# Points and lines configurations for perpendicular bisectors of convex cyclic polygons

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Joint work with:

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Applied Probability Workshop 2020
Online / Novosibirsk State University

# 1 Combinatorial results

# Perpendicular bisectors of 3 points

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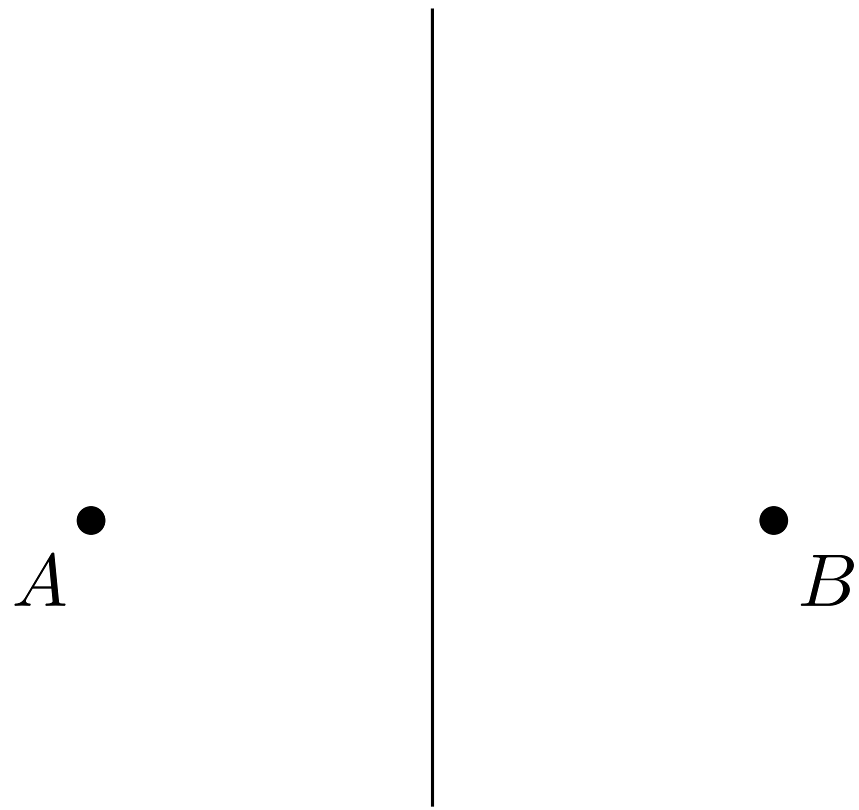
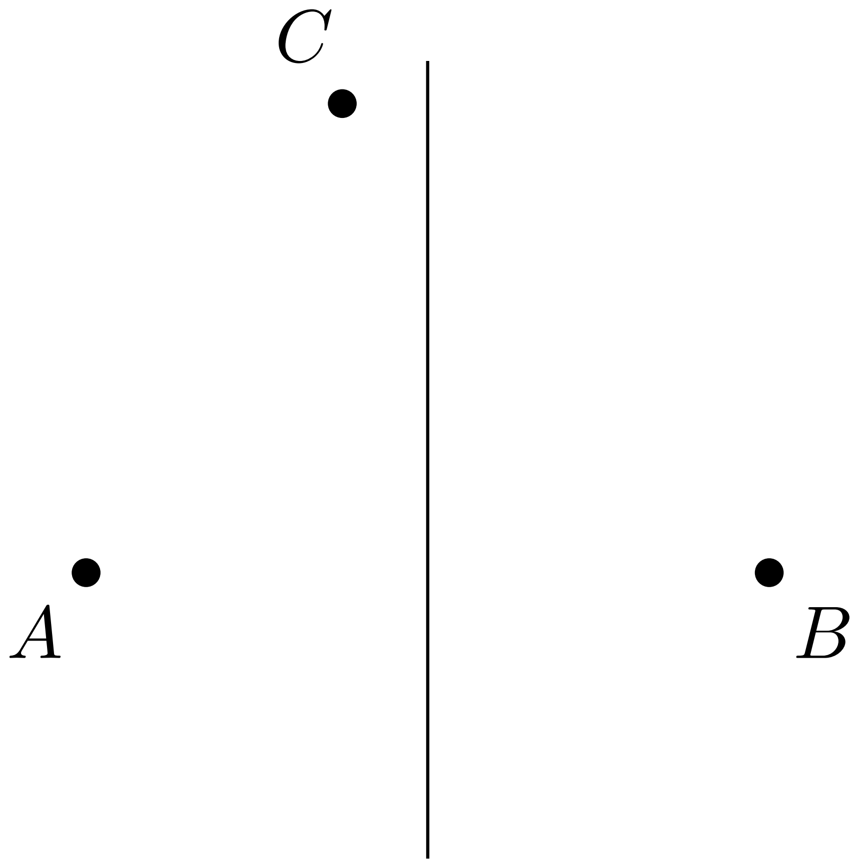


# Perpendicular bisectors of 3 points



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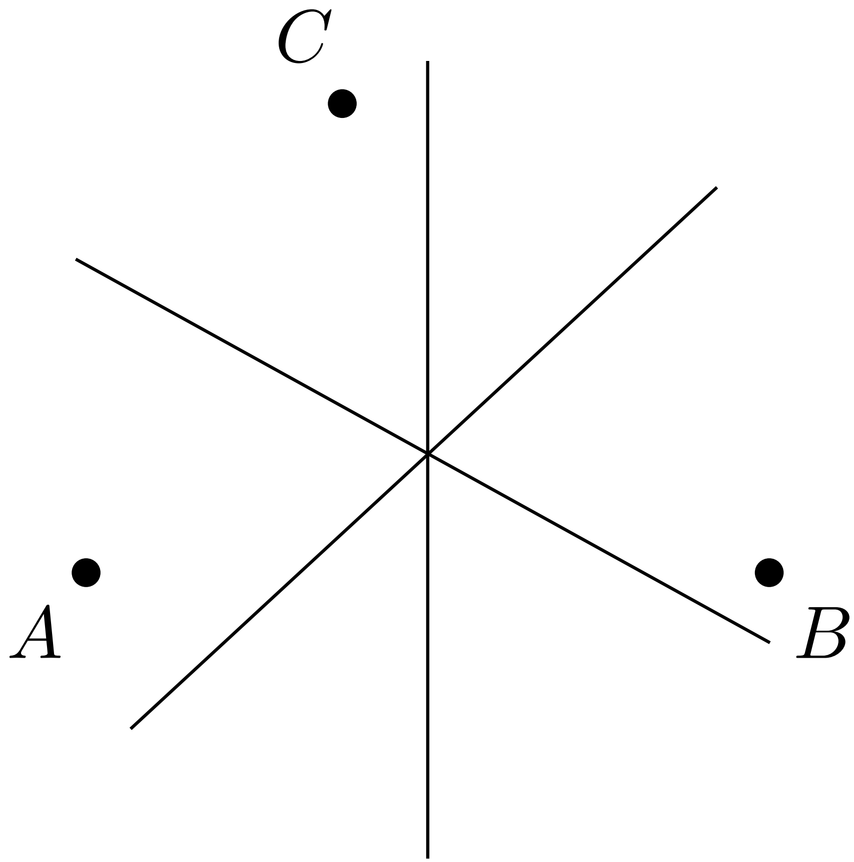

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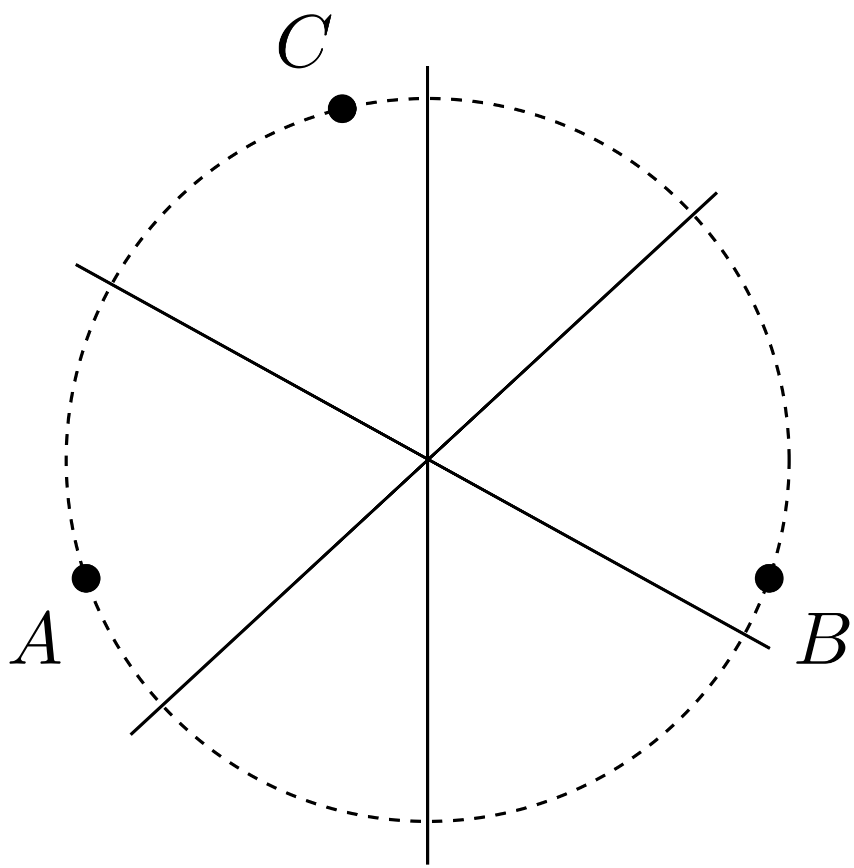

