Dimers on Rail Yard Graphs

Sanjay Ramassamy
Brown University
(visiting the Statistical Laboratory)

Joint work with : Cédric Boutillier
Jérémie Bouttier
Guillaume Chapuy
Sylvie Corteel

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- The dimer model is a well-studied model in statistical mechanics.

- Oustanding examples include the Aztec diamond (dimers on square lattice) and plane partitions (dimers on hexagonal lattice).

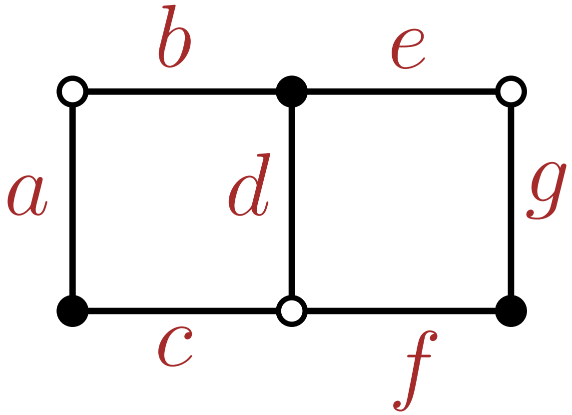
- Rail yard graph setting : common generalisation of these two models (and several others).

- Realises a general Schur process as a dimer model, extending work of Okounkov-Reshetikhin.

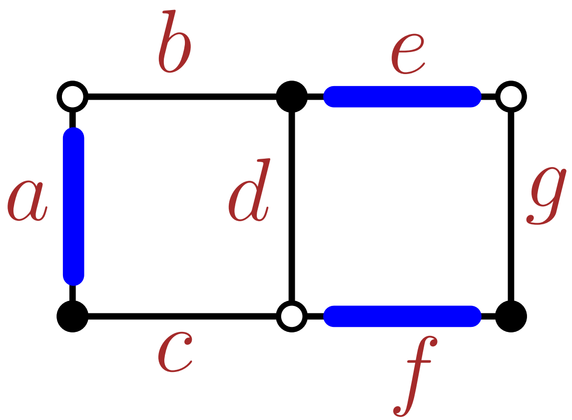
Outline :

1. The dimer model
2. Two examples : Aztec diamond and plane partitions
3. Definition of rail yard graphs
4. Main results
5. Proof ideas
6. Summary and perspectives

1 The dimer model



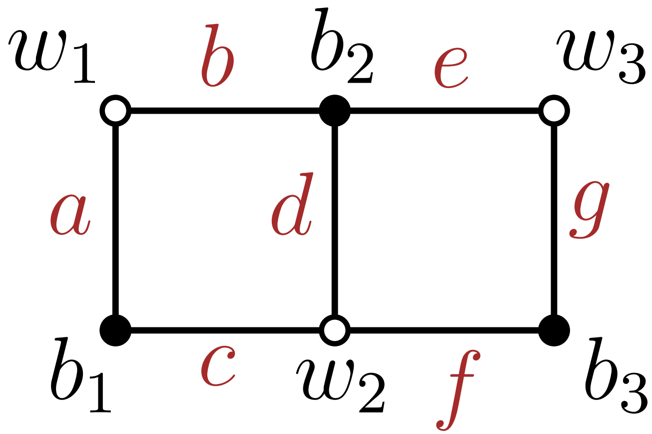
- A dimer configuration is a subset of edges such that each vertex belongs to exactly one edge.



- A dimer configuration is a subset of edges such that each vertex belongs to exactly one edge.

- Weight of a configuration = product of the weights of the edges used. Here, weight(σ) $= aef$.

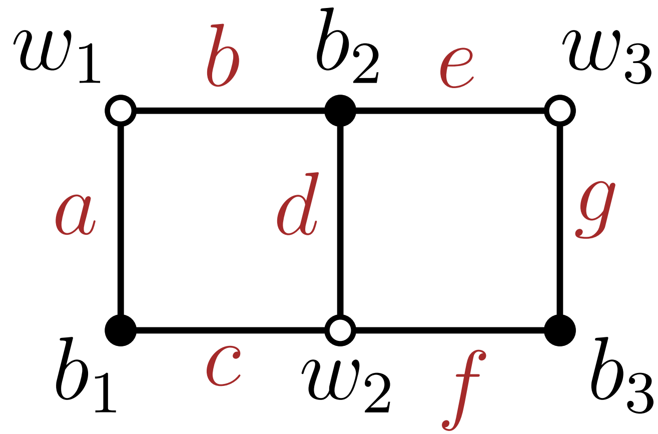
- Partition function Z = sum of the weights of all the dimer configurations.



- For finite planar bipartite graphs, define the Kasteleyn matrix K, a weighted signed adjacency matrix.

- Rows indexed by white vertices and columns indexed by black vertices.

$$K = \begin{pmatrix} a & b & 0 \\ c & -d & f \\ 0 & e & g \end{pmatrix} \quad \begin{matrix} \text{columns: } b_1, b_2, b_3 \\ \text{rows: } w_1, w_2, w_3 \end{matrix}$$



- For finite planar bipartite graphs, define the Kasteleyn matrix K, a weighted signed adjacency matrix.
- Rows indexed by white vertices and columns indexed by black vertices.

Theorem (Kasteleyn, Temperley-Fisher).

$$Z = |\det K|$$

- Probabilistic model : draw a dimer configuration at random, with probability proportional to its weight.

$$\mathbb{P}(\sigma) = \text{weight}(\sigma)/Z$$

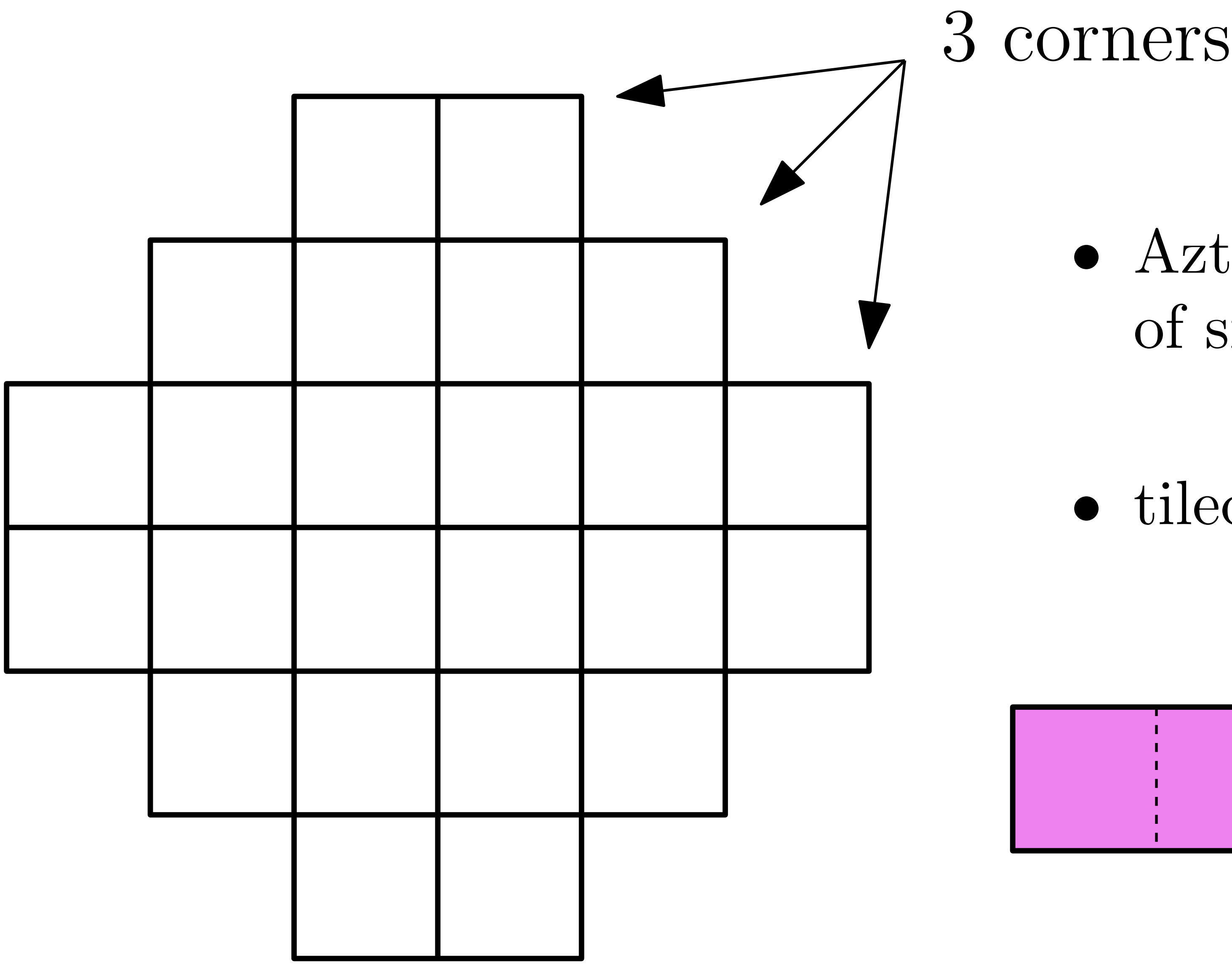
- Local statistics : fix $e_1 = (b_1, w_1), \ldots, e_k = (b_k, w_k)$ to be k edges. What is the probability that a random configuration contains all of these k edges ?

Theorem (Kasteleyn). *This probability is :*

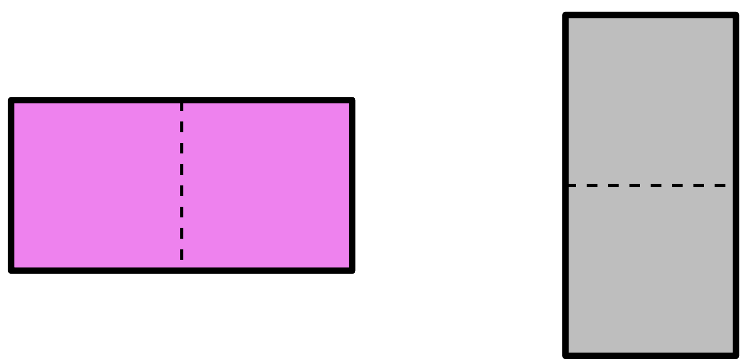
$$\left(\prod_{i=1}^{k} \text{weight}(e_i)\right) \det\left(K^{-1}(b_i, w_j)\right)_{1 \leq i,j \leq k}$$

2 Two examples : the Aztec diamond and plane partitions

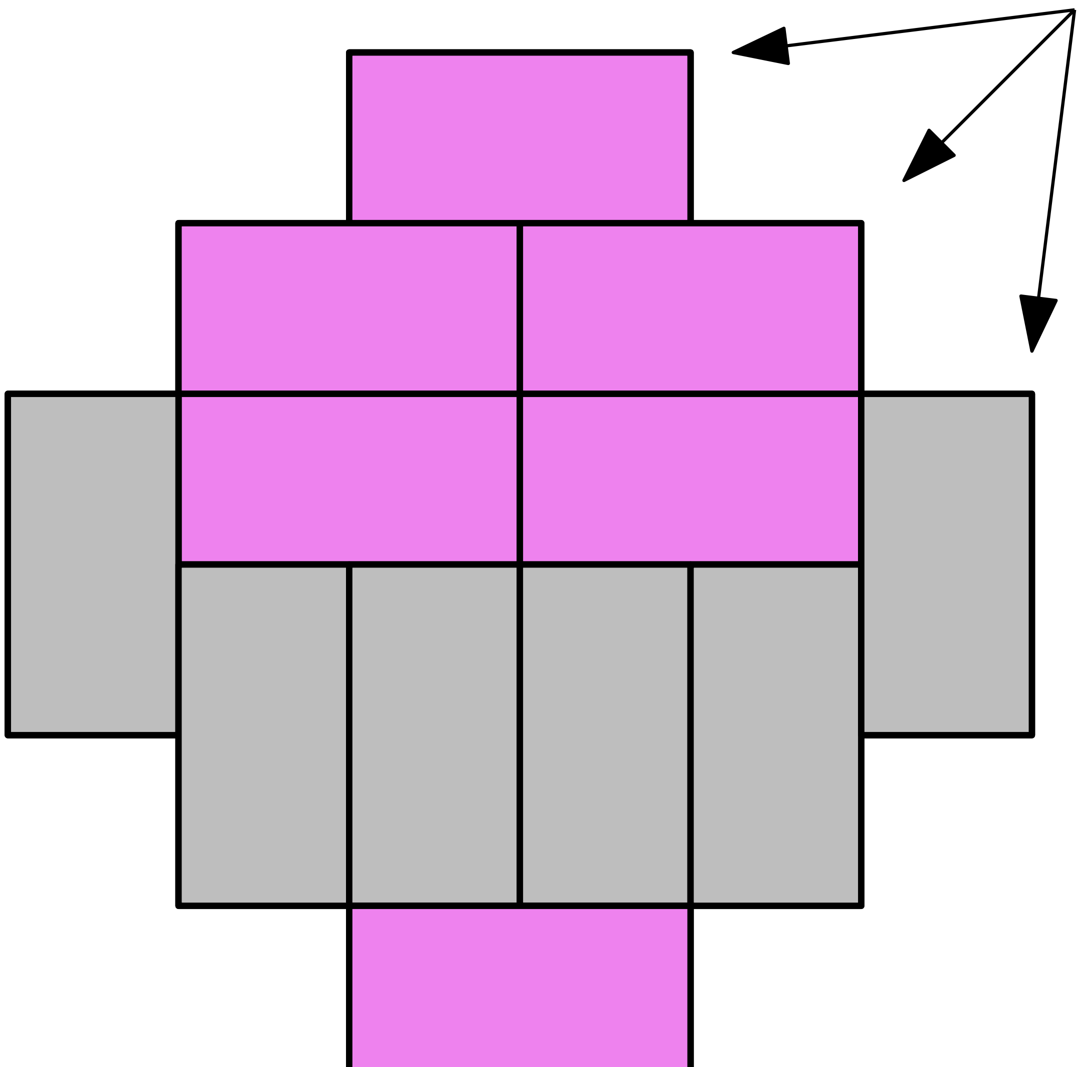
Domino tilings of the Aztec diamond



- Aztec diamond of size 3
- tiled by dominos :



