A first integrability result for Miquel dynamics

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Abstract

Miquel dynamics is a discrete-time dynamical system on the space of square-grid circle patterns. For biperiodic circle patterns with both periods equal to two, we show that the dynamics corresponds to translation on an elliptic curve, thus providing the first integrability result for this dynamics. The main tool is a geometric interpretation of the addition law on the normalization of binodal quartic curves.

1 Introduction

Miquel dynamics was introduced by the second author in [14], following an original idea of Richard Kenyon [10], as a discrete-time dynamical system on the space of square-grid circle patterns. It was then conjectured that for biperiodic circle patterns, Miquel dynamics belongs to the class of discrete integrable systems, which contains among others the dimer model [6] and the pentagram map [12, 13, 15]. In this article, we show that in the particular case when both periods are equal to two, Miquel dynamics corresponds, in the right coordinates, to translation on an elliptic curve. This is the first integrability result established for Miquel dynamics. An important observation we make to prove this is a simple geometric interpretation of the addition law on the normalization of algebraic curves of degree four with two nodes.

1.1 Circle patterns and Miquel dynamics

A *square grid circle pattern* (abbreviated as SGCP) is a collection of points $(S_{i,j})_{(i,j)\in\mathbb{Z}^2}$ in the plane $\mathbb{R}^2$ such that for any $(i,j)\in\mathbb{Z}^2$, the points $S_{i,j}, S_{i+1,j}$, $S_{i,j+1}$ and $S_{i+1,j+1}$ are pairwise distinct and concyclic, with the circle going through them denoted by $C_{i,j}$. The circles are colored in a checkerboard pattern: the circles $C_{i,j}$ with $i+j$ even (resp. odd) are colored black (resp. white). The center of the circle $C_{i,j}$ is denoted by $O_{i,j}$. We define two maps μ_w and μ_b, respectively called *white mutation* and *black mutation*, from the set of SGCPs

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to itself. For any SGCP S, the SGCP $T := \mu_w(S)$ is obtained as follows: for any $(i,j) \in \mathbb{Z}^2$ such that $i+j$ is even (resp. odd), $T_{i,j}$ is obtained by reflecting $S_{i,j}$ through the line $(O_{i,j}O_{i-1,j-1})$. (resp. $(O_{i-1,j}O_{i,j-1})$). It follows from Miquel's six-circles theorem [11] that T is indeed a circle pattern, with the same black circles as S but with potentially different white circles. Similarly, for any SGCP S, the SGCP $T' := \mu_b(S)$ is obtained as follows: for any $(i,j) \in \mathbb{Z}^2$ such that $i+j$ is even (resp. odd), $T'_{i,j}$ is obtained by reflecting $S_{i,j}$ through the line $(O_{i-1,j}O_{i,j-1})$. (resp. $(O_{i,j}O_{i-1,j-1})$). Each mutation is an involution. Miquel dynamics is defined as the discrete-time dynamical system obtained by applying alternately μ_w followed by μ_b. Note that this dynamics is different from the one on circle configurations studied by Bazhanov, Mangazeev and Sergeev [1], which uses a different version of Miquel's theorem.

Given two positive even integers m and n and two non-collinear vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$, an SGCP S is said to be (m,n)-*biperiodic* with monodromies $\vec{u}$ and $\vec{v}$ if for any $(i,j) \in \mathbb{Z}^2$, the following two conditions hold:

1. $S_{i+m,j} = S_{i,j} + \vec{u}$;
2. $S_{i,j+n} = S_{i,j} + \vec{v}$.

We denote by $\mathcal{S}_{m,n}$ the set of all (m,n)-biperiodic SGCPs (with arbitrary monodromies). This set is stable under both black mutation and white mutation. Miquel dynamics on $\mathcal{S}_{m,n}$ is conjectured to be integrable in some sense. In this paper we provide a first integrability result in the case when $m = n = 2$. For the remainder of the paper, all SGCPs will be in $\mathcal{S}_{2,2}$.

Let $S \in \mathcal{S}_{2,2}$ be an SGCP. We will denote its vertices in the fundamental domain $\{0,1,2\}^2$ as follows (see Figure 1 for an illustration):

$$\begin{array}{lll} A = S_{0,0} & B = S_{1,0} & C = S_{2,0} \\ D = S_{0,1} & E = S_{1,1} & F = S_{2,1} \\ G = S_{0,2} & H = S_{1,2} & I = S_{2,2} \end{array}$$

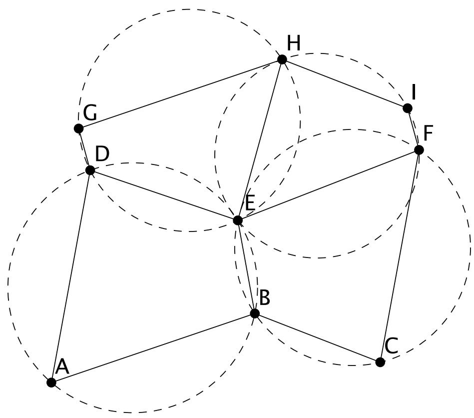

Figure 1: Illustration of the notation for a biperiodic circle pattern with both periods equal to two.