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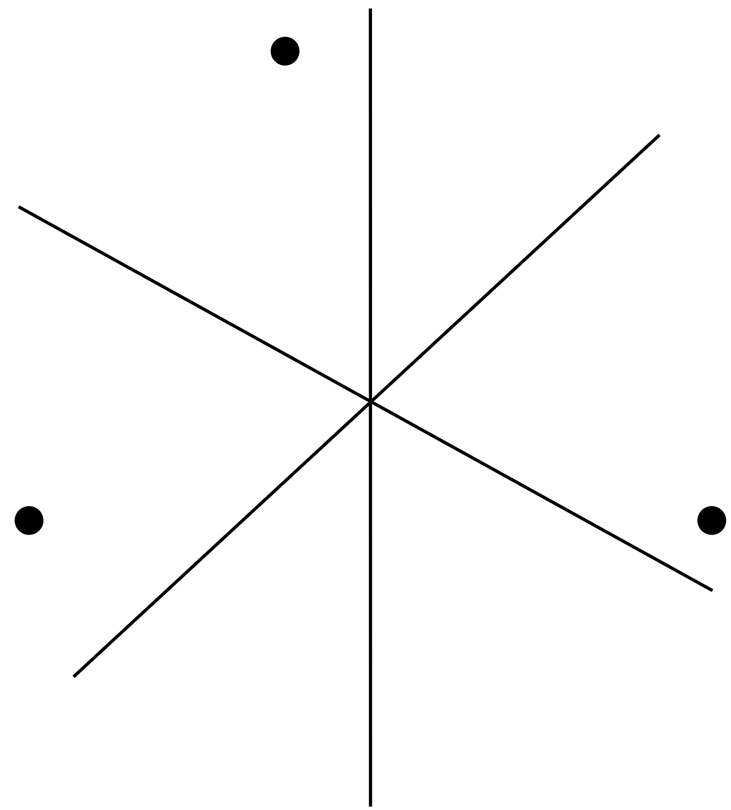
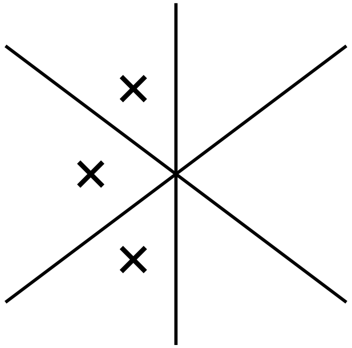
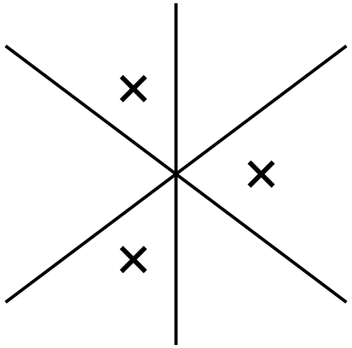
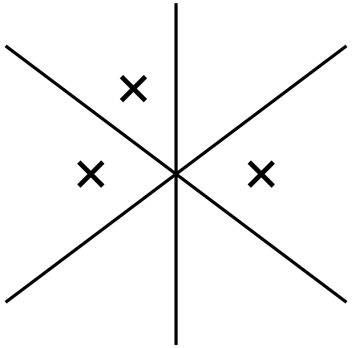


Occupancy word $(1, 0, 1, 1, 0, 0)$

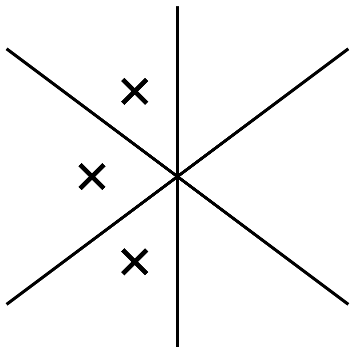
- What occupancy words are realizable ?

- The set of realizable words is stable under cyclic shift and mirror image.

- Eg $(1,0,1,1,0,0)$ realizable implies $(0,1,0,1,1,0)$ and $(0,0,1,1,0,1)$ realizable.

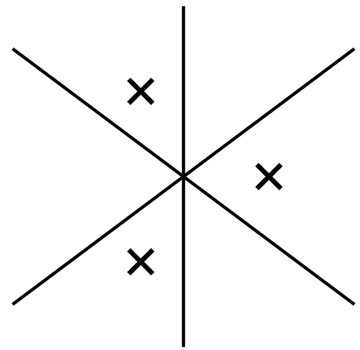
  

$(0, 0, 1, 1, 1, 0)$ $(1, 0, 1, 0, 1, 0)$ $(1, 0, 1, 1, 0, 0)$



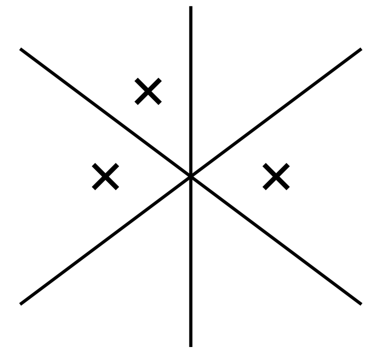
$(0,0,1,1,1,0)$

not realizable



$(1,0,1,0,1,0)$

?



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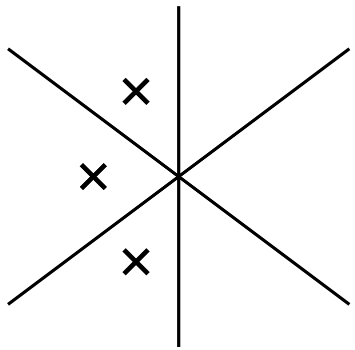

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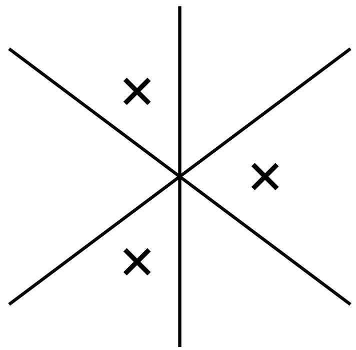

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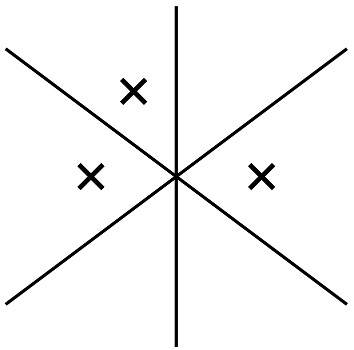
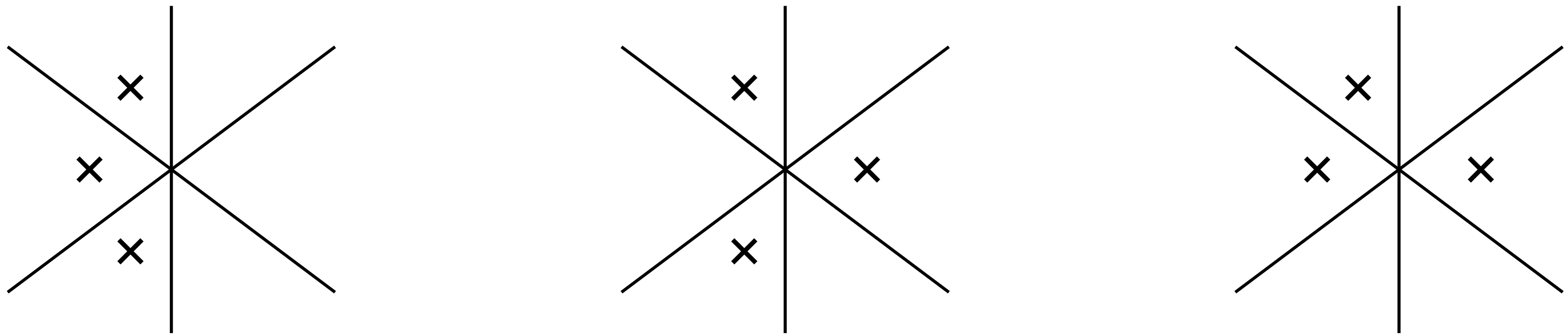


$(1, 0, 1, 1, 0, 0)$

realizable

**Theorem** (Melotti-R.-Thévenin, 2020)**.** *Up to cyclic shifts and reversal, the only realizable occupancy word for* $3$ *points is* $(1, 0, 1, 1, 0, 0)$.

