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Vector-relation configurations and plabic graphs

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Abstract. We study a simple geometric model for local transformations of bipartite graphs. The state consists of a choice of a vector at each white vertex made in such a way that the vectors neighboring each black vertex satisfy a linear relation. Evolution for different choices of the graph coincides with many notable dynamical systems including the pentagram map, Q-nets, and discrete Darboux maps. On the other hand, for plabic graphs we prove unique extendability of a configuration from the boundary to the interior, an elegant illustration of the fact that Postnikov's boundary measurement map is invertible. In all cases there is a cluster algebra operating in the background, resolving the open question for Q-nets of whether such a structure exists.

Keywords: Pentagram map, plabic graphs, dimer model

1 Introduction

The dynamics of local transformations on weighted networks plays a central role in a number of settings within algebra, combinatorics, and mathematical physics. In the context of the dimer model on a torus, these local moves give rise to the discrete cluster integrable systems of Goncharov and Kenyon [11]. Meanwhile, for plabic graphs in a disk, Postnikov transformations relate different parametrizations of positroid cells [18] which in turn define a stratification of the totally non-negative Grassmannian.

The dimer model also manifests itself in many geometrically defined dynamical systems. We focus on projective geometry and draw our initial motivation from the pentagram map. The pentagram map was defined by Schwartz [20] and related to coefficient-type cluster algebra dynamics in [9]. Gekhtman, Shapiro, Tabachnikov, and Vainshtein

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[7, 8] placed the pentagram map and certain generalizations in the context of weighted networks.Although considerable work in various directions of the subject has been undertaken, most relevant to our work is a further generalization termed Y-meshes [10].

We propose a simple but versatile geometric model for the space of edge weights of any bipartite graph modulo gauge equivalence. The induced dynamics of local transformations includes in special cases pentagram maps, Q-nets, and discrete Darboux maps. This common generalization resolves a long standing question of how the pentagram map and Q-nets relate. Moreover, our systems come with a cluster dynamics, which is new in the Q-net case. Lastly, in the setting of plabic graphs we define a geometric version of the boundary measurement map and its inverse. In this language, properties of the boundary measurement map imply unique solvability of a certain family of geometric realization problems.

Let G be a planar bipartite graph with vertex set $B \cup W$. For $b \in B$ let $N(b) \subseteq W$ denotes its set of neighbors. Fix a vector space V. A *circuit* is a subset $S \subseteq V$ such that S is linearly dependent but each of its proper subsets is linearly independent.

Definition 1.1. A *vector-relation configuration* on G consists of choices of

- a nonzero vector $v_w \in V$ for each $w \in W$ and
- a non-trivial linear relation R_b among the vectors $\{v_w : w \in N(b)\}$ for each $b \in B$.

In particular, each set $\{v_w : w \in N(b)\}$ must be linearly dependent. If each set $\{v_w : w \in N(b)\}$ is a circuit, say the configuration satisfies the *circuit condition*.

Definition 1.2. Consider a vector-relation configuration on a graph G as above and suppose $\lambda \neq 0$. The *gauge transformation* by λ at a black vertex $b \in B$ scales the relation R_b by λ (and keeps all other vectors and relations the same). The *gauge transformation* by λ at a white vertex $w \in W$ scales v_w by $1/\lambda$ and scales the coefficient of v_w by λ in each relation in which it appears to compensate. Two vector-relation configurations are called *gauge equivalent* if they are related by a sequence of gauge transformations.

Gauge equivalence classes of vector-relation configurations serve as our main object of study. For the remainder of the introduction, we define both local transformations and a boundary measurement type map for such configurations, and present our main results.

1.1 Local transformations

Goncharov and Kenyon [11] study line bundles with connection on bipartite graphs. This model amounts to a choice of edge weights modulo a notion of gauge equivalence. They also define local transformations of the graph and edge weights which are closely related to Postnikov moves [18] in weighted directed networks.

Let $G = (B \cup W, E)$ be a planar bipartite graph. Let $(\mathbf{v}, \mathbf{R})$ be a vector-relation configuration on G. We can define edge weights

$$\mathrm{wt}(\overline{bw}) = \pm K_{bw}$$

where K_{bw} is the coefficient of v_w in R_b and the signs satisfy the Kasteleyn condition, namely the product of signs on edges around each $2m$-gon face equals $(-1)^{m-1}$. A choice of Kasteleyn signs always exists and, in the plane, any two such choices are gauge equivalent (see e.g. [12]).