Set $S_w := \mu_w(S)$ and $S_b := \mu_b(S)$. We will denote their vertices in the fundamental domain $\{0,1,2\}^2$ respectively by $A_w, \dots, I_w$ and $A_b, \dots, I_b$. Instead of looking at the absolute motion of the points, we consider the relative motion of points with respect to one another. To do so, we introduce the pattern S'_w (resp. S'_b) which is obtained from S_w (resp. S_b) by applying the translation of vector $\overrightarrow{A_w A}$ (resp. $\overrightarrow{A_b A}$). We call *renormalized white mutation* μ'_w (resp. *renormalized black mutation* μ'_b) the map which to S associates S'_w (resp. S'_b). We denote the vertices of S'_w and S'_b in the fundamental domain $\{0,1,2\}^2$ respectively by $A'_w, \dots, I'_w$ and $A'_b, \dots, I'_b$. It was shown in [14] that both points E'_w and E'_b lie on some explicit quartic curve Q_S, which also contains the points A, C, E, G and I (see Section 2 for a precise definition of Q_S). In other words, the relative motion of the point in position $(1,1)$ with respect to the point in position $(0,0)$ lies on this curve Q_S. The curve Q_S has, in an appropriate coordinate system, an equation of the form

$$(x^2+y^2)^2 + ax^2 + by^2 + c = 0, \tag{1.1}$$

with $(a,b,c) \in \mathbb{R}^3$. See Figure 2 for an example. We call a *Miquel quartic* a quartic curve which has an equation of the form (1.1). As a special case of Miquel quartics, when $a + b = 0$, we obtain the family of Cassini ovals [18].

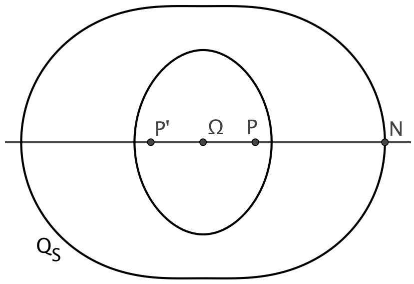

Figure 2: Example of a quartic curve Q_S, with the center Ω, the foci P and P' and the neutral element for addition N all lying on the horizontal coordinate axis. The curve Q_S may have either one or two ovals.

1.2 Addition on binodal quartic curves

A complex quartic curve in $\mathbb{CP}^2$ is called *binodal* if it has two nodes, i.e. singularities at which two regular local branches intersect transversally. A binodal quartic curve is called *non-degenerate* if it has no other singularities. The projective closure in $\mathbb{CP}^2$ of a Miquel quartic is generically a non-degenerate binodal quartic curve, with its two nodes being the *circular points at infinity* (also called isotropic points) with homogeneous coordinates $(1 : \pm i : 0)$, which lie on the infinity line $\overline{\mathbb{C}}_\infty = \mathbb{CP}^2 \setminus \mathbb{C}^2$ (see also Lemma 4.2 for a more precise statement). Every non-degenerate binodal quartic has an elliptic normalization, that is a holomorphic parametrization by an elliptic curve that is bijective outside the nodes. Thus, its normalization has a natural group structure, which is unique once the neutral element is chosen. Everywhere below, whenever we write about the addition law on a non-degenerate binodal quartic, we mean the addition law on its elliptic normalization. By an abuse of notation, the points on a non-degenerate binodal quartic and their lifts to the elliptic normalization will be denoted by the same symbols.

We will show that, for Miquel dynamics, the motion of the point in position $(1,1)$ induced by the composition of a black and a white renormalized mutation corresponds to an explicit translation on the normalization of Q_S. Another geometric interpretation of translation on certain binodal quartic curves, in terms of foldings of quadrilaterals, was introduced by Darboux [4] and studied more recently in [2, 9]. In order to state our translation result, we first give a geometric interpretation of the addition on non-degenerate binodal quartic curves.

The elliptic curve group law has a well-known geometric interpretation in the case of smooth cubic curves. In recent years, other geometric interpretations have arisen in the case of quartic curves, such as Edwards curves, later generalized to twisted Edwards curves [5, 3, 8], and Jacobi quartic curves [17]. For a non-degenerate binodal quartic curve, we obtain the addition law by fixing an arbitrary base point and declaring that, whenever a conic passes through both nodes and the base point, the other three intersection points of the conic with the quartic have zero sum. More specifically, we have the following theorem.