Domino tilings of the Aztec diamond



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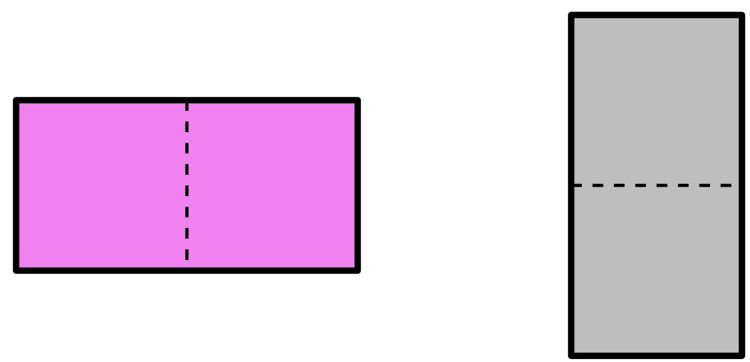
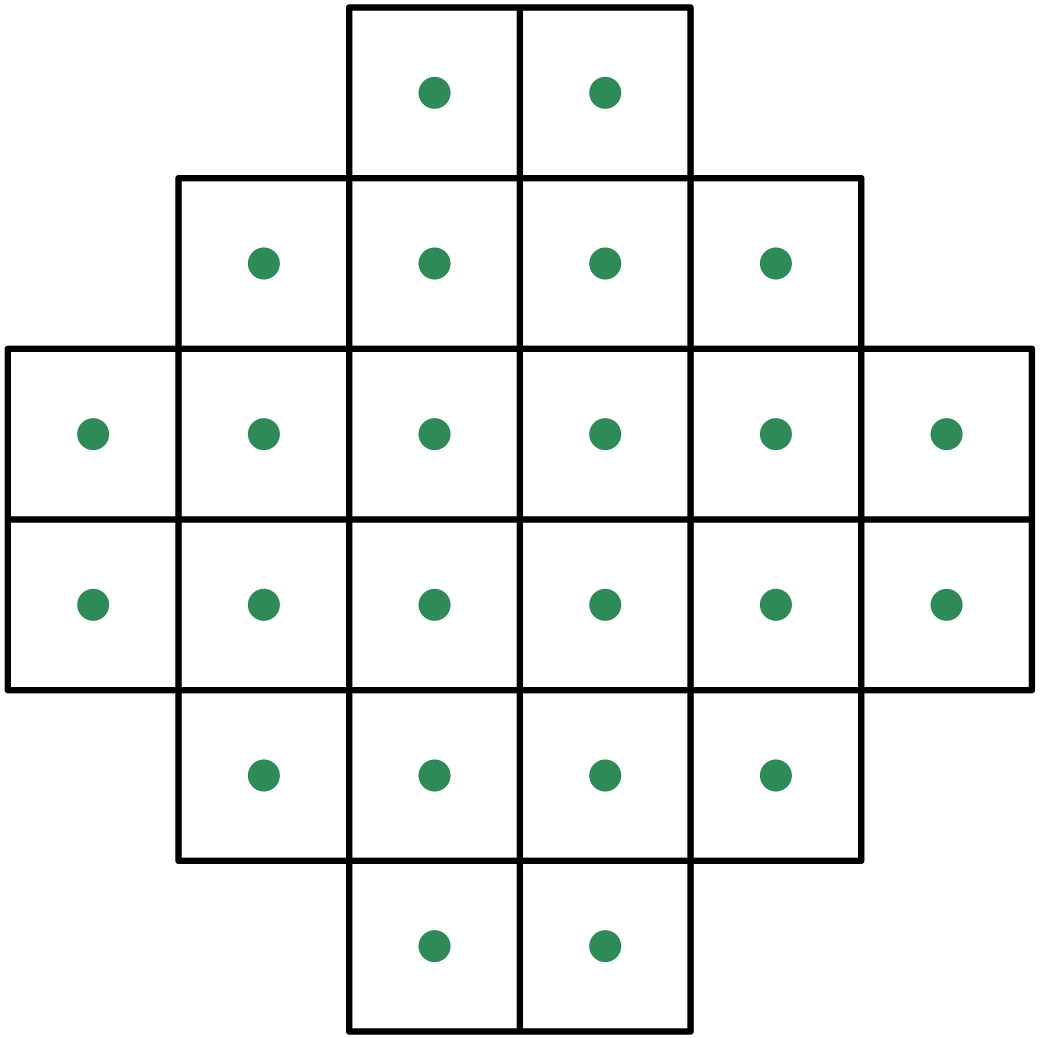
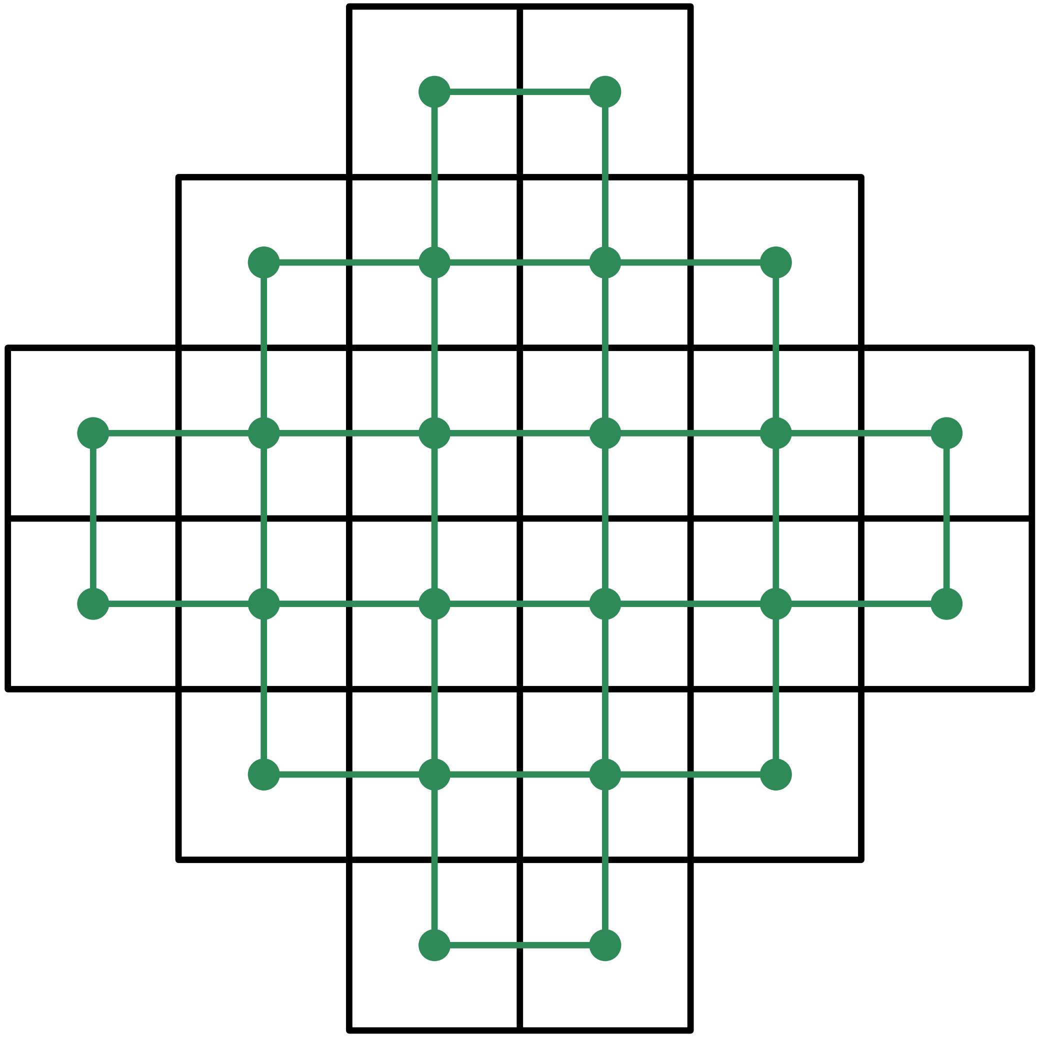