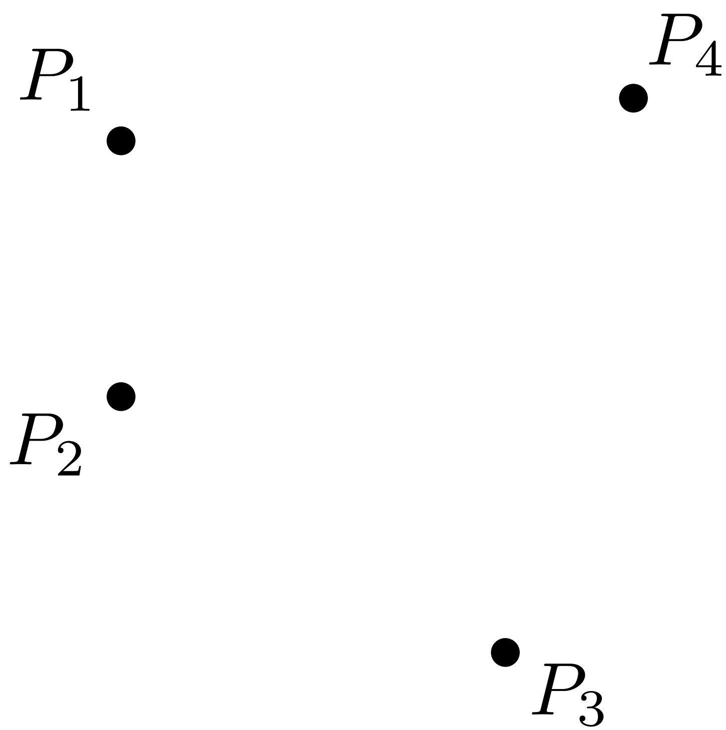


$(0,0,1,1,1,0)$ — not realizable | $(1,0,1,0,1,0)$ — not realizable | $(1,0,1,1,0,0)$ — realizable

earlier reference welcome

**Theorem** (Melotti-R.-Thévenin, 2020)**.** *Up to cyclic shifts and reversal, the only realizable occupancy word for* 3 *points is* $(1,0,1,1,0,0)$.

