Multiple cluster structures for geometric dynamics

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Joint works with:
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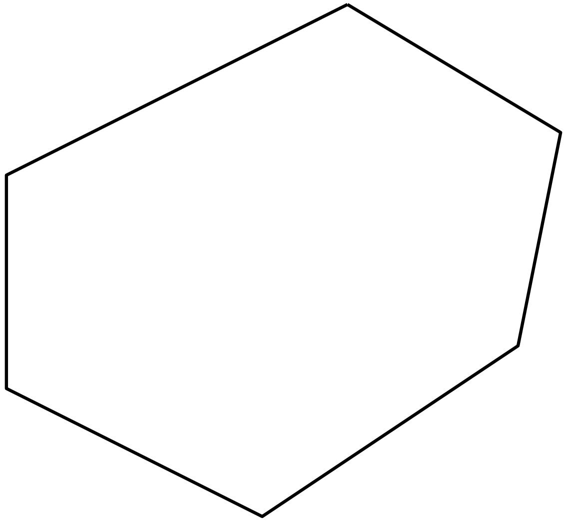
Max Glick (Google)
Pavlo Pylyavskyy (University of Minnesota)

Clusters and geometry online seminar
Yale, April 9 2021

Motivation 1: geometric dynamics

Example: pentagram map
(Schwartz '92)



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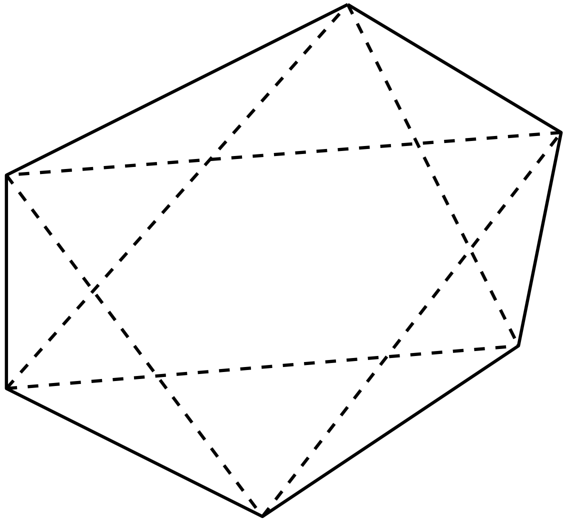

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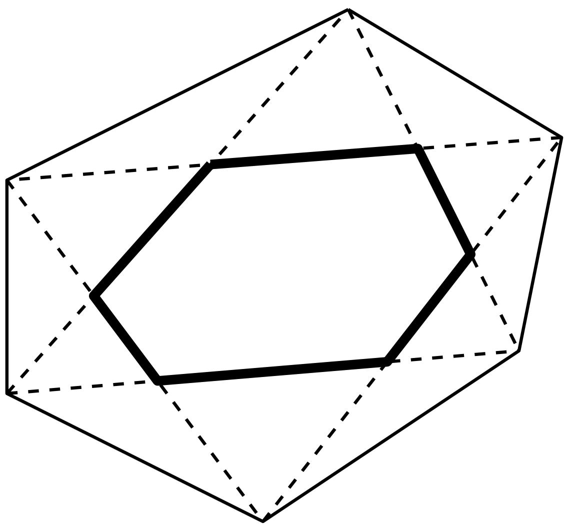

- Integrability and computation of the integrals of motion (Ovsienko-Schwartz-Tabachnikov '10 and Soloviev '13).
- Coordinates that evolve according to cluster algebra mutation rules (Glick '11).

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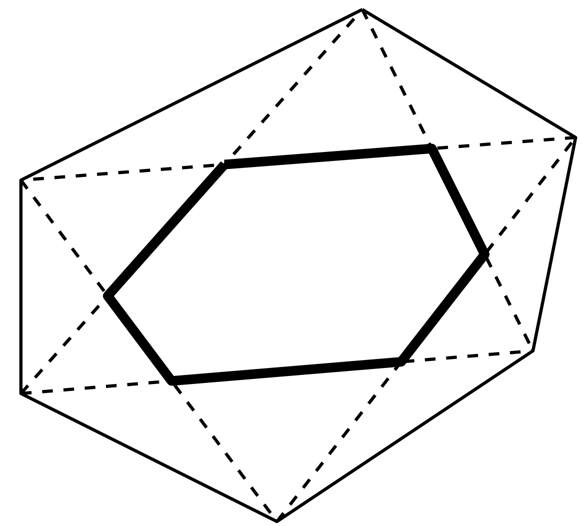


Follows from cluster structure
+
Goncharov-Kenyon '13

- Integrability and computation of the integrals of motion (Ovsienko-Schwartz-Tabachnikov '10 and Soloviev '13).

- Coordinates that evolve according to cluster algebra mutation rules (Glick '11).

Motivation 2: cross-ratios/star-ratios

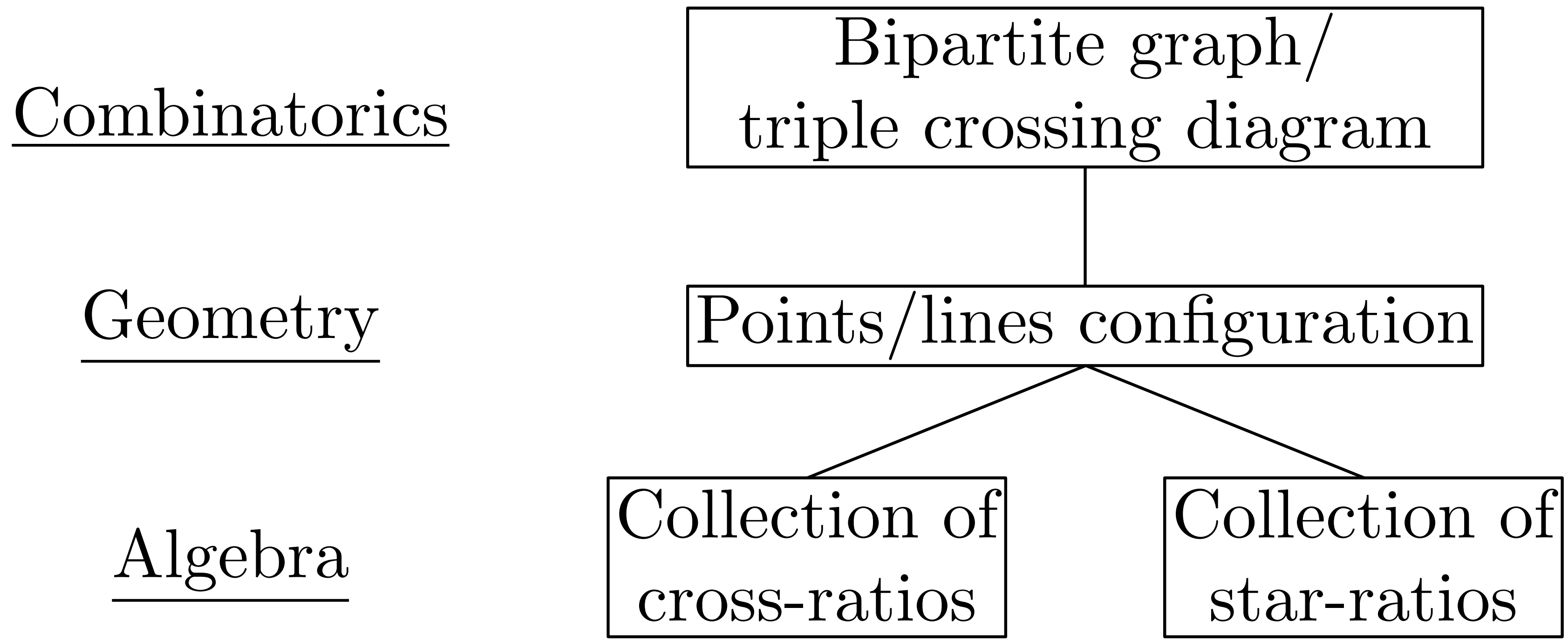
Cross-ratio of 4 points: $\mathrm{cr}(p_1, p_2, p_3, p_4) = \frac{(p_1 - p_2)(p_3 - p_4)}{(p_2 - p_3)(p_4 - p_1)}$

Star-ratio of 5 points: $\mathrm{sr}(p_1, p_2, p_3, p_4; p) = \frac{(p_1 - p)(p_3 - p)}{(p_2 - p)(p_4 - p)}$

- Cluster variables for geometric dynamics were mostly cross-ratios (Fock-Goncharov '06, Glick-Pylyavskyy '16).

- Star-ratio cluster variables were recently discovered for Miquel dynamics (Affolter '18, Kenyon-Lam-R.-Russkikh '18). They also work for the pentagram map.

General framework to encode geometric dynamics yielding their cluster structures (and their integrability):



Geometric dynamics arise as local transformations of the graphs and of the geometric configurations. They induce cluster mutations for both collections of variables.

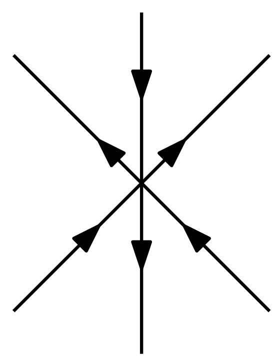
Plan of the talk:

1. TCD maps and their two cluster structures
2. Examples, old and new
3. Operations on TCD maps and more cluster structures

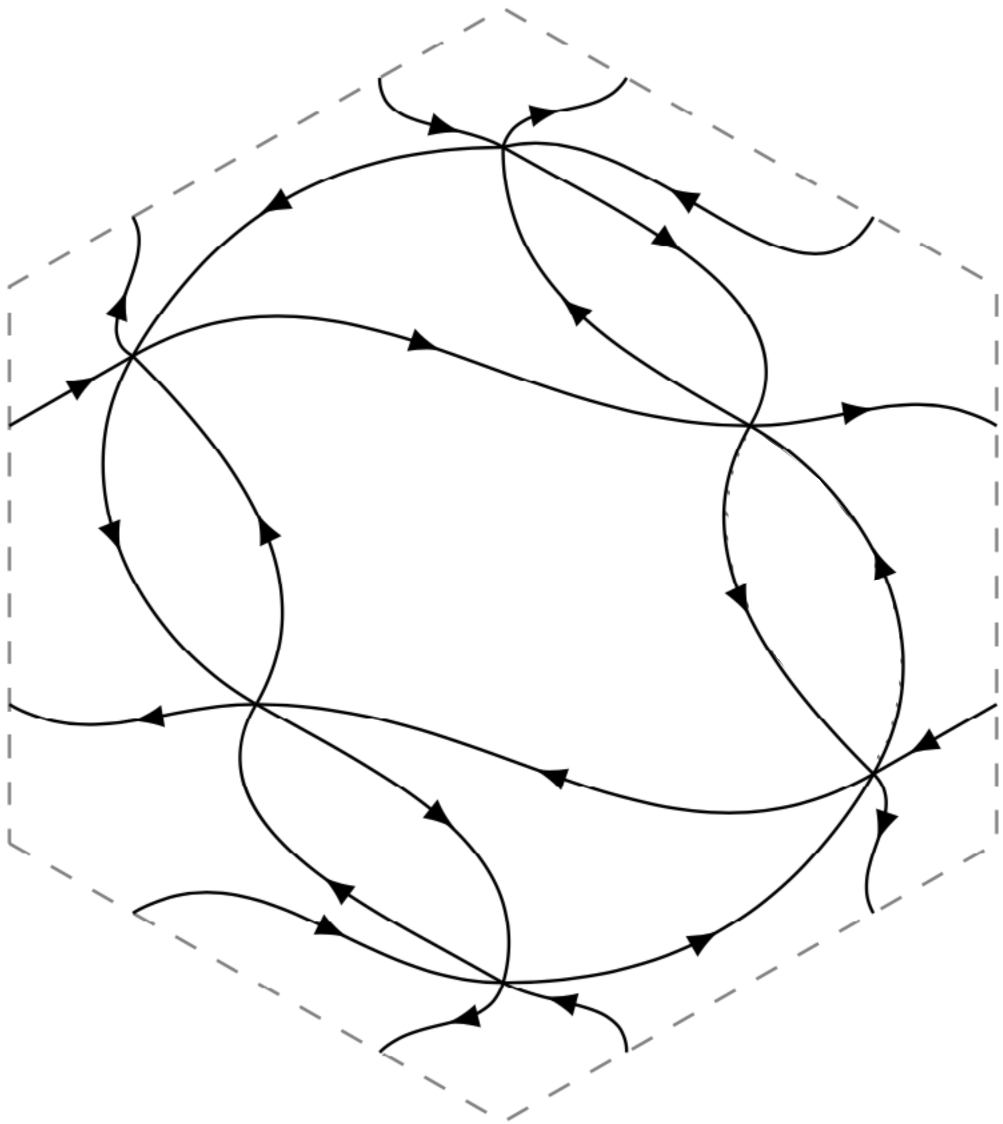
1 TCD maps and their two cluster structures

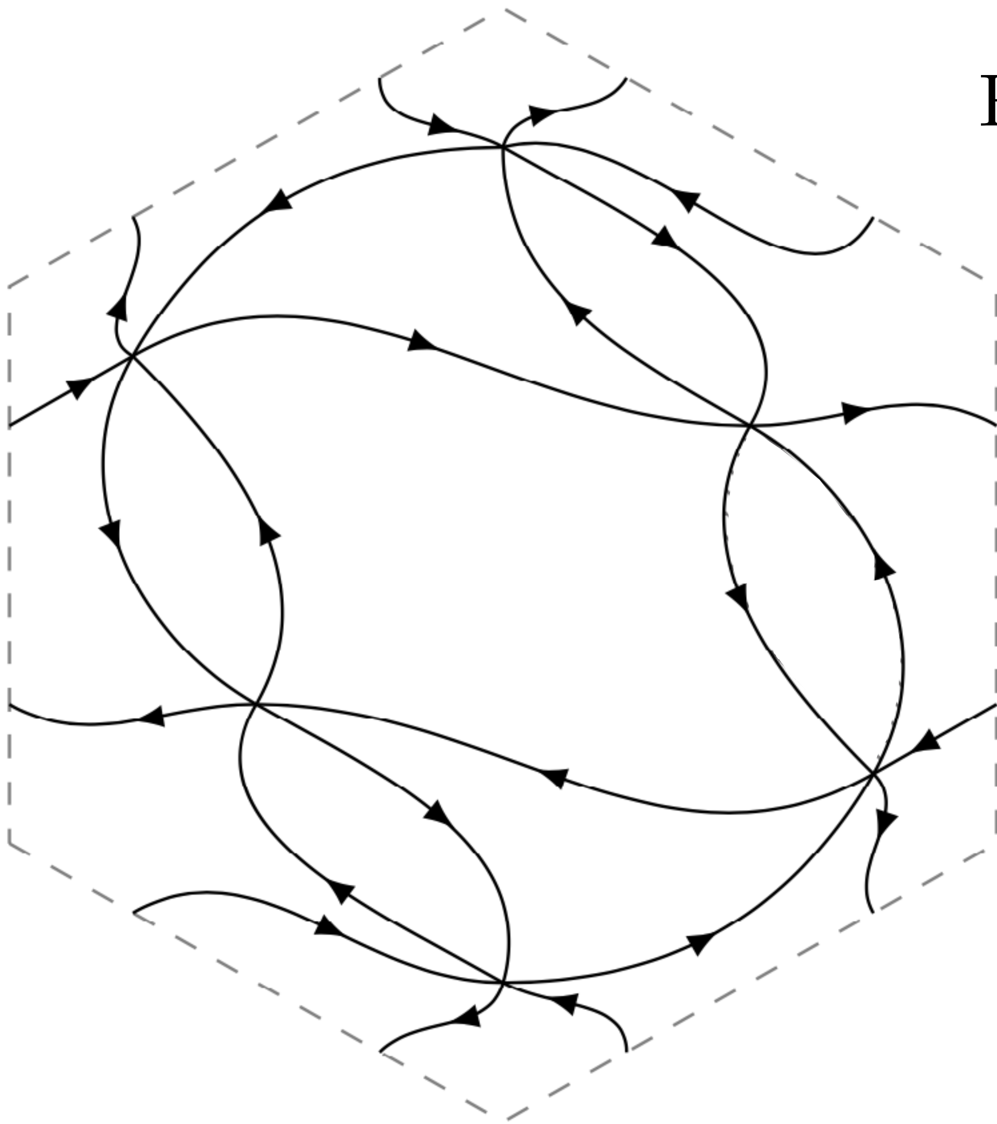
Triple crossing diagrams (TCDs)

- Definition (D. Thurston): collection of directed *strands* which can intersect only 3 at a time alternating in/out.



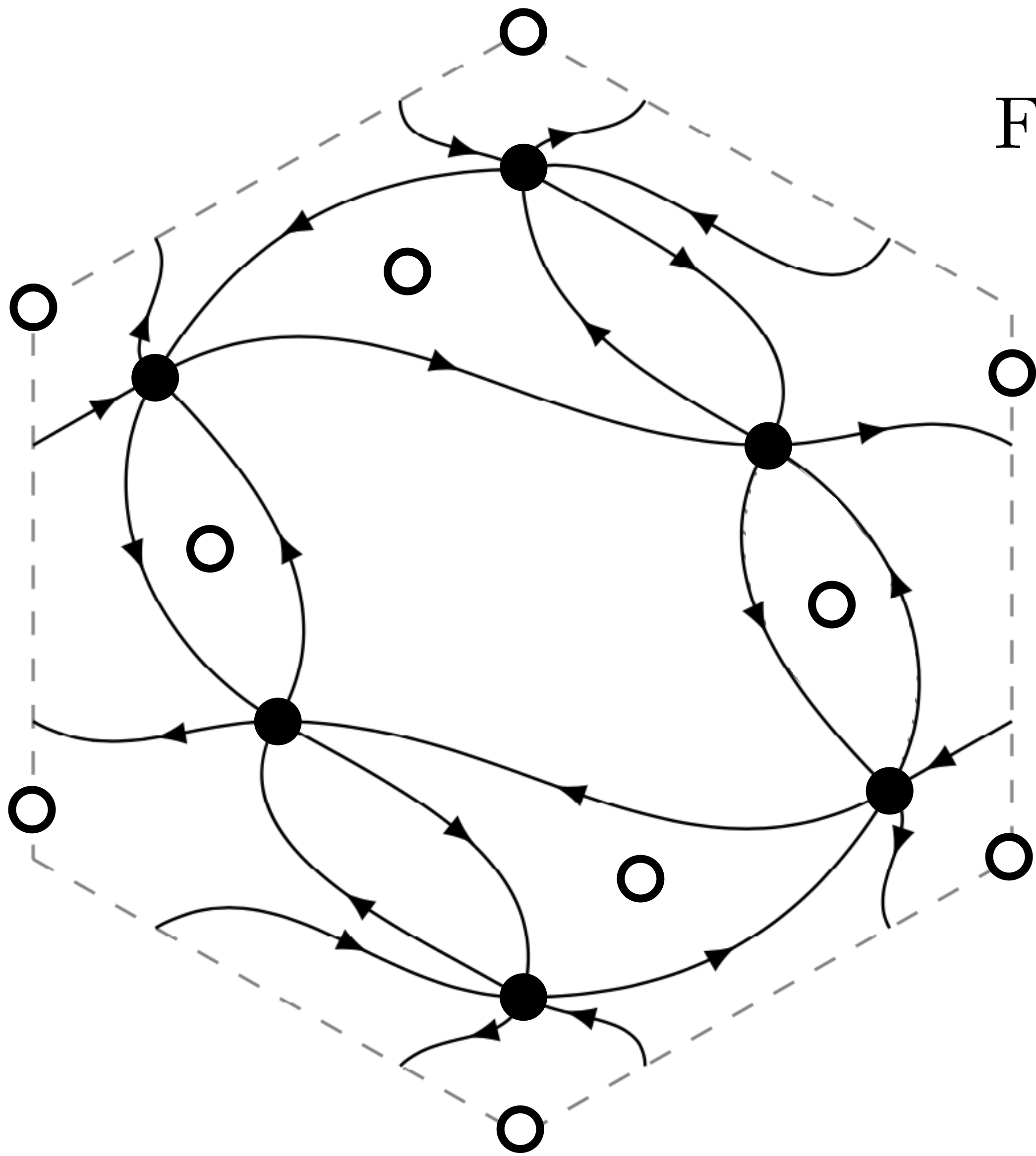
- Each face is oriented either clockwise or counterclockwise.
- May be infinite on the whole plane or inside a disk.