Domino tilings of the Aztec diamond

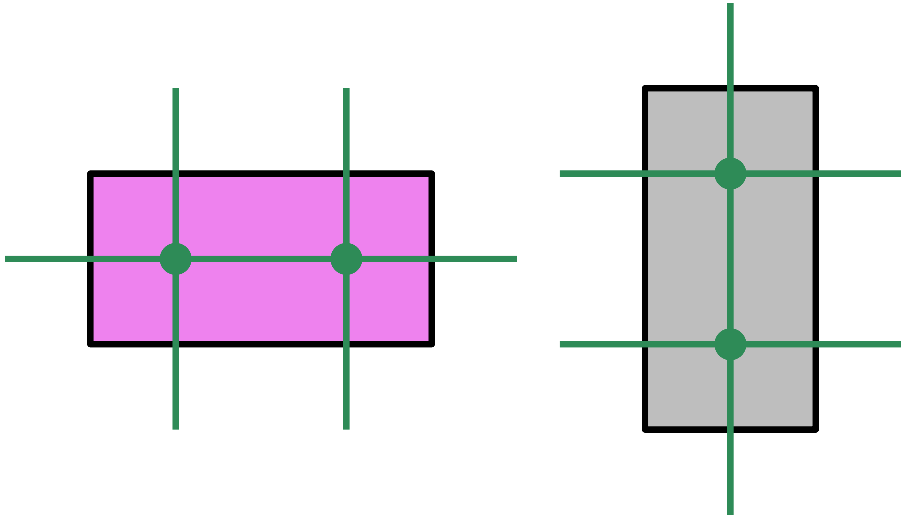


- equivalent to a dimer model on the dual graph

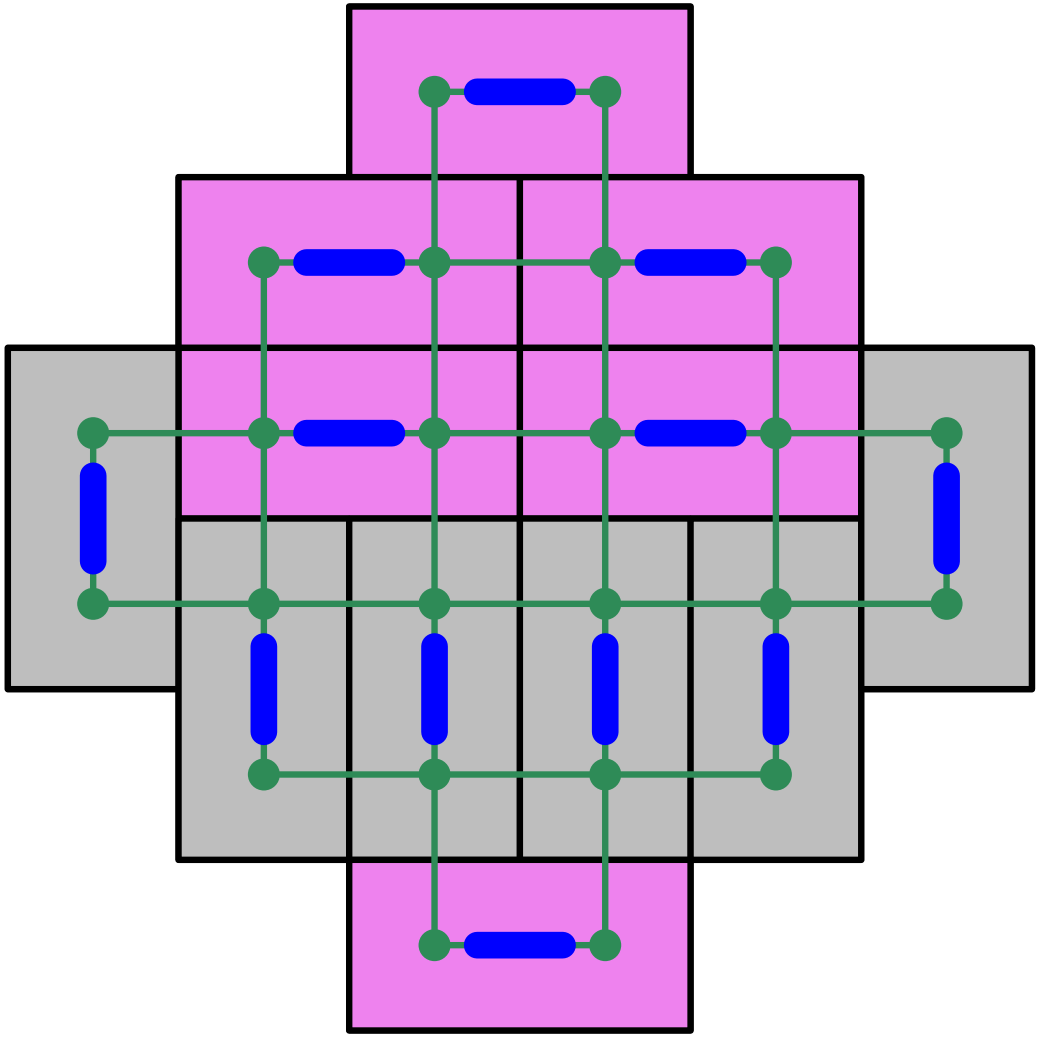
Domino tilings of the Aztec diamond



- equivalent to a dimer model on the dual graph
- domino $\leftrightarrow$ dimer



Domino tilings of the Aztec diamond



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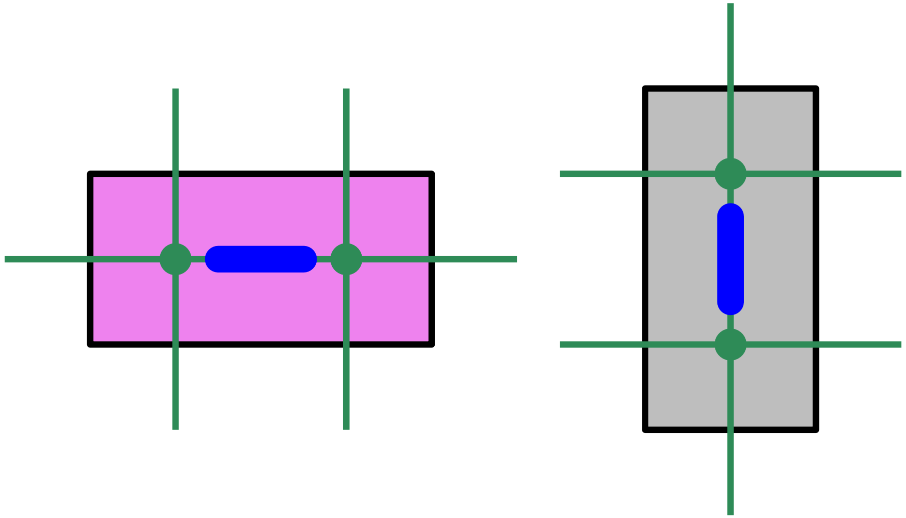
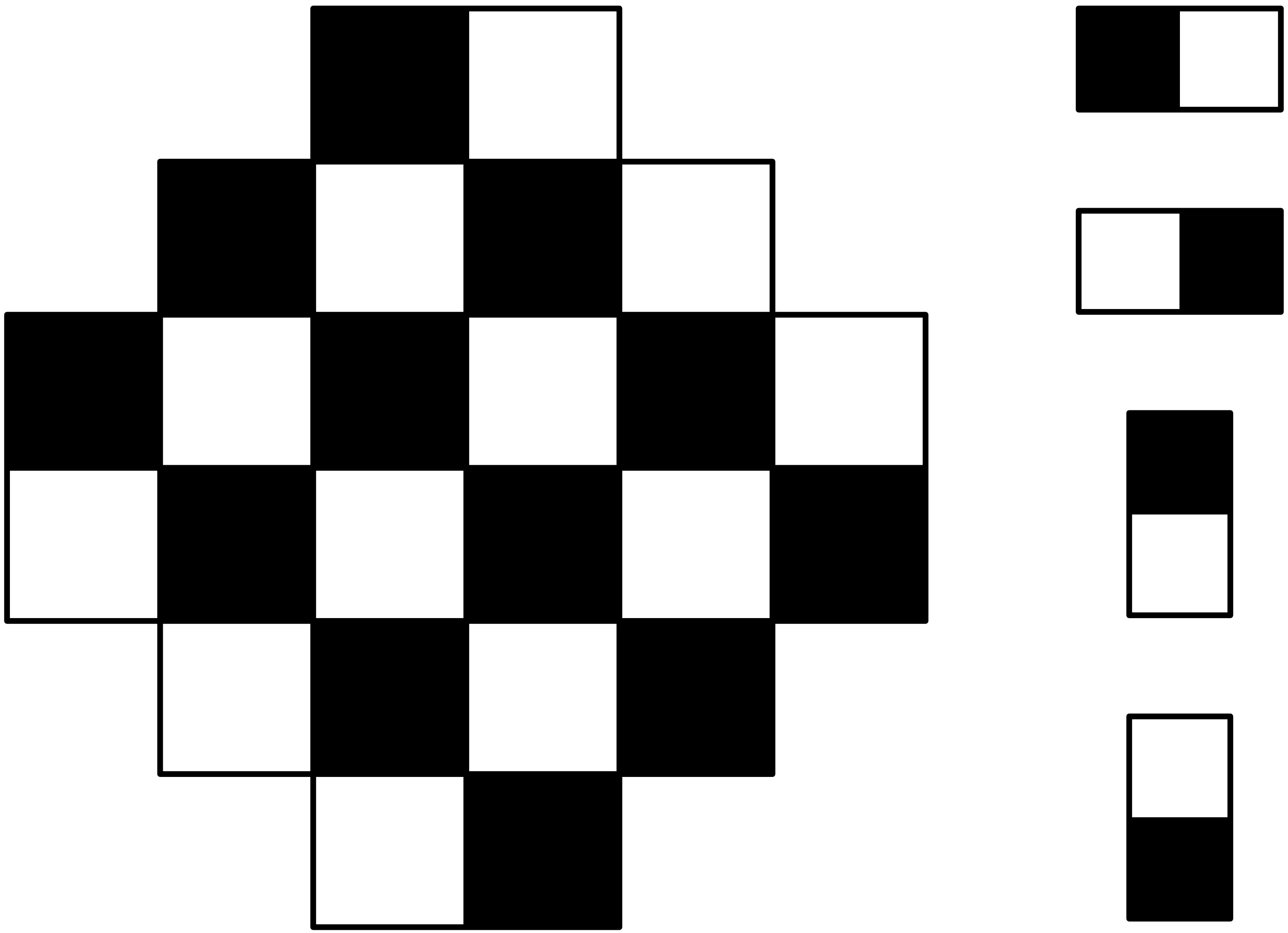
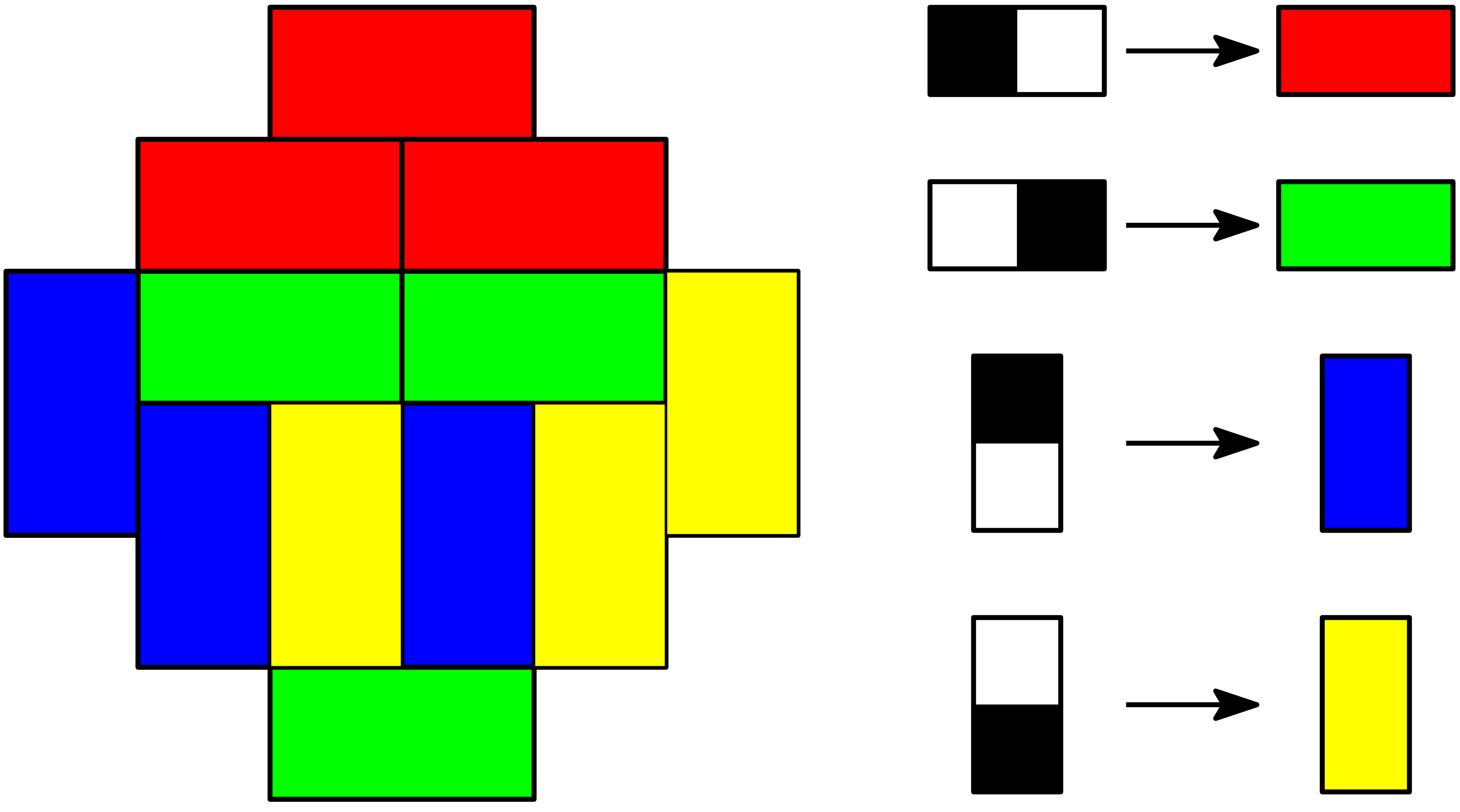


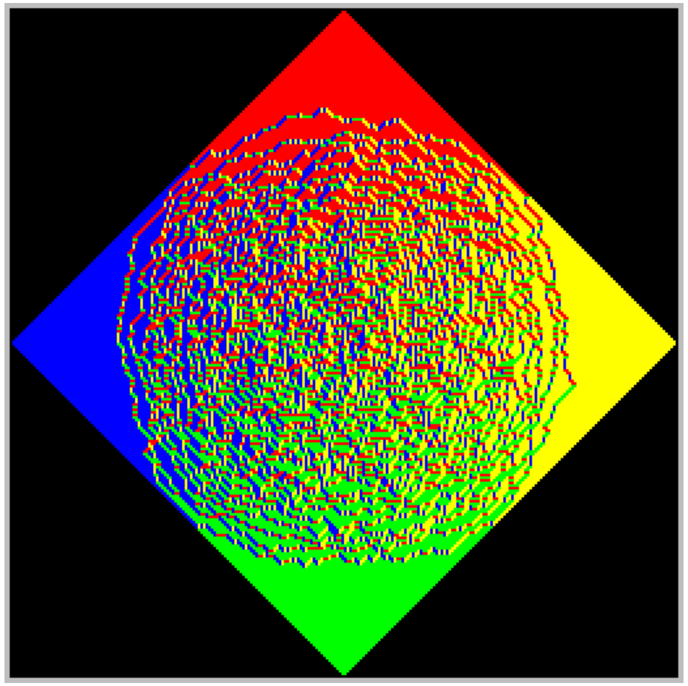
Domino tilings of the Aztec diamond



Domino tilings of the Aztec diamond



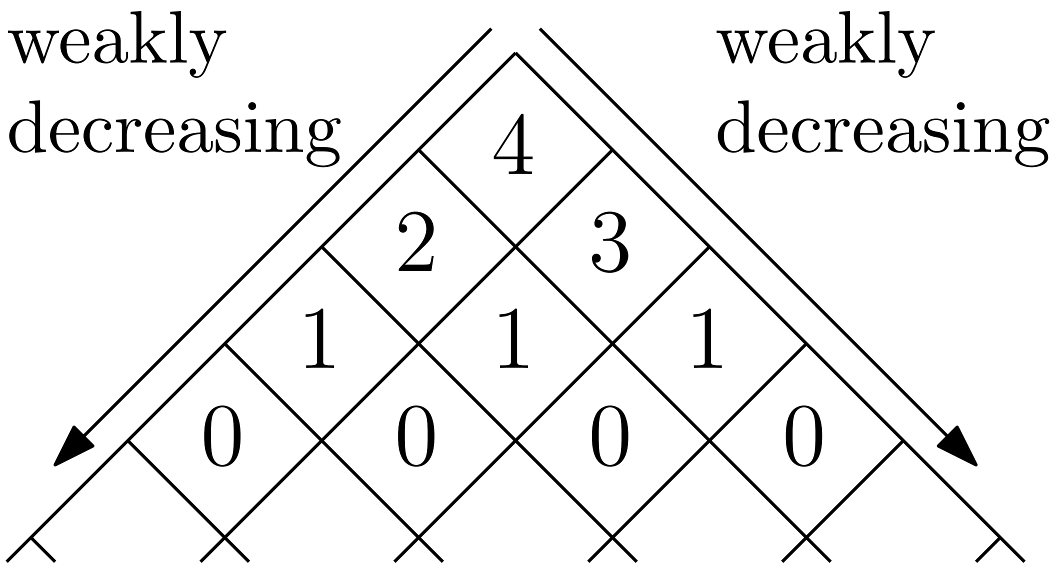
- There are $2^{n(n+1)/2}$ tilings of the Aztec diamond of size n (Elkies-Kuperberg-Larsen-Propp).
- Correlations studied by Johansson then by Chhita-Young (inverse Kasteleyn matrix).
- Arctic circle phenomenon (Jockusch-Propp-Shor) :

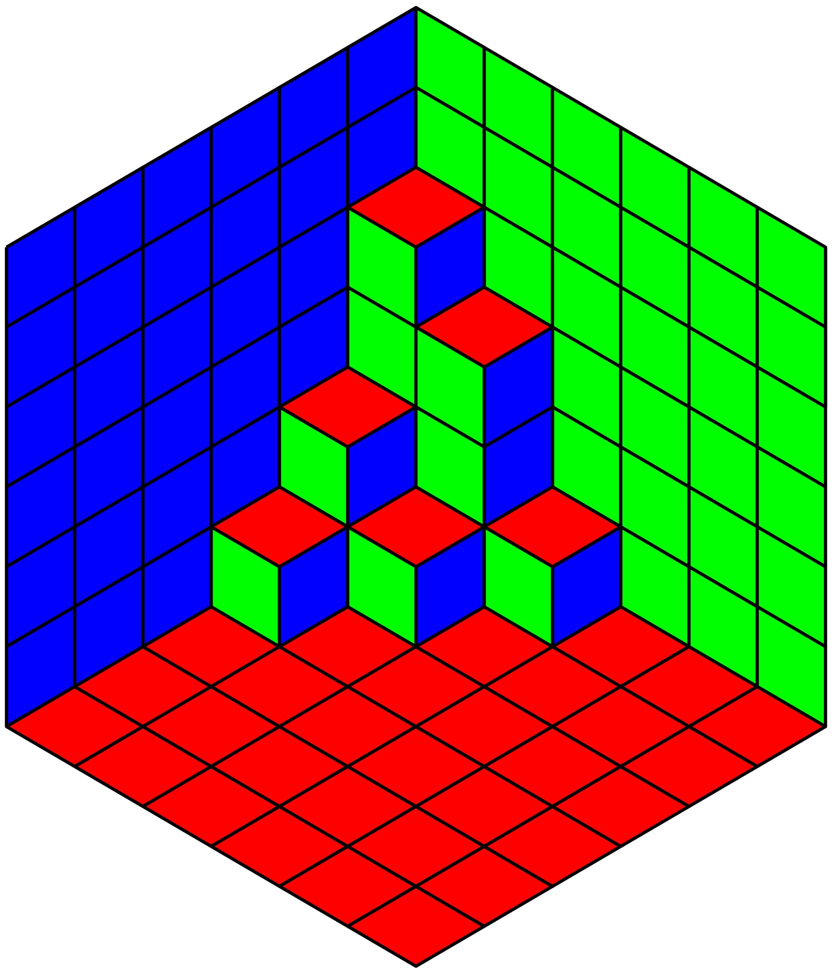


picture by Cris Moore

tuvalu.santafe.edu/
~moore/aztec256.gif

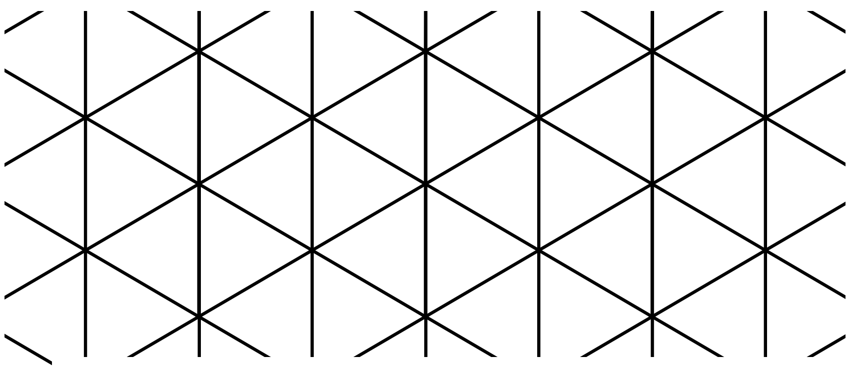
Plane partitions

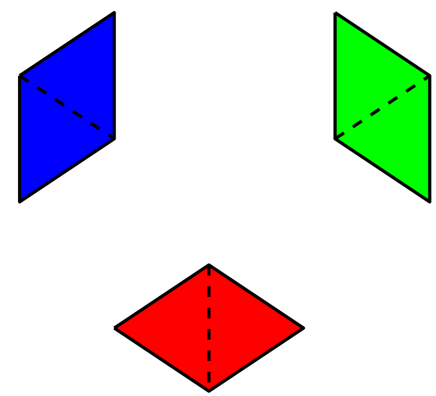




volume = sum of the numbers = number of cubes

- Stack of cubes (3D) $\leftrightarrow$ tiling of the infinite triangular lattice (2D) by rhombi of 3 kinds :





- Corresponds to a dimer model on the infinite hexagonal lattice (dual of the triangular lattice).

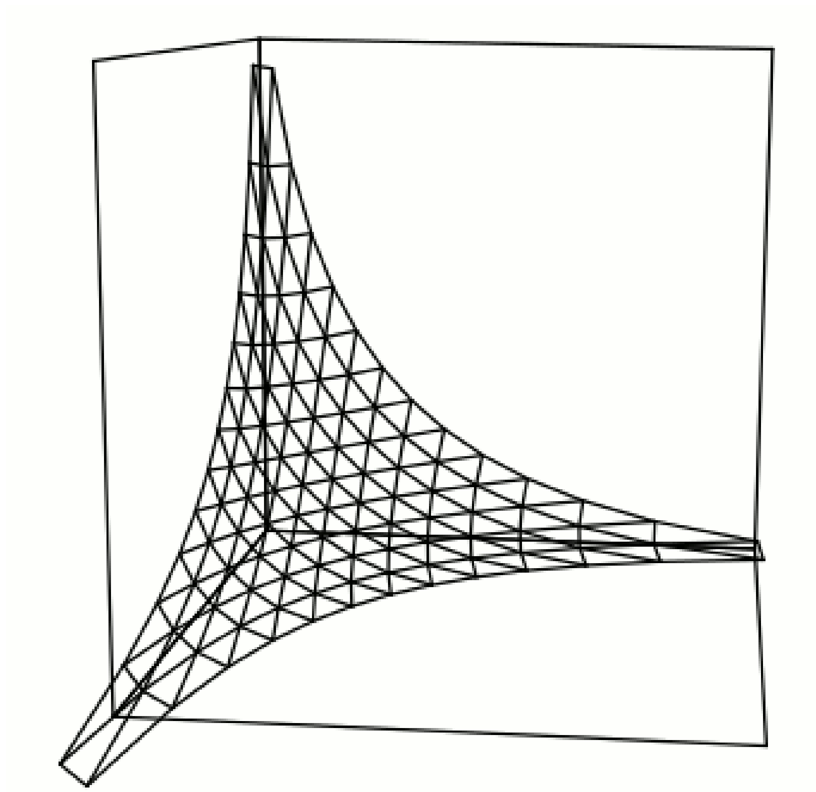
- Measure considered : fix $q < 1$ and weigh each plane partition π by $q^{\mathrm{vol}(\pi)}$.