Theorem 1.1. *Let $\Gamma \subset \mathbb{CP}^2$ be a non-degenerate binodal quartic curve with nodes T_1 and T_2, and let $P \in \Gamma \setminus \{T_1, T_2\}$ be a base point. Consider the two-dimensional family $\mathcal{C}_P$ of conics through the three points T_1, T_2, P. Any conic $c \in \mathcal{C}_P$ intersects Γ at three additional points X_c, Y_c and Z_c, which need not be distinct and may coincide with T_1, T_2 or P. Then one can choose a neutral element N_P for the addition law on the normalization $\hat{\Gamma}$ of Γ such that $X_c + Y_c + Z_c = 0$ for every $c \in \mathcal{C}_P$.*

Theorem 1.1 easily follows from the classical theory of adjoint curves (see for example [16, section 49]). However, it does not seem to be explicitly stated in the literature. This result generalizes the geometric interpretation of addition for twisted Edwards curves in terms of intersections with hyperbolas [8].

We use the above theorem to construct the group law on Miquel quartics. We denote by $\mathcal{S}^0_{2,2} \subset \mathcal{S}_{2,2}$ the subset of patterns S such that the binodal quartic Q_S is non-degenerate. We will see in Lemma 4.2 that a Miquel quartic with an equation of the form (1.1) is non-degenerate if and only if $4c \notin \{0, a^2, b^2\}$ and $a \neq b$, so that $\mathcal{S}^0_{2,2}$ is Zariski-open in $\mathcal{S}_{2,2}$.

Proposition 1.2. *Let $S \in \mathcal{S}^0_{2,2}$. We pick N to be an intersection point of Q_S with the x-axis. One can consider the group law on Q_S as constructed in Theorem 1.1 with N being both the base point and the neutral element. The inverse is given by reflection through the x-axis. The sum of two points P_1 and P_2 is given by taking the circle going through P_1, P_2 and N and reflecting through the x-axis the fourth point of intersection P_3 of this circle with Q_S.*

Theorem 1.3. *Using the group law on Q_S defined in Proposition 1.2 and the notation for A, C, E, E'_w and E'_b defined above, we have for any $S \in \mathcal{S}^0_{2,2}$*

$$E'_w = -E - 2A \tag{1.2}$$

$$E'_b = -E - 2C. \tag{1.3}$$

In particular, the composition of a renormalized white mutation followed by a renormalized black mutation produces a translation by $2(A-C)$.

It follows from [14] that the space $\mathcal{S}_{2,2}$ is of real dimension 9. It was also shown there that, after application of the composition $\mu'_w \circ \mu'_b$ of renormalized black and white mutations, the point A, the vectors $\overrightarrow{AC}$ and $\overrightarrow{AG}$ as well as the angles $\angle CBA$ and $\angle ADG$ do not change (see also Subsection 2.2). These provide eight real conserved quantities. Once we fix the values of these eight conserved quantities, we obtain a one-parameter family of possible patterns, parametrized by the position of E on a quartic curve of Miquel type. Theorem 1.3 shows that Miquel dynamics induces a translation on that quartic curve. This is a sign of integrability of Miquel dynamics for $(2,2)$-biperiodic circle patterns, with the quartic curve playing, in a sense, the role of a Liouville torus. It also reinforces the conjecture about the integrability of Miquel dynamics for general (m,n)-biperiodic circle patterns.

We conclude this introduction by deriving, as a consequence of Theorem 1.3, a measure on Miquel quartics which is invariant under Miquel dynamics.

Corollary 1.4. *Let Q be a non-degenerate Miquel quartic with an equation of the form* (1.1)*. The 1-form*

$$\omega = \frac{d(x^2+y^2)}{xy} \tag{1.4}$$

is invariant under any translation on Q. In particular, for any $S \in \mathcal{S}^0_{2,2}$, the composition $\mu'_b \circ \mu'_w$ induces a map for the motion of E on Q_S which leaves invariant the form ω on Q_S. Furthermore, for such a map on Q_S, the modulus $|\omega|$ is an invariant measure.

Outline of the paper

Section 2 consists in recalling several results needed from [14]: the connection between patterns in $\mathcal{S}_{2,2}$ and five-pointed equilateral hyperbolas, the integrals of motion for Miquel dynamics on $\mathcal{S}_{2,2}$ and the construction of the quartic curve Q_S. In Section 3, we describe a simple geometric construction of E'_w starting from the curve Q_S and two points A and E on it. Finally in Section 4, we prove Theorem 1.1 about the group law on non-degenerate binodal quartics and, combining it with the geometric construction of the previous section, we prove Theorem 1.3 and Corollary 1.4 about Miquel dynamics.

