Polarisations, isogenies, and pairings in abelian varieties

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Abstract. This note is an introduction to the notion of polarisation on an abelian variety and isogenies between polarised abelian varieties.

1. Introduction

These notes are written for my talk "Isogenies between abelian varieties: an overview" for the Leiden isogeny days. These are a work in progress. The aim is to give an overview of polarisations, isogenies and their strong link with the Weil pairing. I intend to update them regularly.

For my HDR thesis [Rob21a], I wrote an (unfinished) set of notes [Rob21b] on the mathematics of abelian varieties. The aim of [Rob21b] is to cover more technical topics on abelian varieties by giving the relevant pointers to the literature [Mum70; Mil91; BL04; MGE12; MFK94; Mum83; Mum84; Mum91; BLR12; FC90]…Indeed, it would be way too ambitious (for me!) to write full proofs of each of these topics, so [Rob21b] summarizes the main results and gives references for the proofs. For now it covers abelian varieties, the basic theory of abelian schemes, degenerations and the theory of Néron's models (the lifting / canonical lift part of the theory is not yet written), and pairings (Weil, Weil-Cartier, Tate, Tate-Cartier, Tate-Lichtenbaum). Further planned topics include the study of various moduli (of abelian varieties, of curves, via theta functions, CM theory). Unfortunately it is a bit stalled.

The planned topics of these notes are more elementary: polarisations, isogenies, and their strong links with the Weil pairing. I also intend to do a comparison with the case of elliptic curves (similarity / differences when going in higher dimension), and describe how these objects behave when working over $\mathbb{C}$ (they have a nice description in terms of linear algebra / quadratic forms).

Throughout these notes we only deal with separable isogenies. In particular, when looking at N-isogenies, we implicitly restrict to the case where N is prime to the characteristic p of the base field (or $p = 0$).

2. Outline

Here are planned topics.

(1) *Complex abelian varieties*: $A = V/\Lambda$, lattices, polarisations, isogenies: explain the different aspect of polarisations: as a Hermitian form on V, a symplectic form on Λ, a morphism $A \to \hat{A}$, an algebraic class of divisors (the Apell-Humbert theorem).
    Jacobians, analytic description of the Theta divisor.

(2) *Abelian varieties*: Isogenies, divisibility. The dual abelian variety. Dual isogenies. Dual abelian variety: $\widehat{A} = \operatorname{Pic}^0(A)$. Dual isogeny as pullback. Alternative interpretation

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$\widehat{A} = \mathrm{Ext}^1(A, \mathbb{G}_m)$ (via Mumford's theta groups), application to define the Weil-Cartier pairing of an isogeny. The Weil pairing on $A[m] \times \widehat{A}[m]$. Compatibility of pairings with isogenies. Weil pairing on the Tate modules $T_\ell A$. Biduality, Poincaré bundle, biduality and dual isogenies behave as expected with respect to the Weil pairing.

(3) *Polarisations*. Algebraic interpretations of the different facets of polarisations in the complex analytic case (Weil pairing, divisors). Characterisation of when $\phi : A \to \widehat{A}$ is a polarisation. The Néron-Severi group. Field of definition of a polarisation vs field of definition of an associated divisor, the case of $k = \mathbb{F}_q$. Polarisations and pairings, Theta group, descent. Contragredient isogeny. Product polarisations, polarisations on product. The contragredient matrix is the transpose of the matrix of contragredient isogenies. The Jacobian of a curve and its theta divisor. The special case of elliptic curves.

(4) *N-isogenies*. Link with maximal isotropic kernels. Contragredient isogeny $\tilde{f}$. Characterisation of a N-isogeny via the contragredient isogeny and via pairings.

(5) *Maximal isotropic kernels*. Elementary theory of symplectic finite abelian groups [Zhm71; PSV10]. Symplectic CRT. Maximal isotropic kernel: rank g, standard kernels.

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