- Partition function (MacMahon) :

$$Z = \sum_{\pi} q^{\mathrm{vol}(\pi)} = \prod_{k=1}^{\infty} \left(1 - q^{k}\right)^{-k}$$

- Determinantal correlation kernel computed by Okounkov-Reshetikhin.

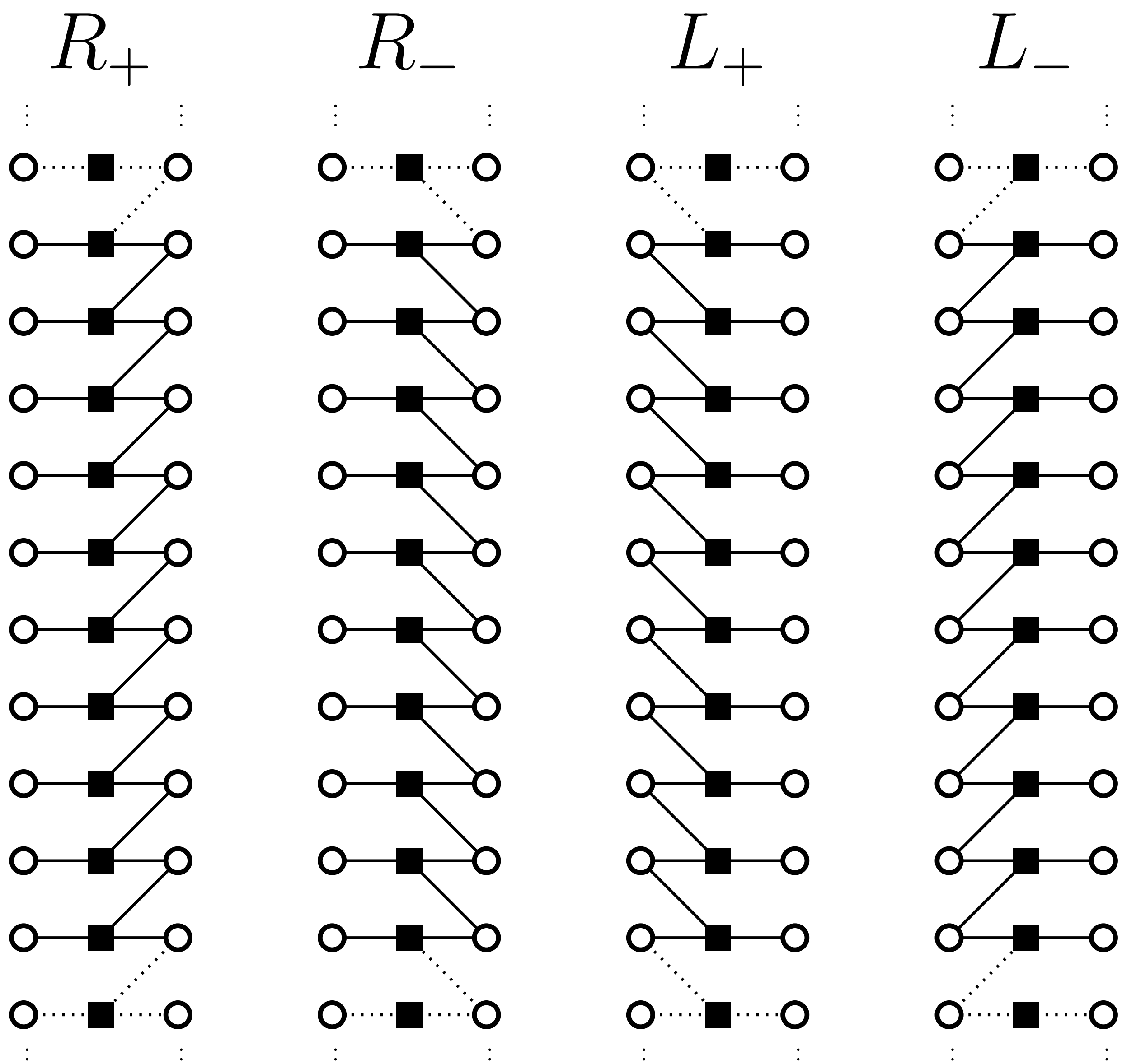
- Limit shape ($q \to 1$, appropriate rescaling) computed by Cerf-Kenyon and Okounkov-Reshetikhin (skew plane partitions).



picture from Cerf-Kenyon,
The low-temperature expansion of the
Wulff crystal in the 3D Ising model.

3 Definition of rail yard graphs

Four elementary graphs :

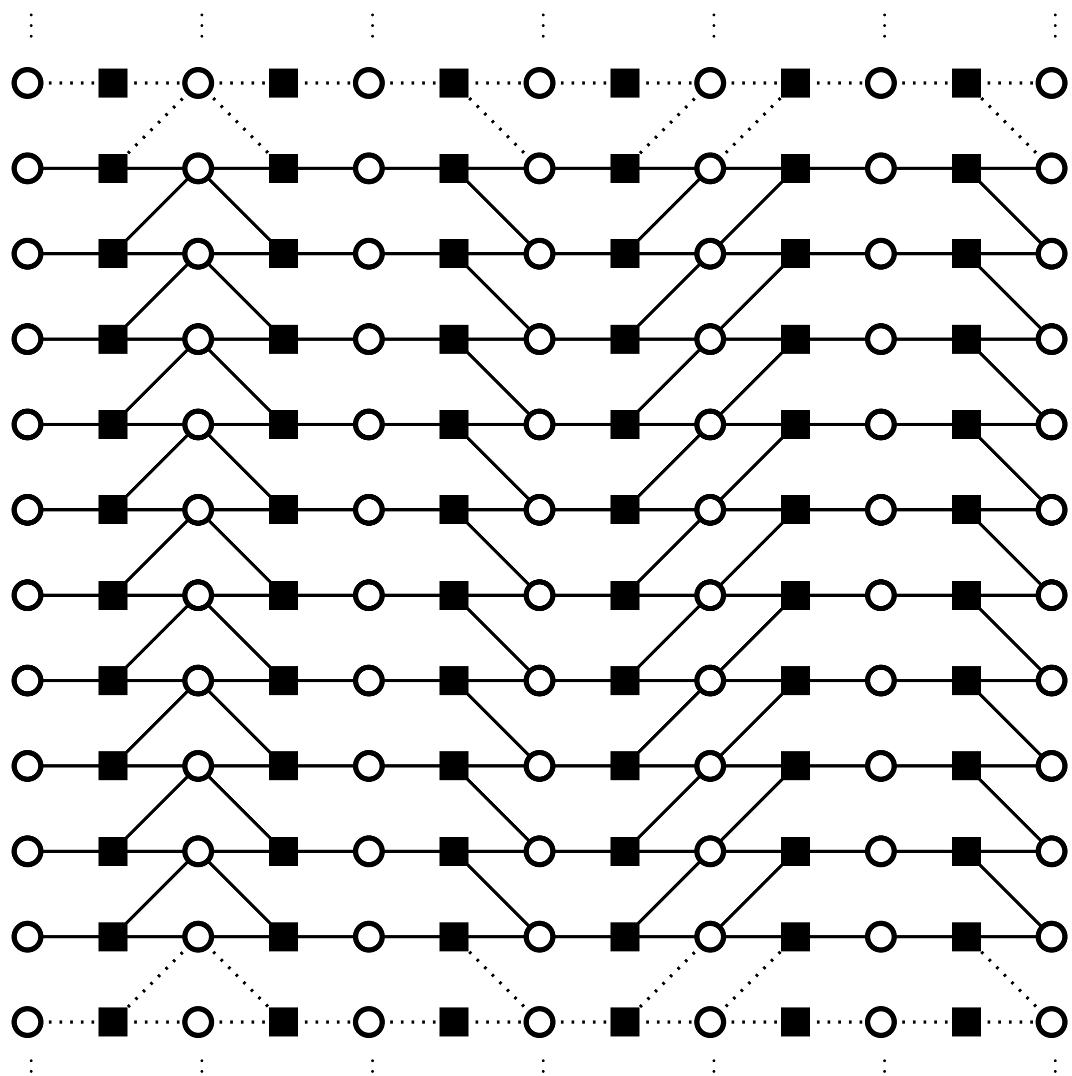


- A rail yard graph (RYG) is obtained by gluing elementary graphs along columns of white vertices.

- For every $n \geq 1$ and every couple of words $(a_1, \ldots, a_n) \in \{L, R\}^n$ and $(b_1, \ldots, b_n) \in \{+, -\}^n$ we define an associated RYG.

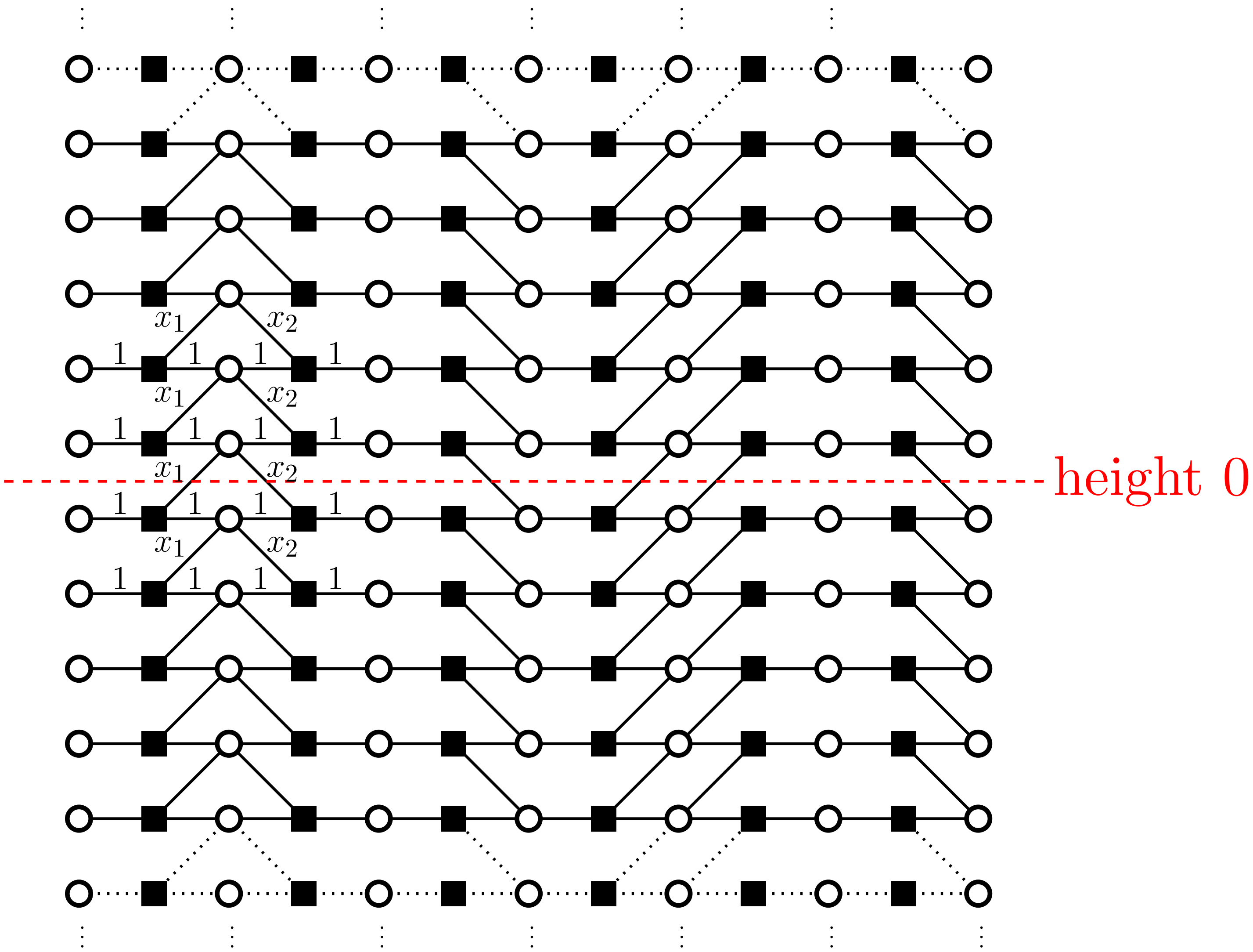
n=6 $(a_1, a_2, a_3, a_4, a_5, a_6) = (R, L, R, R, L, R)$

$(b_1, b_2, b_3, b_4, b_5, b_6) = (+, +, -, +, -, -)$



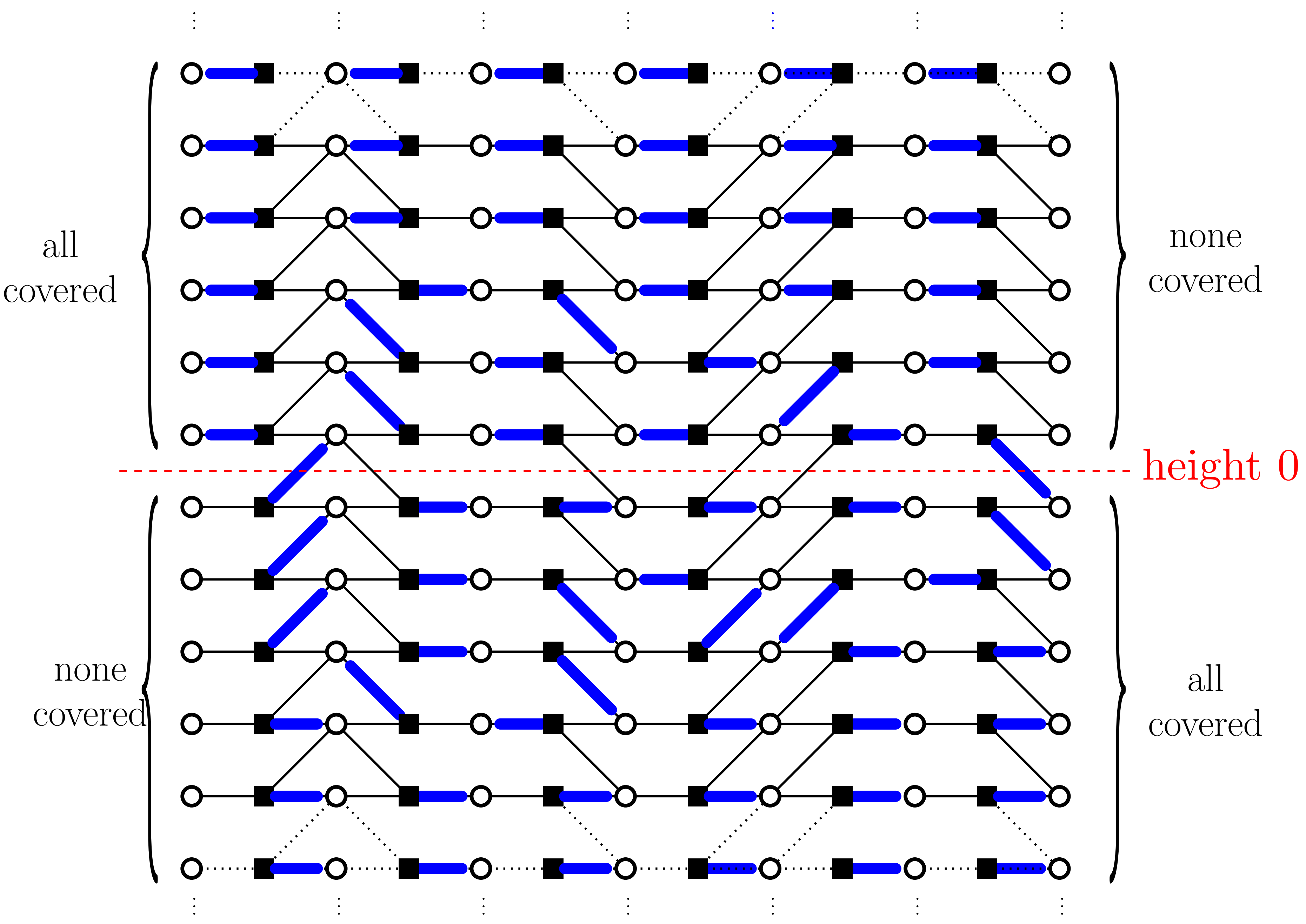
Weights : 1 on horizontal edges

x_i on diagonal edges encoded by (a_i, b_i), $x_i < 1$



A dimer configuration on a RYG is a subset of edges such that :