Theorem 1.3. *The above associates a line bundle with connection on G to each gauge class of vector-relation configurations satisfying the circuit condition. Moreover, the classical local transformations can be implemented on the vector-relation level via local geometric rules.*

In Section 2, we define these local rules and also give geometric definitions of face weights which are the main gauge-invariant coordinates. We also explain the sense in which our model is a common generalization of the pentagram map and Q-nets.

1.2 Configurations on plabic graphs

Plabic graphs are a family of finite planar graphs widely used in the study of positroids and the totally non-negative Grassmannian. Restricted to this case, our model provides a new take on plabic graphs and much of the surrounding theory.

A *plabic graph* is a finite planar graph $G = (B \cup W, E)$ embedded in a disk with the vertices all colored black or white. We assume throughout that G is in fact bipartite and that all of its boundary vertices are colored white. An *almost perfect matching* of G is a matching that uses all internal vertices (and some boundary vertices). Assume always that G has at least one almost perfect matching.

Fix for the moment a plabic graph $G = (B \cup W, E)$. Let $M = |B|$, $N = |W|$, and let n be the number of boundary vertices. As all boundary vertices are white that leaves $N - n$ internal white vertices. Number the elements of B and W respectively 1 through M and 1 through N in such a way that the boundary (white) vertices are numbered 1 through n in clockwise order. We will sometimes use these numbers in place of the vertices, e.g. writing v_i for the vector of a configuration at the white vertex numbered i. Let $k = N - M$. Each almost perfect matching uses exactly $n - k$ boundary vertices.

We make some modifications to our model to better cater to plabic graphs and related geometry. First, the natural ambient dimension is $k = N - M$ so we simply fix as our vector space $V = \mathbb{C}^k$. We also require that the boundary vertices $v_1, \ldots, v_n$ span V. The main result given these assumptions, which in isolation is rather striking, is that a configuration is uniquely determined up to gauge by its boundary vectors. Moreover, the boundary vectors can be viewed as columns of a $k \times n$ matrix $[v_1 \cdots v_n]$ and the subset of the Grassmannian arising this way is dense in the associated positroid variety.

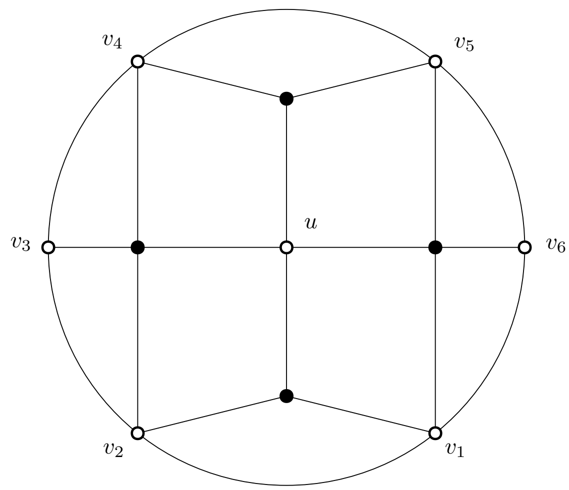

Figure 1: A plabic graph corresponding to the open cell in $Gr(3,6)$

Theorem 1.4. *Fix a plabic graph G with all the preceding conventions and notation.*

1. *If $(\mathbf{v}, \mathbf{R})$ is a vector-relation configuration on G satisfying the circuit condition then $A = [v_1 \cdots v_n]$ lies in the positroid variety $\Pi_{\mathcal{M}}$ where $\mathcal{M}$ is the positroid of G.*

2. *Suppose G is reduced. For generic $A \in \Pi_{\mathcal{M}}$, the columns $v_1, \ldots, v_n$ of A can be extended to a vector-relation configuration on G that is unique up to gauge at internal vertices. In particular, each internal vector is determined up to scale.*

In Section 3.1 we review the definitions of the positroid $\mathcal{M}$, the variety $\Pi_{\mathcal{M}} \subseteq Gr_{k,n}$, and the reducedness condition for plabic graphs.

