MODULAR PERIODICITY OF THE EULER NUMBERS AND A SEQUENCE BY ARNOLD

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ABSTRACT. For any positive integer q, the sequence of the Euler up/down numbers reduced modulo q was proved to be ultimately periodic by Knuth and Buckholtz. Based on computer simulations, we state for each value of q precise conjectures for the minimal period and for the position at which the sequence starts being periodic. When q is a power of 2, a sequence defined by Arnold appears, and we formulate a conjecture for a simple computation of this sequence.

1. INTRODUCTION

The sequence of Euler up/down numbers $(E_n)_{n\geq 0}$ is the sequence with exponential generating series

$$\sum_{n=0}^{\infty} \frac{E_n}{n!} x^n = \sec x + \tan x. \tag{1}$$

It is referenced as sequence A000111 in [Slo17] and its first terms are

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, \ldots$$

The numbers E_n were shown by André [And79] to count up/down permutations on n elements (see Section 3).

Knuth and Buckholtz [KB67] proved that for any integer $q \geq 1$, the sequence $(E_n \mod q)_{n\geq 0}$ is ultimately periodic. For any $q \geq 1$ we define :

- $s(q)$ to be the minimum number of terms one needs to delete from the sequence $(E_n \mod q)_{n\geq 0}$ to make it periodic ;
- $d(q)$ to be the smallest period of the sequence $(E_n \mod q)_{n\geq s(q)}$.

For example, the sequence $(E_n \mod 3)$ starts with

$$1, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, \ldots$$

so one might expect to have $s(3) = 1$ and $d(3) = 4$. Clearly $s(1) = 0$ and $d(1) = 1$. In the remainder of this paper, we formulate precise conjectures for the values of $s(q)$ and $d(q)$ for any $q \geq 2$.

Organisation of the paper. In Section 2 we reduce the problem to the case when q is a prime power and we conjecture the values of $s(q)$ and $d(q)$ when q is an odd prime power. In Section 3 we conjecture the values of $s(q)$ and $d(q)$ when q is a power of 2, after having introduced the Entringer numbers and a sequence defined by Arnold describing the 2-adic valuation of the Entringer numbers. In Section 4, we provide a simple construction which conjecturally yields the Arnold sequence.

2. Case when q is not a power of 2

The following lemma implies that it suffices to know the values of $s(q)$ and $d(q)$ when q is a prime power in order to know the values of $s(q)$ and $d(q)$ for any $q \geq 2$.

Lemma 1. *Fix $q \geq 2$ and write its prime number decomposition as*

$$q = \prod_{i=1}^{k} p_i^{\alpha_i}, \tag{2}$$

where $k \geq 1$, $p_1, \ldots, p_k$ are distinct prime numbers and $\alpha_1, \ldots, \alpha_k$ are positive integers. Then

$$s(q) = \max_{1 \leq i \leq k} s(p_i^{\alpha_i}) \tag{3}$$

$$d(q) = \mathrm{lcm}(d(p_1^{\alpha_1}), \ldots, d(p_k^{\alpha_k})). \tag{4}$$

The proof is elementary and uses the Chinese remainder theorem.

When q is an odd prime power, Knuth and Buckholtz [KB67] found the following :

Theorem 2 ([KB67])**.** *Let p be an odd prime number.*

(1) If $p \equiv 1 \mod 4$, then

$$d(p) = p - 1.$$

(2) If $p \equiv 3 \mod 4$, then

$$d(p) = 2p - 2.$$

(3) For any $k \geq 1$,

$$s(p^k) \leq k.$$

(4) For any $k \geq 2$,

$$d(p^k) | p^{k-1} d(p).$$

We conjecture the following for the exact values of $s(q)$ and $d(q)$ when q is an odd prime power :

Conjecture 1. *Let p be an odd prime number.*

(1) For any $k \geq 1$,

$$s(p^k) = k.$$

(2) For any $k \geq 2$,

$$d(p^k) = p^{k-1} d(p).$$

Conjecture 1 is supported by Mathematica simulations done for all odd prime powers $q < 1000$.

3. Entringer numbers and case when q is a power of 2

Formulating a conjecture analogous to Conjecture 1 for powers of 2 requires to define, following Arnold [Arn91], a sequence describing the behavior of the 2-adic valuation of the Entringer numbers.

3.1. **The Seidel-Entringer-Arnold triangle.** The Entringer numbers are a refined version of the Euler numbers, enumerating some subsets of up/down permutations. For any $n \geq 0$, a permutation $\sigma \in \mathcal{S}_n$ is called *up/down* if for any $2 \leq i \leq n$, we have $\sigma(i-1) < \sigma(i)$ (resp. $\sigma(i-1) > \sigma(i)$) if i is even (resp. i is odd). André [And79] showed that the number of up/down permutations on n elements is E_n. For any $1 \leq i \leq n$, the *Entringer number* $e_{n,i}$ is defined to be the number of up/down permutations $\sigma \in \mathcal{S}_n$ such that $\sigma(n) = i$. The Entringer numbers are usually displayed in a triangular array called the Seidel-Entringer-Arnold triangle, where the numbers $(e_{n,i})_{1 \leq i \leq n}$ appear from left to right on the n-th line (see Figure 1).

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 0 & & \\
 & 0 & & 1 & & 2 & & 2 & \\
5 & & 5 & & 4 & & 2 & & 0
\end{array}$$

Figure 1. First five lines of the Seidel-Entringer-Arnold triangle.