1. on the left boundary, all the vertices below height 0 are unmatched ;
2. on the right boundary, all the vertices above height 0 are unmatched ;
3. all the other vertices are matched to exactly one neighbour ;
4. the dimer configuration contains finitely many diagonal edges.


all covered
none covered
height 0
none covered
all covered

4 Main results

Theorem (Boutillier-Bouttier-Chapuy-Corteel-R). *The partition function for dimer configurations on a RYG specified by* $n, \underline{a}$ *and* $\underline{b}$ *is*

$$Z\left(n, \underline{a}, \underline{b}\right) = \prod_{\substack{1 \le i < j \le n \\ b_i = +, b_j = -}} z_{ij},$$

where

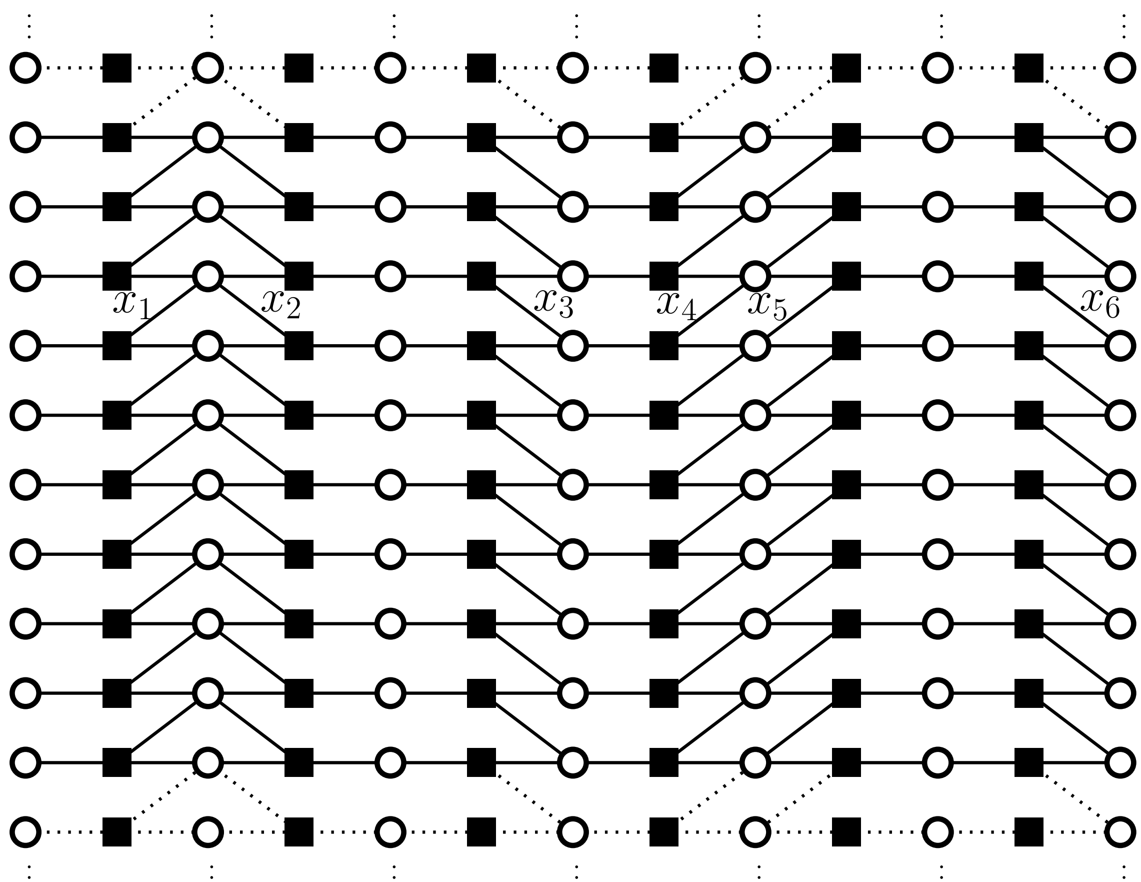
$$z_{ij} = \begin{cases} 1 + x_i x_j & \text{if } a_i \neq a_j \\ \frac{1}{1 - x_i x_j} & \text{if } a_i = a_j. \end{cases}$$

n=6

$$(a_1, a_2, a_3, a_4, a_5, a_6) = (R, L, R, R, L, R)$$

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (+, +, -, +, -, -)$$

$$Z = \frac{(1+x_1x_5)(1+x_2x_3)(1+x_2x_6)(1+x_4x_5)}{(1-x_1x_3)(1-x_1x_6)(1-x_2x_5)(1-x_4x_6)}$$



Theorem (Boutillier-Bouttier-Chapuy-Corteel-R)**.** *Let* $E = \{e_1, \ldots, e_s\}$ *be* s *edges, with* $e_i = (\alpha_i, \beta_i)$*.*

black ↗ ↖ white

The probability that all the edges in E *appear in a random dimer configuration is*

$$\mathbb{P}(E) = (-1)^{H(E)} \left(\prod_{i=1}^{s} \mathrm{weight}(e_i) \right) \det(C_{\alpha_i, \beta_j})_{1 \leq i,j \leq s}$$

where $H(E)$ *is the number of horizontal edges in* E *with a black right endpoint.*

If α is a black vertex at position (α^x, α^y)
and β is a white vertex at position (β^x, β^y)

$$C_{\alpha\beta} = \left[z^{\alpha^y} w^{-\beta^y}\right] \frac{F_{\alpha^x}(z)}{F_{\beta^x}(w)} \frac{\sqrt{zw}}{z-w}$$

where

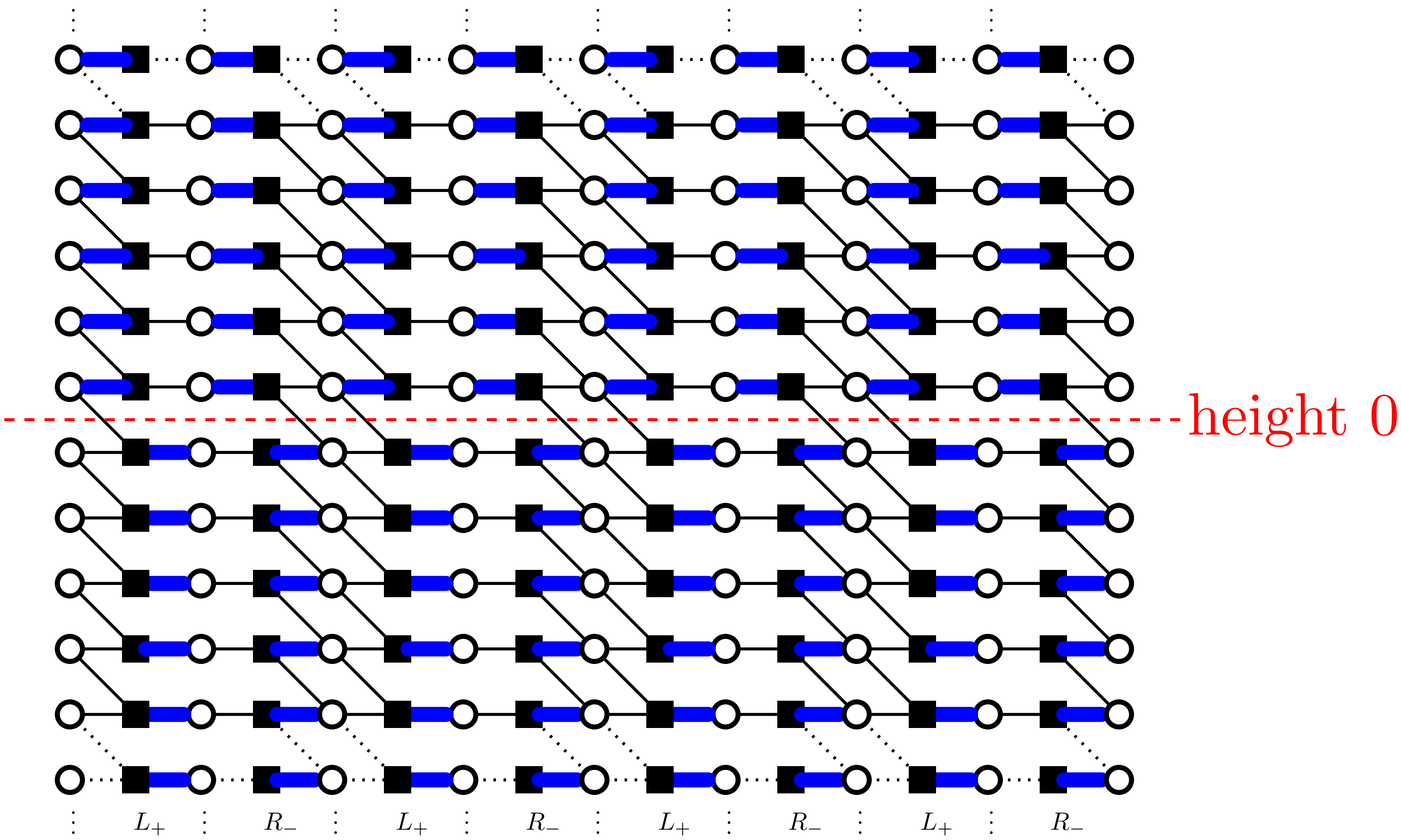
$$F_k(z) = \frac{\displaystyle\prod_{\substack{i:(a_i,b_i)=(R,+)\\ 2i<k}} (1+x_i z) \prod_{\substack{j:(a_j,b_j)=(L,-)\\ 2j>k}} (1-x_j/z)}{\displaystyle\prod_{\substack{i:(a_i,b_i)=(L,+)\\ 2i\le k}} (1-x_i z) \prod_{\substack{j:(a_j,b_j)=(R,-)\\ 2j\ge k}} (1+x_j/z)}$$

- Skew plane partitions (hexagonal lattice), the Aztec diamond, pyramid partitions and more generally steep tilings of Bouttier-Chapuy-Corteel (square lattice) arise as particular instance of RYGs.

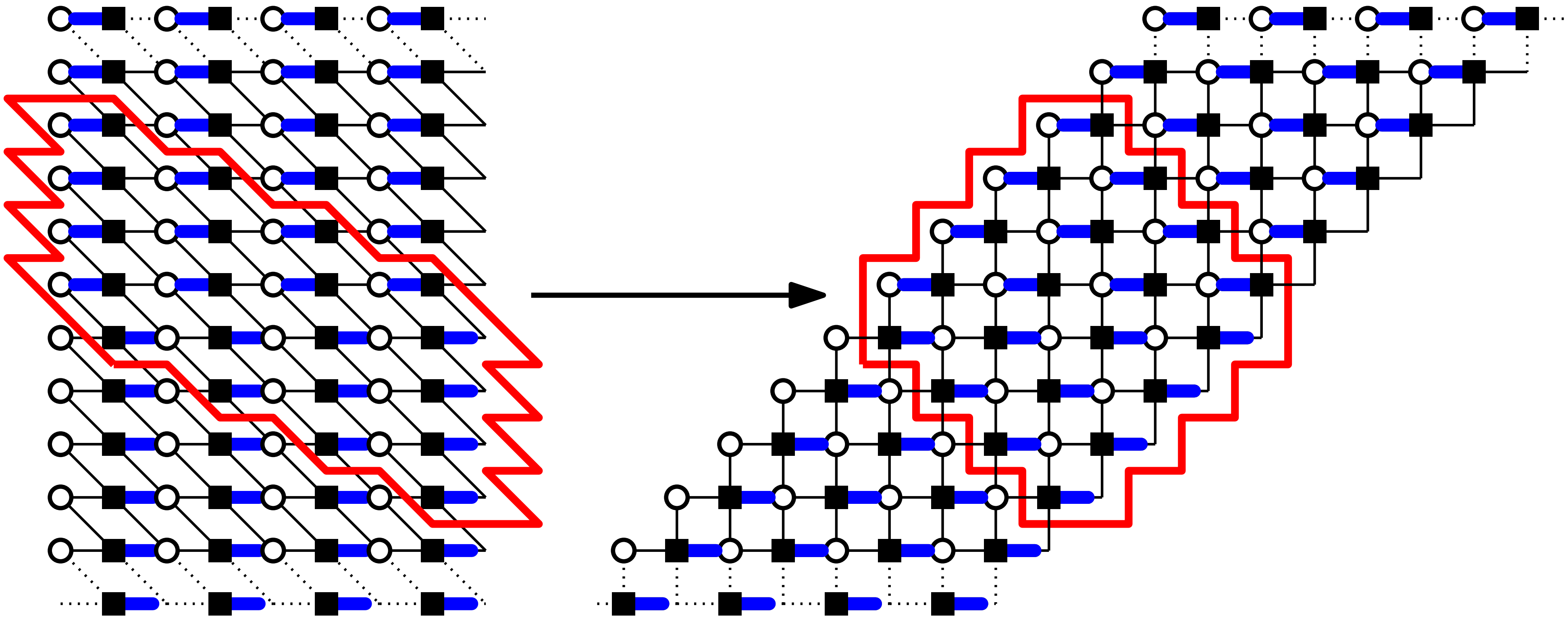
- We recover the results of Okounkov-Reshetikhin (resp. Chhita-Young) for correlations of skew plane partitions (resp. the Aztec diamond).

- We show that the correlation kernel $(C_{\alpha\beta})$ is an inverse to the Kasteleyn operator on the RYGs.