Example 1.5. Consider the plabic graph G in Figure 1. It has $k = 3$ and $n = 6$. In fact, the associated positroid is the uniform matroid, meaning all 3 element subsets of $\{1, \ldots, 6\}$ are included. So the positroid variety is the whole Grassmannian $Gr_{3,6}$. In short, the boundary vectors $v_1, \ldots v_6 \in \mathbb{C}^3$ of a configuration can be generic.

Now suppose $v_1, \ldots v_6 \in \mathbb{C}^3$ are given and consider the possibilities for the internal vector u. The lower black vertex forces u, v_1, v_2 to be dependent while the top black vertex forces u, v_4, v_5 to be dependent. If the v_i are generic then u must lie on the line of intersection of the planes $\langle v_1, v_2 \rangle$ and $\langle v_4, v_5 \rangle$. Hence u is determined up to scale. The other two black vertices have degree 4. It is always possible to find a linear relation among 4 vectors in $\mathbb{C}^3$, so there are no added conditions imposed on u.

1.3 Relation to previous work

Our model of vector-relation configurations has substantial precedent in the literature. In fact, a main selling point of our specific formulation is that it is versatile enough to tie into previously studied ideas in a variety of areas. We outline some of the relevant previous work here for the interested reader's convenience.

In the plabic graph setting, Lam's *relation space* [16, Section 14] is in a sense dual to our model. In light of this connection, Lam's main result [16, Theorem 14.6] demonstrates that the operation of boundary restriction as in our Theorem 1.4 agrees up to sign with the boundary measurement map [18]. Our only claims to originality in Theorem 1.4 are the elegance and the elementary geometric nature both of the formulation and of our proofs. Another related model is provided by Postnikov [19]. Both [16, Section 14] and [19] are attempts to put on more mathematical footing on-shell diagrams [2], and in fact physicists have informed us that vector-relation configurations are one way they think about said diagrams.

In the case of the dimer model on the torus, special cases and other hints of vector-relation configurations appear in the following settings: higher pentagram maps [7, 8], discrete differential geometry [3], and the inverse spectral transform [14, 5]. We close this section by noting that the full version [1] of this extended abstract contains full proofs of all results as well as investigations in a number of related directions.

2 Vector-relation configurations

In this section we develop the theory of vector-relation configurations on general planar bipartite graphs as in Definitions 1.1 and 1.2. To that end, let $G = (B \cup W, E)$ be a planar bipartite graph. We will denote a vector-relation configuration on G by $(\mathbf{v}, \mathbf{R})$ (or sometimes just $\mathbf{v}$ for short) where $\mathbf{v} = (v_w)_{w \in W}$ and $\mathbf{R} = (R_b)_{b \in B}$.

For each $w \in W$, let P_w denote the projection of $v_w \in V$ to the corresponding projective space $\mathbb{P}V$.

Proposition 2.1. *If $(\mathbf{v}, \mathbf{R})$ satisfies the circuit condition then the points $(P_w)_{w \in W}$ uniquely determine the gauge class of the configuration.*

We work on the level of the point configurations $(P_w)_{w \in W}$ for the remainder of this section. They are characterized by the property that for each $b \in B$ the set $\{P_w : w \in N(b)\}$ is a circuit in projective space (e.g. 4 coplanar points, no 3 of which are collinear). Note that for any partition of a circuit into two subsets, there is a unique point on the intersection of the affine hulls of the subsets. Fix $b \in B$. Let F be a face containing b and let w_1, w_2 be the neighbors of b along F. Define $P(F, b)$ to be the point

$$P(F, b) = \langle P_{w_1}, P_{w_2} \rangle \cap \langle \{P_w : w \in N(b) \setminus \{w_1, w_2\}\} \rangle \tag{2.1}$$

where $\langle \cdot \rangle$ denotes an affine span.