From TCD to bipartite graph:

Place a black vertex at each triple point, a white vertex at each counter-clockwise face. Add an edge whenever a counter-clockwise face is adjacent to a triple point.



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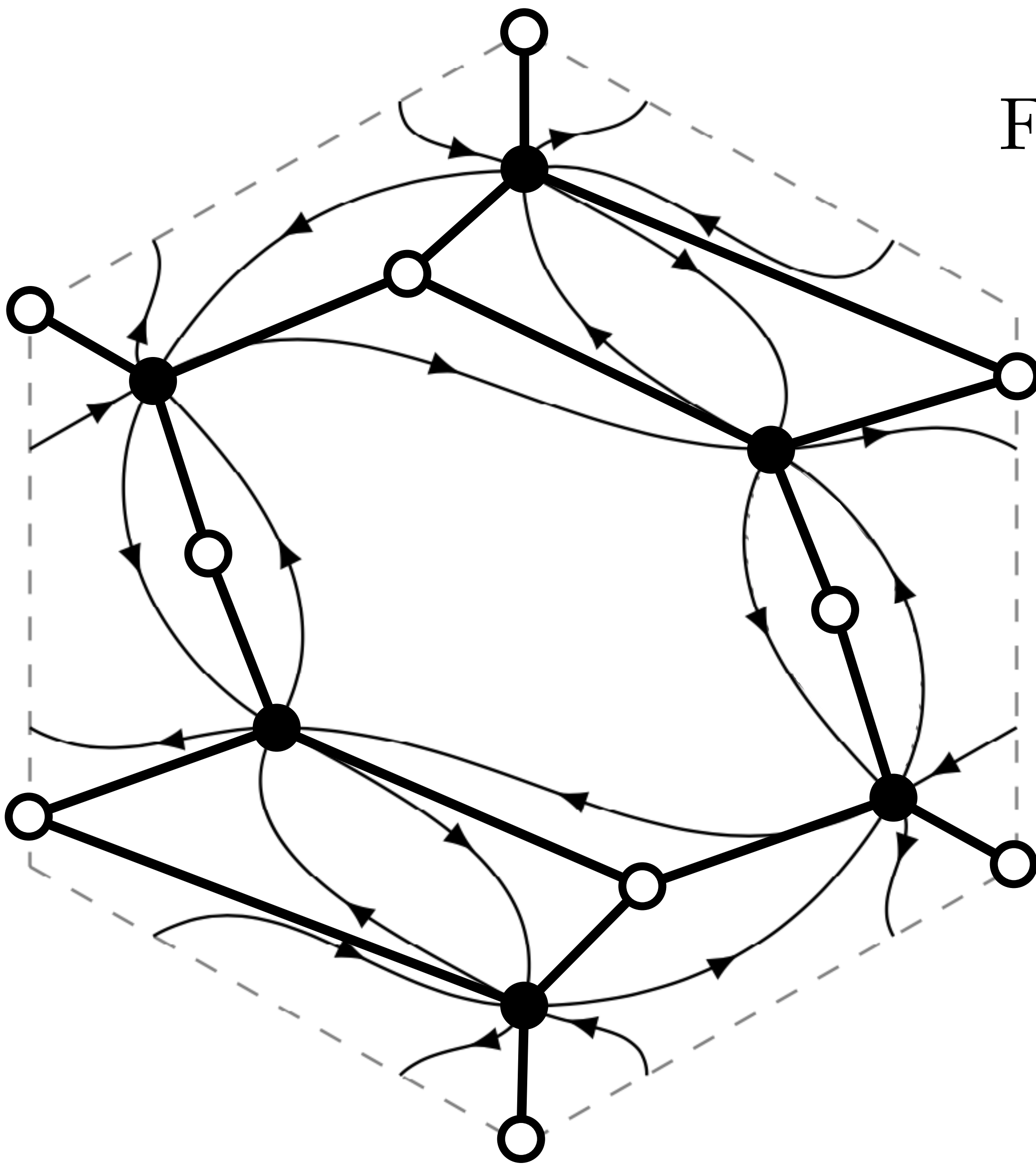
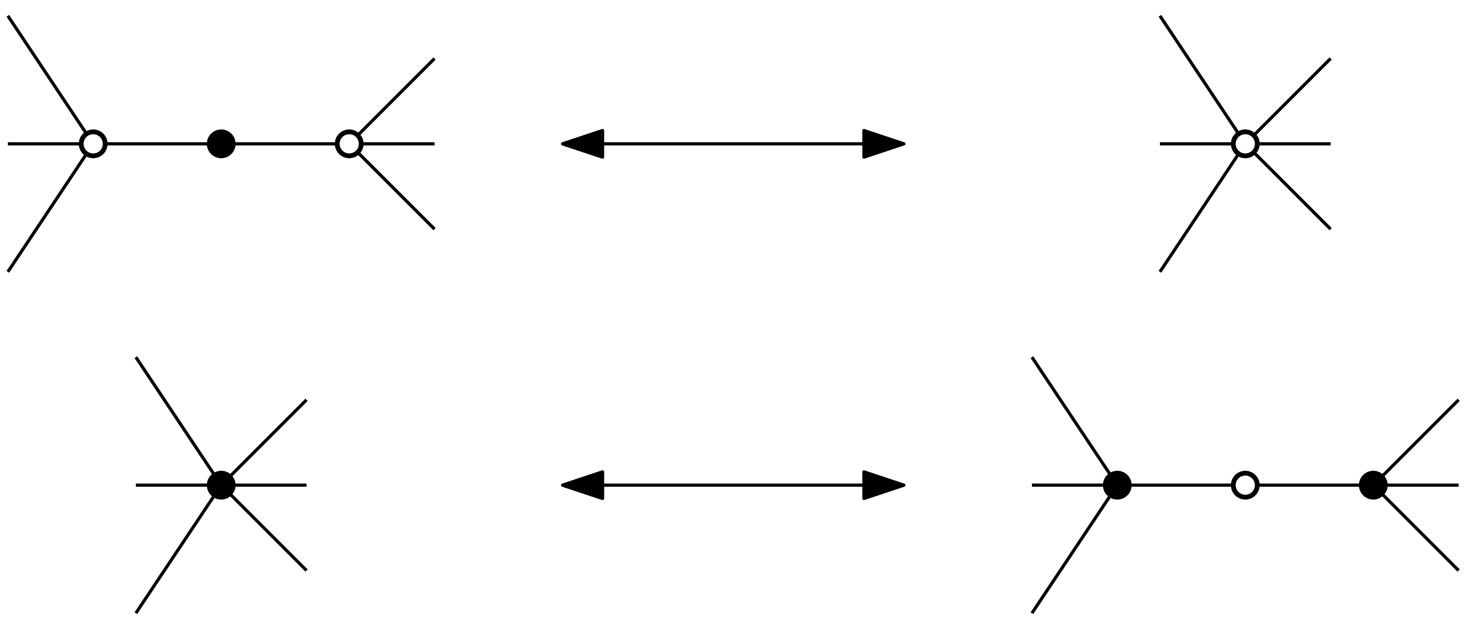


From TCD to bipartite graph:

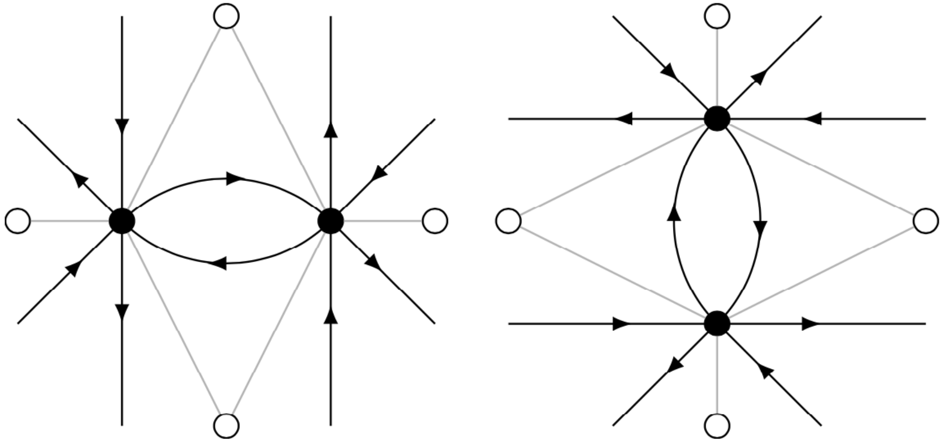
Place a black vertex at each triple point, a white vertex at each counter-clockwise face. Add an edge whenever a counter-clockwise face is adjacent to a triple point.

- TCDs are a special case of bipartite graphs where all black vertices must have degree 3.

- Starting from an arbitrary planar bipartite graph, get a TCD by contracting degree 2 black vertices and iteratively splitting black vertices of degree more than 3:



Local moves for TCDs



2-2 move at a clockwise face

For TCDs

Spider move
Square move
Urban renewal

For bipartite graphs

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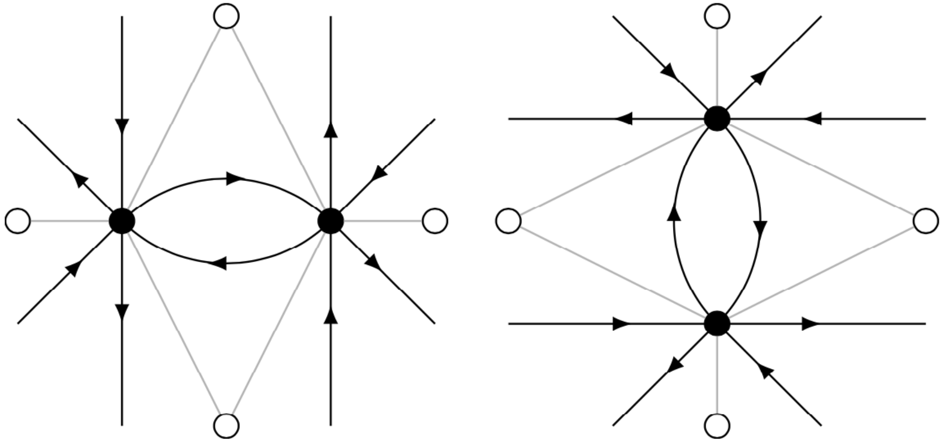

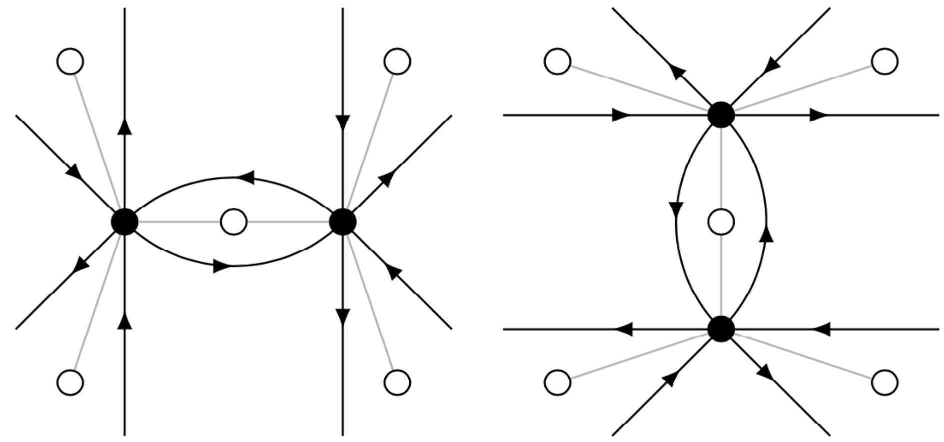

2-2 move at a clockwise face

For TCDs

2-2 move at a counterclockwise face

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For bipartite graphs

Resplit around a white vertex

Vector-relation configurations (AGPR)

- Fix $n \geq 1$ and a bipartite graph G.

- A vector-relation configuration (VRC) for G assigns to each white vertex a point in $\mathbb{C}P^n$ and to each black vertex of degree $d \geq 2$ a subspace of dimension $d-2$ in $\mathbb{C}P^n$, such that each edge corresponds to an incidence relation between a point and a subspace.

- Equivalently, attach to each white vertex w a vector v_w in $\mathbb{C}^{n+1}$ and to each black vertex b a non-trivial linear relation $\sum_{w \sim b} \mu_{bw} v_w = 0$. Attach μ_{bw} to the edge (b, w).

Gauge choices

The VRC is invariant if one:

- multiplies a given v_{w_0} by $1/\lambda$ and all the μ_{bw_0} by λ;
- multiplies all $\mu_{b_0 w}$ by λ for a given black vertex b_0.

- Let H be a hyperplane of $\mathbb{C}^{n+1}$ containing none of the v_w. Pick coordinates on $\mathbb{C}^{n+1}$ such that points in H have last coordinate 0. Then scale each v_w such that its last coordinate is 1.

- This is called an *affine gauge* and it satisfies around every black vertex b: $\sum_{w\sim b} \mu_{bw} = 0$.

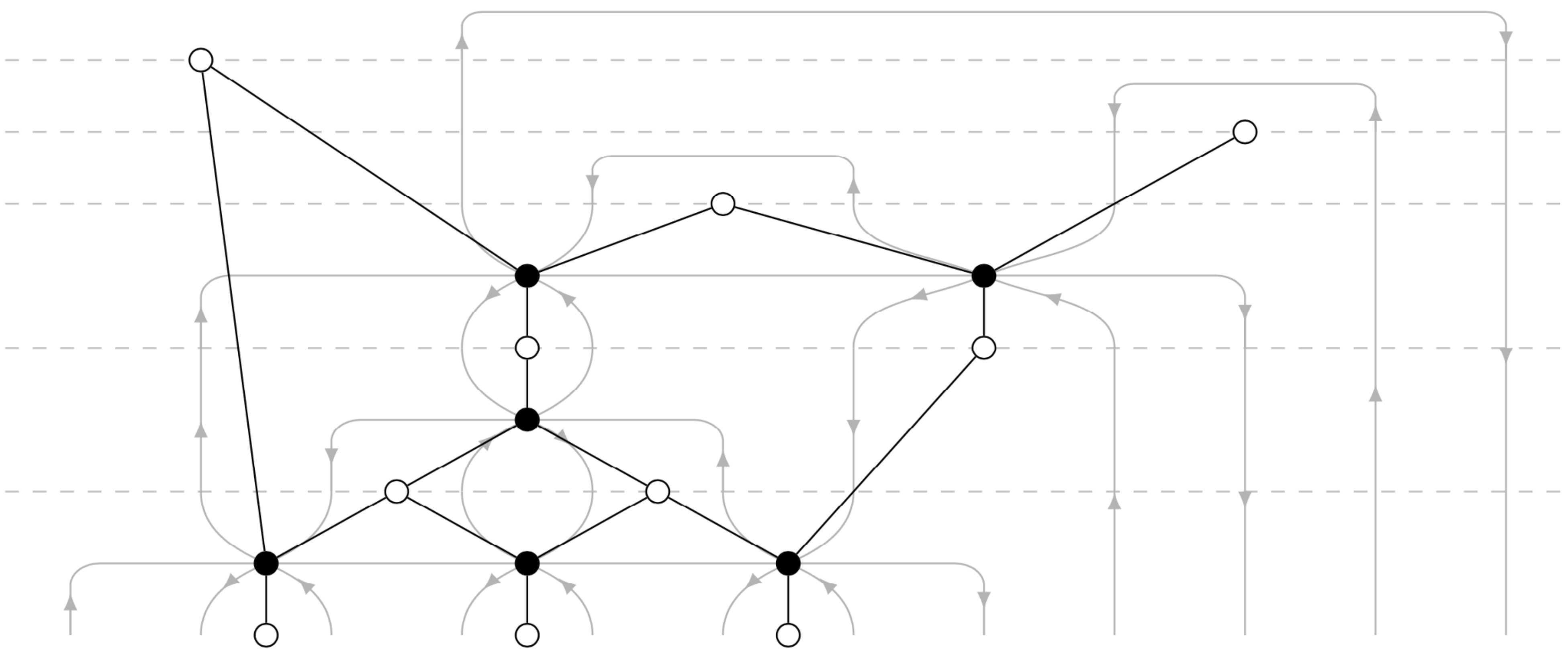
TCD maps (AGR '21)

- Fix $n \geq 1$ and a TCD T.

- A TCD map associated with T assigns to each white vertex a point in $\mathbb{C}P^n$ and to each black vertex a line in $\mathbb{C}P^n$, such that each edge corresponds to an incidence relation between a point and a line.

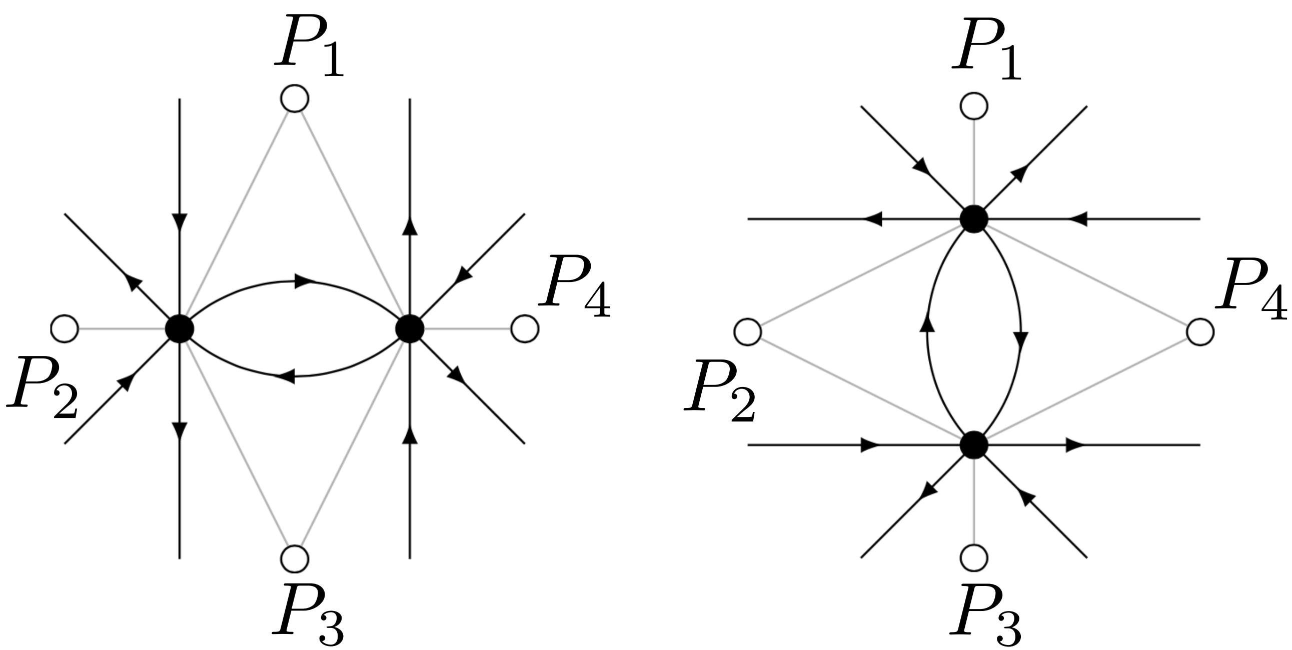
- TCD maps are special cases of VRCs, but they are more flexible and give rise to a richer theory.