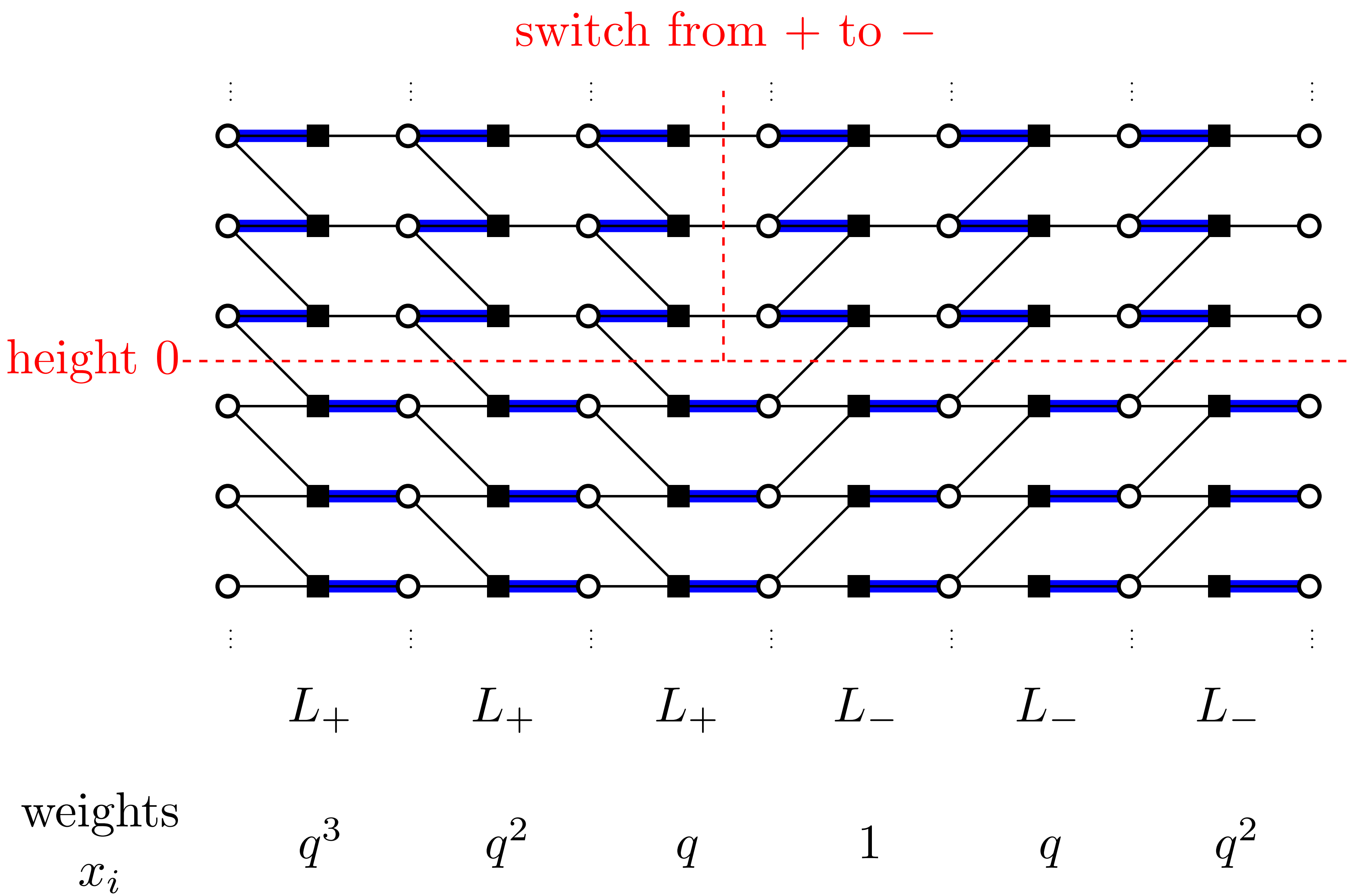
- Aztec diamond of size $n \leftrightarrow$ alternate L_+ and R_- (n times each)

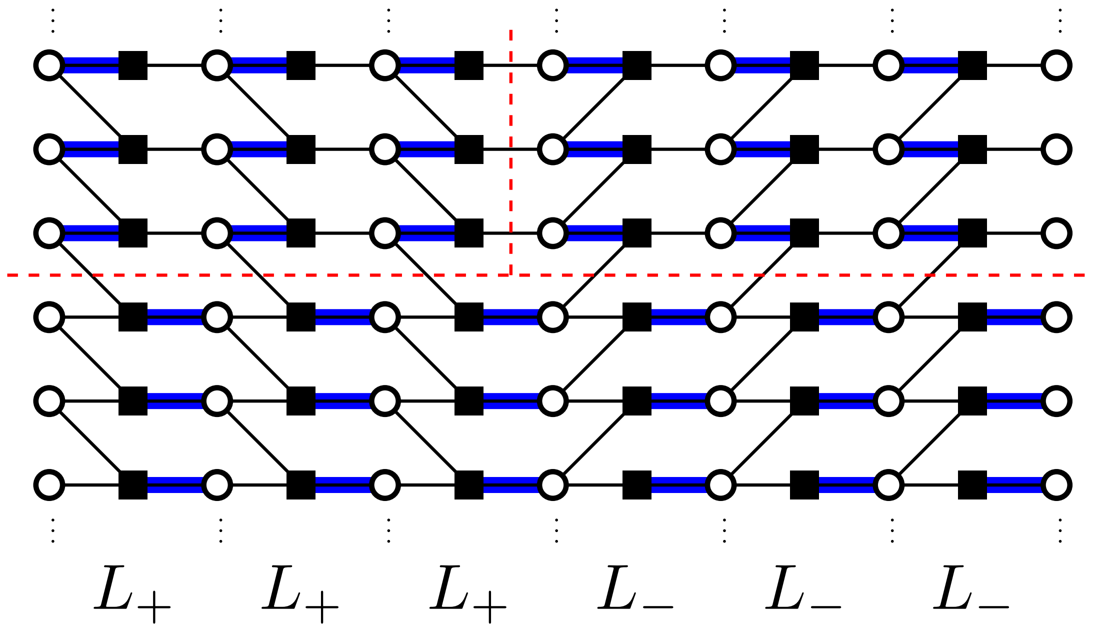


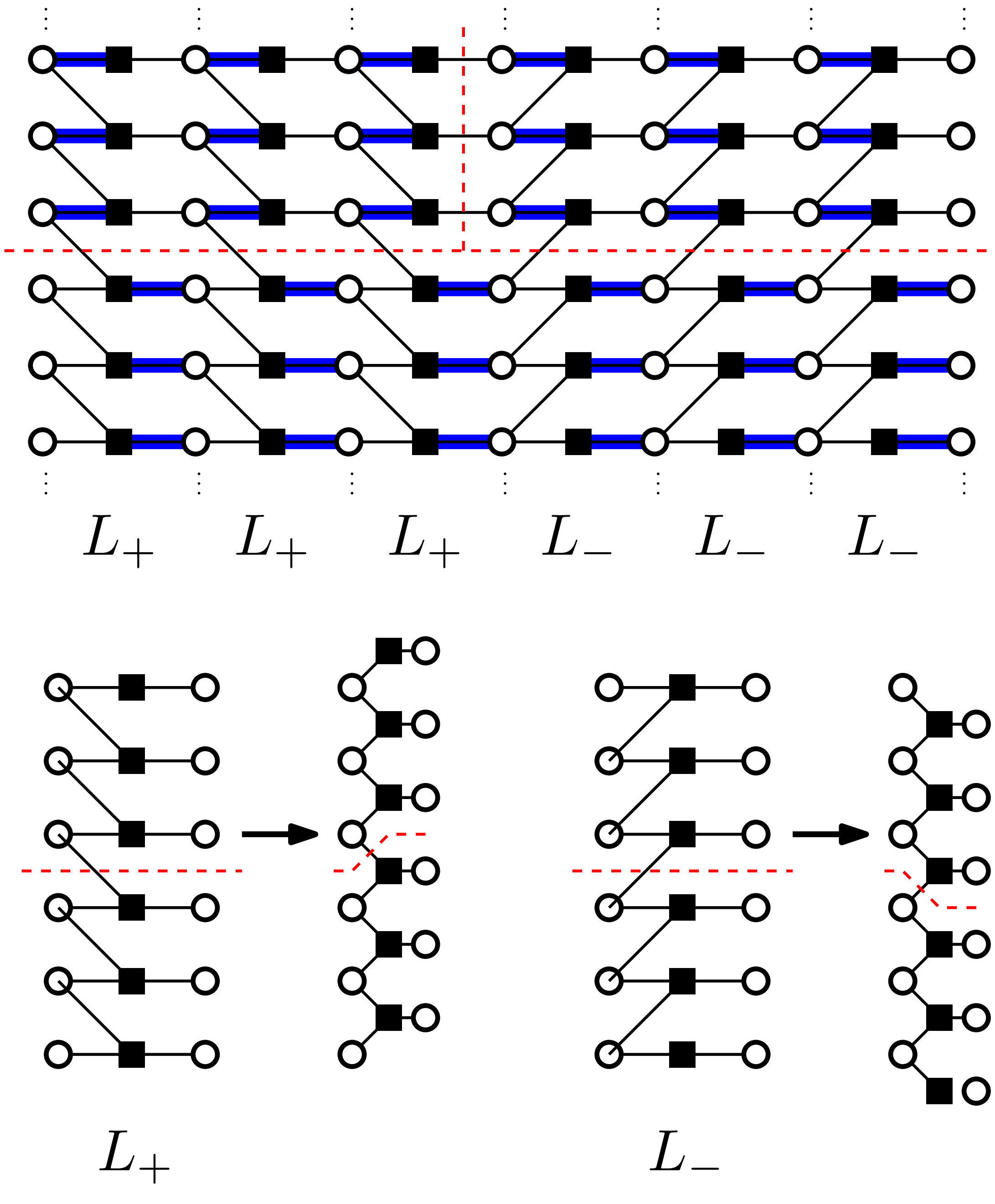
- Contract vertices of degree 2.

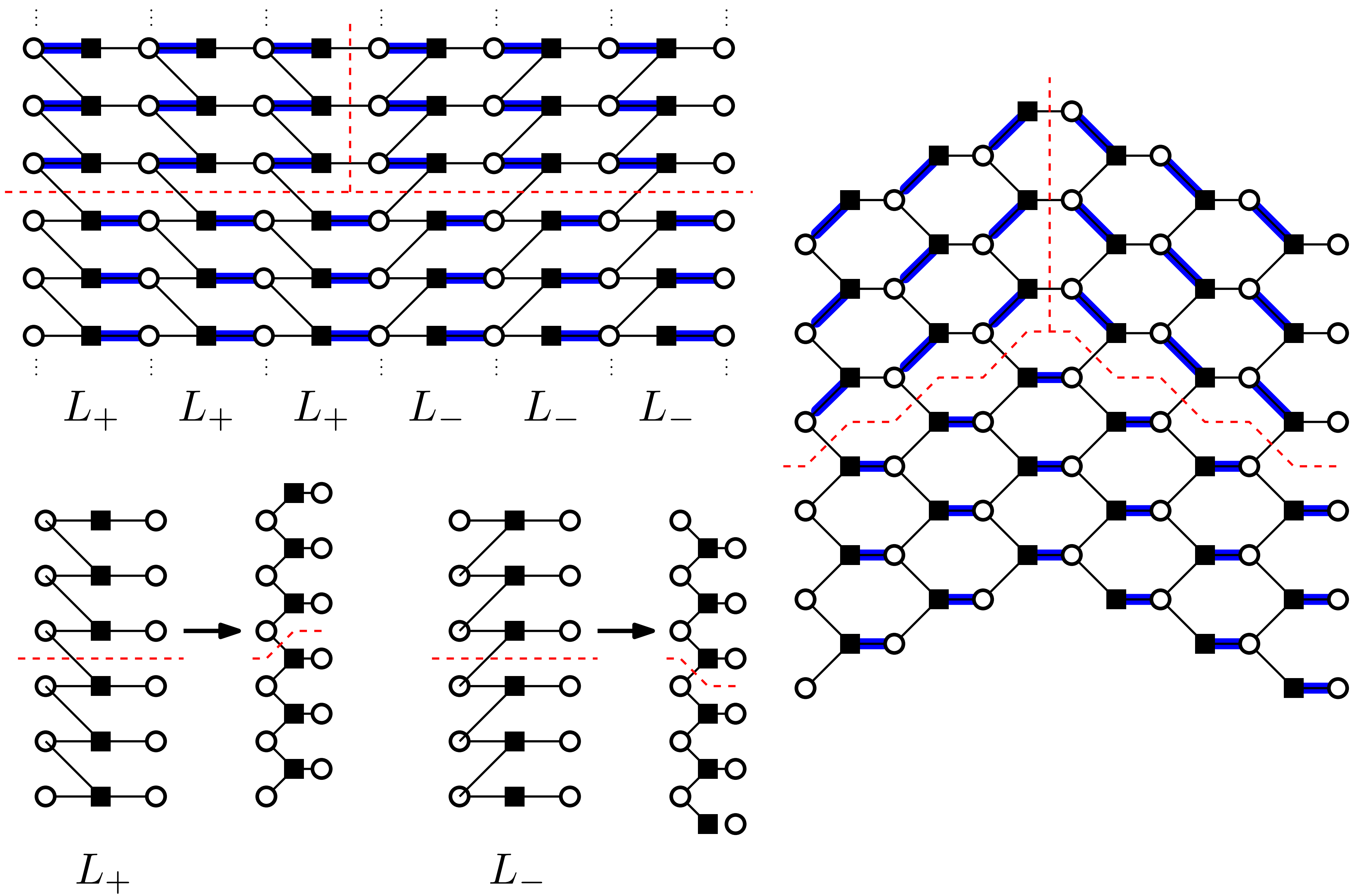


Plane partition (with $n \times n$ floor) $\longleftrightarrow$ n times L_+ followed by n times L_-