Figure 2 depicts the local transformations that we consider on bipartite graphs and sets notation for some of the graphs' vertices. We extend each transformation to a map-

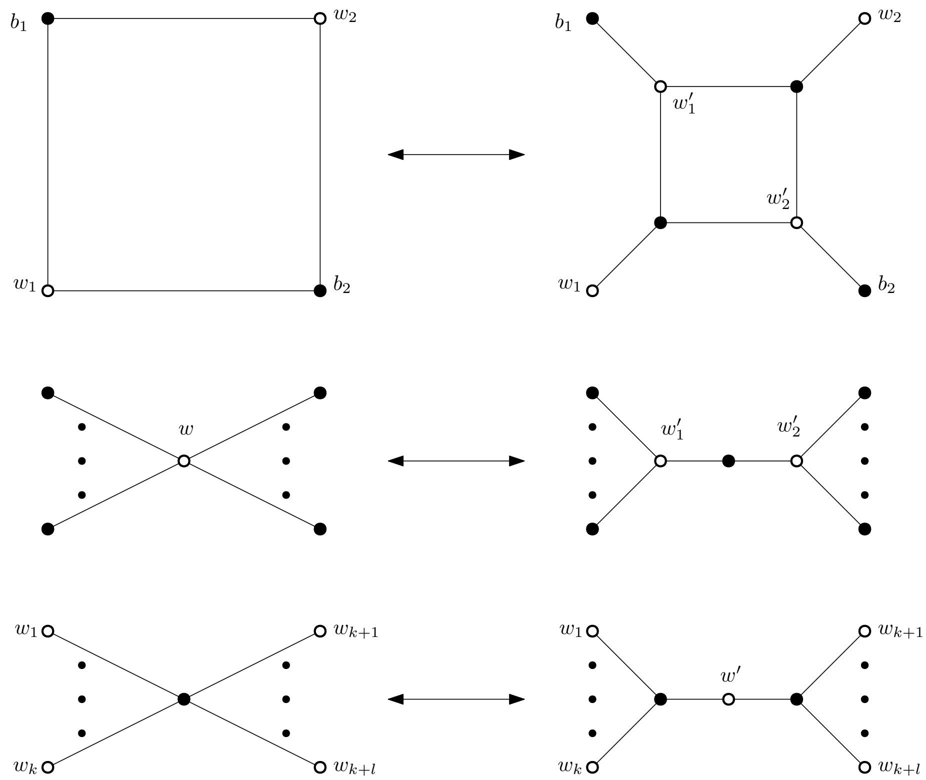

Figure 2: Combinatorial local moves on planar bipartite graphs.

ping of point configurations. The first move is *urban renewal* which operates on a quadrilateral face F of the original graph. Define points at the new white vertices w_1', w_2' to be $P_{w_1'} = P(F, b_1)$ and $P_{w_2'} = P(F, b_2)$. The other two moves split a vertex into two new vertices of the same color, which are then attached by a new degree two vertex of the opposite color. If the vertex being split is a white vertex w, them put the same point $P_{w_1'} = P_{w_2'} = P_w$ at each new white vertex. Lastly, if the vertex being split is black there is a new degree two white vertex w'. Put

$$P_{w'} = \langle P_{w_1}, \ldots, P_{w_k} \rangle \cap \langle P_{w_{k+1}} \ldots P_{w_{k+l}} \rangle.$$

Proposition 2.2. *The above definitions yield valid point configurations on the graph resulting from each move.*

It is a direct verification that the dynamics on the points P_w correspond to classical local transformations on edge weights as claimed by Theorem 1.3.

For a line bundle with connection on G, the basic gauge invariant functions are the *monodromies* around individual faces. These face weights evolve under local transformations via Y-pattern dynamics [6] of the cluster algebra associated to the dual quiver. As

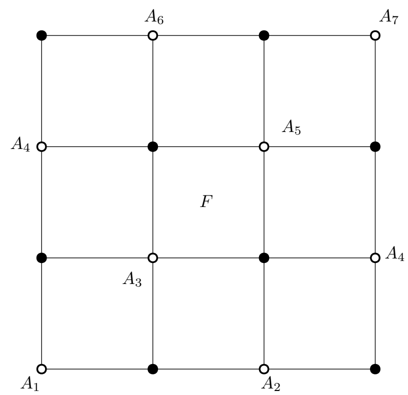

Figure 3: A portion of the bipartite graph whose vector-relation dynamics coincide with the pentagram map.

such, the same formulas apply to our geometrically defined systems. Moreover, the face weights have simple projectively invariant formulas.