Theorem (AGR). *For a TCD T on a disk with $|W|$ white vertices and $|B|$ black vertices, the maximal dimension spanned by points of a TCD map for T is $|W| - |B| - 1$.*



Local moves for TCD maps 1/2

The spider move



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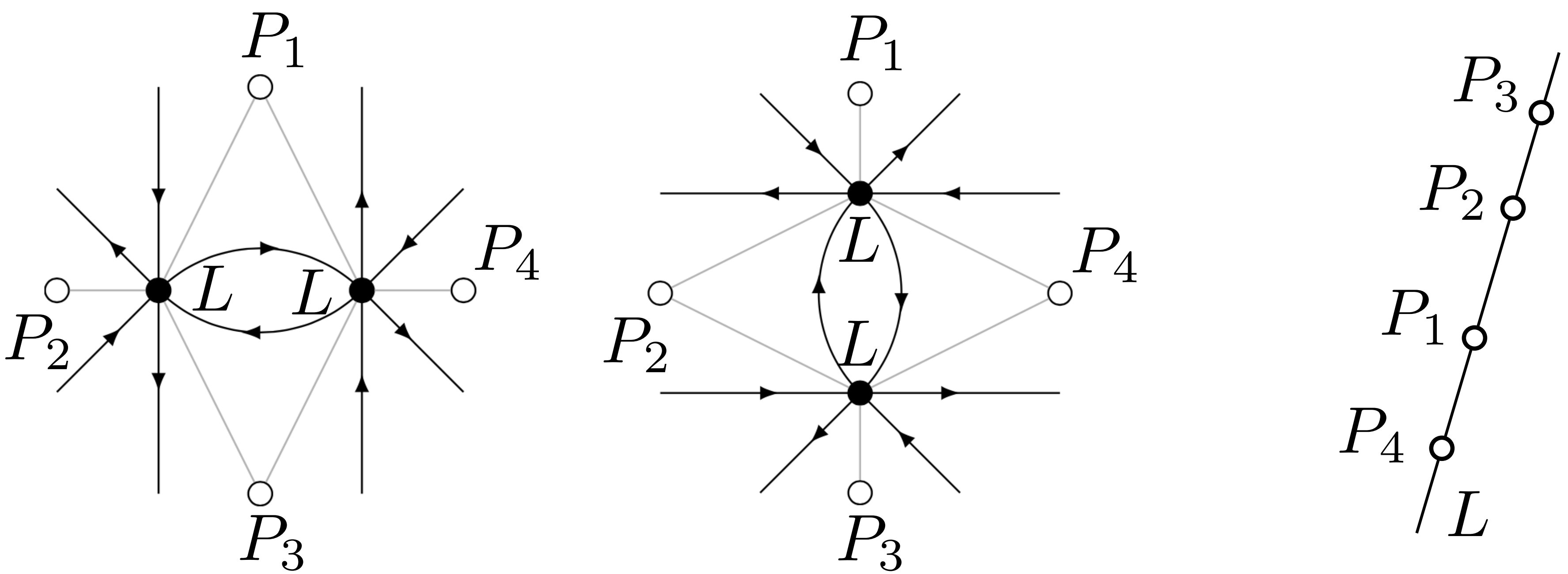
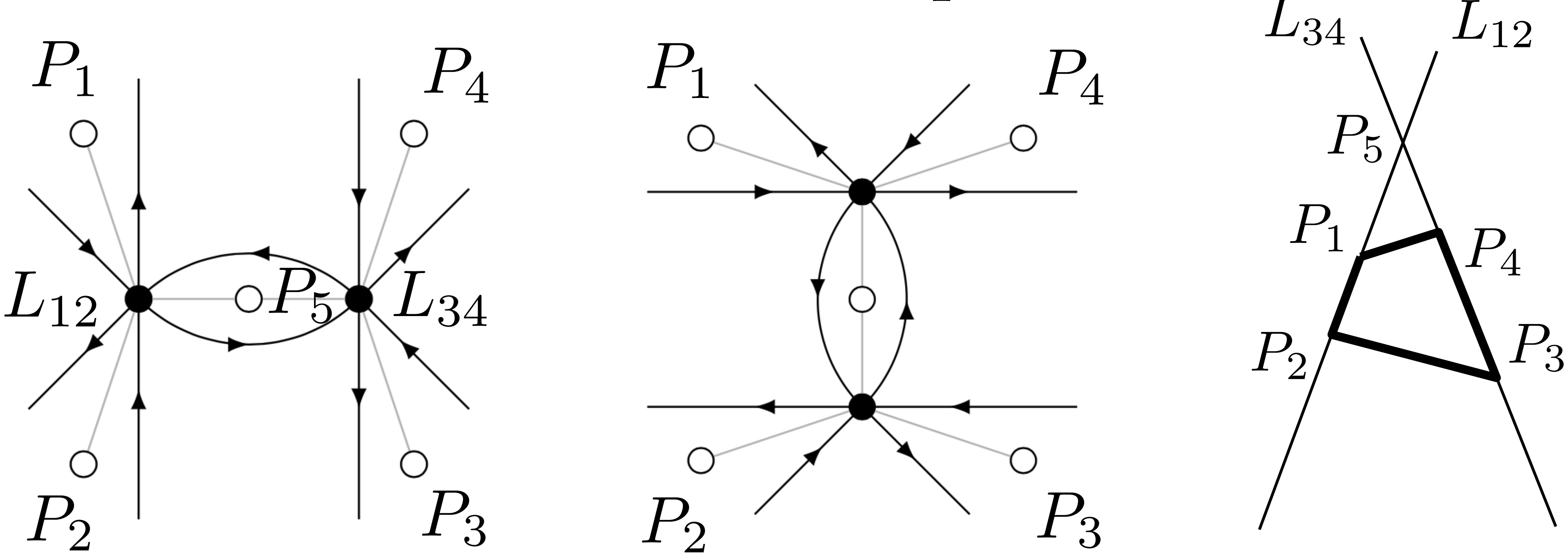


Reparametrization move, no change in geometry

Local moves for TCD maps 2/2

The resplit



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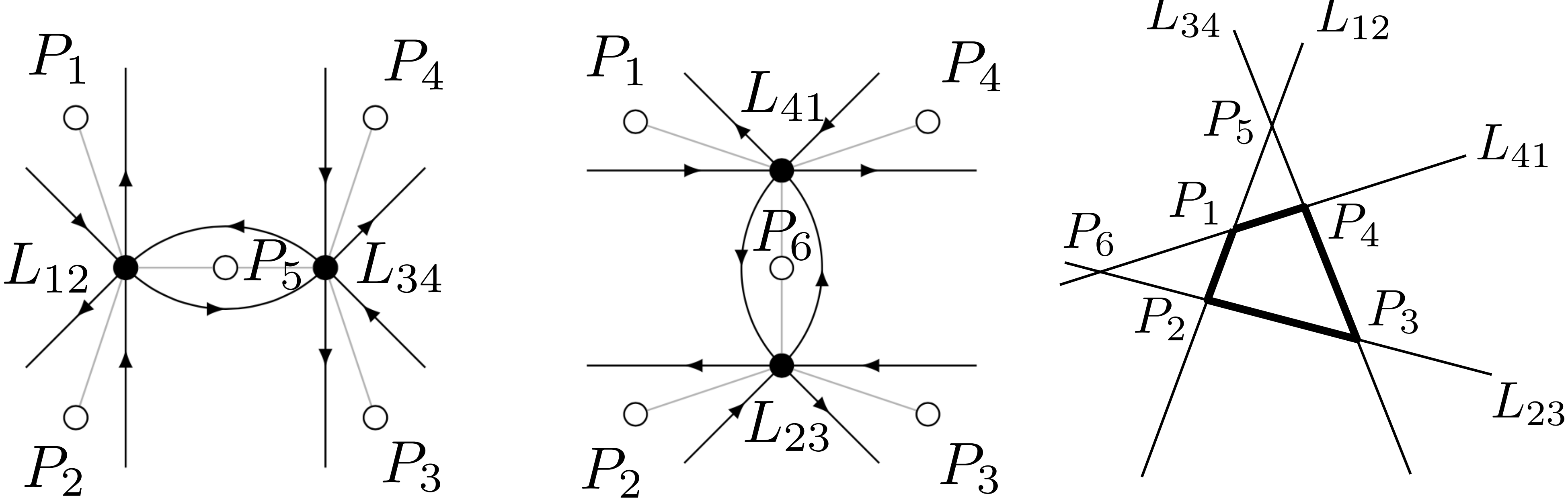


Change in geometry: exchanges the two focal points P_5 and P_6 of the quadrilateral $P_1P_2P_3P_4$.

Menelaus theorem (ca. 100 AD):

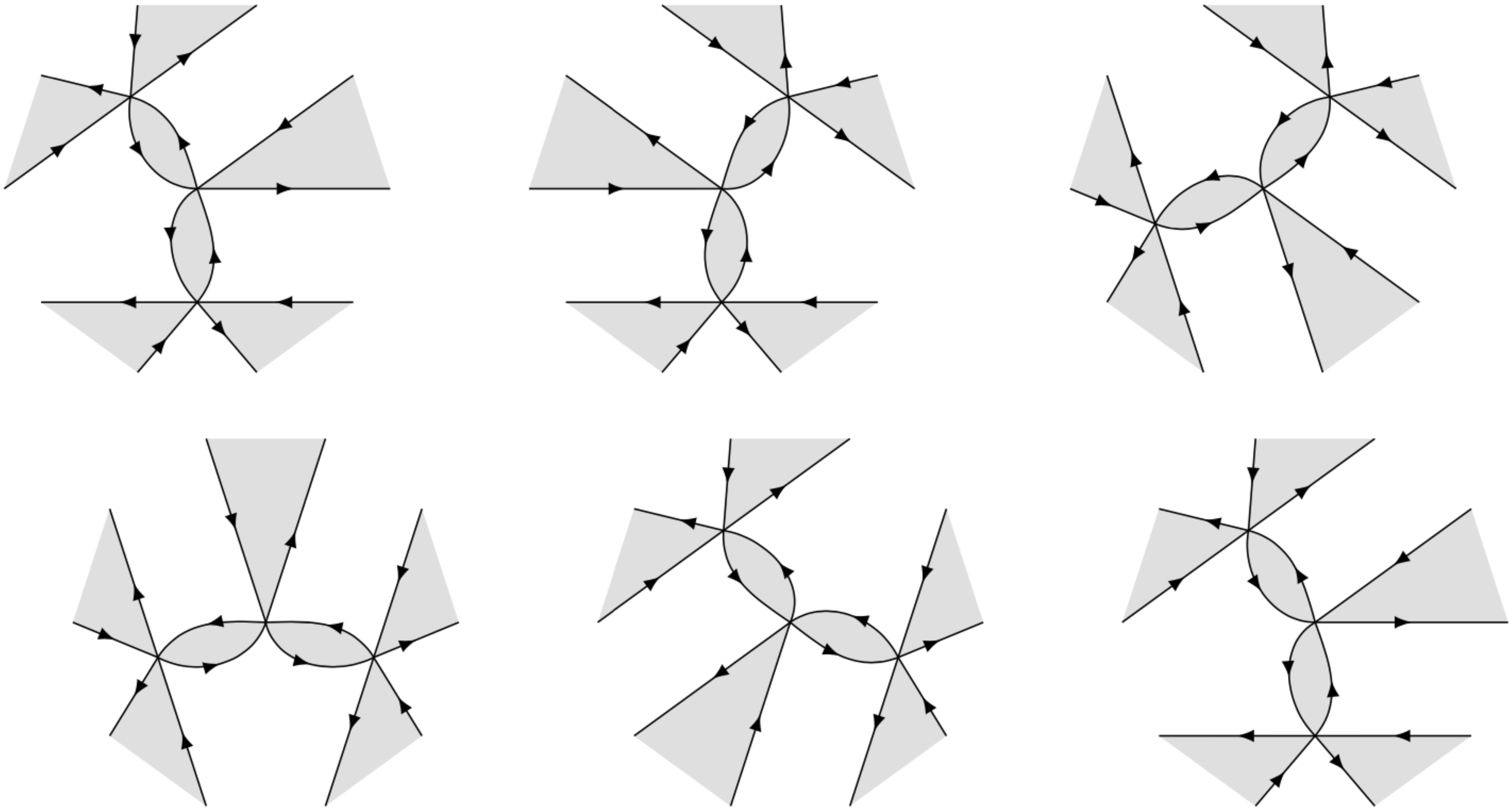
$$\frac{(P_3-P_4)(P_5-P_1)(P_2-P_6)}{(P_4-P_5)(P_1-P_2)(P_6-P_3)} = -1.$$

Multidimensional consistency

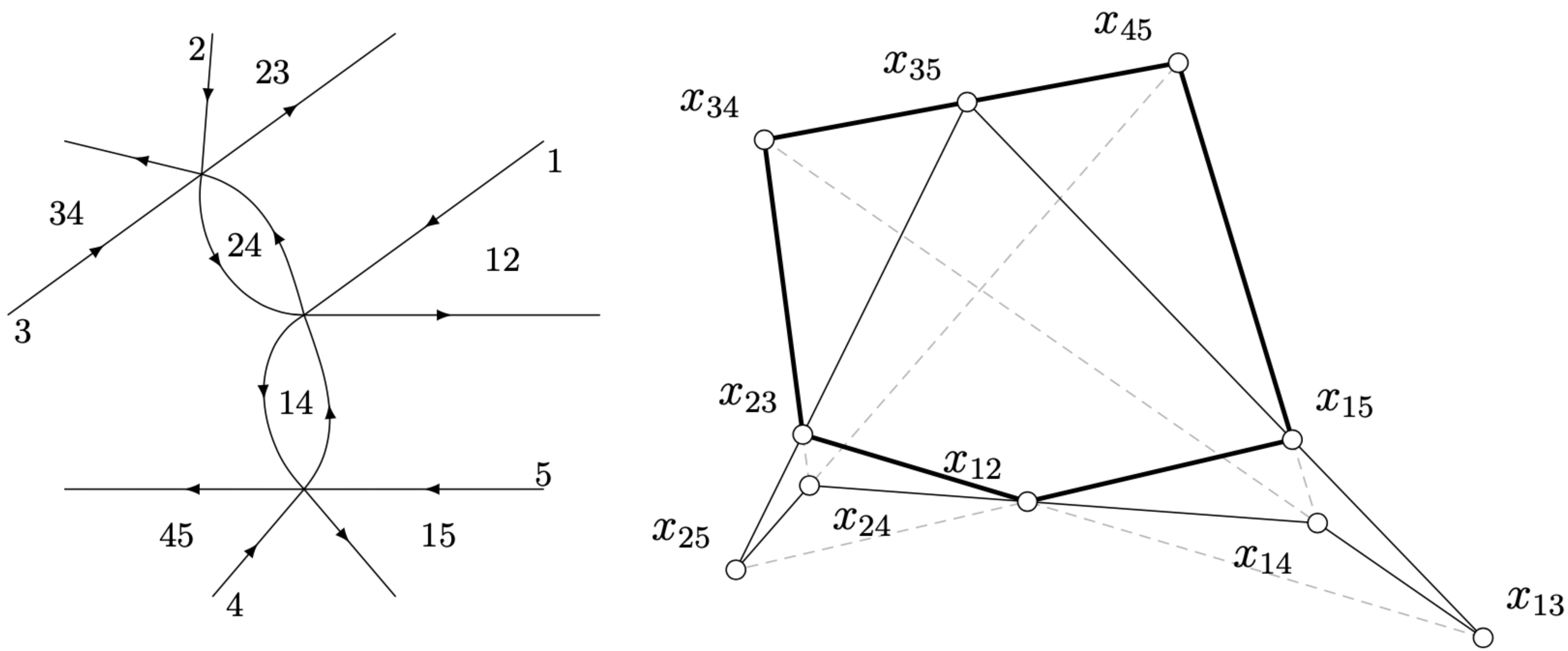
Theorem (AGR). *Consider a TCD map with labeled strands. After a sequence of 2-2 moves leaving the labeled TCD invariant, the TCD map will also be unchanged.*

- The name comes from discrete differential geometry, where lattice equations are called multidimensionally consistent if they can be unambiguously defined on any higher dimensional lattice.

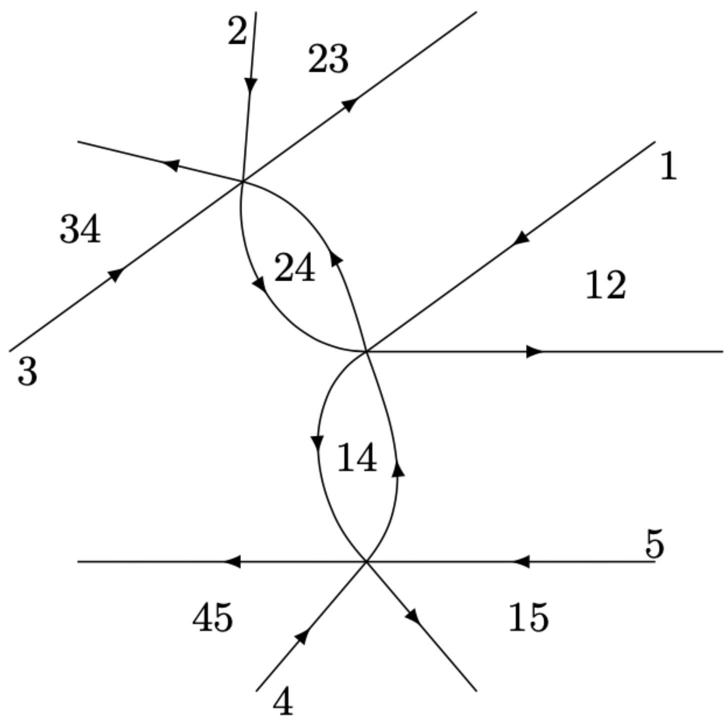
- It follows from Balitsky-Wellman '20 that it suffices to prove it for cycles of 4,5 or 10 2-2 moves.

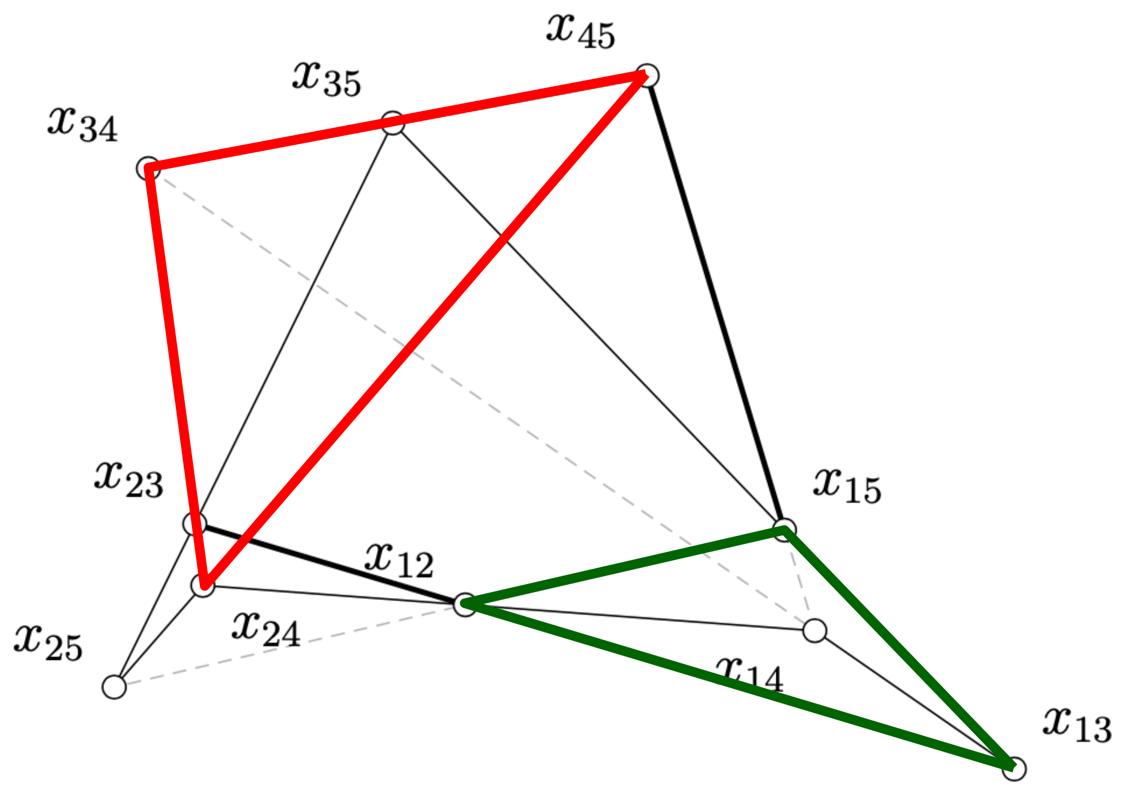


The case of a 5-cycle corresponds to Desargues theorem



Desargues' theorem (ca. 1648): Consider two triangles ABC and $A'B'C'$ in $\mathbb{R}P^3$. Then the three points $AB \cap A'B'$, $AC \cap A'C'$ and $BC \cap B'C'$ are aligned iff the lines AA', BB' and CC' intersect at a common point.





