_{l'+t}(x) \right).$$

where  $A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

$$\chi \in \hat{\mathbb{Z}}_2, i, j, k, l \in \mathbb{Z}_n$$

$$(i', j', k', l') = A(i, j, k, l)$$

# Addition and isogenies

- Let  $x \in A$ .  $x = (\vartheta_i(x))_{i \in \mathcal{Z}_{\ell n}}$ . Let  $\tilde{\pi}$  be the affine cone of  $\pi$ .

$$\tilde{\pi}(x) = (\vartheta_{\ell j}(x))_{j \in \mathcal{Z}_n}$$

- We have an isomorphism  $\mathcal{Z}_n \times \mathcal{Z}_\ell \rightarrow \mathcal{Z}_{\ell n}$ ,  $(j, i) \mapsto \ell j + ni$ . If  $i \in \mathcal{Z}_\ell$  we define :

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- We remark that  $x$  is entirely determined by  $\{\tilde{\pi}_i(x)\}_{i \in \mathcal{Z}_\ell}$

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$$\tilde{\pi}_{i+j}(x + y) = \text{chaine\_add}(\tilde{\pi}_i(x), \tilde{\pi}_j(y), \tilde{\pi}_{i-j}(x - y))$$

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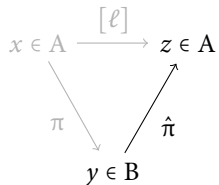
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# The dual isogeny



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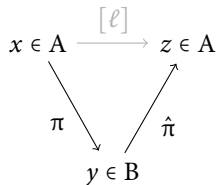
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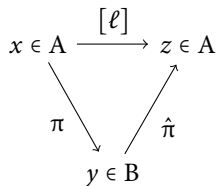
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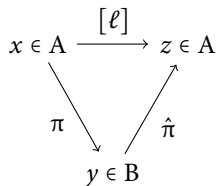
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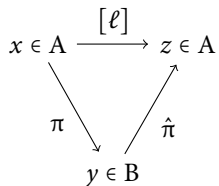
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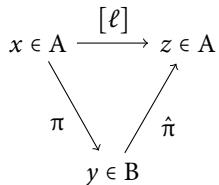
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- Each choice of the  $\ell$ -roots of unity give a different theta structure on  $A$ , hence a different decomposition  $A[\ell] = A[\ell]_1 \oplus K$ . All the  $\ell^2$ -isogenies  $B \rightarrow C$  containing  $K$  comes from these choices. Moreover we know the kernel of the dual isogeny  $C \rightarrow A$ , this is helpfull for computing isogeny graphs.
- Let  $T_{e_1}, \dots, T_{e_{2g}}$  be a basis for  $B[\ell]$ . If  $x, y$  and  $x - y$  are true points of  $\ell$ -torsion, so is  $x + y := \text{chaîne\_add}(x, y, x - y)$ . This means we can compute “true” representatives of  $B[\ell]$  with  $g(2g + 1)$   $\ell$ -root of unity,  $g(2g - 1)$  additions and  $\ell^{2g}$  chain additions.
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