# Generalization to $n > 3$ points ?

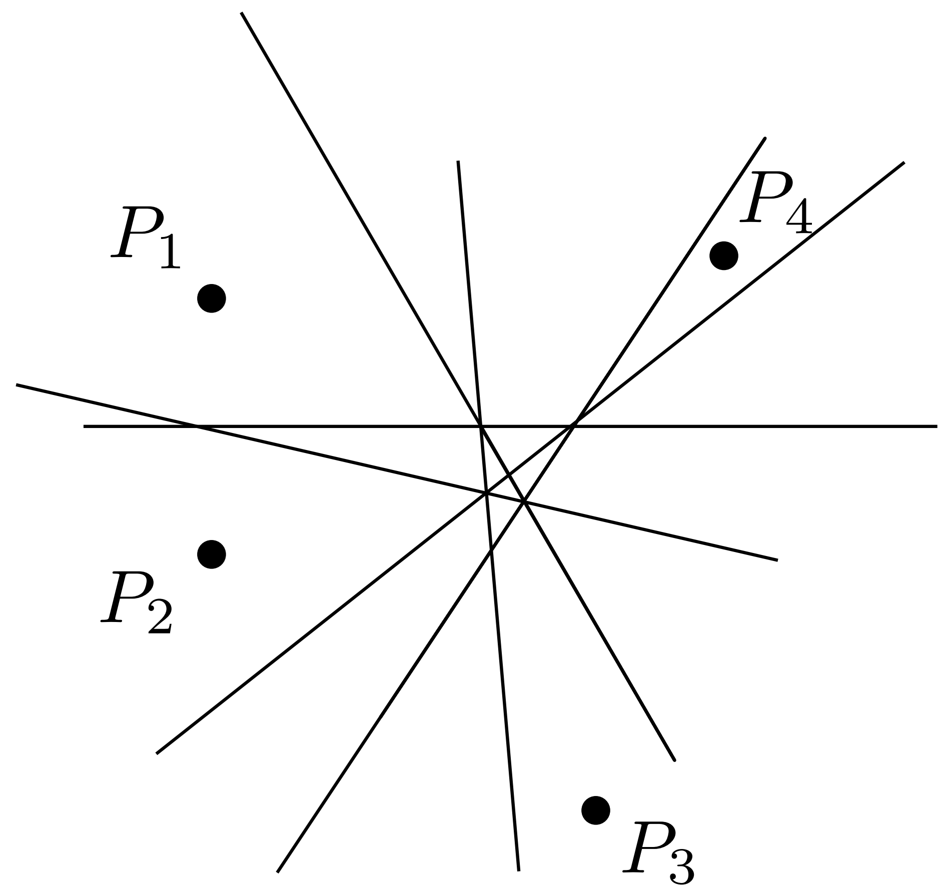


# Generalization to $n > 3$ points ?

Draw all $N = \binom{n}{2}$ perpendicular bisectors

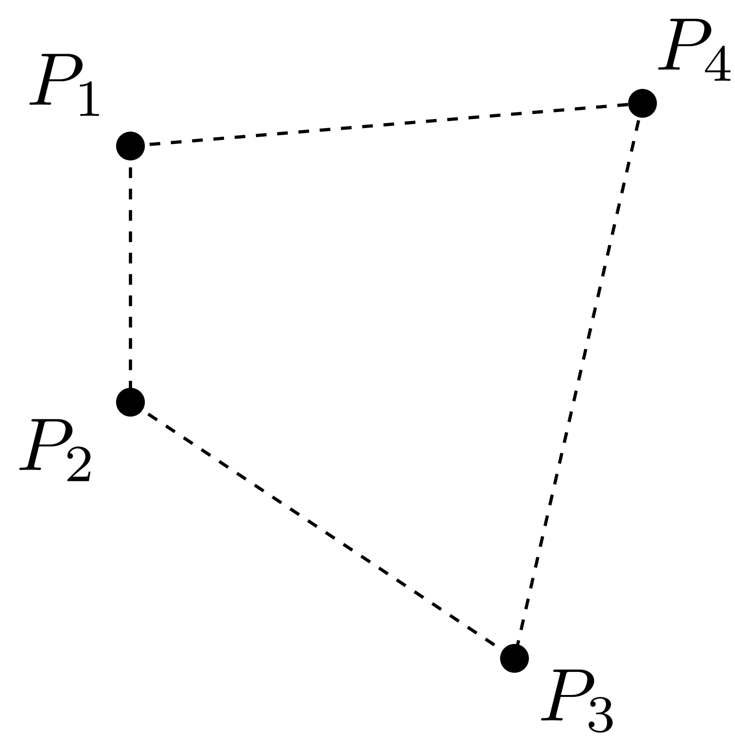
$N$ lines

$\binom{N+1}{2} + 1$ regions



# Generalization to $n > 3$ points ?

Draw the $n$ perpendicular bisectors of the $n$-gon



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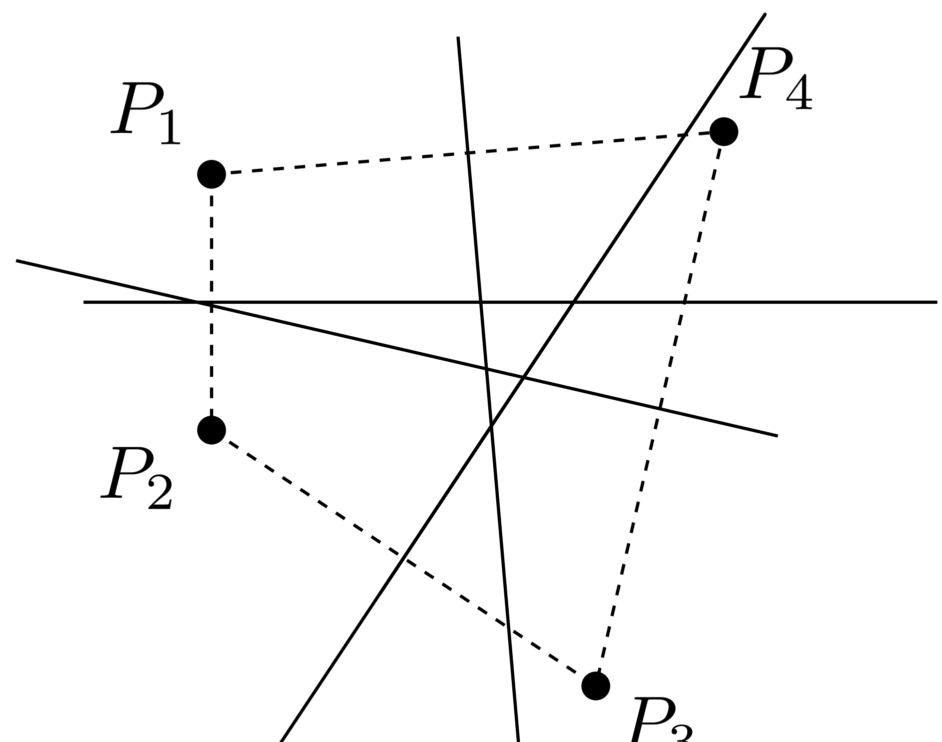
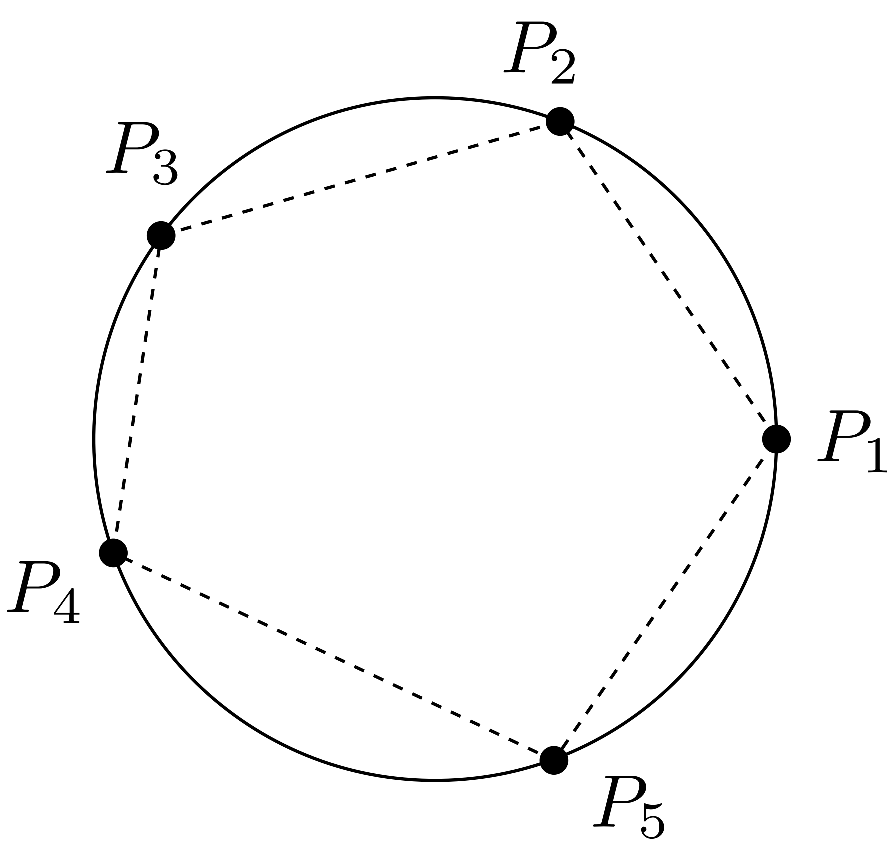


- Draw the $n$ perpendicular bisectors of a convex cyclic $n$-gon: $n$ empty regions and $n$ occupied regions.



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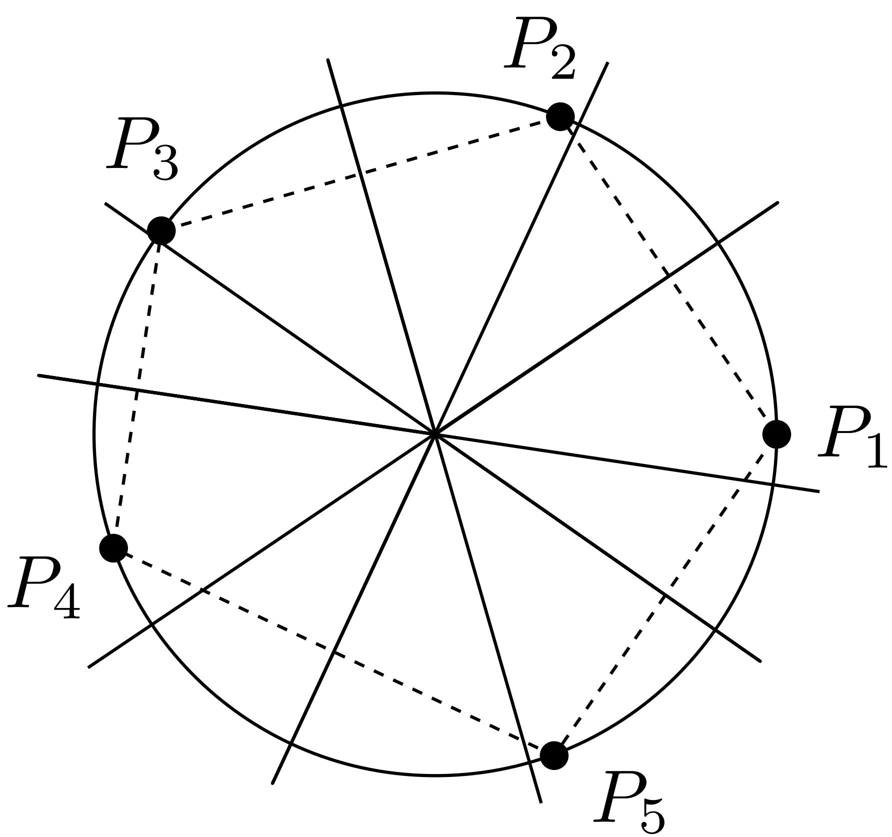
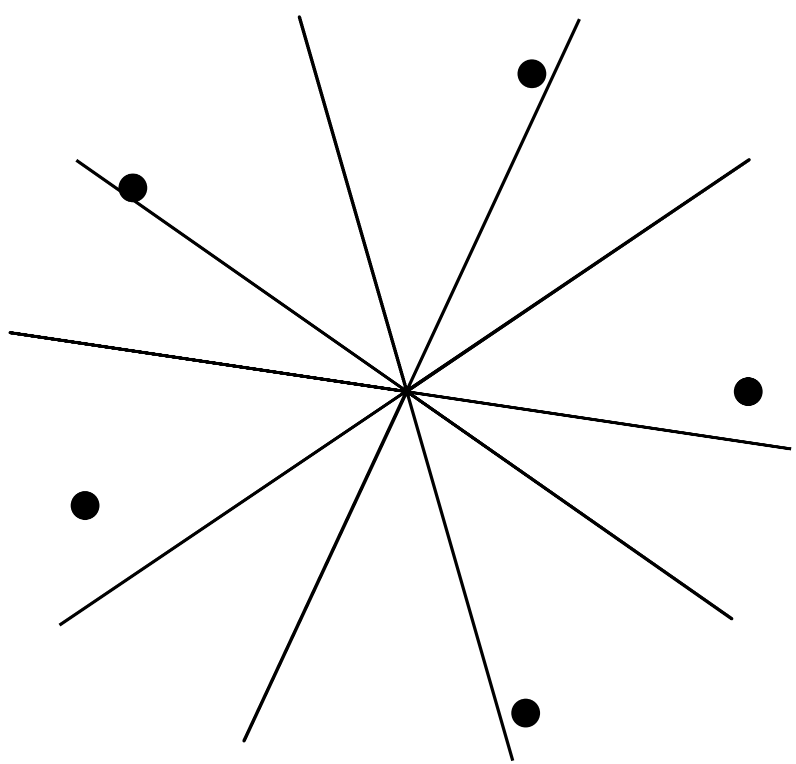


- Draw the $n$ perpendicular bisectors of a convex cyclic $n$-gon: $n$ empty regions and $n$ occupied regions.