Suppose points $P_1, \dots, P_{2k}$ in an affine space are given with the triples $\{P_1, P_2, P_3\}$, $\{P_3, P_4, P_5\}$, ..., $\{P_{2k-1}, P_{2k}, P_1\}$ all collinear. The *multi-ratio* (called a *cross ratio* for $k = 2$ and a *triple ratio* for $k = 3$) of the points is

$$[P_1, \dots, P_{2k}] = \frac{P_1 - P_2}{P_2 - P_3} \frac{P_3 - P_4}{P_4 - P_5} \cdots \frac{P_{2k-1} - P_{2k}}{P_{2k} - P_1}.$$

Each individual fraction involves 3 points on a line and is interpreted as a ratio of signed distances.

Proposition 2.3. *Let F be a face with boundary cycle $w_1, b_1, w_2, b_2, \dots, w_m, b_m$ in clockwise order. In terms of the points P_w, the face weight of F equals*

$$Y_F = (-1)^{m-1}[P_{w_1}, P(F, b_1), P_{w_2}, P(F, b_2), \dots, P_{w_m}, P(F, b_m)]^{-1}.$$

Example 2.4. The *pentagram map* [20] inputs a polygon in the plane with vertices A_i and outputs the polygon with vertices $B_i = \langle A_{i-1}, A_{i+1} \rangle \cap \langle A_i, A_{i+2} \rangle$. Let G be an infinite square grid and associate an A_i to each white vertex of G in the manner suggested by Figure 3. The two new points produced by urban renewal at F are precisely B_3 and B_4. In fact, urban renewal at half the faces (in a checkerboard pattern) followed by some degree 2 vertex removals results in a full application of the pentagram map.

Example 2.5. A Q-net [4] is a map from $\mathbb{Z}^3$ to 3-space such that each primitive square maps to four coplanar points. Sufficient initial data for a Q-net lives on a so-called stepped surface which when projected to a plane looks like an infinite tiling by regular rhombi. A point of the Q-net lives at each vertex of the tiling and the four points associated to each rhombus are coplanar. This setup can be modeled by a bipartite graph G with a white vertex at each vertex of the tiling, a black vertex in the center of each rhombus, and edges connecting them in the natural way. A certain sequence of four urban renewals studied in [13] realizes one step of Q-net dynamics which on the tiling level amounts to flipping three rhombi inside a regular hexagon.

3 Configurations on plabic graphs

3.1 Background on positroid varieties

We now return to the plabic graph case. The proof of Theorem 1.4 utilizes a significant amount of the theory of positroid varieties. We begin by reviewing the relevant material, generally following [17] and [16].

Given a $k \times n$ matrix A, and $J \subseteq \{1, \ldots, n\}$ of size k, let $\Delta_J(A)$ denote the maximal minor of A using column set J. The totally nonnegative Grassmannian is the set of $A \in Gr(k, n)$ for which $\Delta_J(A)$ is real and nonnegative for all J. The *matroid* of any $A \in Gr(k, n)$ is

$$\mathcal{M} = \{J : \Delta_J(A) \neq 0\}.$$

A *positroid* is a set of k-element subsets of $\{1, \ldots, n\}$ that arises as the matroid of a point in the totally nonnegative Grassmannian. We also denote a positroid by $\mathcal{M}$ even though this is a more restrictive notion than a matroid.

Let $\mathcal{M}$ be a positroid. For $j = 1, \ldots, n$, consider the column order $j < j + 1 < \ldots < n < 1 < \ldots < j - 1$. Let I_j be the lexicographically minimal element of $\mathcal{M}$ relative to this order. The collection of sets $(I_1, \ldots, I_n)$ is called the *Grassmann necklace* of $\mathcal{M}$. The positroids index a decomposition of the complex Grassmannian by *open positroid varieties* $\Pi^\circ_{\mathcal{M}}$, defined as intersections of cyclic shifts of Schubert cells encoded by $(I_1, \ldots, I_n)$. The *positroid variety* $\Pi_{\mathcal{M}}$ is defined to be the Zariski closure of $\Pi^\circ_{\mathcal{M}}$. In order to give quicker definitions, we fall back on the literature.