The Entringer numbers can be computed using the following recurrence formula (see for example [Sta97]). For any $n \geq 2$ and for any

$1 \leq i \leq n$, we have

$$e_{n,i} = \begin{cases} \sum_{j<i} e_{n-1,j} & \text{if n is even} \\ \sum_{j\geq i} e_{n-1,j} & \text{if n is odd} \end{cases}. \tag{5}$$

3.2. **Arnold's sequence.** Replacing each entry of the Seidel-Entringer-Arnold triangle by its 2-adic valuation, we obtain an infinite triangle denoted by T (see Figure 2).

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & \infty & & 0 & & & \\ & & 0 & & 0 & & \infty & & \\ & \infty & & 0 & & 1 & & 1 & \\ 0 & & 0 & & 2 & & 1 & & \infty \end{array}$$

FIGURE 2. First five lines of the triangle T of 2-adic valuations of the Entringer numbers.

We read this triangle T diagonal by diagonal, with diagonals parallel to the left boundary. For any $i \geq 1$, denote by D_i the i-th diagonal of the triangle T parallel to the left boundary. For example D_1 starts with $0, \infty, 0, \infty, 0, \ldots$. For any $i \geq 1$, denote by m_i the minimum entry of diagonal D_i. Arnold [Arn91] observed that the further away one moves from the left boundary, the higher the 2-adic valuation of the Entringer numbers becomes. In particular, he observed (without proof) that the sequence $(m_i)_{i\geq 1}$ was weakly increasing to infinity. He defined the following sequence : for any $k \geq 1$,

$$u_k := \max \{i \geq 1 | m_i < k\}.$$

In other words, u_k is the number of diagonals containing at least one entry that is not zero modulo 2^k. The sequence $(u_k)_{k\geq 1}$ is referenced as the sequence A108039 in OEIS [Slo17] and its first few terms are given in Table 1.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
u_k	2	4	4	4	8	8	8	8	10	12	12	16	16	16	16	16	18	20

TABLE 1. The first few values of u_k.

Note that the first few terms given by Arnold were incorrect, because the entry 4 appeared four times, whereas it should be appearing only three times. We also remark that we cannot define any sequence analogous to (u_k) when studying the p-adic valuations of the Entringer numbers for odd primes p. Indeed, the p-adic valuation 0 seems to appear in diagonals of arbitrarily high index.

3.3. **Case when q is a power of 2.** Using the sequence $(u_k)_{k\geq 1}$, we formulate the following conjecture for $s(q)$ and $d(q)$ when q is a power of 2 :

Conjecture 2. *For any $k \geq 1$, we have*

$$s(2^k) = u_k. \tag{6}$$

Furthermore, if $k \geq 1$ and $k \neq 2$, we have

$$d(2^k) = 2^k. \tag{7}$$

Finally, we have $d(4) = 2$.

Numerical simulations performed on Mathematica for $k \leq 12$ support Conjecture 2.

4. Construction of Arnold's sequence

In this section we provide a construction which conjecturally yields Arnold's sequence $(u_k)_{k\geq 1}$.

We denote by $\mathbb{Z}_+$ the set of nonnegative integers and we denote by

$$S := \bigsqcup_{d\geq 1} \mathbb{Z}_+^d$$

the set of all finite sequences of nonnegative integers. We define a map $f : S \to S$, which maps each $\mathbb{Z}_+^d$ to $\mathbb{Z}_+^{2d}$, as follows. Fix $\underline{x} = (x_1, \ldots, x_d) \in S$. If all the x_i's are equal to x_d, we set

$$f(\underline{x}) = (x_d, \ldots, x_d, 2x_d, \ldots, 2x_d),$$

where x_d and $2x_d$ both appear d times on the right-hand side. Otherwise, define

$$s := \max\{1 \leq i \leq d-1 | x_i \neq x_d\}$$

and set

$$f(\underline{x}) = (x_1, \ldots, x_d, x_1 + x_d, \ldots, x_{s-1} + x_d, 2x_d, \ldots, 2x_d),$$

where $2x_d$ appears $d - s + 1$ times on the right-hand side. For example, we have

$$f((2, 4, 4, 4)) = (2, 4, 4, 4, 8, 8, 8, 8) \tag{8}$$

and
(9)
$$f(2,4,4,4,8,8,8,8) = (2,4,4,4,8,8,8,8,10,12,12,16,16,16,16,16).$$

By iterating this function f indefinitely, one produces an infinite sequence :

Lemma 3. *Fix $d \geq 1$ and $\underline{x} \in \mathbb{Z}_+^d$. There exists a unique (infinite) sequence $(X_k)_{k\geq 1}$ such that for any $k \geq 1$ and for any $n \geq \log_2(k/d)$, X_k is the k-th term of the finite sequence $f^n(\underline{x})$.*

This infinite sequence is called the *f-transform* of $\underline{x}$. The lemma follows from the observation that for any $\ell \geq 1$ and for any $\underline{y} \in \mathbb{Z}_+^\ell$, $\underline{y}$ and $f(\underline{y})$ have the same first ℓ terms.

We can now formulate a conjecture about the construction of the sequence $(u_k)_{k\geq 1}$:

Conjecture 3. *Arnold's sequence $(u_k)_{k\geq 1}$ is the f-transform of the quadruple $(2,4,4,4)$.*

Conjecture 3 is supported by the estimation on Mathematica of u_k for every $k \leq 512$.

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