- What occupancy words are realizable ?



occupancy word

$$(1, 0, 1, 1, 0, 1, 0, 0, 1, 0)$$

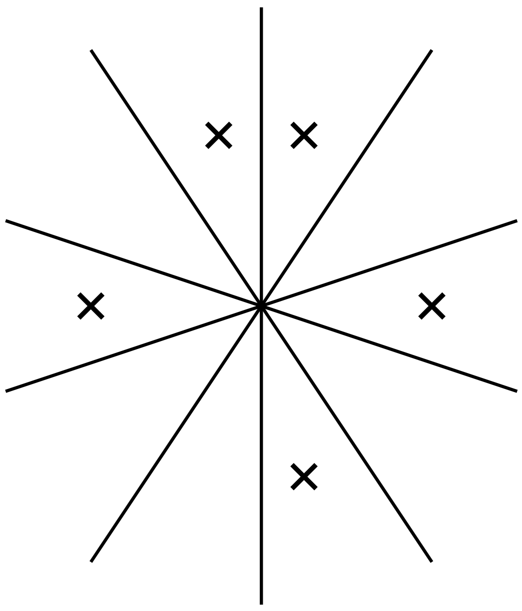
- The *signature* of a word composed of $n$ zeros and $n$ ones is obtained by adding the first half of the word to the second half.

- The signature is *interlacing* if, after erasing the 1's, it is non-empty and the 0's and the 2's alternate.

**Theorem** (Melotti-R.-Thévenin). *A word composed of $n$ zeros and $n$ ones arises as the occupancy word for a points and lines configuration of perpendicular bisectors of a convex cyclic $n$-gon if and only if its signature is interlacing.*

# Examples

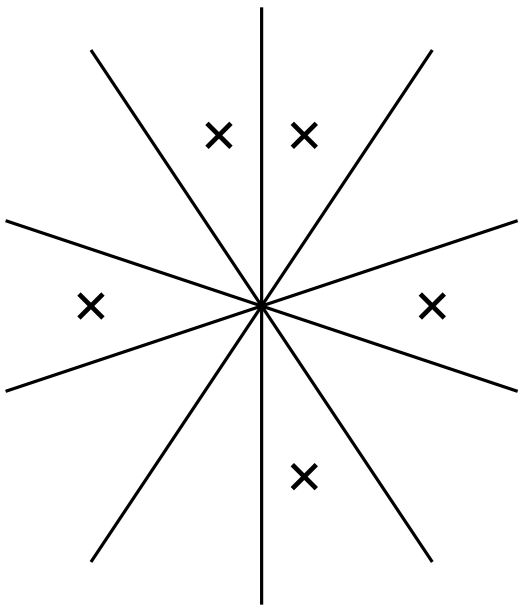
word $(1, 0, 1, 1, 0, 1, 0, 0, 1, 0)$



# Examples

word $(1, 0, 1, 1, 0, \mid 1, 0, 0, 1, 0)$

signature $(2, 0, 1, 2, 0)$

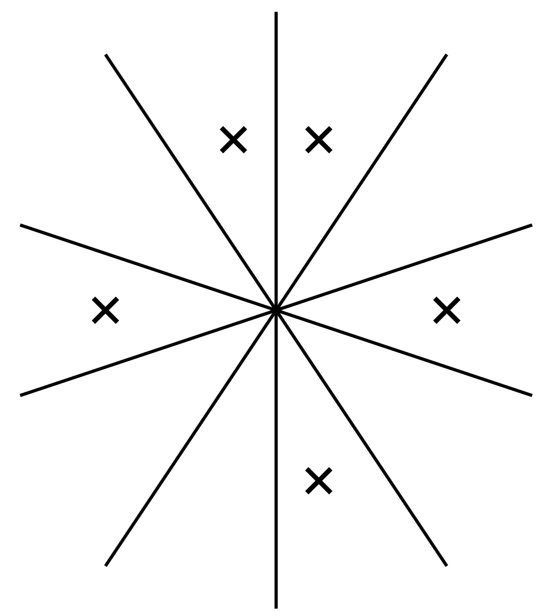


# Examples

word $(1, 0, 1, 1, 0, \Big| 1, 0, 0, 1, 0)$

signature $(2, 0, \quad , 2, 0)$

interlacing signature $\Rightarrow$ realizable word



# Examples

Green: the region and its opposite region are both occupied.

Red: the region and its opposite region are both empty.

Uncolored otherwise.

word $(1, 0, 1, 1, 0, | 1, 0, 0, 1, 0)$

signature $(2, 0, \quad , 2, 0)$

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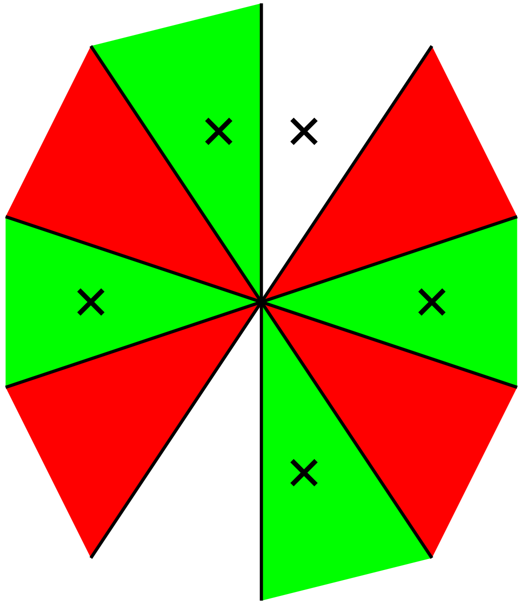


# Examples

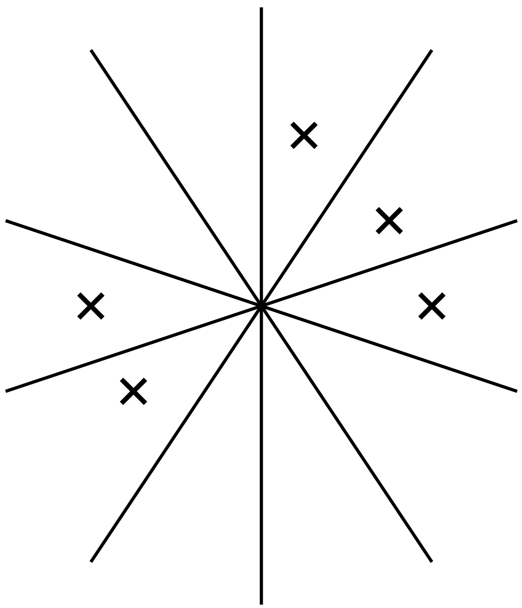
Green: the region and its opposite region are both occupied.

Red: the region and its opposite region are both empty.

Uncolored otherwise.

word $(1, 1, 1, 0, 0, 1, 1, 0, 0, 0)$

signature $(2, 2, 1, 0, 0)$



# Examples

Green: the region and its opposite region are both occupied.

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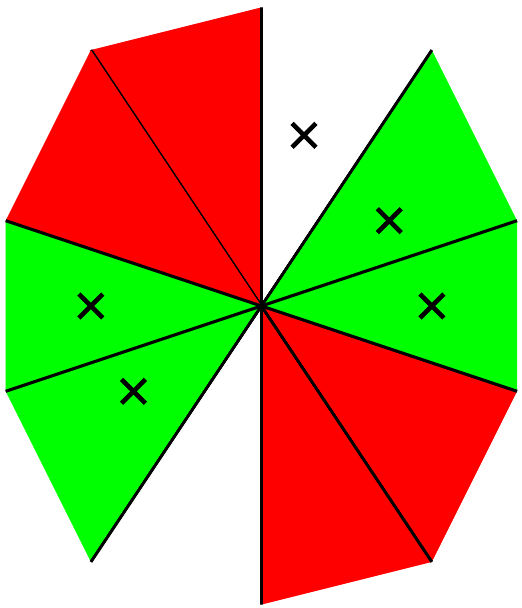
Uncolored otherwise.

word $(1, 1, 1, 0, 0, 1, 1, 0, 0, 0)$

signature $(2, 2, \ \ , 0, 0)$

word not realizable



# Examples

Green: the region and its opposite region are both occupied.

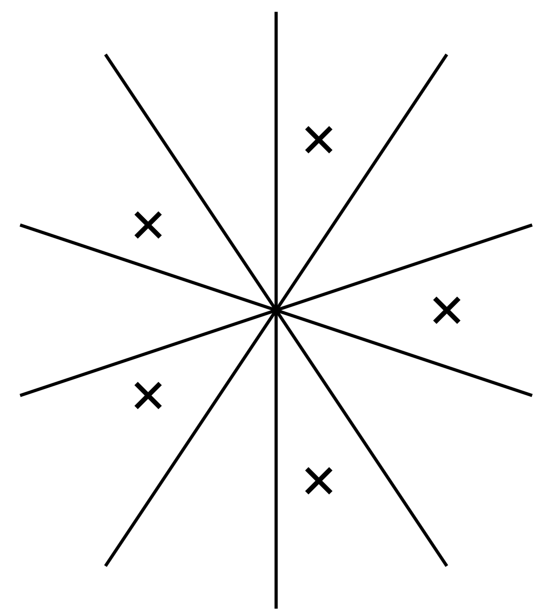
Red: the region and its opposite region are both empty.