Theorem 3.1 (Knutson–Lam–Speyer [15])**.** *The positroid variety $\Pi_{\mathcal{M}}$ is a closed irreducible variety defined in the Grassmannian by*

$$\Pi_{\mathcal{M}} = \{A \in Gr_{k,n} : \Delta_J(A) = 0 \textit{ for all } J \notin \mathcal{M}\}.$$

Let $G = (B \cup W, E)$ be a plabic graph. Following our conventions, all boundary vertices are white. Let n be the number of boundary vertices and $k = |W| - |B|$. An almost perfect matching is a matching in G that uses all internal vertices. Hence it is a matching of B with $W \setminus J$ where $J \subseteq \{1, \ldots, n\}$ (identified with the boundary vertices) and $|J| = k$. The *positroid* of G, denoted $\mathcal{M}_G$ is the set of J that arise this way as the unused vertices of an almost perfect matching.

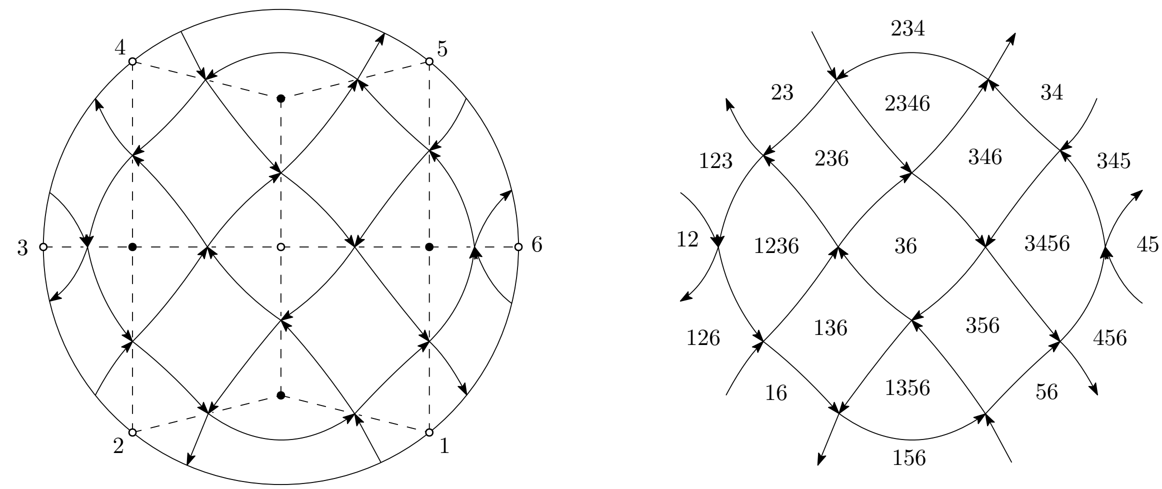

Figure 4: The alternating strand diagram for a plabic graph (left) and the associated labeling by sets of the faces and vertices of the graph (right)

An *arc* in a plabic graph is a direct path connecting the midpoints of two edges which meet at a vertex and are also part of a common face. The arc looks something like a circular arc centered at the common vertex, and it is oriented clockwise (resp. counterclockwise) if the vertex is white (resp. black). Near a boundary vertex an arc is also allowed to begin or end at a nearby boundary point. A *strand* is a maximal oriented path that decomposes into arcs. The collection of strands together is called the *alternating strand diagram* of the graph. One example of such a diagram is given in Figure 4.

Say a plabic graph G is reduced if its alternating strand diagram has the following properties:

- All strands start and end at the boundary.
- No strand has a self intersection unless it corresponds to a degree 1 boundary vertex whose neighbor is also degree 1.
- No pair of strands have a pair of intersections that they encounter in the same order.

If G is reduced then there are exactly n strands, one starting near each boundary vertex. Call j the number of the strand starting near vertex j. The strands divide the disk into regions. For each j, add a label of j to each region lying to the left of strand number j as it is traversed. Each region corresponds naturally to a white vertex, black vertex, or face of G. Use the notations S_w, S_b, and S_F to denote the set of labels received by the region in the strand diagram associated to w, b, or F (see the right of Figure 4 for an example).