2 The space $\mathcal{S}_{2,2}$ and the curve Q_S

In this section we describe the dichotomy for patterns in $\mathcal{S}_{2,2}$, between generic patterns and trapezoidal patterns. We provide the integrals of motion for renormalized mutations, which include a quartic curve. We explain how to construct this quartic curve in both the generic and trapezoidal cases. All the statements in this section were proved in [14].

2.1 Generic and trapezoidal patterns

Let $S \in \mathcal{S}_{2,2}$. Denote its vertices by $A, \ldots, I$ as on Figure 1. The pattern S has four cyclic faces, $ABED$, $BCFE$, $DEHG$ and $EFIH$. Then either all the faces of S are trapezoids (trapezoidal case) or no face of S is a trapezoid (generic case). The trapezoidal case is subdivided into two subcases :

- each triple of points $\{A, B, C\}$, $\{D, E, F\}$ and $\{G, H, I\}$ is aligned (horizontal trapezoidal case) ;
- each triple of points $\{A, D, G\}$, $\{B, E, H\}$ and $\{C, F, I\}$ is aligned (vertical trapezoidal case).

We denote respectively by $\mathcal{G}$, $\mathcal{T}_h$ and $\mathcal{T}_v$ the classes of generic, horizontal trapezoidal and vertical trapezoidal patterns. It was shown that each of these three classes is stable under Miquel dynamics.

Patterns in $\mathcal{S}_{2,2}$ enjoy a special property, formulated in terms of an equilateral hyperbola. Recall that an equilateral hyperbola is a hyperbola with orthogonal asymptotes. Degenerate cases of equilateral hyperbolas correspond to the union of two orthogonal lines.

Proposition 2.1 ([14]). *Fix $S \in \mathcal{S}_{2,2}$. There exists an equilateral hyperbola $\mathcal{H}$ going through the points B, D, E, F, and H. Furthermore, $\mathcal{H}$ is non-degenerate if and only if $S \in \mathcal{G}$. The pattern S is in $\mathcal{T}_h$ (resp. $\mathcal{T}_v$) if and only if the points D, E and F (resp. B, E and H) lie on one line of $\mathcal{H}$ and the points B and H (resp. D and F) lie on the other line of $\mathcal{H}$.*

It was actually shown in [14] that the space $\mathcal{S}_{2,2}$ is parametrized by five-pointed equilateral hyperbolas: pick an equilateral hyperbola $\mathcal{H}$ (four degrees of freedom) then pick five distinct points B, D, E, F, H on $\mathcal{H}$, there is a unique way to reconstruct a pattern $S \in \mathcal{S}_{2,2}$ from this data, inverting the construction of Proposition 2.1.

The vertical trapezoidal case is handled in a similar fashion as the horizontal trapezoidal case. Thus from now on, among the trapezoidal patterns we shall only consider the horizontal trapezoidal ones and refer to the horizontal trapezoidal case simply as the trapezoidal case, omitting "horizontal".

2.2 Conserved quantities under renormalized mutations

We recall the results about the invariants under Miquel dynamics found in [14]. Fix $S \in \mathcal{S}_{2,2}$ and write $S'_w = \mu'_w(S)$ and $S'_b = \mu'_b(S)$. Denote by $A'_w, \ldots, I'_w$ (resp. $A'_b, \ldots, I'_b$) the vertices of S'_w (resp. S'_b). The following statements hold :

- $(A, C, G, I) = (A'_w, C'_w, G'_w, I'_w) = (A'_b, C'_b, G'_b, I'_b)$;
- $\angle CBA = -\angle C'_w B'_w A'_w = -\angle C'_b B'_b A'_b$;
- $\angle ADG = -\angle A'_w D'_w G'_w = -\angle A'_b D'_b G'_b$.

Furthermore, there exists a quartic curve Q_S, the construction of which is explained in the next two subsections, verifying the following properties.