Uncolored otherwise.

word $(1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$

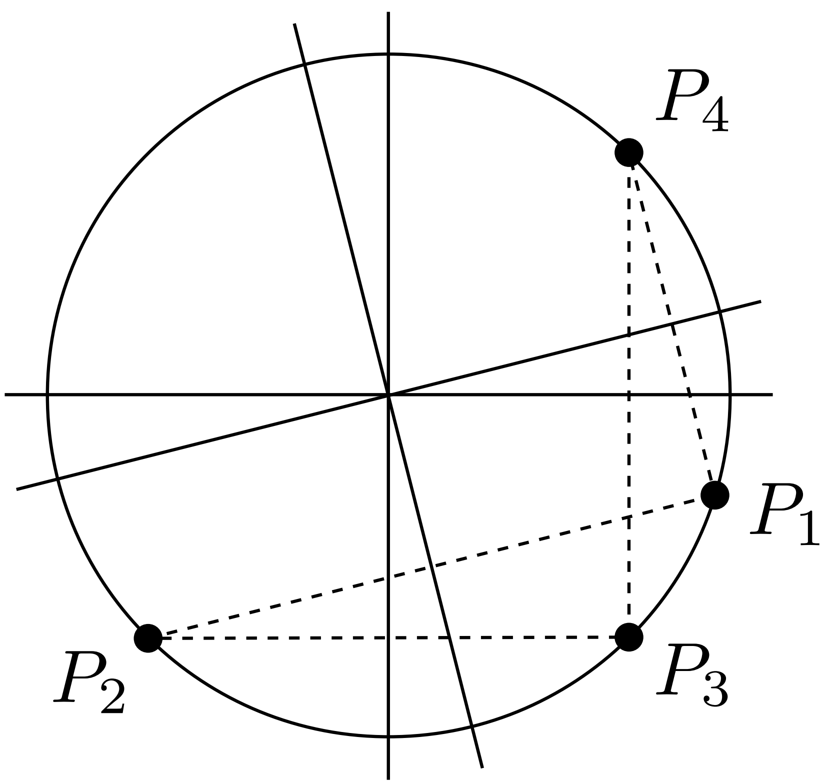
signature $(1, 1, 1, 1, 1)$

word not realizable



- Open question: characterize the occupancy words arising from a cyclic (not necessarily convex) polygon.

- This time the entries of the word may be larger than 1.




occupancy word

$(0, 1, 0, 0, 0, 1, 0, 2)$

# Realizable bracelets

- A bracelet is an equivalence class of words up to cyclic shifts and reversal.

- Denote by $B_n$ the number of bracelets composed of $n$ ones and $n$ zeros that can be realized as an occupancy word constructed from a convex cyclic polygon.

**Corollary.**

$$\lim_{n\to\infty} \frac{1}{n} \log B_n = 3.$$

# 2 Probabilistic results

# Model 1: uniformly random points

- Fix $n \geq 3$. Sample $n$ iid uniform random points on the circle and draw the perpendicular bisectors of the associated convex cyclic $n$-gon.

**Proposition.** *The probability of achieving a given bracelet is a rational number.*

- For example, for $n$ points, the probability of the bracelet $(1, 0, 1, \ldots, 1, 0, \ldots, 0)$ is $\frac{n}{3.2^{2n-6}}$.

# Number of regions of each color

- Denote respectively by $H_g$, $H_r$ and $H_u$ the number of green, red and uncolored regions.

- We have $H_r = H_g$ and $H_u = 2n - 2H_g$.

**Theorem** (Melotti-R.-Thévenin)**.**

$$\mathbb{E}[H_g] = \frac{n}{2}\left(1 + \frac{1}{3^{n-2}}\right).$$

# Fraction of space for each color

- Denote respectively by $F_g$, $F_r$ and $F_u$ the fraction of space covered by the green, red and uncolored regions.

**Theorem** (Melotti-R.-Thévenin)**.**

$$\mathbb{E}[F_g] = \frac{3^n + 2n + 11}{8.3^{n-1}}$$

$$\mathbb{E}[F_r] = \frac{3^{n-1} + 2n - 7}{8.3^{n-1}}$$

$$\mathbb{E}[F_u] = \frac{3^{n-1} - n - 1}{2.3^{n-1}}.$$

**Corollary.** *Denoting by $F_e$ the fraction of space covered by empty regions, we have*

$$\mathbb{E}[F_e] = \frac{3}{8}\left(1 - \frac{1}{3^{n-2}}\right).$$

- All the above exact formulas for fixed $n$ are relatively simple but their proofs are very computational.

- Can one find simpler and more conceptual proofs providing a nice interpretation of each summand ?

- Green, red and uncolored regions are asymptotically equidistributed around the circle, whether we consider their cardinality or their size.

- For $t \in [0,1]$, denote by $h_{g,n}(t)$, $h_{r,n}(t)$ and $h_{u,n}(t)$ the number of green, red and uncolored regions contained entirely between the angles 0 and $2\pi t$.

- For $t \in [0,1]$, denote by $f_{g,n}(t)$, $f_{r,n}(t)$ and $f_{u,n}(t)$ the angle (divided by $2\pi$) covered by green, red and uncolored regions contained entirely between the angles 0 and $2\pi t$.

**Theorem** (Melotti-R.-Thévenin)**.** *We have the following functional convergences:*

$$\left(\frac{h_{g,n}(t)}{2n}, \frac{h_{r,n}(t)}{2n}, \frac{h_{u,n}(t)}{2n}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} \left(\frac{t}{4}, \frac{t}{4}, \frac{t}{2}\right)_{0\leq t\leq 1}$$

$$(f_{g,n}(t), f_{r,n}(t), f_{u,n}(t))_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} \left(\frac{3t}{8}, \frac{t}{8}, \frac{t}{2}\right)_{0\leq t\leq 1}.$$

# Model 2: uniformly random bracelet

- Fix $n \geq 3$ and among all $B_n$ realizable bracelets, pick one uniformly at random.

- The colors are again asymptotically equidistributed, but with a different distribution than for the uniform random points model.

- For $x \in [0, n]$, denote by $K_{g,x}$ the number of occurrences of the letter 2 between positions 1 and $\lfloor x \rfloor$ in the signature of a uniformly random bracelet.

**Theorem** (Melotti-R.-Thévenin)**.** *Denote by $W$ the standard Brownian motion. We have the functional convergence*

$$3\sqrt{\frac{2}{n}}\left(K_{g,cn} - \frac{cn}{6}\right)_{0 \le c \le 1} \xrightarrow[n\to\infty]{(d)} (W_c)_{0\le c\le 1}\,.$$