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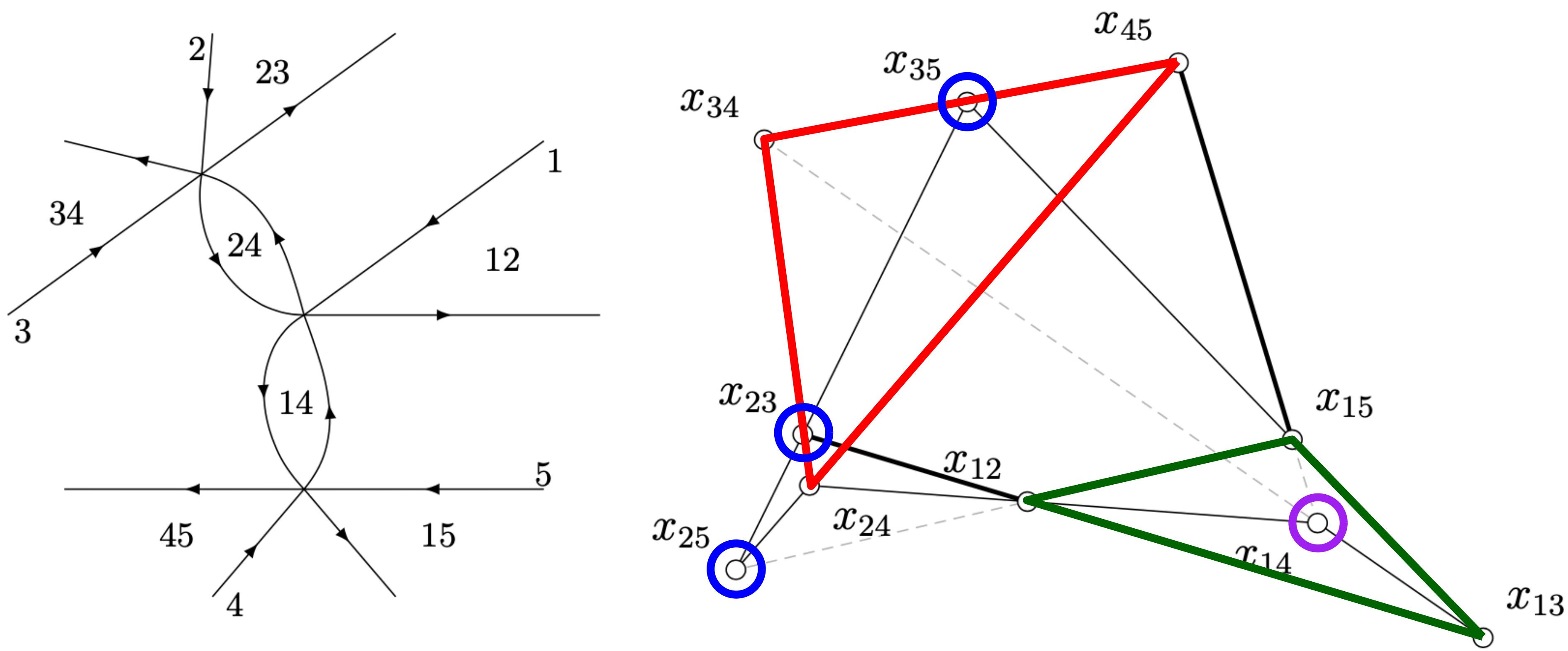

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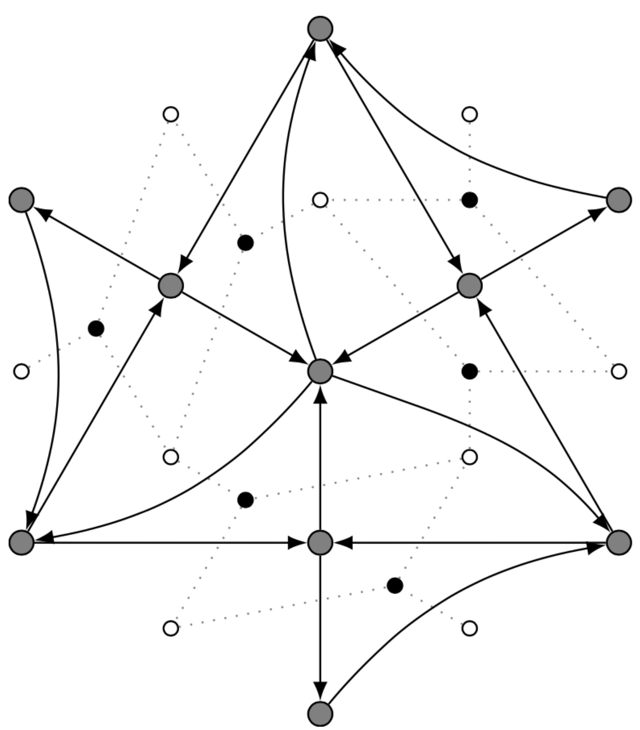
Projective cluster structure (AGPR)

- Works for both TCD maps and VRCs.
- The projective quiver is the dual graph of the bipartite graph, with dual edges oriented so that they turn counterclockwise around black vertices.



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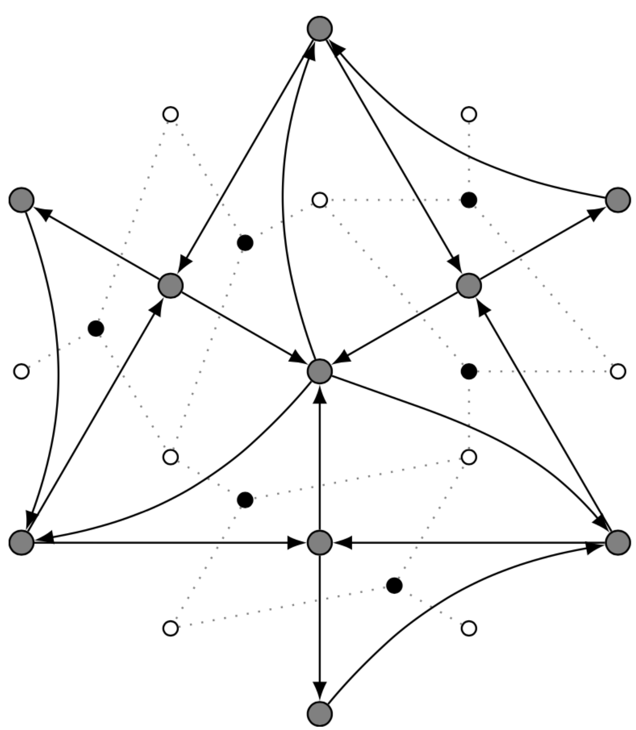

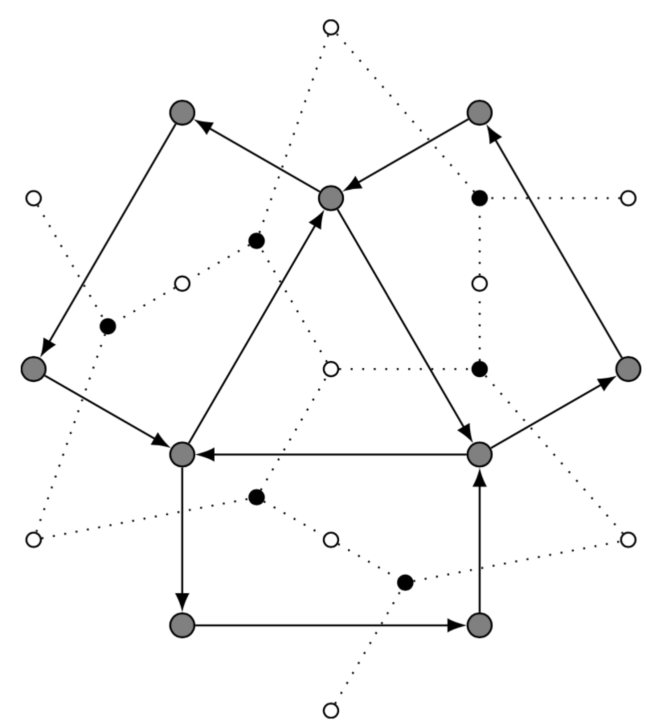

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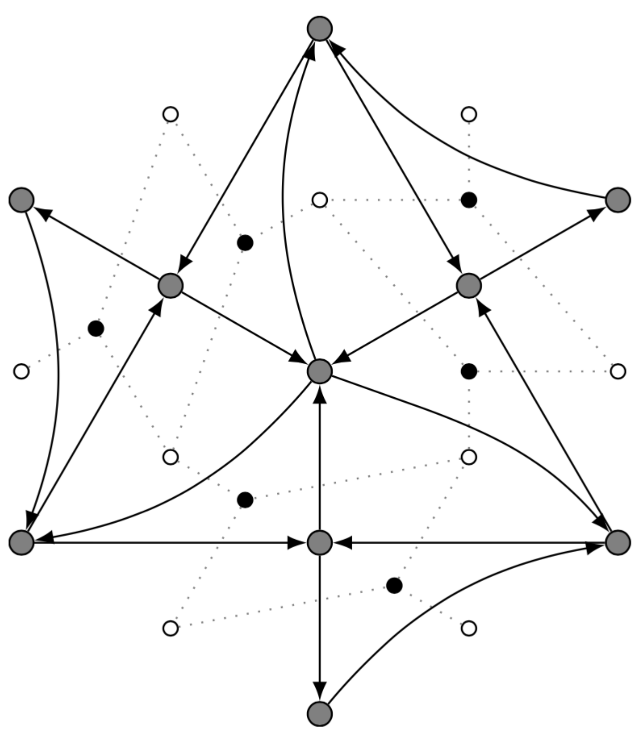

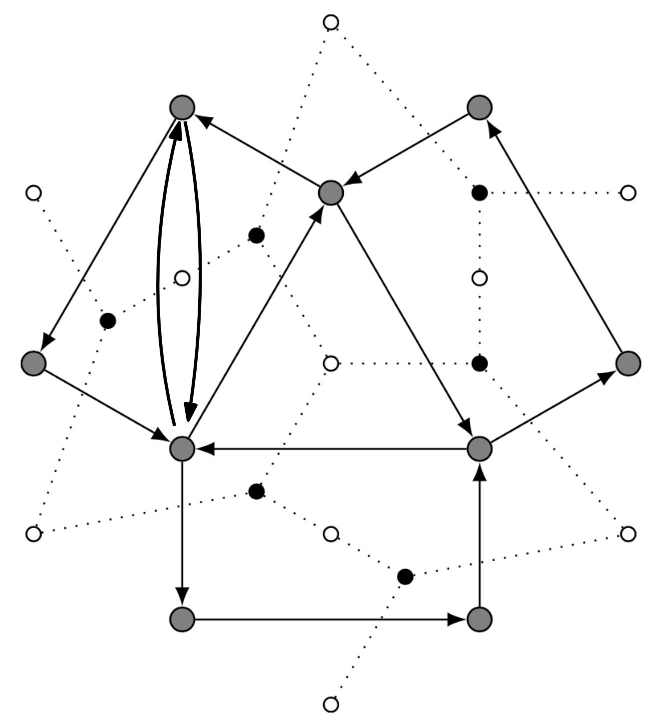

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  $$\mathrm{mr}(w_1, v_{12}, w_2, v_{23}, \ldots, w_d, v_{d1}) = \prod_{i=1}^{d} \frac{w_i - v_{i,i+1}}{v_{i,i+1} - w_{i+1}}.$$

Projective cluster structure (AGPR)

- For a face $f = (b_1, w_1, \ldots, b_d, w_d)$ of degree $2d$ of a bipartite graph, define $v_{i,i+1}$ to be the other white neighbor of b_{i+1}.

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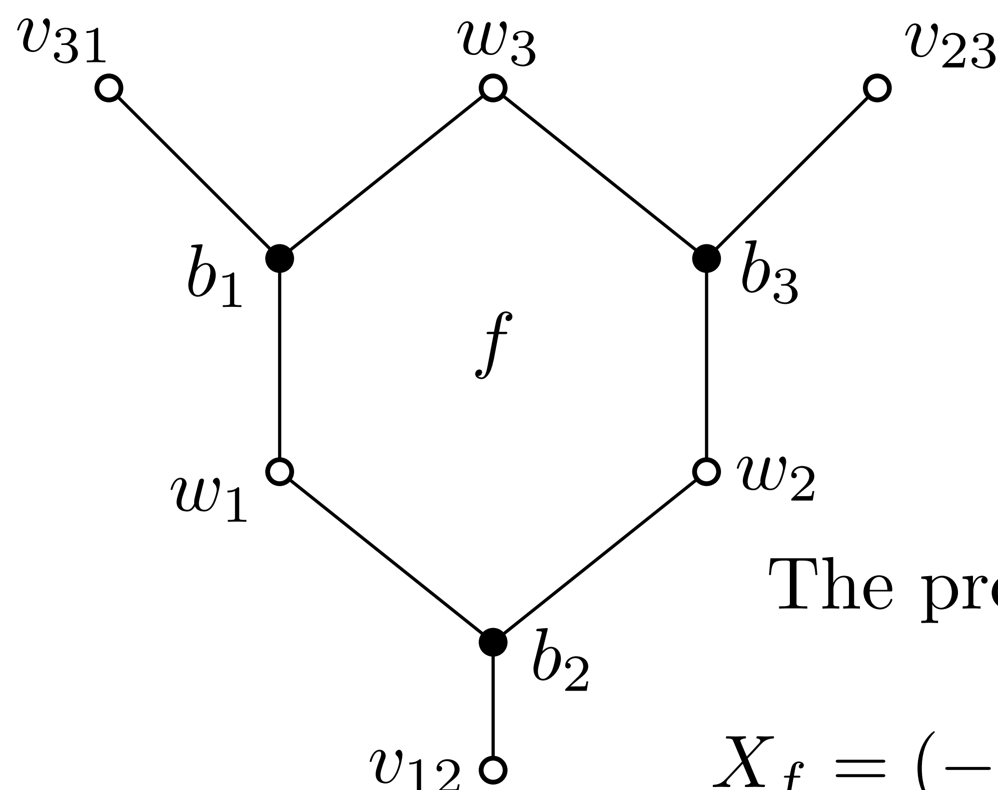

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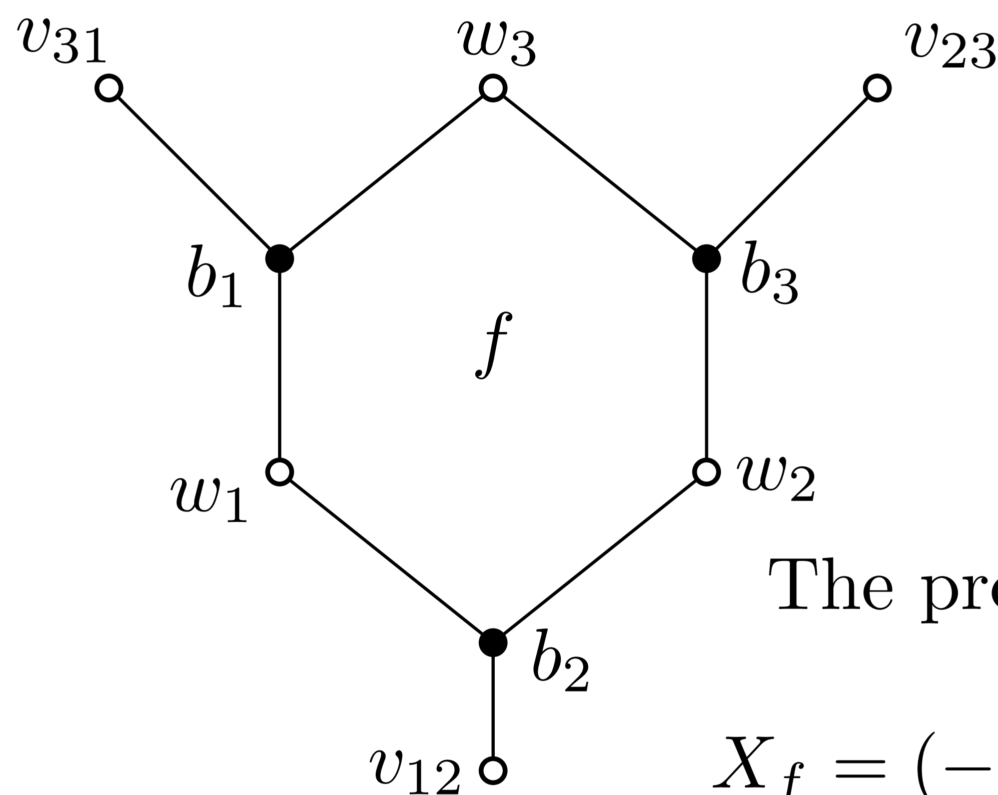

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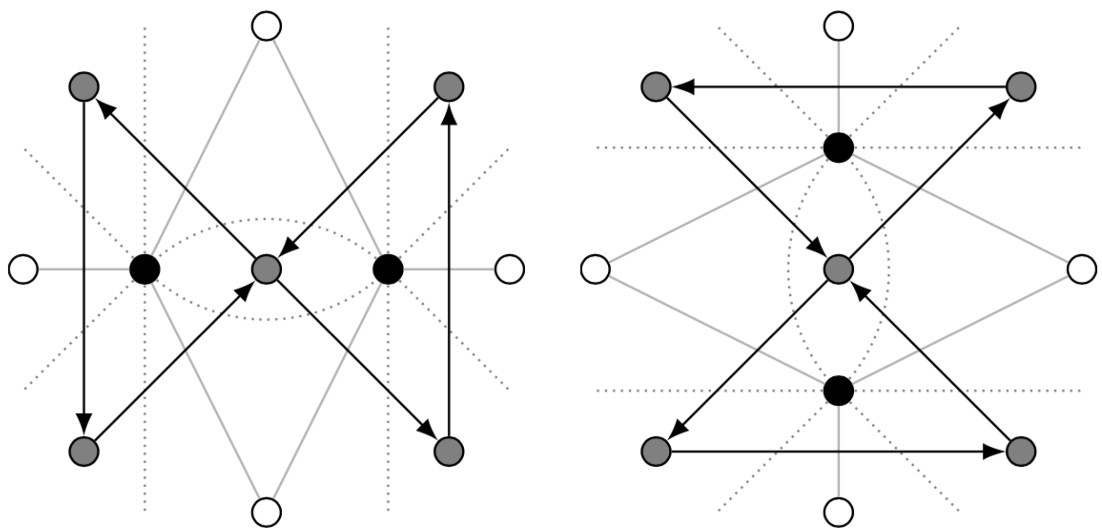
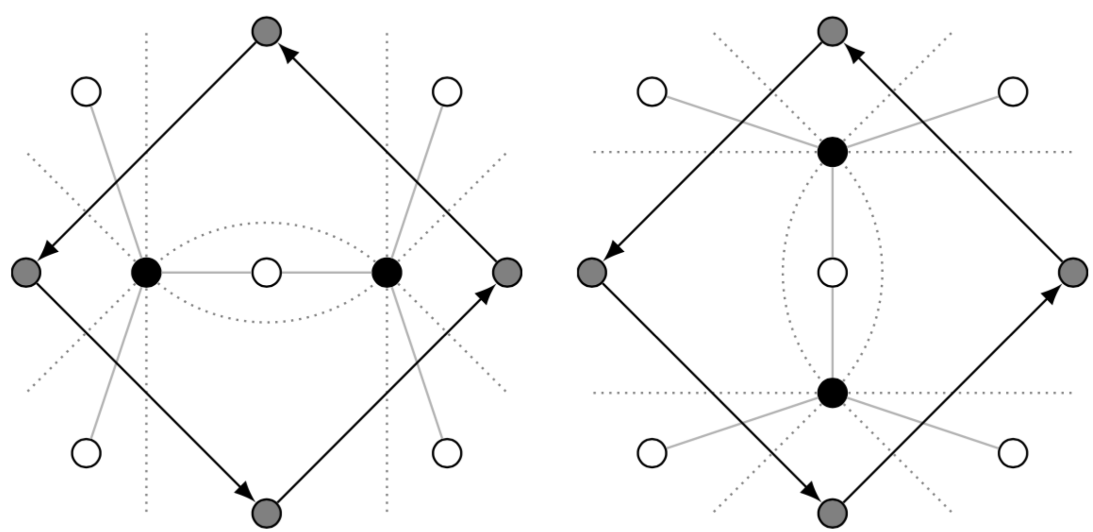


The spider move induces a mutation of the projective quiver and of the projective cluster variables.