Proposition 2.2 ([14]). *For any $S \in \mathcal{S}_{2,2}$, we have $Q_S = Q_{S'_w} = Q_{S'_b}$. Furthermore, the points A, C, G, I, E, E'_w and E'_b lie on Q_S.*

2.3 Construction of Q_S in the generic case

Assume $S \in \mathcal{G}$. In this case it was shown in [14] that the angles $\angle CBA$ and $\angle ADG$ are not flat. Denote by O_B (resp. O_D) the center of the circle through A, B and C (resp. A, D and G). Let P be the intersection point of the parallel to (AG) through O_B and the parallel to (AC) through O_D. Let P' be the symmetric of P across Ω, where Ω is the center of the parallelogram $ACIG$. Define

$$\lambda := \frac{PA^2P'A^2 - PC^2P'C^2}{\Omega A^2 - \Omega C^2}$$

and

$$k := \frac{\Omega A^2 PC^2 P'C^2 - \Omega C^2 PA^2 P'A^2}{\Omega A^2 - \Omega C^2}$$

Then we define Q_S be to the following locus of points in $\mathbb{R}^2$:

$$Q_S := \left\{ M \in \mathbb{R}^2 | PM^2P'M^2 - \lambda \Omega M^2 = k \right\}. \tag{2.1}$$

The points P and P' are called the foci of the quartic Q_S (see Figure 2). Taking coordinates centered at Ω and such that P lies on the x-axis, we obtain for Q_S an equation of the form (1.1).

2.4 Construction of Q_S in the trapezoidal case

Assume $S \in \mathcal{T}_h$. Take coordinates centered at the center Ω of the parallelogram $ACIG$ (which is actually a rectangle in the trapezoidal case), with the x-axis parallel to (AC). The points C, D and E have respective coordinates $(x_C, y_C), (x_D, y_E)$ and (x_E, y_E). Define the quantities

$$\alpha = x_C^2 + y_C^2 + x_E^2 + y_E^2 + \frac{(x_D + x_C)^2(x_C^2 + y_C^2 - x_E^2 - y_E^2)}{y_C^2 - y_E^2} \tag{2.2}$$

$$\beta = x_C^2 + y_C^2 + x_E^2 + y_E^2 + \frac{(x_D + x_C)^2(x_E^2 - x_C^2)(x_C^2 + y_C^2 - x_E^2 - y_E^2)}{(y_C^2 - y_E^2)^2} \tag{2.3}$$

$$\gamma = (x_C^2 + y_C^2)(x_E^2 + y_E^2) + \frac{(x_D + x_C)^2(x_E^2 y_C^2 - x_C^2 y_E^2)(x_C^2 + y_C^2 - x_E^2 - y_E^2)}{(y_C^2 - y_E^2)^2} \tag{2.4}$$

Then Q_S is the curve of equation

$$(x^2 + y^2)^2 - \alpha x^2 - \beta y^2 + \gamma = 0. \tag{2.5}$$

It would be interesting to have a coordinate-free geometric construction of Q_S in the trapezoidal case, as we had in the generic case.

3 Another construction of renormalized mutation

The proof of statement (1.2) in Theorem 1.3 relies on a direct construction of E'_w from the points A, E, I and the quartic curve Q_S. Denote by O_A (resp. O_I) the center of the circle $\mathcal{C}_A$ (resp. $\mathcal{C}_I$) going through A (resp. I) and E and tangent to Q_S at A (resp. I). Note that the points O_A and O_I must be distinct, hence the line (O_AO_I) is well-defined. Otherwise the circle $\mathcal{C}_A = \mathcal{C}_I$ would intersect the quartic curve Q_S in five real points counted with multiplicity, which would contradict Bézout's theorem, since there are already intersection points with the circular points at infinity which both count twice. We have the following simple construction for E'_w :

Proposition 3.1. *The point E'_w is obtained by reflecting E through the line (O_AO_I). In particular, the circle going through A, E and E'_w is tangent to Q_S at A.*

Proof. We distinguish two cases, whether S is generic or trapezoidal. All the computations mentioned below are easily performed on a computer algebra software, but we do not display in this paper all the formulas obtained, since some of them would take up to ten lines.

Generic case. Assume $S \in \mathcal{G}$. Up to applying a similarity, one may assume that the equilateral hyperbola going through B, D, E, F and H has equation $xy = 1$. Denote respectively by b, d, e, f and h the abscissas of B, D, E, F and H. We will successively compute several quantities in terms of b, d, e, f and h. We first compute the coordinates of O_1, O_2 and O_4, which are the respective circumcenters of the triangles BDE, DEH and EFH. The points A and B are the two intersection points of the following two circles :

- the circle centered at O_1 going through B ;
- the image under the translation of vector $\overrightarrow{HB}$ of the circle centered at O_2 going through H.