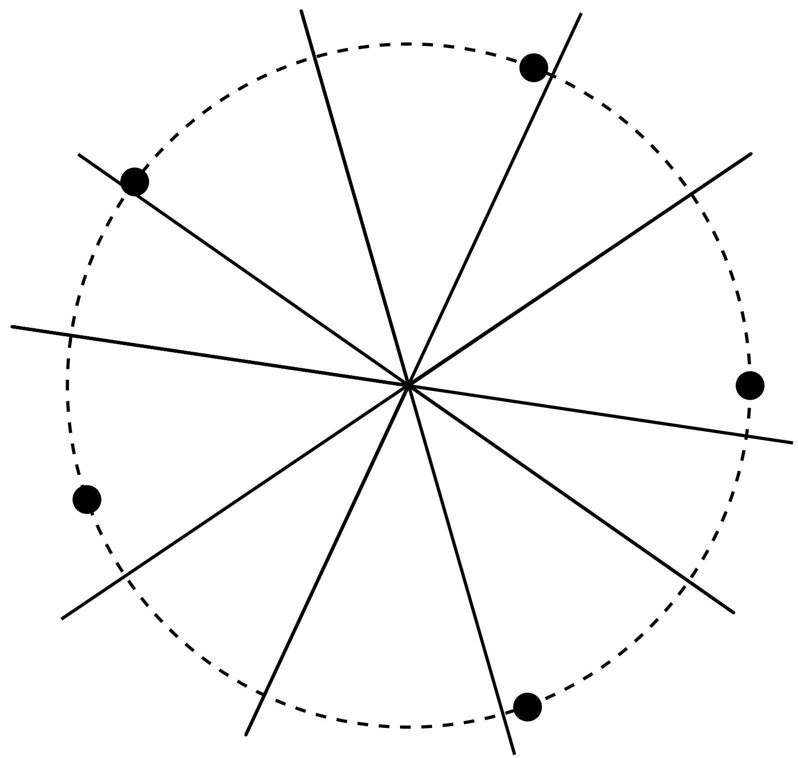
# 3 Proof ingredients

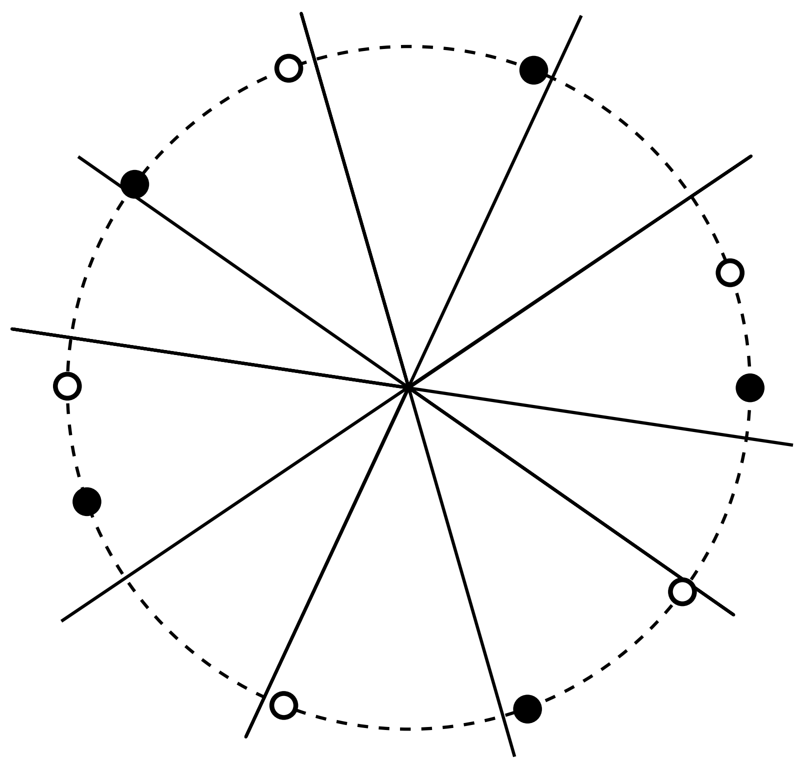
# Black and white dots

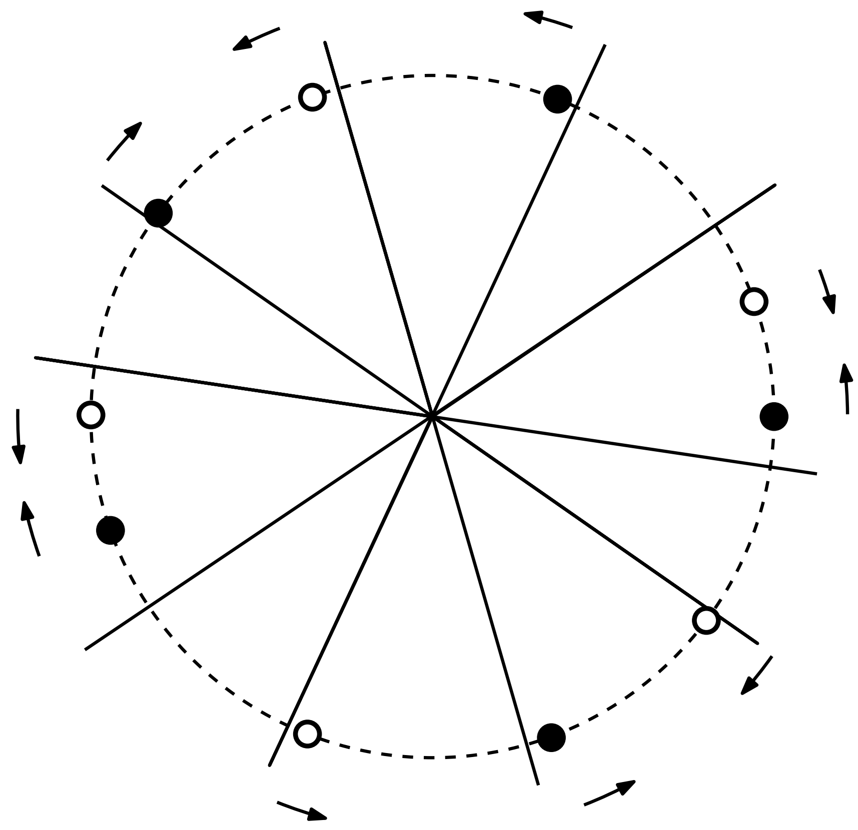
- The $n$ points on the circle are depicted by black dots. Add a white dot at the antipodal position of each black dot.

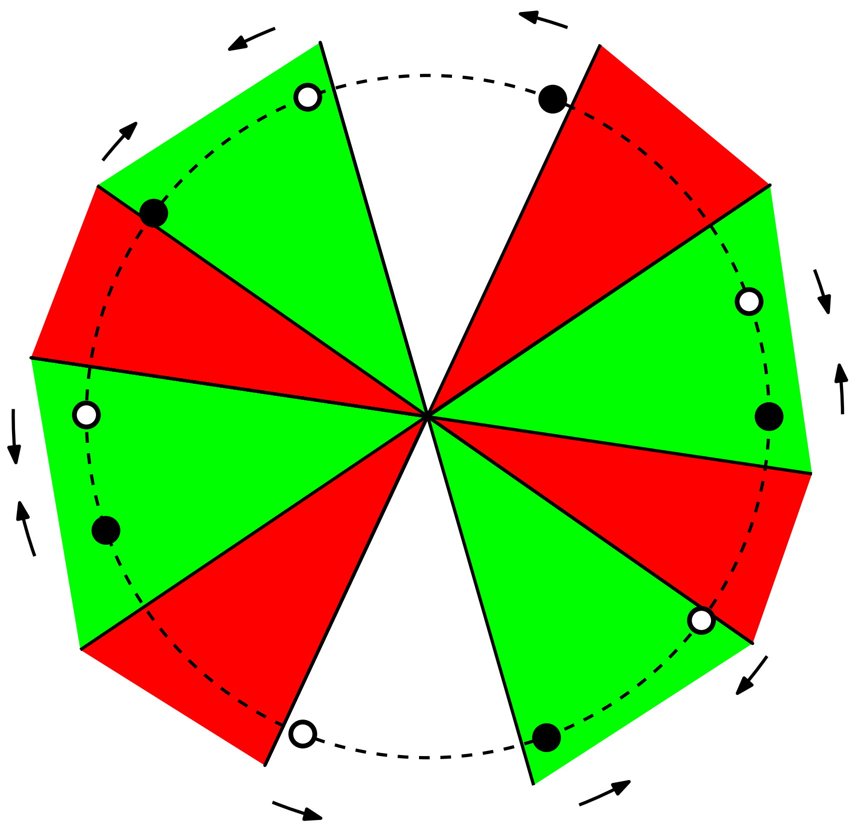
- Attach to each dot an arrow pointing in the direction of the dot of the opposite color which is the closest.

- The sequence of arrows determines the sequence of colors of the regions.





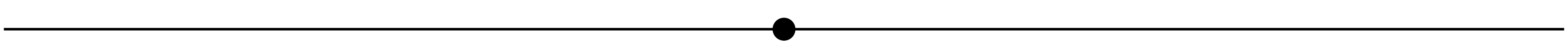




# Exponential spacings for model 1

- Represent the upper semi-circle by an interval, containing $n$ iid uniform random dots, with iid colors.

- Making the spacings between two consecutive dots iid exponential variables makes the total length of the interval random, but allows for a *local* construction of arrows, hence of colors of regions.



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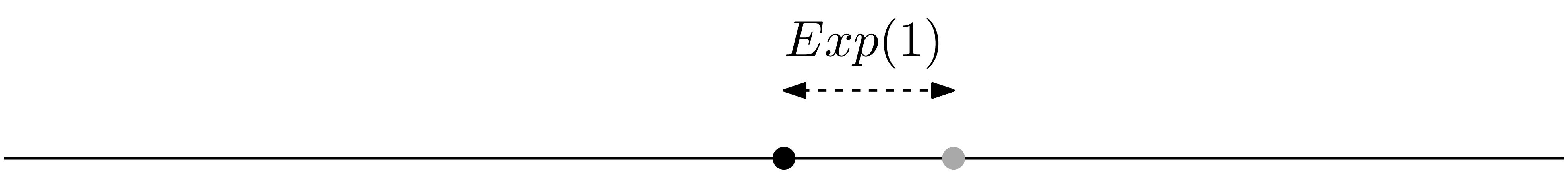