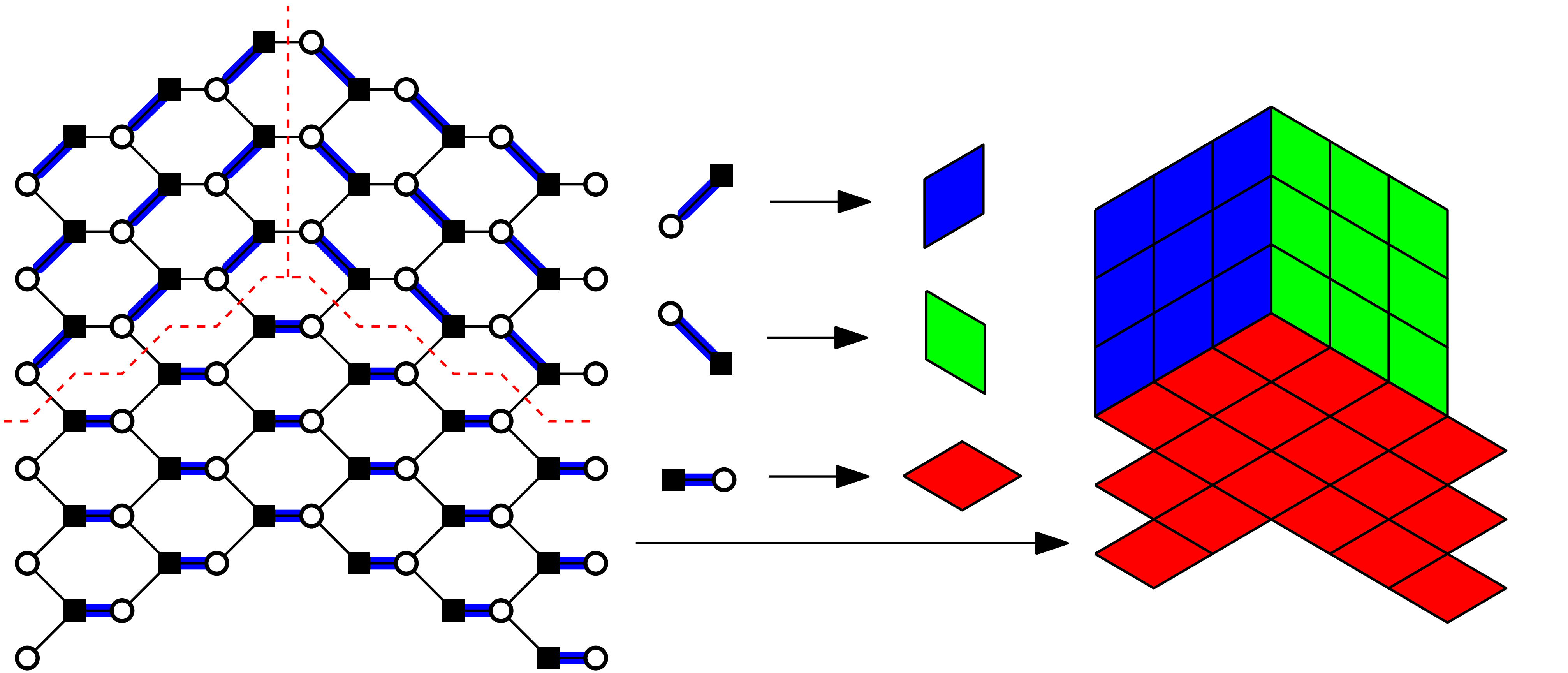


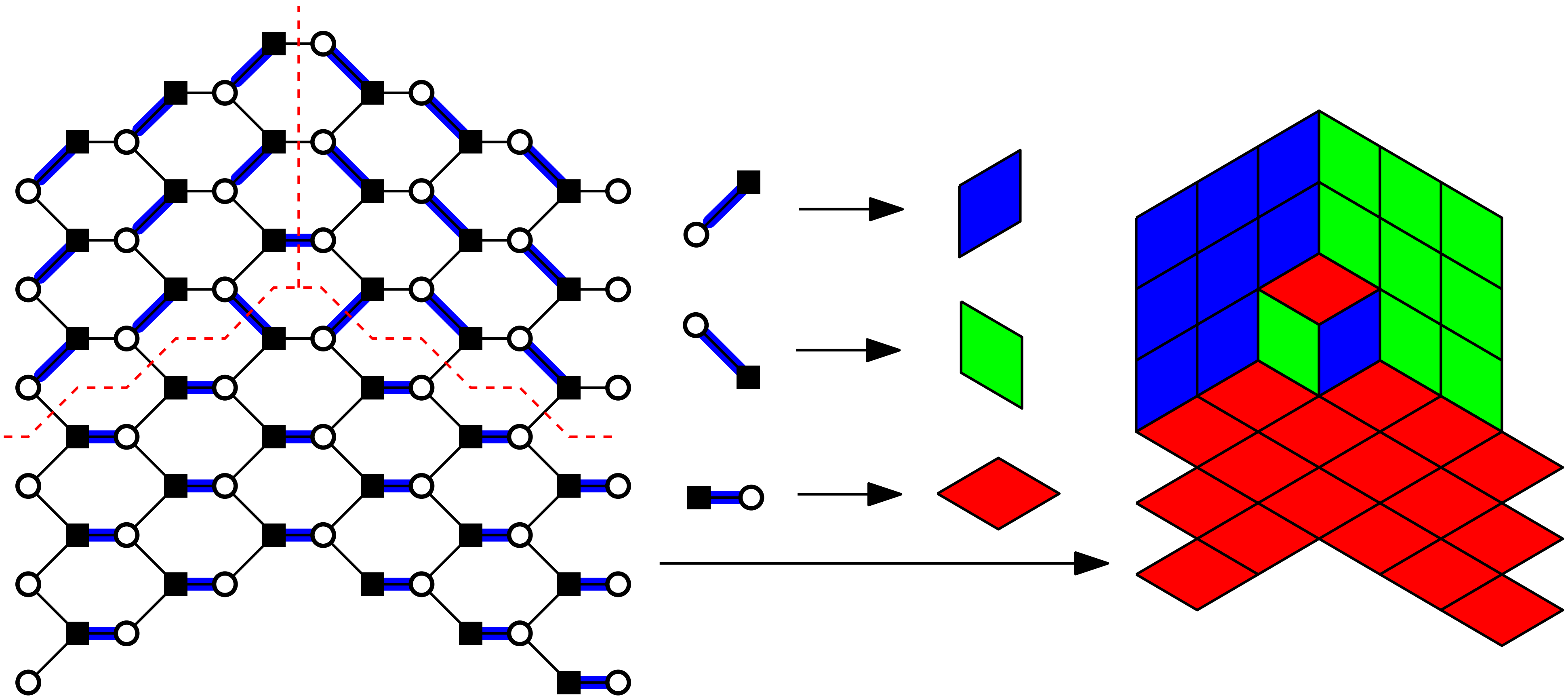

L_+ L_+ L_+ L_- L_- L_-


L_+ L_+ L_+ L_- L_- L_-
L_+
L_-



L_+ L_+ L_+ L_- L_- L_-
L_+
L_-





5 Proof ideas

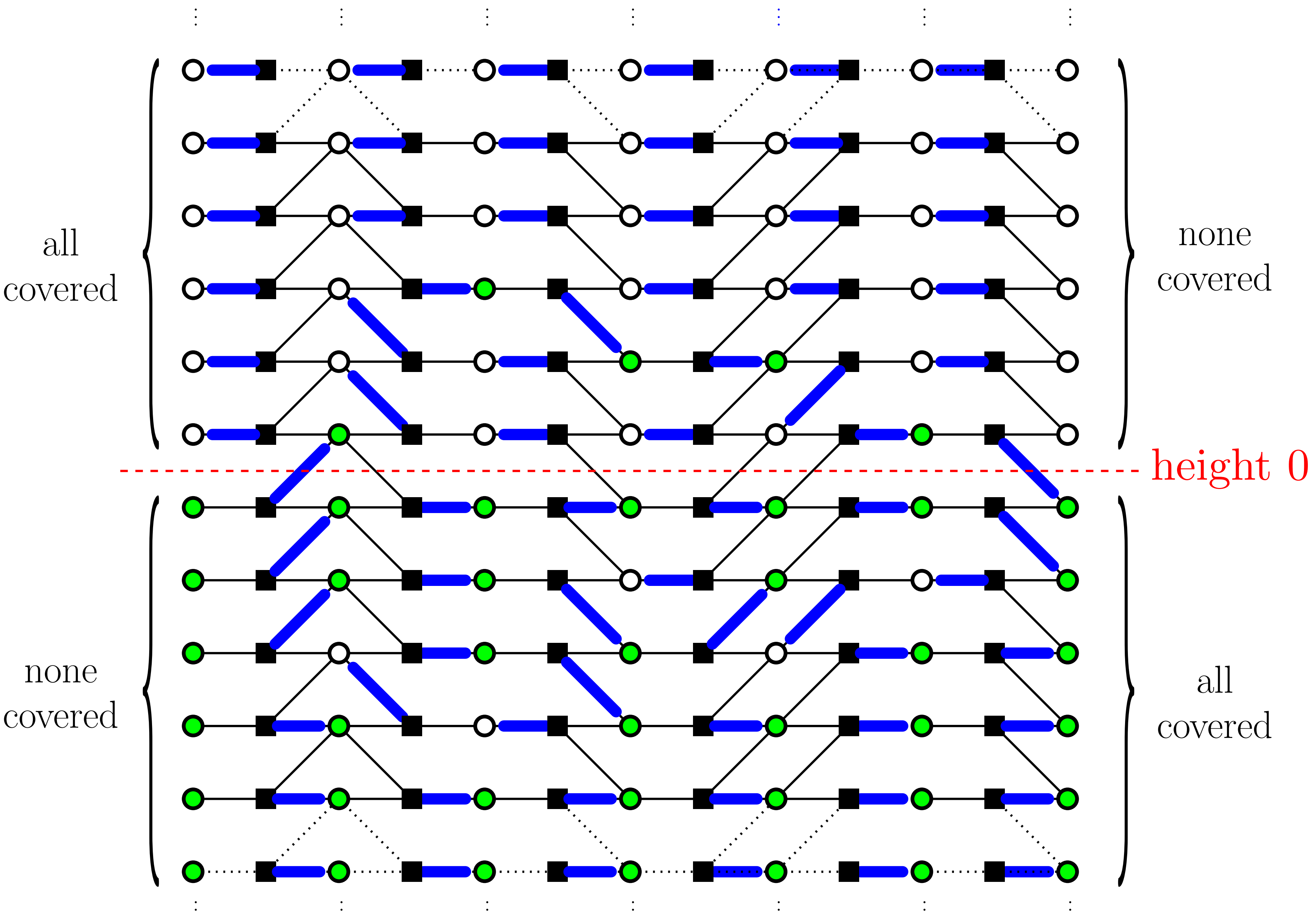
- Key idea : realise the RYG dimer model as a Schur process (Okounkov-Reshetikhin, Johansson, Bouttier-Chapuy-Corteel).

- Bijection between dimer configurations and configurations of particles/holes on each column of white vertices.

Dimers $\longrightarrow$ particles/holes

On each white vertex v, put :

- a particle if v is matched to its left
- a hole if v is matched to its right
- a particle if v is unmatched and on the left boundary
- a hole if v is unmatched and on the right boundary.


all
covered
none
covered
height 0
none
covered
all
covered

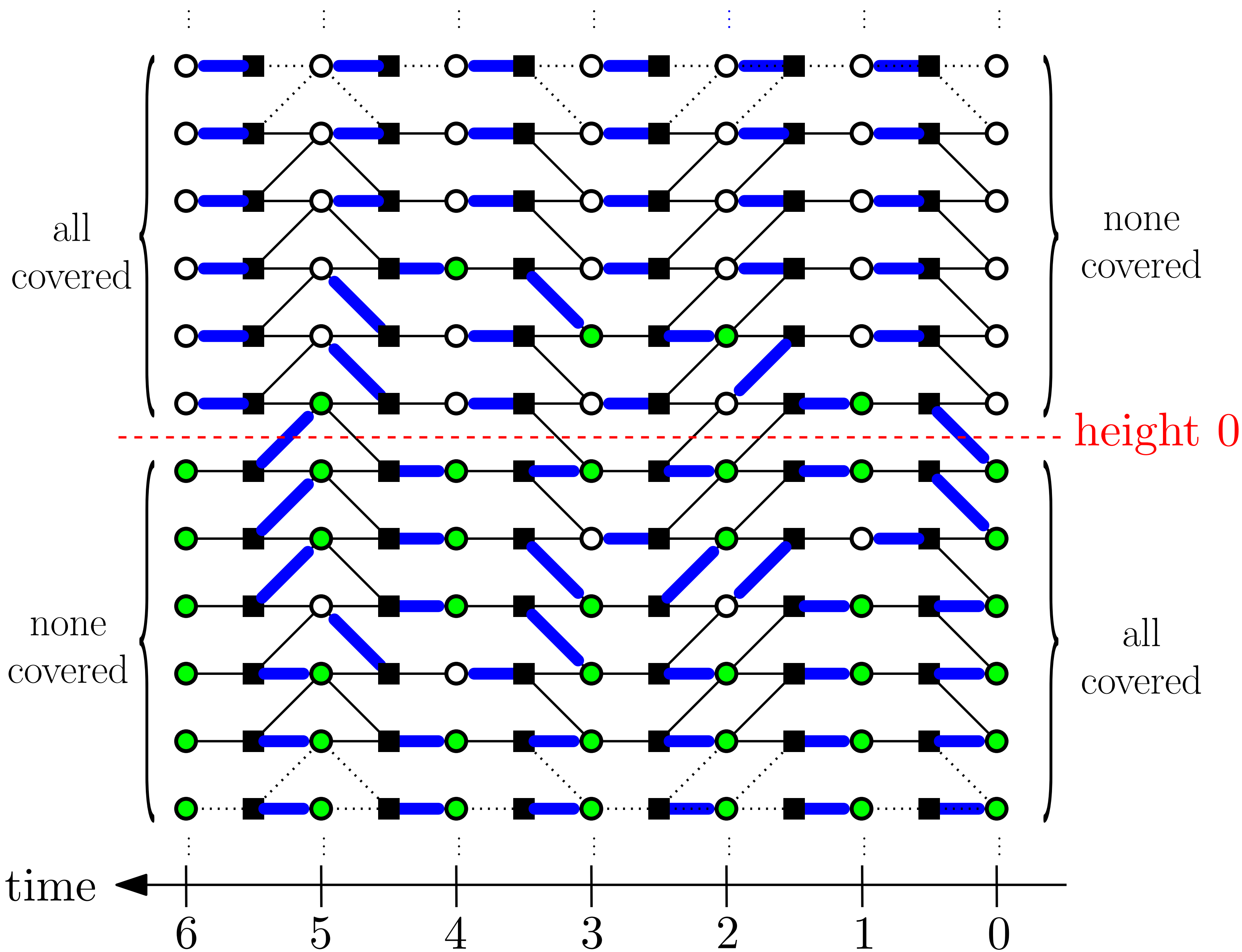
- A dimer configuration can be seen as the evolution of a particle system on a vertical line (Maya diagram) between times 0 and n. The initial and final states are imposed.

- Particles and holes move from right to left.

- For columns of type R_- or R_+, particles select the path indicated by the dimer if they have a choice. Holes fill the gaps.

- For columns of type L_- or L_+, holes select the path indicated by the dimer if they have a choice. Particles fill the gaps.

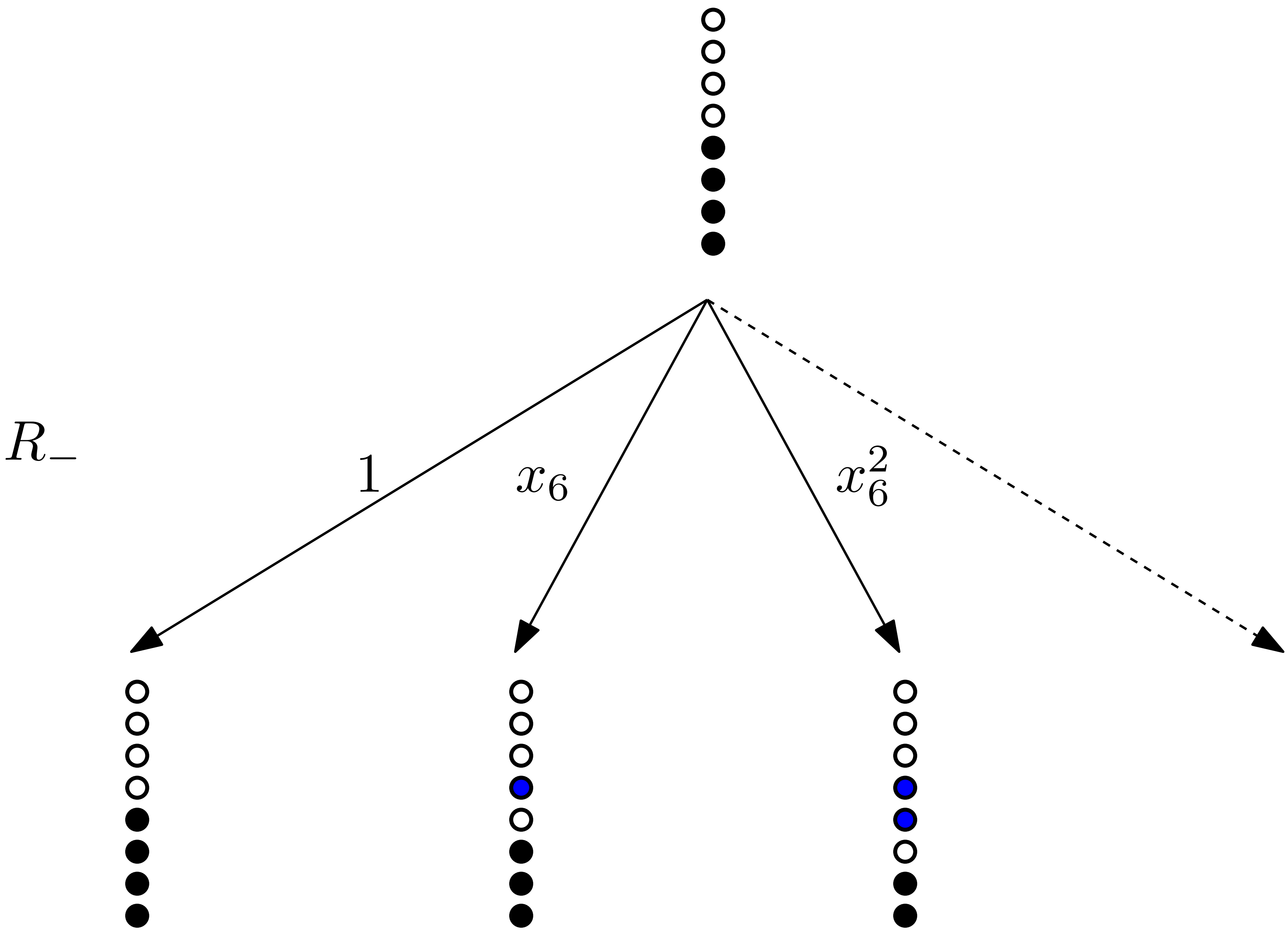
“Particles/holes slide along dimers.”