The statement about the second circle follows from the fact that $S \in \mathcal{S}_{2,2}$ has vertical monodromy equal to $\overrightarrow{BH}$. Denoting by O'_2 the image of O_2 under the translation of vector $\overrightarrow{HB}$, we obtain A as the reflection of B through the line $(O_1O'_2)$:

$$A = (b + d + e, b^{-1} + d^{-1} + e^{-1})$$

Applying translations of respective vectors $\overrightarrow{DF}$, $\overrightarrow{BH}$ and $\overrightarrow{DF} + \overrightarrow{BH}$, we obtain the coordinates of the points C, G and I :

$$\begin{aligned} C &= (b + e + f, b^{-1} + e^{-1} + f^{-1}) \\ G &= (d + e + h, d^{-1} + e^{-1} + h^{-1}) \\ I &= (e + f + h, e^{-1} + f^{-1} + h^{-1}). \end{aligned}$$

Next, we compute the coordinates of E_w (the reflection of E through the line (O_1O_4)), A_w (the reflection of A through the line going through O_1 and $O_4+\overrightarrow{IA}$) and finally $E'_w = E_w + \overrightarrow{A_wA}$. Then we compute Ω (midpoint of $[AI]$) and the foci P and P' of the quartic curve Q_S (which requires to first compute the respective circumcenters O_B and O_D of the triangles ABC and ADG, as explained in Subsection 2.3):

$$\Omega = (\frac{b+d+2e+f+h}{2}, \frac{b^{-1}+d^{-1}+2e^{-1}+f^{-1}+h^{-1}}{2})$$
$$P = (\frac{b+d+e+f+h-bdefh}{2}, \frac{b^{-1}+d^{-1}+e^{-1}+f^{-1}+h^{-1}-(bdefh)^{-1}}{2})$$
$$P' = (\frac{b+d+3e+f+h+bdefh}{2}, \frac{b^{-1}+d^{-1}+3e^{-1}+f^{-1}+h^{-1}+(bdefh)^{-1}}{2})$$

We also compute the real number

$$\lambda = \frac{PA^2P'A^2 - PC^2P'C^2}{\Omega A^2 - \Omega C^2}.$$

The quartic Q_S then has an equation of the form

$$Q_S = \left\{ M \in \mathbb{R}^2 | PM^2P'M^2 - \lambda \Omega M^2 = k \right\}$$

for some real number k. Next, we compute the intersection point O_A (resp. O_I) of the normal to the quartic at A (resp. I) and the perpendicular bisector of the segment $[AE]$ (resp. $[IE]$). We finally check that E'_w is indeed the reflection of E through the line (O_AO_I).

Trapezoidal case. Assume $S \in \mathcal{T}_h$. Up to applying a similarity, we may assume that the degenerate equilateral hyperbola going through B, D, E, F and H has equation $xy = 0$ with D, E and F lying on the line $y = 0$ and B and H lying on the line $x = 0$. As in the generic case, we successively compute as functions of the coordinates of B, D, E, F and H the following quantities :

1. the coordinates of A, C, I and E'_w ;
2. the quantities α and β using formulas (2.2) and (2.3) ;
3. the coordinates of O_A and O_I.

We finally check that E'_w is indeed the reflection of E through the line (O_AO_I). □

4 The group law on non-degenerate binodal quartics

We first prove Theorem 1.1 about the group law on general non-degenerate binodal quartics in Subsection 4.1, before applying it to prove Theorem 1.3 about Miquel dynamics in Subsection 4.2. The proof of Corollary 1.4 is in Subsection 4.3.

4.1 General non-degenerate binodal quartics

Recall that two divisors on a Riemann surface are said to be linearly equivalent if their difference is the divisor of a meromorphic function. A *complete linear system* is defined to be any set of all the divisors linearly equivalent to a given divisor. We will deal with divisors on the elliptic normalization $\hat{\Gamma}$ of a non-degenerate binodal quartic Γ.

Lemma 4.1. *Let $\Gamma \subset \mathbb{CP}^2$ be a non-degenerate binodal quartic curve with nodes T_1 and T_2. Consider the three-dimensional family $\mathcal{C}$ of all the conics (including degenerations to unions of lines) passing through the nodes T_1 and T_2. A conic $c \in \mathcal{C}$ intersects Γ at four additional points X_c, Y_c, Z_c and W_c, which need not be distinct and may coincide with T_1 or T_2, and we denote by $\mathcal{D}_c$ the divisor $[X_c] + [Y_c] + [Z_c] + [W_c]$ on $\hat{\Gamma}$. The set $\{\mathcal{D}_c\}_{c\in\mathcal{C}}$ forms a complete linear system.*