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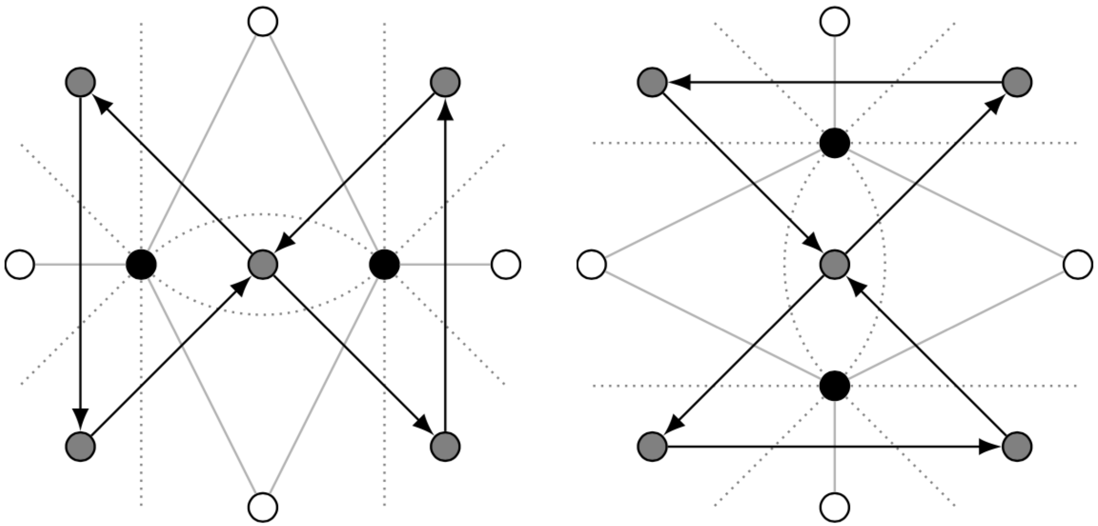

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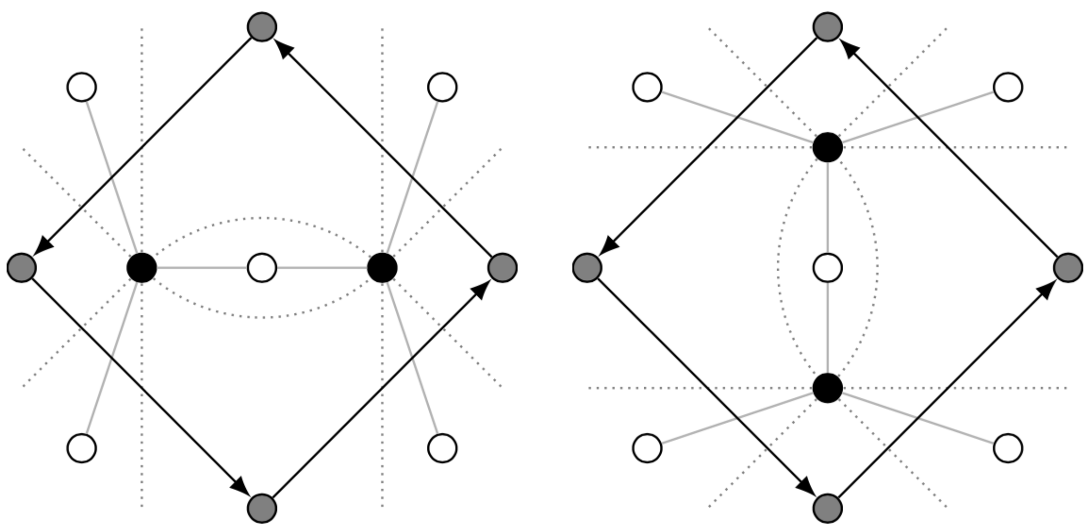
FG X variables



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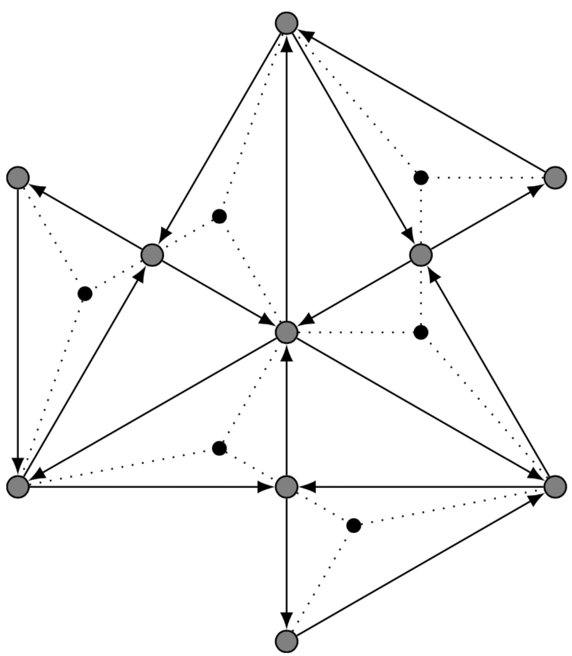
Affine cluster structure (AGR)

- Defined only for TCD maps.

- The vertices of the affine quiver of a TCD are its white vertices. The edges form counterclockwise oriented triangles around each black vertex.



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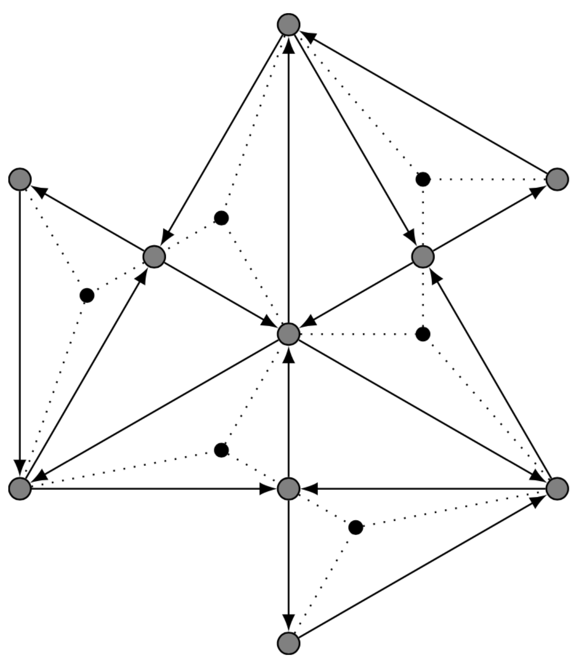

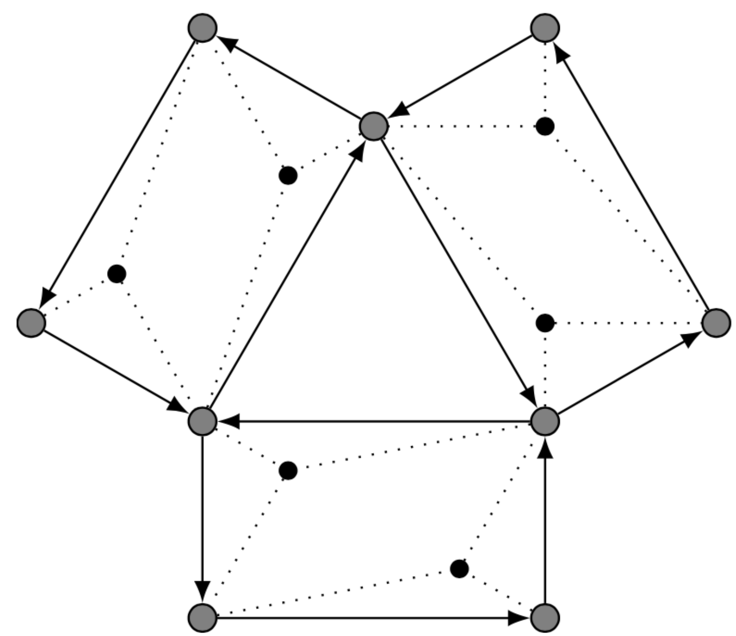

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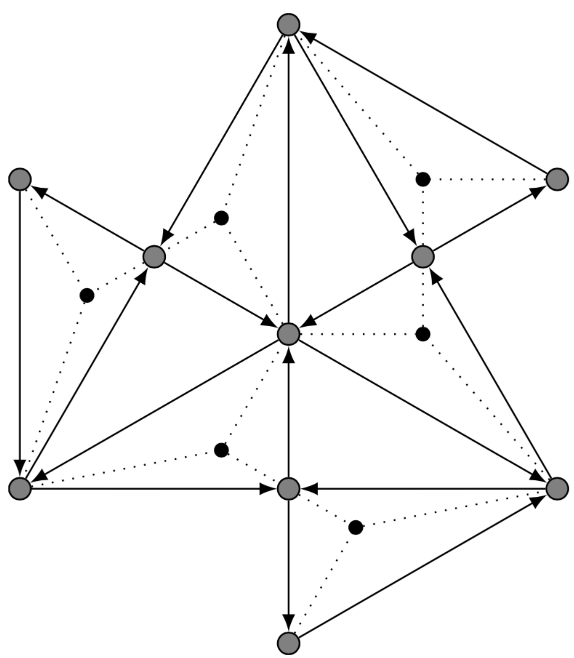

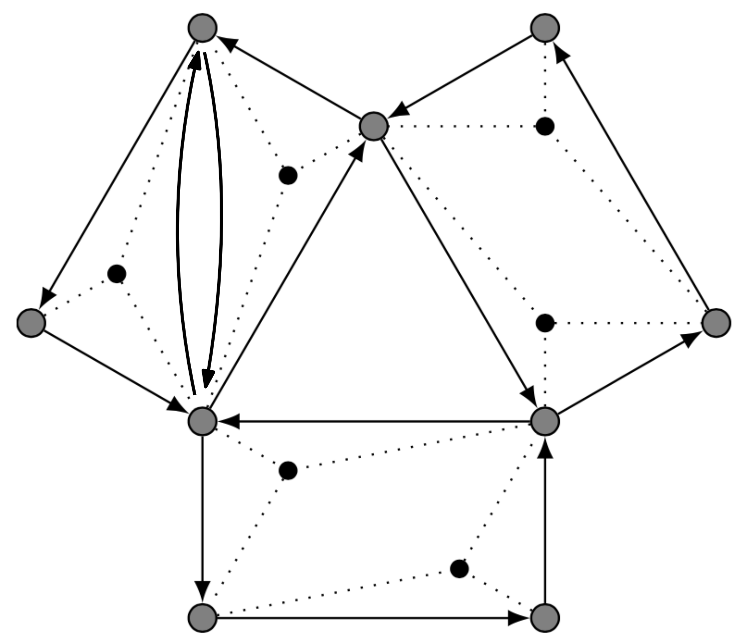

Affine cluster structure (AGR)

- If $w_1, w'_1 \ldots, w_d, w'_d, w$ are $2d+1$ points in $\mathbb{C}P^n$ such that w is the intersection point of the d lines $w_i w'_i$, define the star-ratio of these points as
$$\mathrm{sr}_H(w_1, w'_1 \ldots, w_d, w'_d; w) = \prod_{i=1}^{d} \frac{w_i - w}{w'_i - w}.$$

- Its value depends on the choice of a hyperplane H at infinity.

Affine cluster structure (AGR)

- If a degree d white vertex w has neighbors $b_1, \ldots, b_d$, define the other neighbors of b_i to be w_i and w'_i such that $ww_iw'_i$ is oriented counterclockwise around b_i.

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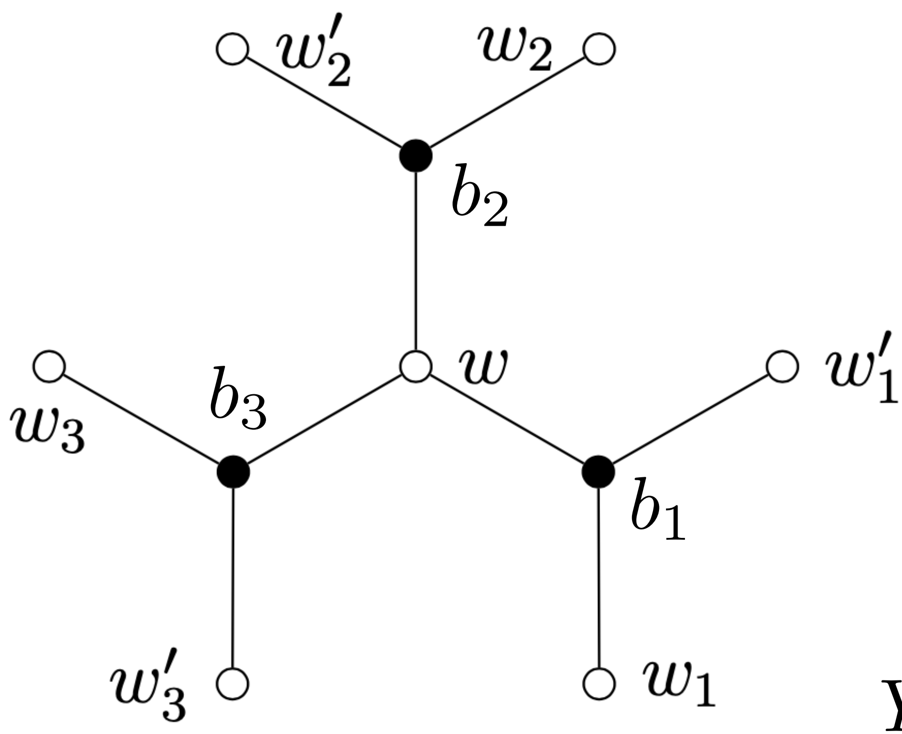

The affine cluster variable Y_w relative to H is defined as

$$Y_w = -\operatorname{sr}_H(w_1, w_1' \ldots, w_d, w_d'; w)$$

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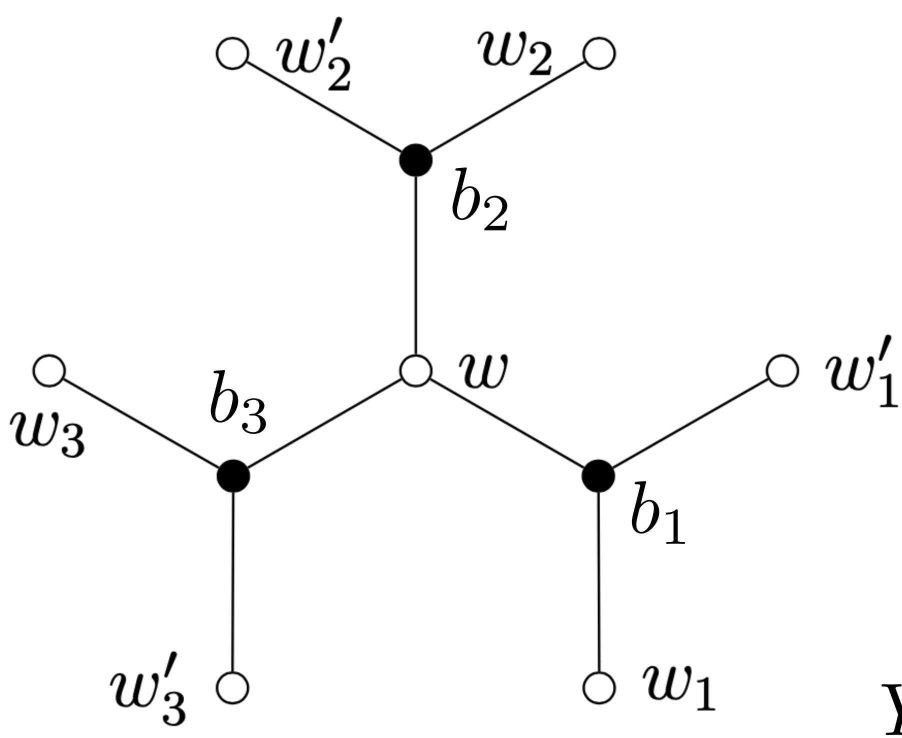

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$$Y_w = (-1)^{d+1} \prod_{i=1}^{d} \frac{\mu(b_i, w_i')}{\mu(b_i, w_i)}$$

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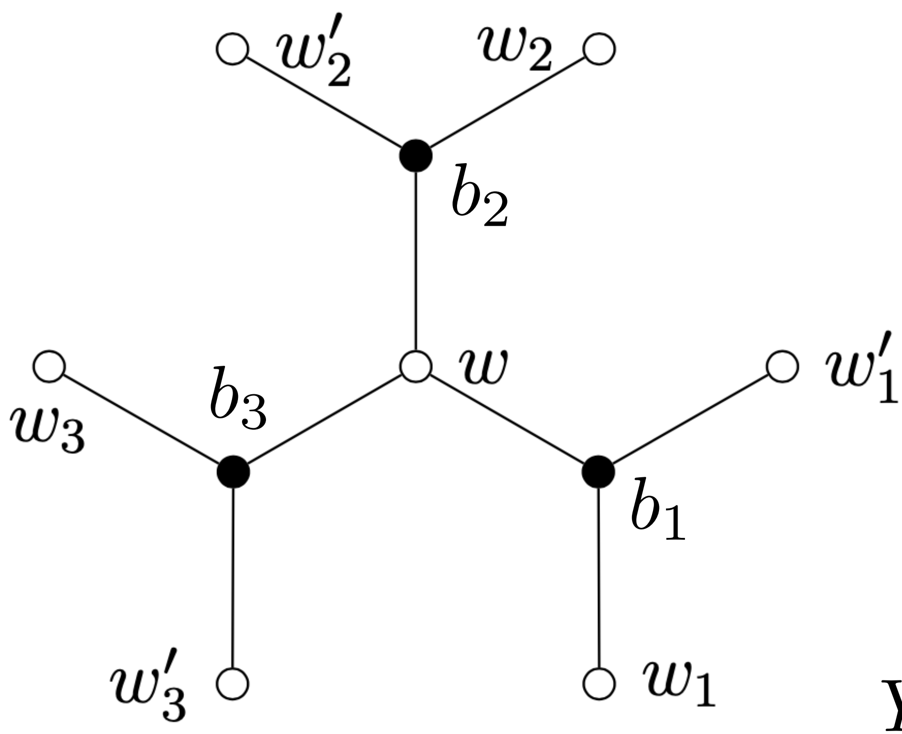
$$Y_w = -\operatorname{sr}_H(w_1, w_1' \ldots, w_d, w_d'; w)$$

Affine cluster structure (AGR)

- If a degree d white vertex w has neighbors $b_1, \ldots, b_d$, define the other neighbors of b_i to be w_i and w_i' such that ww_iw_i' is oriented counterclockwise around b_i.