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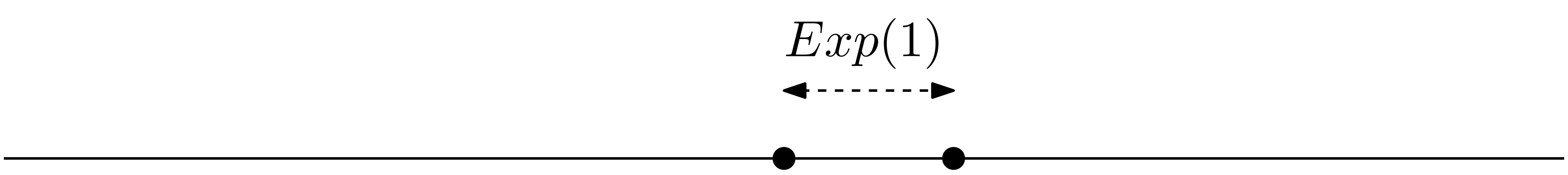

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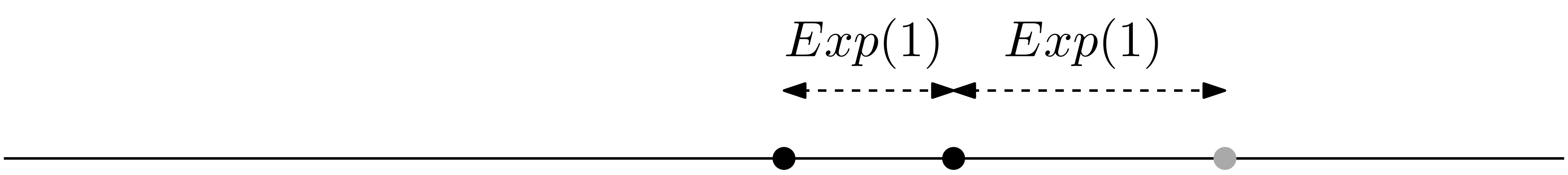

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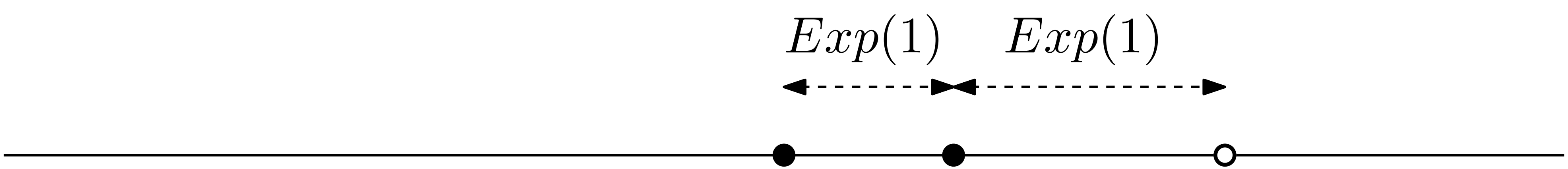

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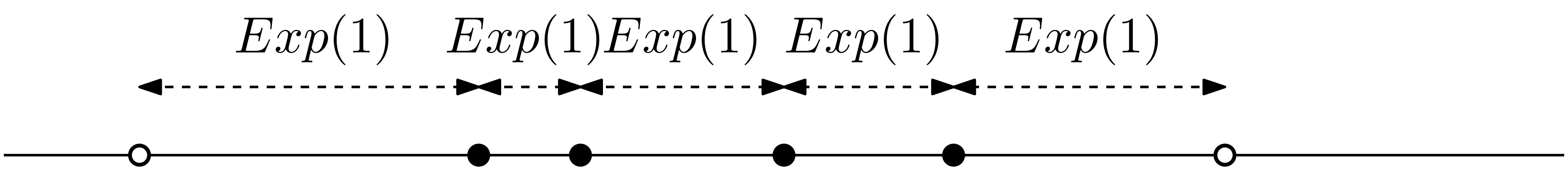

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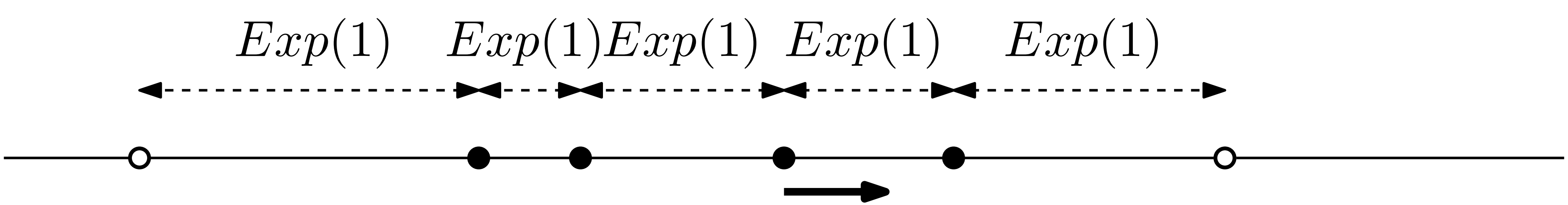

# Walk associated to a uniform word

- Associate to a realizable word a walk in $\mathbb{Z}^2$.

$$(1, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0)$$

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- Associate to a realizable word a walk in $\mathbb{Z}^2$.

$$(1, 1, 0, 0, 0, 1, \big| 0, 1, 1, 1, 0, 0)$$

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{matrix}$$

# Walk associated to a uniform word

- Associate to a realizable word a walk in $\mathbb{Z}^2$.
- Left for $\begin{smallmatrix}0\\1\end{smallmatrix}$, right for $\begin{smallmatrix}1\\0\end{smallmatrix}$ and up for $\begin{smallmatrix}0\\0\end{smallmatrix}$ or $\begin{smallmatrix}1\\1\end{smallmatrix}$.

$$(1, 1, 0, 0, 0, 1, \big| 0, 1, 1, 1, 0, 0)$$

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$(0, 0)$

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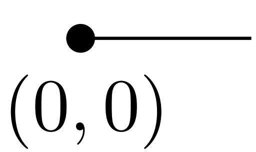
$$(1,1,0,0,0,1,\,\big|\,0,1,1,1,0,0)$$

$$\begin{matrix}1 & 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 1 & 1 & 0 & 0\end{matrix}$$



# Walk associated to a uniform word

- Associate to a realizable word a walk in $\mathbb{Z}^2$.
- Left for $\begin{matrix}0\\1\end{matrix}$, right for $\begin{matrix}1\\0\end{matrix}$ and up for $\begin{matrix}0\\0\end{matrix}$ or $\begin{matrix}1\\1\end{matrix}$.

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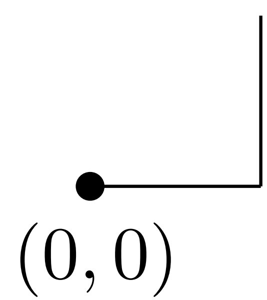
$$\begin{matrix}1 & 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 1 & 1 & 0 & 0\end{matrix}$$



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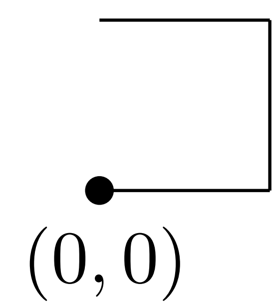
$$(1, 1, 0, 0, 0, 1, \,|\, 0, 1, 1, 1, 0, 0)$$

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{matrix}$$



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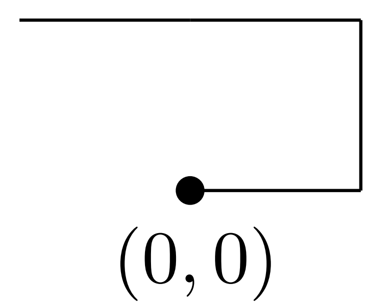

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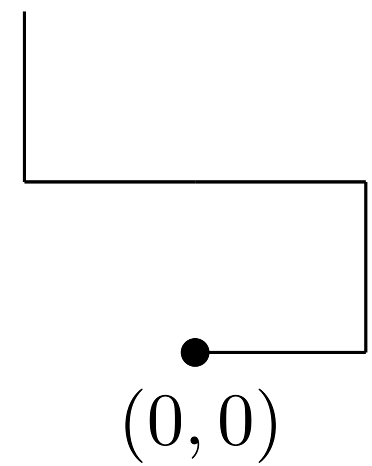

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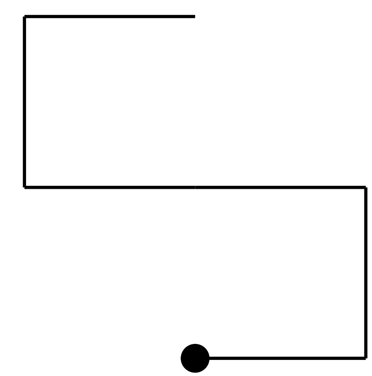

$(0,0)$

- 2-to-1 map from realizable words to walks taking steps left, right or up with a positive even number of up steps.

P. Melotti, S. Ramassamy, P. Thévenin, Points and lines configurations of perpendicular bisectors of convex cyclic polygons, arXiv:2003.11006 (2020).

THANK YOU !