Proof. This lemma is an immediate consequence of the theory of adjoint curves developed for example in [16, section 49]. For an algebraic curve $\mathcal{G}$ with singularities that are all nodes, an adjoint curve $\mathcal{A}$ to $\mathcal{G}$ is defined to be any curve passing through all the nodes of $\mathcal{G}$. Let $\mathcal{D}_{\mathcal{A}}$ be the divisor corresponding to all the intersections of $\mathcal{A}$ with $\mathcal{G}$ outside of the nodes. The Brill-Noether residue theorem [16, p.216] states that the set $\{\mathcal{D}_{\mathcal{A}}\}$, where $\mathcal{A}$ ranges over all the adjoint curves to $\mathcal{G}$ of a fixed degree d, forms a complete linear system. The lemma corresponds to the special case with two nodes and adjoint curves of degree $d = 2$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For any conic $c \in \mathcal{C}_P$, denote by $\mathcal{D}_{c,P}$ the divisor $\mathcal{D}_c - [P] = [X_c] + [Y_c] + [Z_c]$. The linear equivalence of two effective divisors on $\hat{\Gamma}$ containing the base point P is equivalent to the linear equivalence of their differences with the single-point divisor $[P]$, since linear equivalence classes of divisors form an additive group. Together with Lemma 4.1, this implies that the set $\{\mathcal{D}_{c,P}\}_{c\in\mathcal{C}_P}$ forms a complete linear system.

For every two divisors $\sum_{j=1}^{d}[S_j]$ and $\sum_{j=1}^{d}[T_j]$ of the same degree d on an elliptic curve the equality

$$\sum_{j=1}^{d} S_j = \sum_{j=1}^{d} T_j \tag{4.1}$$

for the addition in the group law is independent on the choice of neutral element: if it holds for the group law defined by one neutral element, then it holds for every other neutral element. Equality (4.1) holds if and only if the corresponding divisors are linearly equivalent. This is a particular case of Abel's Theorem [7, chapter 2, section 2].

Since $\{\mathcal{D}_{c,P}\}_{c\in\mathcal{C}_P}$ forms a complete linear system, for any $(c, c') \in (\mathcal{C}_P)^2$, $[X_c] + [Y_c] + [Z_c]$ is linearly equivalent to $[X_{c'}] + [Y_{c'}] + [Z_{c'}]$, thus $X_c + Y_c + Z_c$ equals $X_{c'} + Y_{c'} + Z_{c'}$ for the group law on the elliptic normalization of Γ, regardless of the choice of neutral element. By surjectivity of the tripling map, there exists N such that $3N$ equals the quantity $X_c + Y_c + Z_c$, which is independent of $c \in \mathcal{C}_P$. Taking this point N as the neutral element, we obtain a group law on $\hat{\Gamma}$ such that for any $c \in \mathcal{C}_P$, $X_c + Y_c + Z_c = 0$. □

4.2 The group law on non-degenerate Miquel quartics

We first identify when a Miquel quartic is non-degenerate.

Lemma 4.2. *A Miquel quartic Q with an equation given by* (1.1) *is a non-degenerate binodal quartic curve if and only if $a \neq b$ and $4c \notin \{0, a^2, b^2\}$. In that case, its nodes are the two circular points at infinity.*

Proof. The equation of Q in homogeneous coordinates is

$$(x^2+y^2)^2+ax^2z^2+by^2z^2+cz^4=0. \tag{4.2}$$

A singular point of the quartic of homogeneous coordinates $(x:y:z)$ must satisfy (4.2) as well as the following three equations, corresponding to the vanishing of the three partial derivatives with respect to x, y and z of the left-hand side of (4.2):

$$x(2x^2+2y^2+az^2)=0 \tag{4.3}$$
$$y(2x^2+2y^2+bz^2)=0 \tag{4.4}$$
$$z(ax^2+by^2+2cz^2)=0 \tag{4.5}$$

One checks that the two circulars points at infinity T_1 and T_2 of respective coordinates $(1:i:0)$ and $(1:-i:0)$ satisfy equations (4.2) to (4.5) for any choice of a,b,c. Furthermore, the Hessian of the left-hand side of (4.2) at T_1 is given by

$$\begin{pmatrix} 8 & 8i & 0 \\ 8i & -8 & 0 \\ 0 & 0 & 2(a-b) \end{pmatrix},$$

which has rank 2 if and only if $a \neq b$. Hence T_1 is a node if and only if $a \neq b$. The same holds for T_2.

Equation (4.2) has no solution on the line at infinity $z=0$ besides T_1 and T_2. Distinguishing when x or y vanish, it is easy to check that for $a \neq b$, the only other singularities are

- the point $(0:0:1)$ when $c=0$;
- the points $(\pm\sqrt{-a/2}:0:1)$ when $c=a^2/4$;
- the points $(0:\pm\sqrt{-b/2}:1)$ when $c=b^2/4$.

This concludes the proof. □

We now use Theorem 1.1 to provide a geometric construction of the group law on a non-degenerate Miquel quartic.