The affine cluster variable Y_w relative to H is defined as

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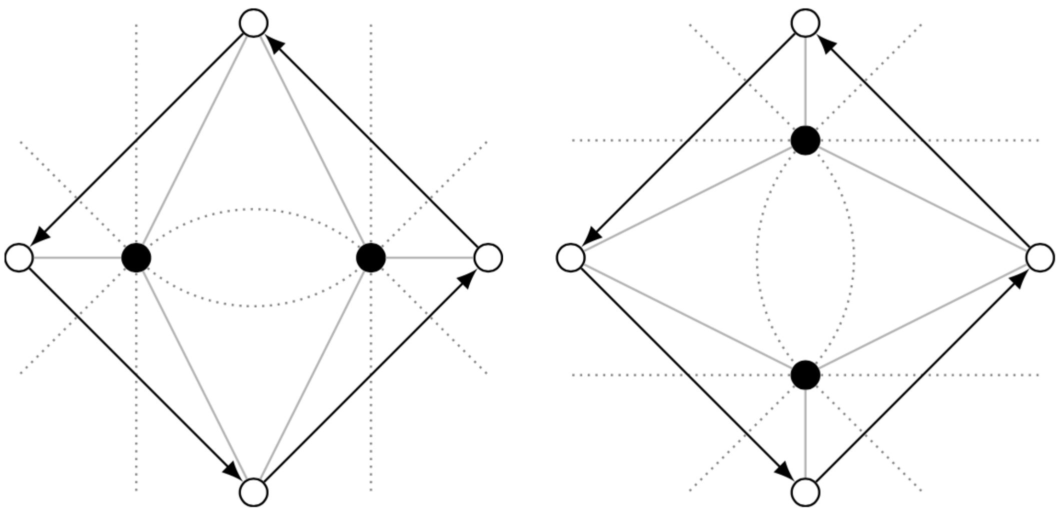
Equivalently,

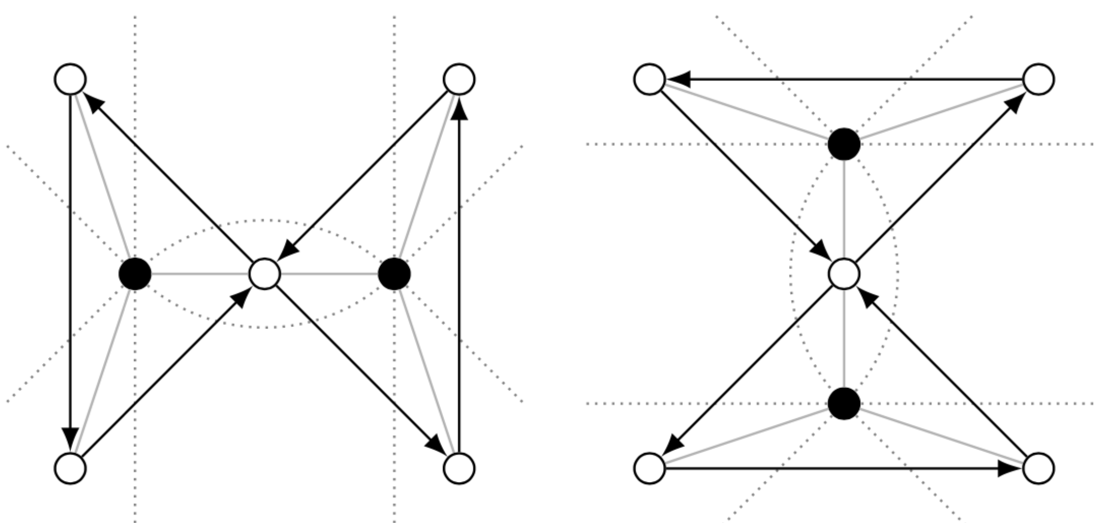
$$Y_w = (-1)^{d+1} \prod_{i=1}^{d} \frac{\mu(b_i, w_i')}{\mu(b_i, w_i)}$$

affine gauge wrt H

Affine cluster structure (AGR)



The spider move leaves the affine quiver and the affine cluster variables invariant.



The resplit induces a mutation of the affine quiver and of the affine cluster variables.

2 Examples, old and new

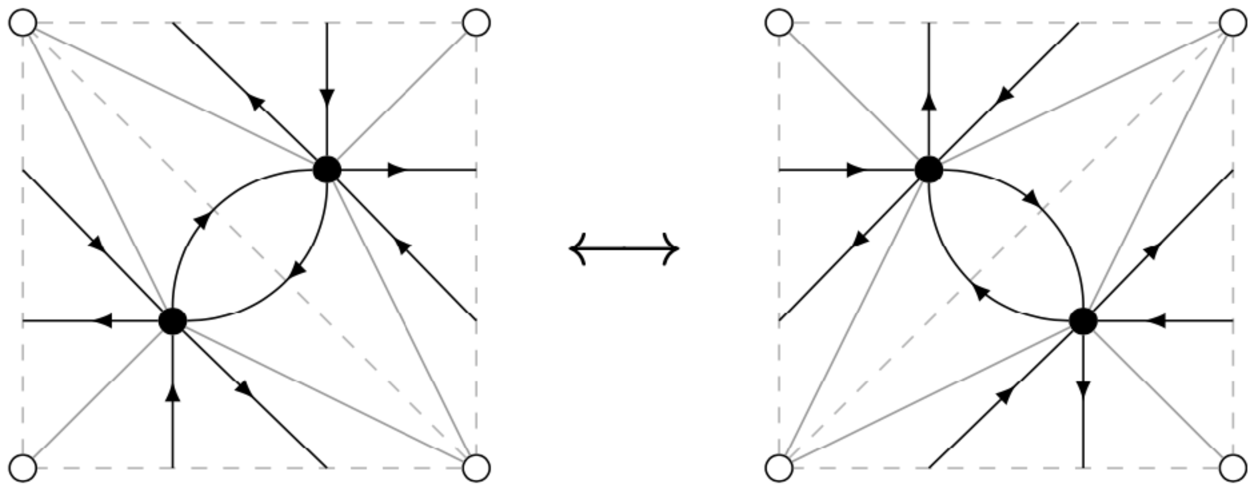
Triangulations
(Fomin-Shapiro-Thurston, Fock-Goncharov)

- TCD map with target space $\mathbb{C}P^1$ associated to the triangulation of a polygon.

- White vertices are placed at the vertices of the triangulation. One black vertex is placed inside each triangle.

- The projective quiver has one vertex per edge of the triangulation. Projective cluster variables are cross-ratios of 4 points around an edge.

Triangulations
(Fomin-Shapiro-Thurston, Fock-Goncharov)



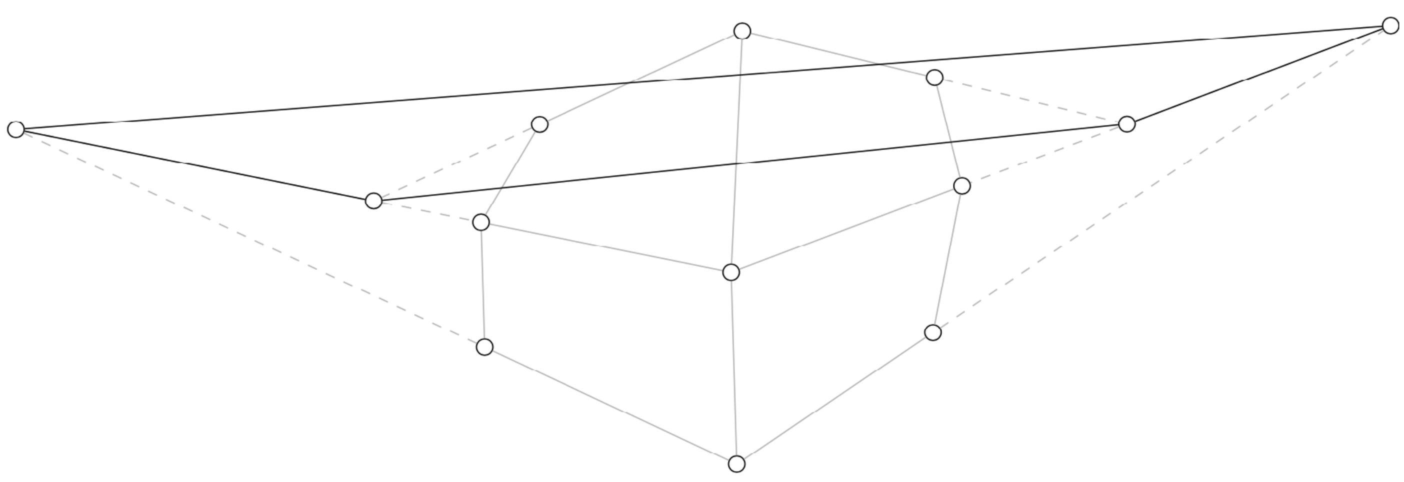
- White vertices are placed at the vertices of the triangulation. One black vertex is placed inside each triangle.

- The projective quiver has one vertex per edge of the triangulation. Projective cluster variables are cross-ratios of 4 points around an edge.

$\mathbb{Z}^2$ Q-nets

- Fix $n, N \geq 2$. A $\mathbb{Z}^N$ Q-net is a map from the vertices of $\mathbb{Z}^N$ to $\mathbb{C}P^n$ such that the images of any 4 points around a 2-cell are coplanar.

- The (horizontal) Laplace transform $\Delta_h(T)$ of a $\mathbb{Z}^2$ Q-net T is obtained by assigning to each quad the intersection of its horizontal sides.



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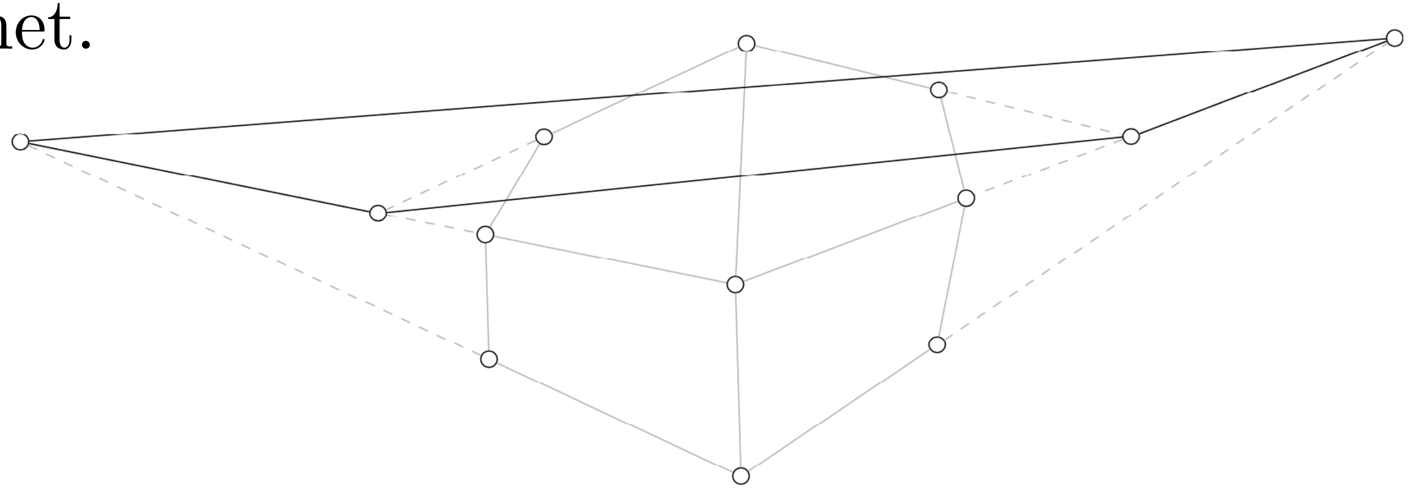
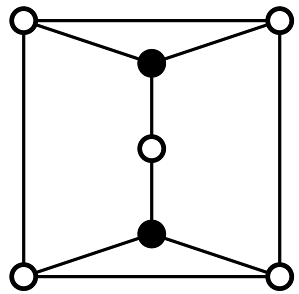
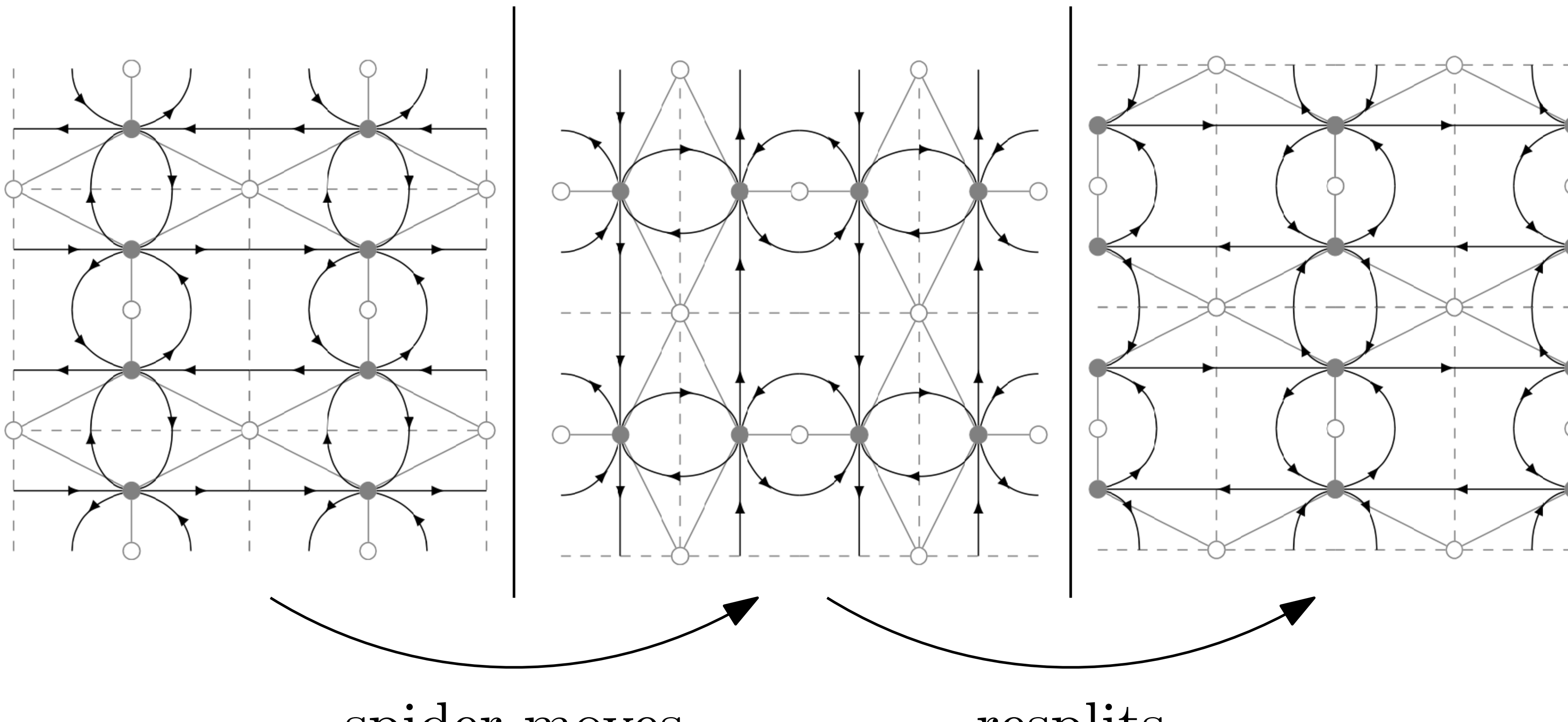
- The (horizontal) Laplace transform $\Delta_h(T)$ of a $\mathbb{Z}^2$ Q-net T is obtained by assigning to each quad the intersection of its horizontal sides. $\Delta_h(T)$ is again a $\mathbb{Z}^2$ Q-net.



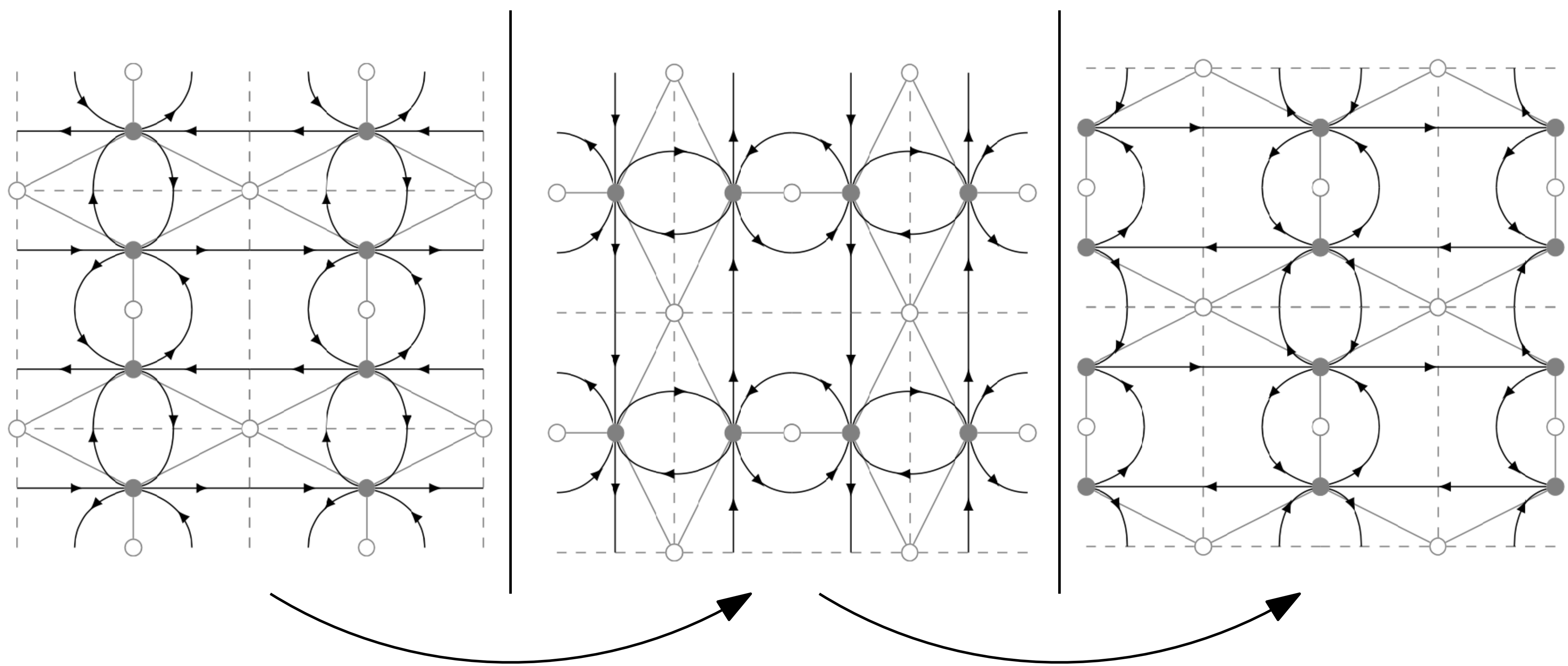
$\mathbb{Z}^2$ Q-nets

Draw  inside each quad



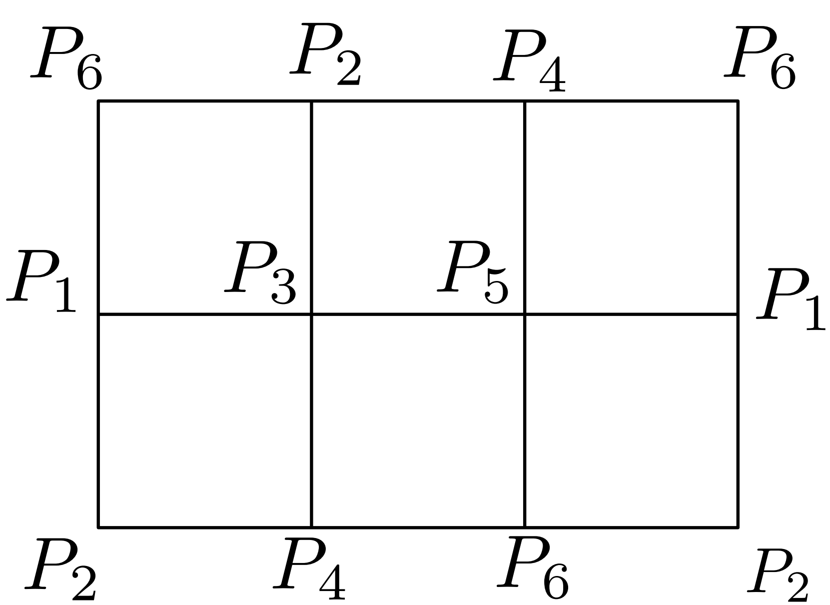
$\mathbb{Z}^2$ Q-nets

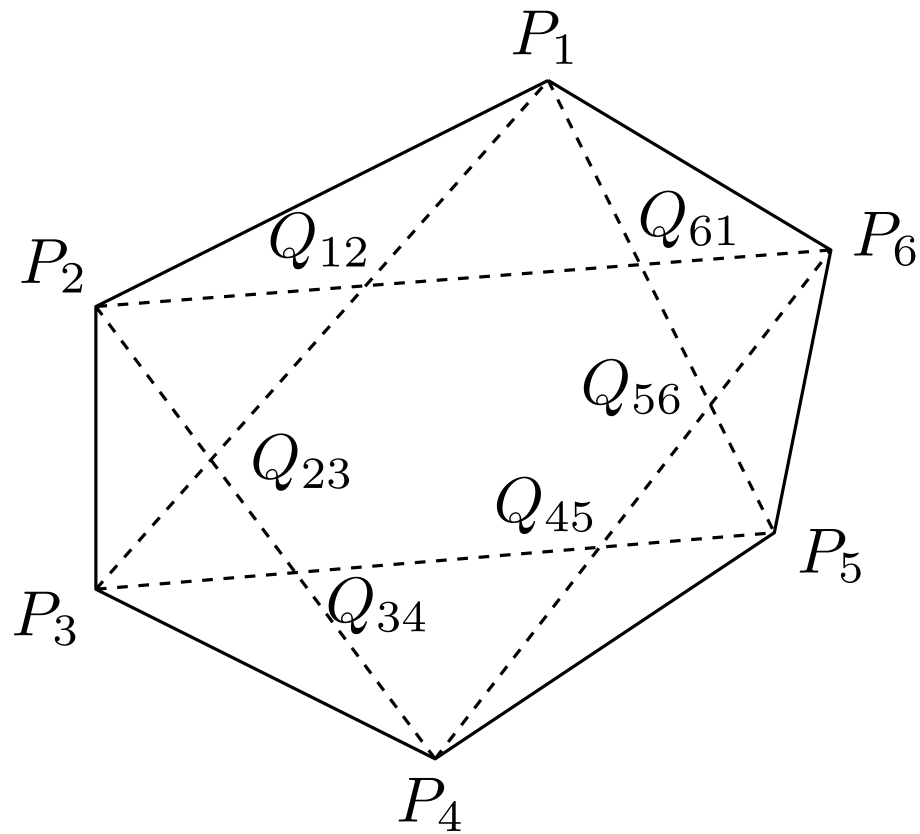
Theorem (AGPR,AGR)**.** *The Laplace transform dynamics for* $\mathbb{Z}^2$ *Q-nets is captured both by the projective and the affine cluster structure.*