Proposition 3.2. *Let F be a face of G, $b \in B$, and $w \in W$.*

- $|S_F| = k$, $|S_w| = k-1$, *and* $|S_b| = k+1$.
- *If b and w are on the boundary of F then $S_w \subseteq S_F \subseteq S_b$.*

3.2 Recasting the main result

The remainder of this section is devoted to outlining the proof of Theorem 1.4, and we begin by reformulating this result. Let $G = (B \cup W, E)$ be a plabic graph with all the notation of Section 1.2. Recall in this setting we only allow gauge transformations at the internal vertices. Let $\mathcal{C}_G^\circ$ denote the space of gauge equivalence classes of vector-relation configurations on G satisfying the circuit condition modulo the action of $GL_k(\mathbb{C})$.

If $(\mathbf{v}, \mathbf{R}) \in \mathcal{C}_G$ then by assumption $v_1, \ldots, v_n$ span $V = \mathbb{C}^k$. The v_i are defined up to a common change of basis so $A = [v_1 \cdots v_n]$ is a well-defined point of $Gr_{k,n}$. We use Φ to denote the map $\Phi : \mathcal{C}_G^\circ \to Gr_{k,n}$ taking $(\mathbf{v}, \mathbf{R})$ to A, and we call Φ the *boundary restriction map*. In this language Theorem 1.4 can be stated as follows.

Theorem 3.3. *Let G be a reduced plabic graph.*

1. *The map Φ is injective.*
2. *The Zariski closure of the image of Φ is the positroid variety $\Pi_{\mathcal{M}}$.*

3.3 The reconstruction map

In this subsection, we discuss the proof of Theorem 3.3. Specifically, we define a rational map $\Psi : \Pi_{\mathcal{M}} \to \mathcal{C}_G^\circ$ which turns out to be the inverse of Φ. As Ψ has the effect of reconstructing the entire configuration from just the boundary vectors, we term it the *reconstruction map*.

Fix $A \in \Pi_{\mathcal{M}}$ and in fact fix a particular matrix representative so that the columns $v_1, \ldots, v_n$ of A all live in V. Let $H_j \subseteq V$ denote the linear span of $\{v_i : i \in I_j \setminus \{j\}\}$. For each $w \in W$, define

$$L_w = \bigcap_{j \in S_w} H_j. \tag{3.1}$$

Lemma 3.4. *For generic $A \in \Pi_{\mathcal{M}}$:*

1. *Each H_j is a hyperplane.*
2. *The set $\{H_j : j \in S_F\}$ of k hyperplanes are in general position for each face F.*
3. *Each L_w is a line.*

Proposition 3.5. *Let $b \in B$ and choose nonzero vectors $v_w \in L_w$ for each neighbor w of b. Then these v_w satisfy a unique linear relation up to scale, and this relation has all coefficients nonzero.*

Proposition 3.6. *Let $A \in \Pi_G$ be generic. Then there exists a unique configuration $(\mathbf{v}, \mathbf{R}) \in \mathcal{C}_G^\circ$ such that $\Phi(\mathbf{v}, \mathbf{R}) = A$ and $v_w \in L_w$ for all $w \in W$.*

We now have our definition of the reconstruction map $\Psi : T_G \to \mathcal{C}_G^\circ$, namely it maps A to the configuration given by Proposition 3.6. Clearly $\Phi \circ \Psi$ is the identity. In plainer terms we have existence of an extension of generic $A \in \Pi_{\mathcal{M}}$ to a full configuration. The following result establishes uniqueness.

Proposition 3.7. *Let $(\mathbf{v}, \mathbf{R}) \in \mathcal{C}_G^\circ$ and suppose $A = \Phi(\mathbf{v})$. Then $(\mathbf{v}, \mathbf{R}) = \Psi(A)$.*

Example 3.8. Consider the plabic graph G in Figure 4. As previously discussed, G corresponds to the uniform matroid in $Gr_{3,6}$, and it follows that $I_j = \{j, j+1, j+2\}$ with indices modulo 6. Given $A = [v_1 \cdots v_6]$ then, $H_i = \langle v_{i+1}, v_{i+2} \rangle$. The unique internal white vertex w has $S_w = \{3, 6\}$, so

$$L_w = H_3 \cap H_6 = \langle v_4, v_5 \rangle \cap \langle v_1, v_2 \rangle.$$

One obtains a full configuration $\Psi(A)$ by picking $v_w \in L_w$. On the other hand, Example 1.5 explains that any extension must have v_w on this line verifying Proposition 3.7 in this case.

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