Proof of Proposition 1.2. Here N is an intersection point of the non-degenerate Miquel quartic Q_S with the x-axis. Since the nodes of Q_S are the circular points at infinity and the (complex) circles are exactly the conics going through both circular points, the set $\mathcal{C}_N$ consists in all the circles going through N. By symmetry with respect to the x-axis, the osculating circle to Q_S at N has an intersection of order 4 with Q_S at N. Thus N can be taken as the neutral element for a group law on Q_S with base point N as constructed in Theorem 1.1. Since both Q_S and any tangent circle to Q_S at N are symmetric with respect to the x-axis, we deduce that the inverse for this group law is given by reflection through the x-axis. The statement about the sum of two points follows immediately. □

Theorem 1.3 is then an immediate consequence of Proposition 3.1.

Proof of Theorem 1.3. By symmetry, it suffices to prove (1.2). We consider two circles :

- the osculating circle to Q_S at N ;
- the circle going through the points A, E and E'_w, which is tangent to Q_S at A by Proposition 3.1.

By Lemma 4.1, the divisors $[E'_w] + [E] + 2[A]$ and $4[N]$ are linearly equivalent. For the group law on Q_S with N as base point and neutral element described in Proposition 1.2, we have $N = 0$, hence $E'_w + E + 2A = 0$, by an argument similar to the one used in the proof of Theorem 1.1. □

4.3 The invariant measure on Miquel quartics

Proof of Corollary 1.4. Let Q be a non-degenerate binodal Miquel quartic, with an equation of the form (1.1). We show below that the pullback of the form ω to the elliptic normalization of the quartic Q is a holomorphic differential. Firstly, ω is meromorphic.

Secondly, it has no poles at the intersection points of Q with the coordinate axes. For example, consider a point $M \in \mathbb{C}^2$ of its intersection with the y-axis. The germ at M of the quartic is the graph of an even function $y = g(x)$, hence $g'(0) = 0$ and $g(x) = g(0) + O(x^2)$ with $g(0) \neq 0$, since $(0, 0) \notin Q$ (which follows from the fact that $c \neq 0$ by Lemma 4.2). Thus, in a neighborhood of the point M one has $dy = O(x)dx$ and

$$\omega = \frac{2dx}{y} + \frac{2dy}{x} = O(1)dx.$$

The case of an intersection point with the x-axis is treated analogously.

Thirdly, we show below that the circular points at infinity are not poles of the restriction of the form ω to local branches of the quartic Q at these points. There are two local branches of the quartic at each circular point, since each circular point is a node. Each of these local branches is regular and transverse to the infinity line, otherwise the intersection index of the quartic with the infinity line would be greater than 4, which would contradict Bézout's theorem. Take an arbitrary local branch φ at a circular point, say T_1 with homogeneous coordinates $(1 : i : 0)$, and consider the restriction to it of the form ω. The function xy is meromorphic on $\mathbb{CP}^2$ with a pole of order 2 along the infinity line and the local branch φ is transverse to it, thus the denominator $xy|_\varphi$ has a pole of order 2 at T_1. The primitive $(x^2 + y^2)|_\varphi$ of the numerator has at most first order pole at T_1, since the circular points satisfy the equality $x^2 + y^2 = 0$. In more detail, let us introduce new affine coordinates (u, v) on $\mathbb{C}^2$ so that $x^2 + y^2 = uv$ and the u-axis intersects the infinity line at the point T_1. Define $\widetilde{u} = \frac{1}{u}$ and $\widetilde{v} = \frac{v}{u}$ and observe that they are local affine coordinates centered at T_1. The coordinate $\widetilde{u}$ can be taken as a local parameter of the branch φ since φ is transverse to the infinity line and $\widetilde{u}$ vanishes on the infinity line with order 1. Furthermore, $\widetilde{v}$ is holomorphic and vanishes at T_1 thus $\widetilde{v} = O(\widetilde{u})$ in a neighborhood of T_1 on φ. Therefore, one has

$$x^2 + y^2 = uv = \frac{\widetilde{v}}{\widetilde{u}^2} = O(\frac{1}{\widetilde{u}})$$

in a neighborhood of T_1 on φ. Hence the restriction of the form ω to a local branch of Q has a pole of order at most $2 - 2 = 0$ at a circular point.

Thus the pullback of the form ω is a holomorphic differential on the normalization of Q. Recall that any holomorphic differential on an elliptic curve is invariant by any translation defined by the group structure on the curve. Its modulus induces a measure on the real part, which is invariant under every translation, hence under the translation induced on E by the composition $\mu'_b \circ \mu'_w$. Note in passing that, since ω has no zeros, it induces a form of constant sign on each component of the real part of the quartic. $\square$

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