The pentagram map

Schief ('09) observed that the pentagram map could be obtained as the following specialization of a $\mathbb{Z}^2$ Q-net:





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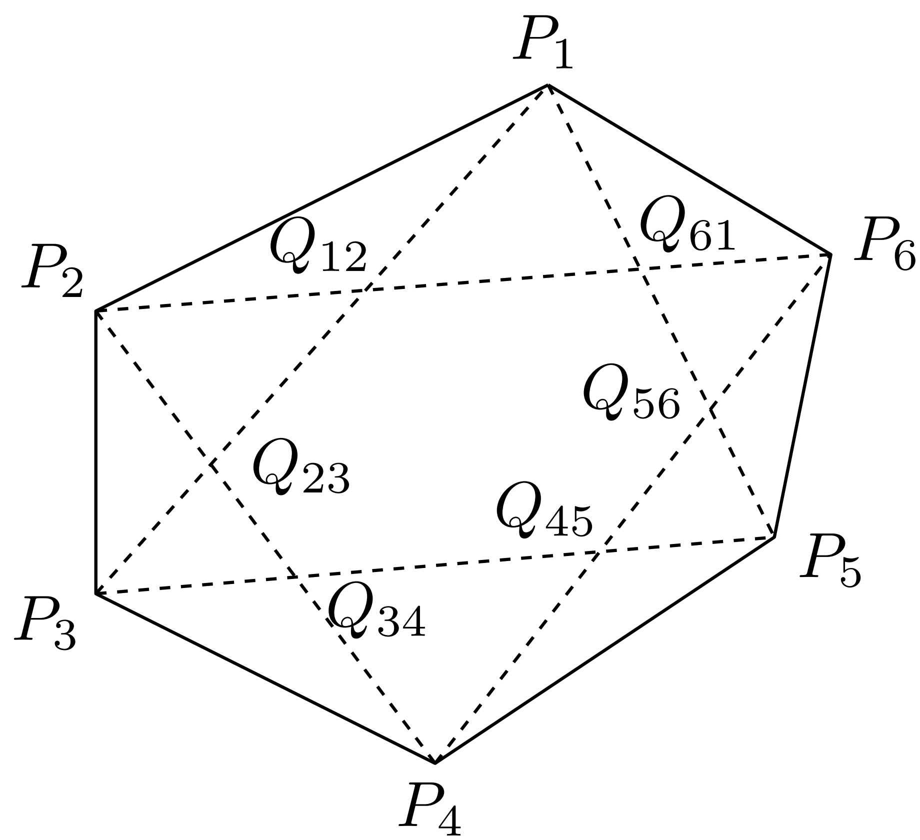
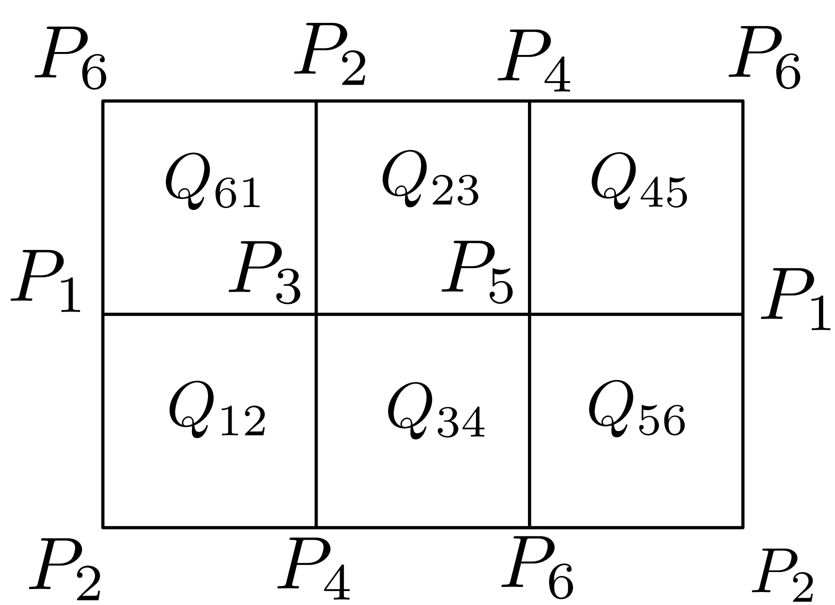
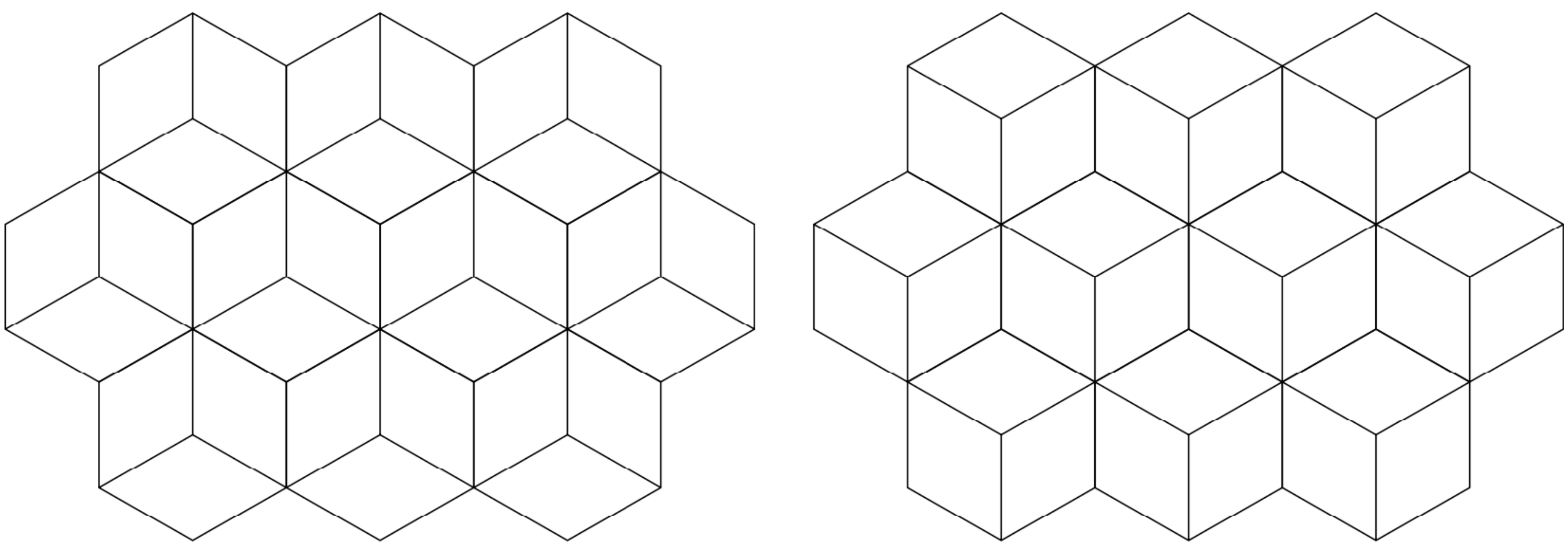




$\mathbb{Z}^3$ Q-nets

- The dynamics is obtained by prescribing the points of the $\mathbb{Z}^3$ Q-net along a stepped surface (Cauchy data) and propagating to obtain the points on the next layer.



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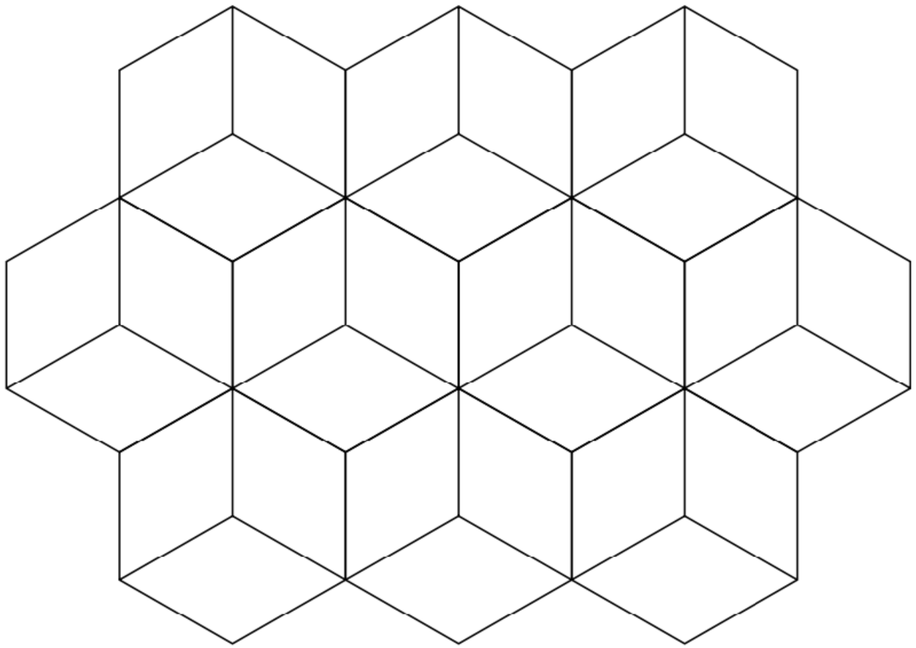

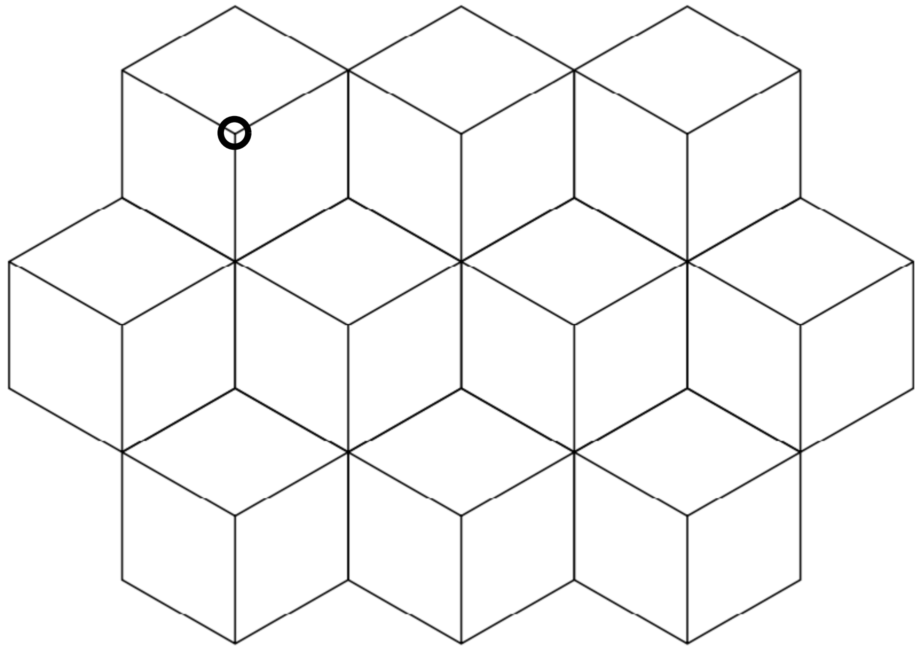

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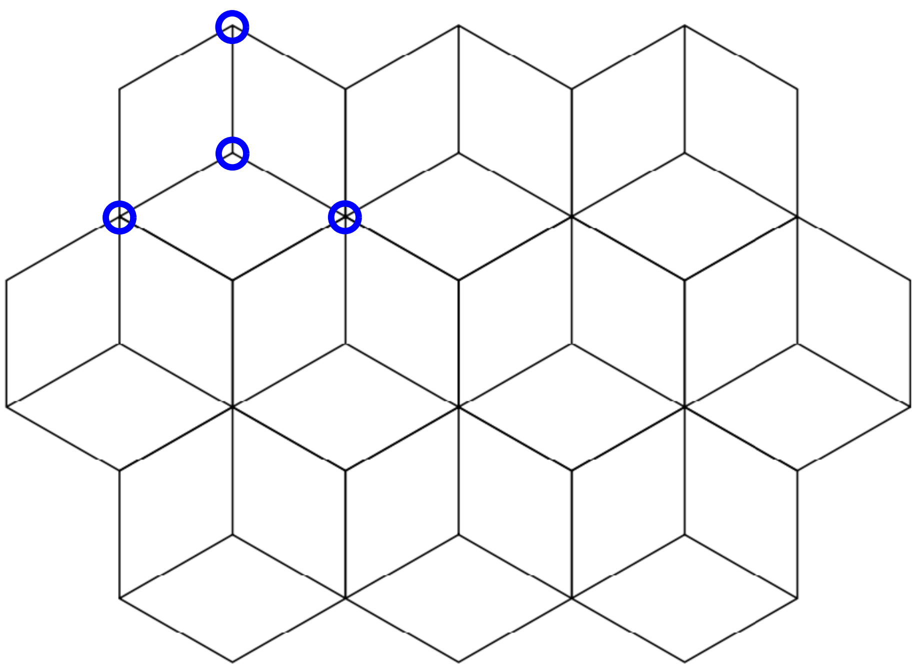

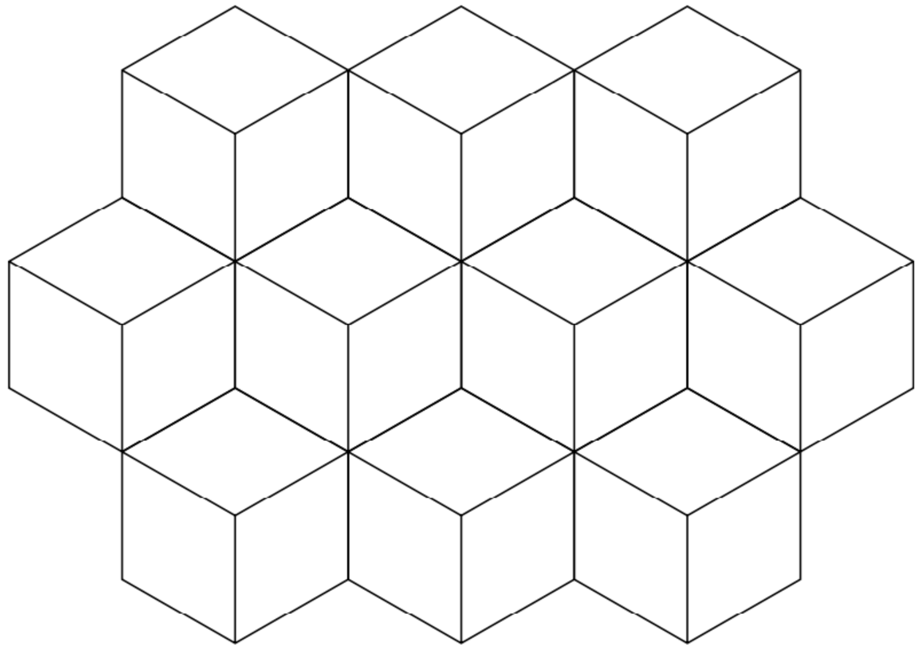

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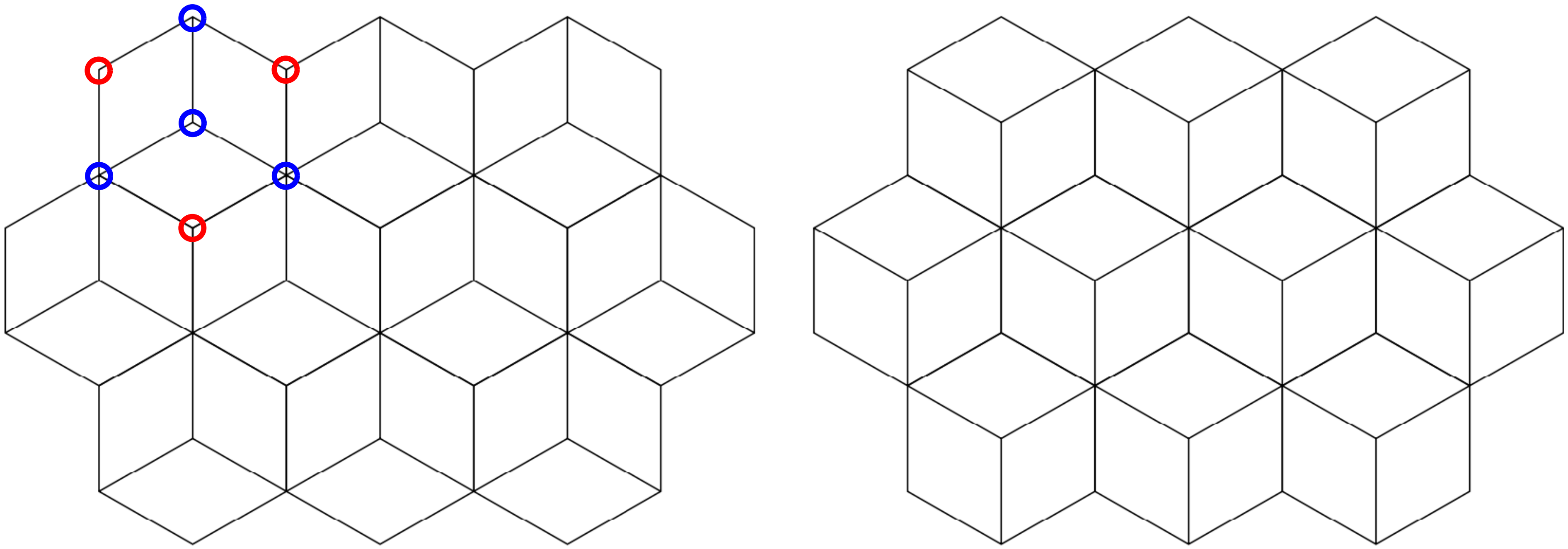