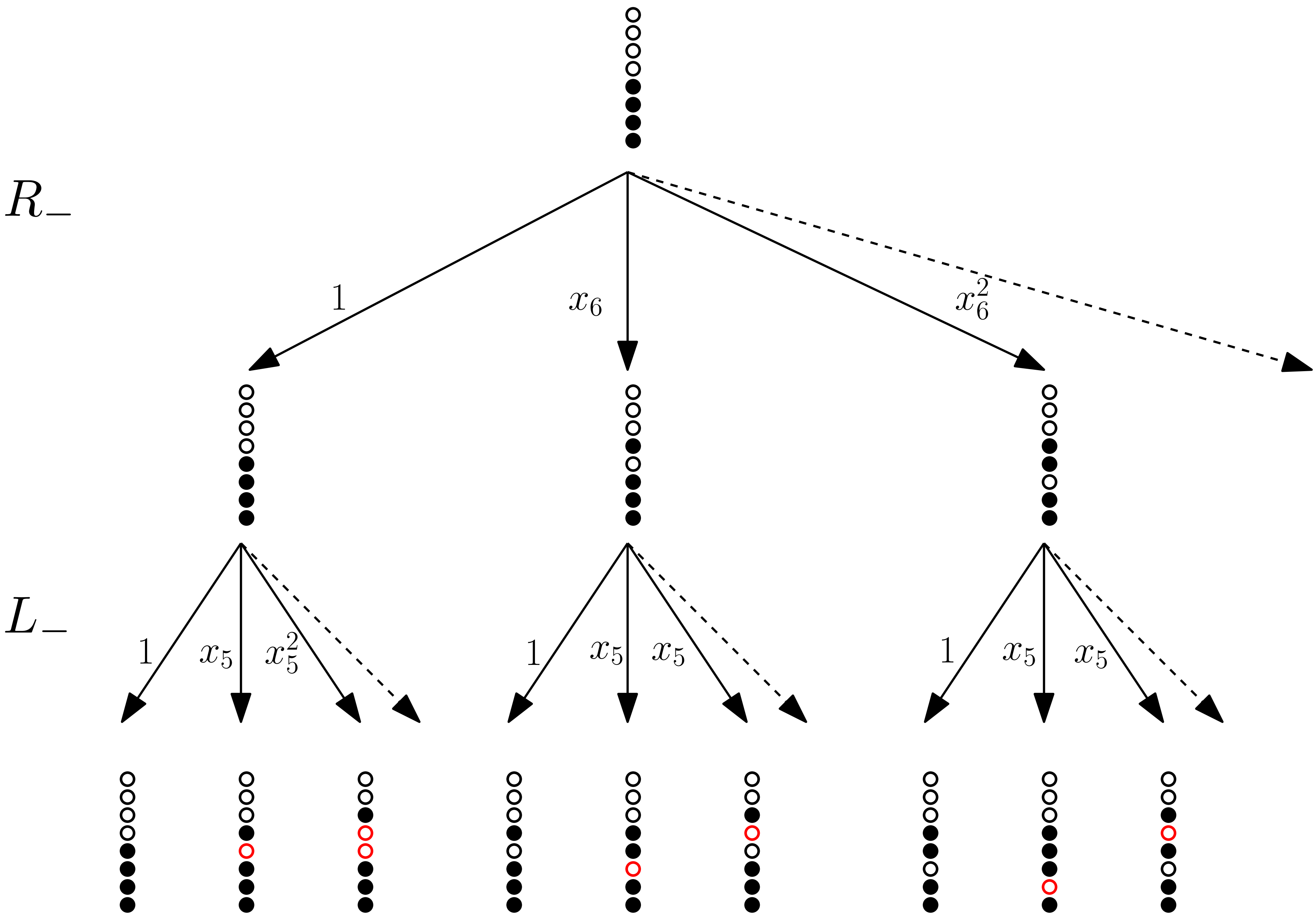
Rail yard at Maradana Station, Colombo, Sri Lanka.
Wikipedia illustration by Shamli071, used under Creative Commons license.

- R_- (resp. R_+) : particles can move up (resp. down) by one unit if possible.

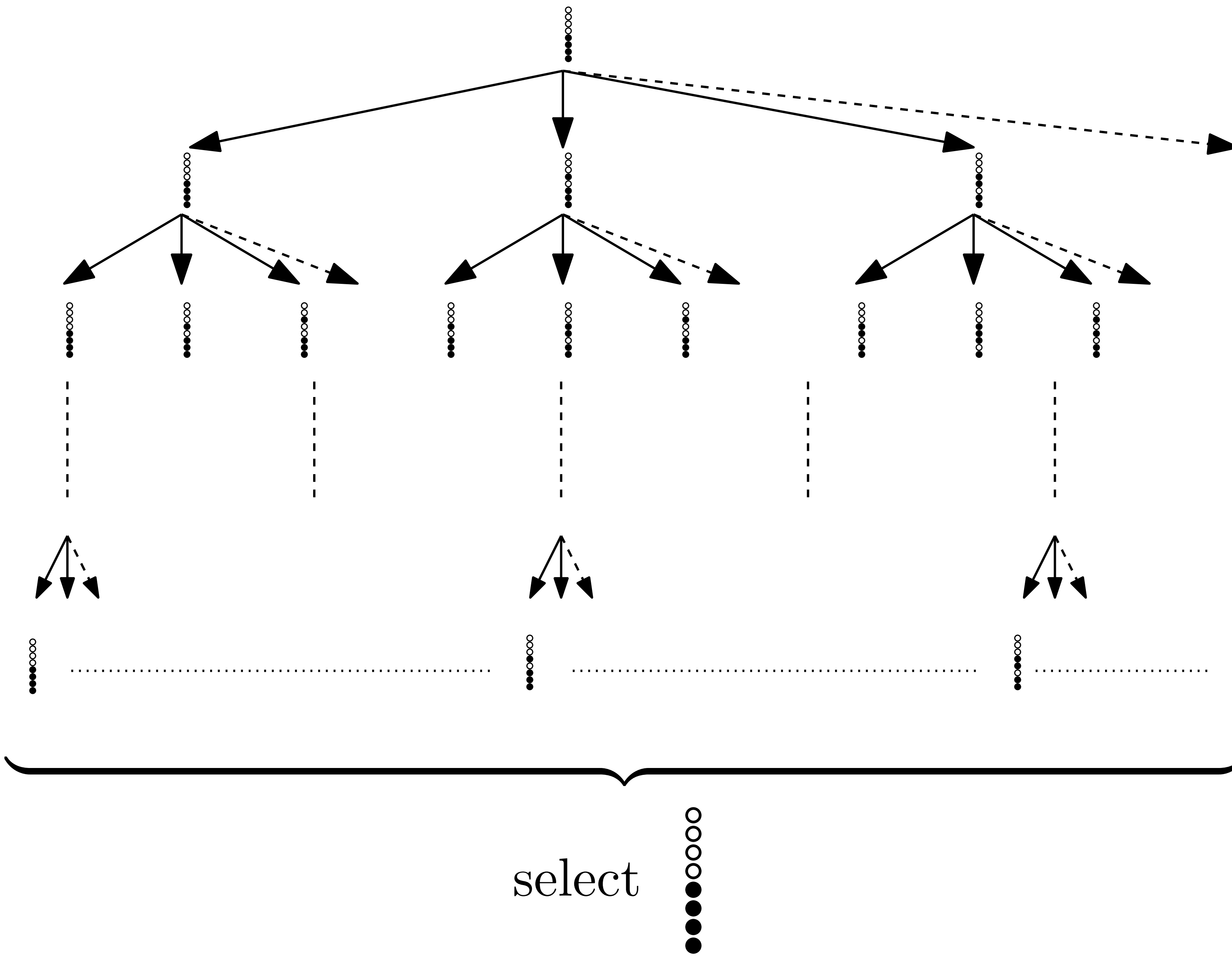
- L_- (resp. L_+) : holes can move down (resp. up) by one unit if possible.

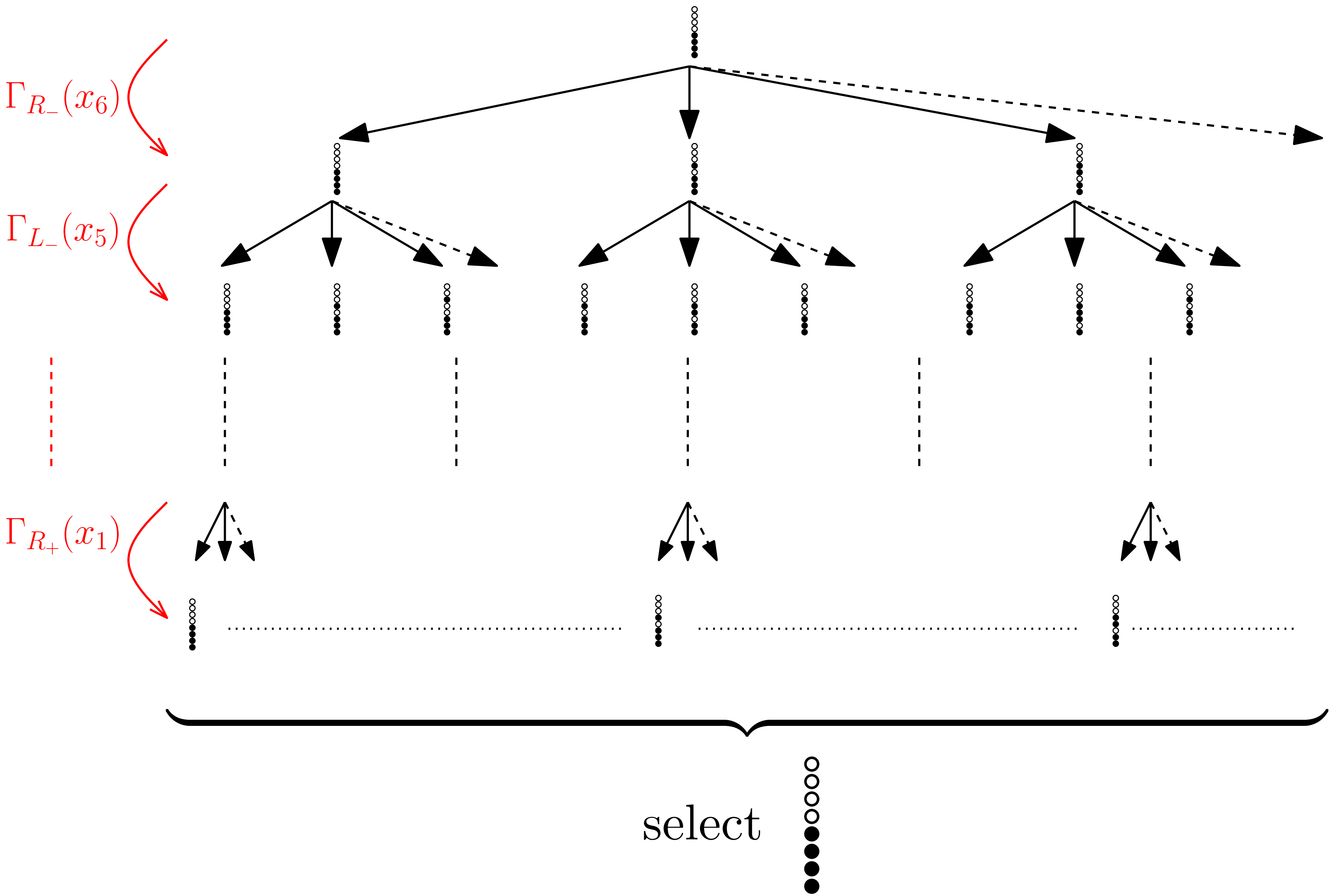


R_-
1
x_6
x_6^2



R_-
L_-
1
x_6
x_6^2
1
x_5
x_5^2
1
x_5
x_5
1
x_5
x_5


select

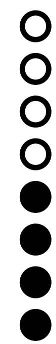


$\Gamma_{R_-}(x_6)$
$\Gamma_{L_-}(x_5)$
$\Gamma_{R_+}(x_1)$
select

- Dimer partition function :

$$Z = \langle \emptyset \mid \Gamma_{R_+}(x_1)\Gamma_{L_+}(x_2)\Gamma_{R_-}(x_3)\Gamma_{R_+}(x_4)\Gamma_{L_-}(x_5)\Gamma_{R_-}(x_6) \mid \emptyset \rangle$$

where the symbol $\emptyset$ denotes



- Using :

  - $\Gamma_{R_+}(x) \mid \emptyset\rangle = \Gamma_{L_+}(x) \mid \emptyset\rangle = \mid \emptyset\rangle$
  - $\langle \emptyset \mid \Gamma_{R_-}(x) = \langle \emptyset \mid \Gamma_{L_-}(x) = \langle \emptyset \mid$
  - commutation relations between Γ's

we obtain the formula for the partition function.

- To localise particles/holes/dimers on RYGs, we express constrained partition functions as a product of Γ and ψ operators.

- Using Wick's formula, express n-point correlations for dimers as the determinant of a matrix, whose entries are 2-point correlation functions.

- To evaluate these 2-point correlations, use commutation relations between the Γ's and ψ's.

6 Summary and perspectives

Summary

- RYG model provides a dimer realisation of a general Schur process.

- It is a common generalisation of skew plane partitions, the Aztec diamond and steep tilings.

- Using a transfer-matrix approach and dimer-localising operators, we can compute the partition function and the determinantal correlation kernel.

Open questions

- Using the sampling algorithm studied by Betea-Boutillier-Bouttier-Chapuy-Corteel-Vuletić, we observe some limit shapes. Can we compute them ?

- Study more general weights ?

- Study more general graphs ?

THANK YOU !