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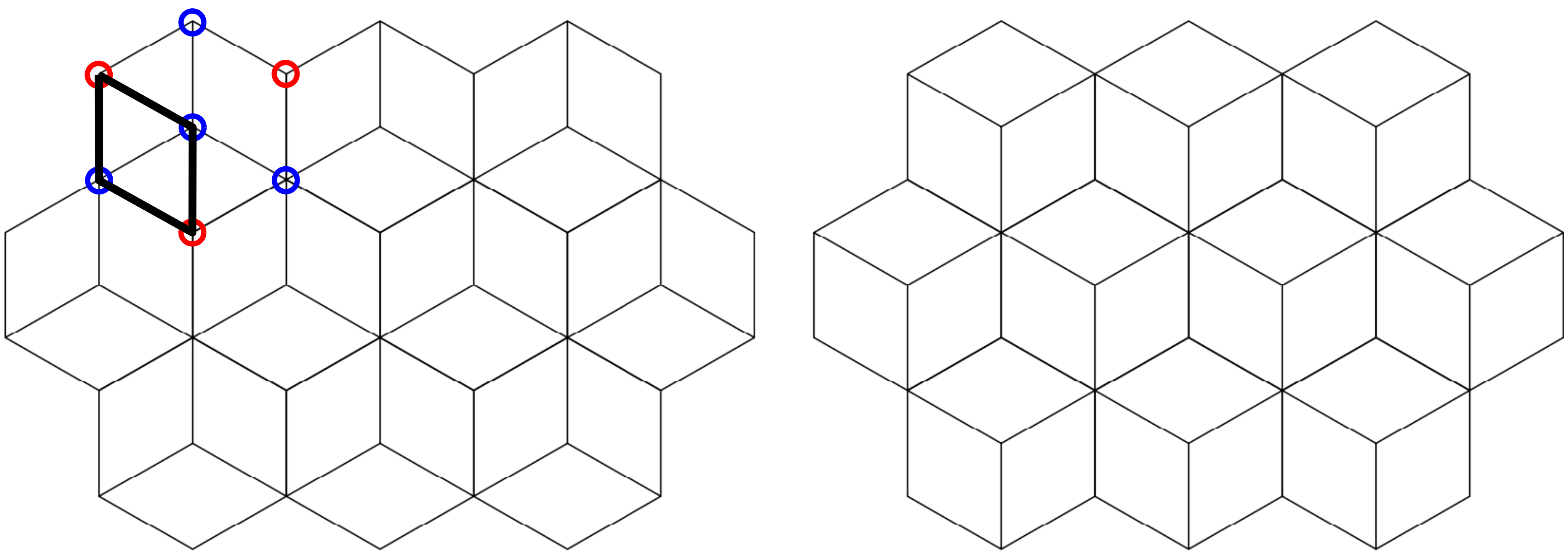

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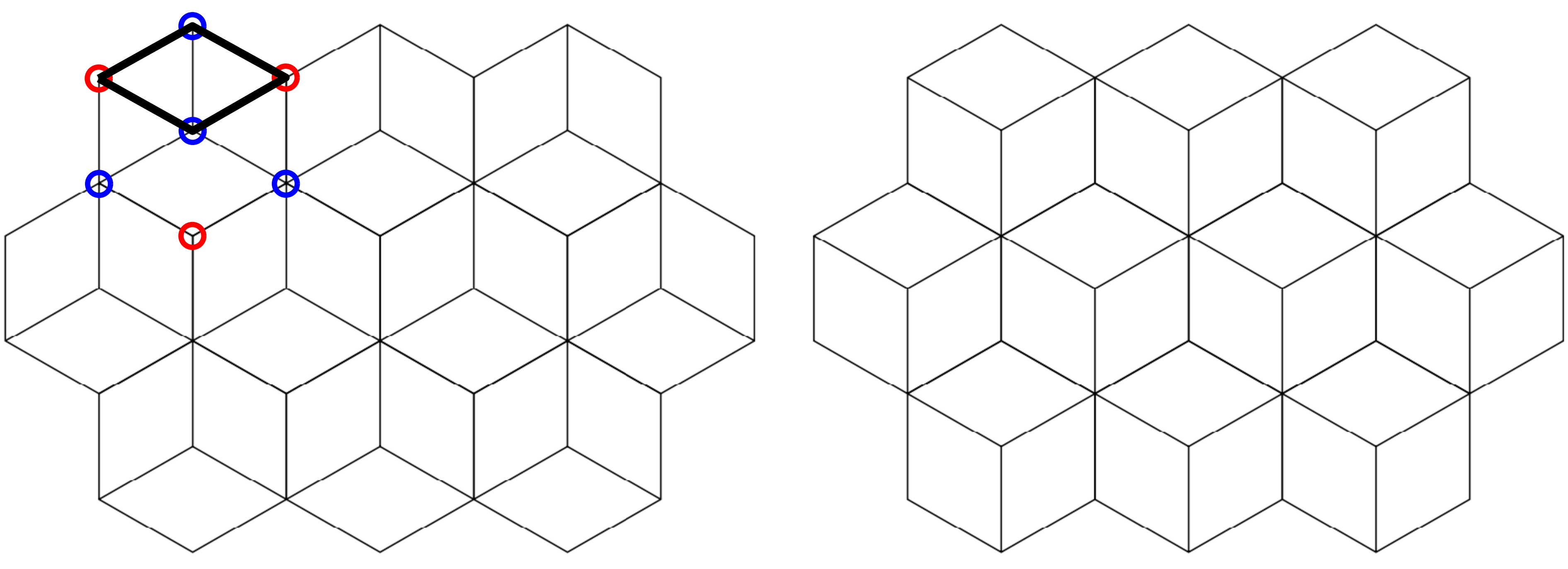

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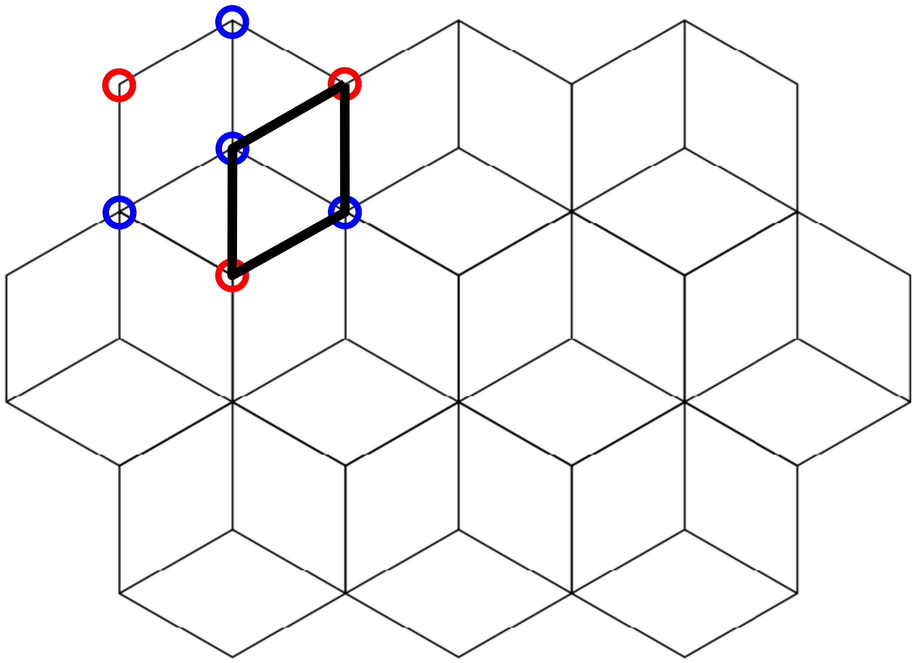

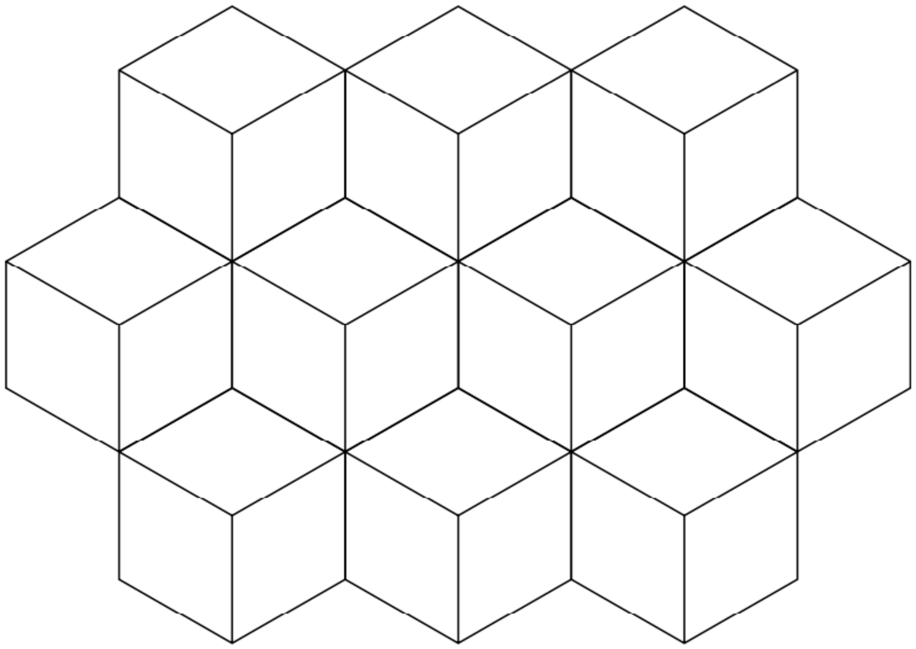

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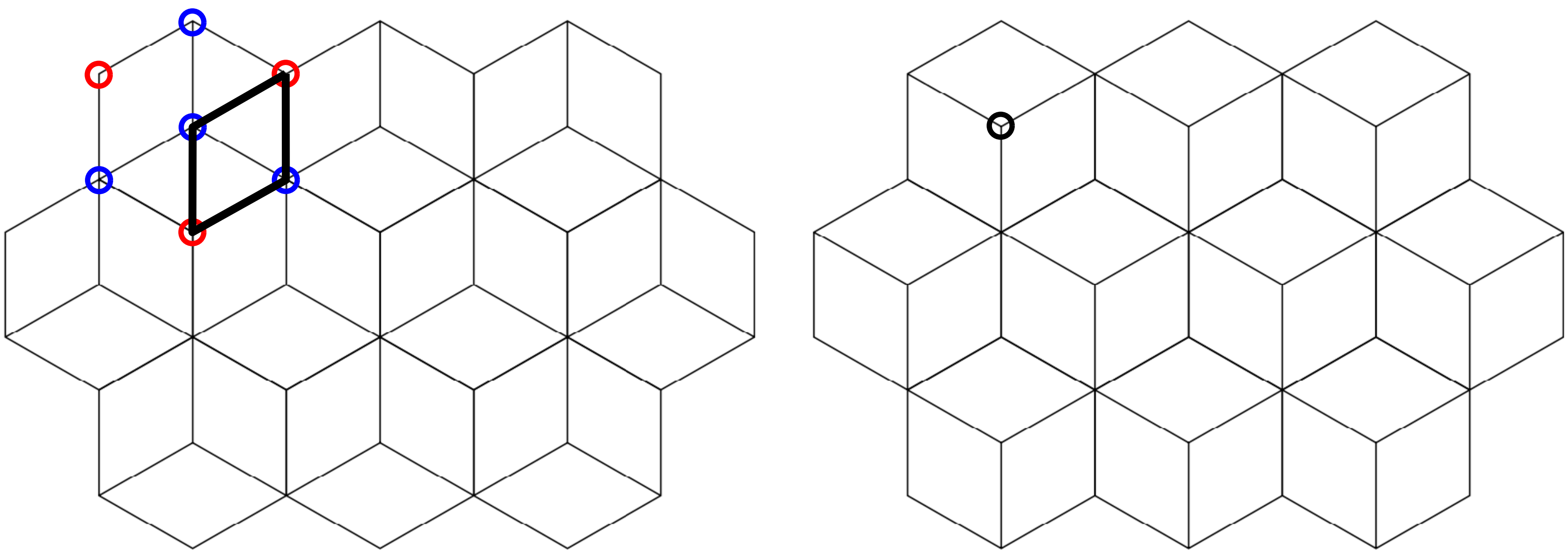
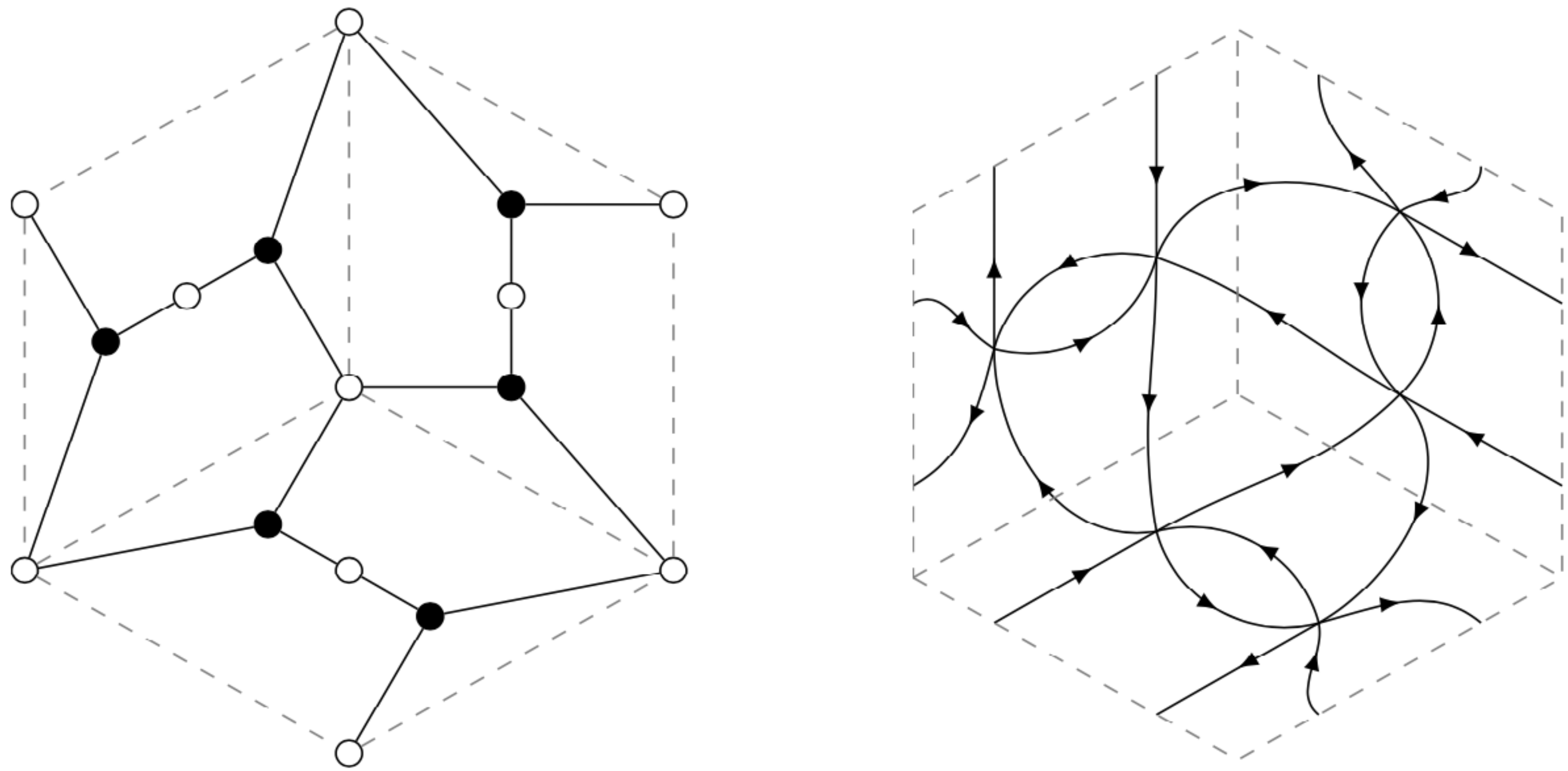
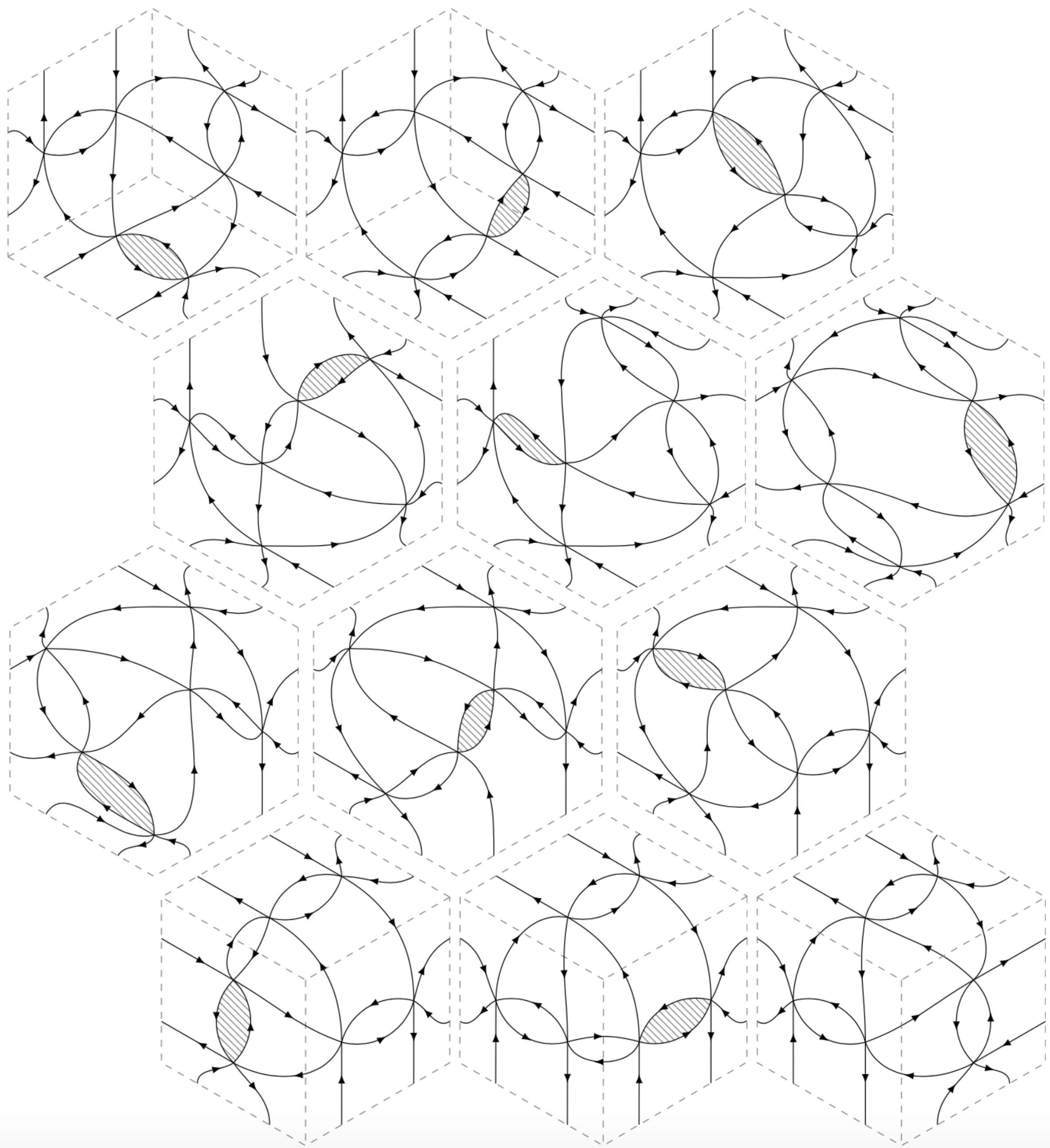


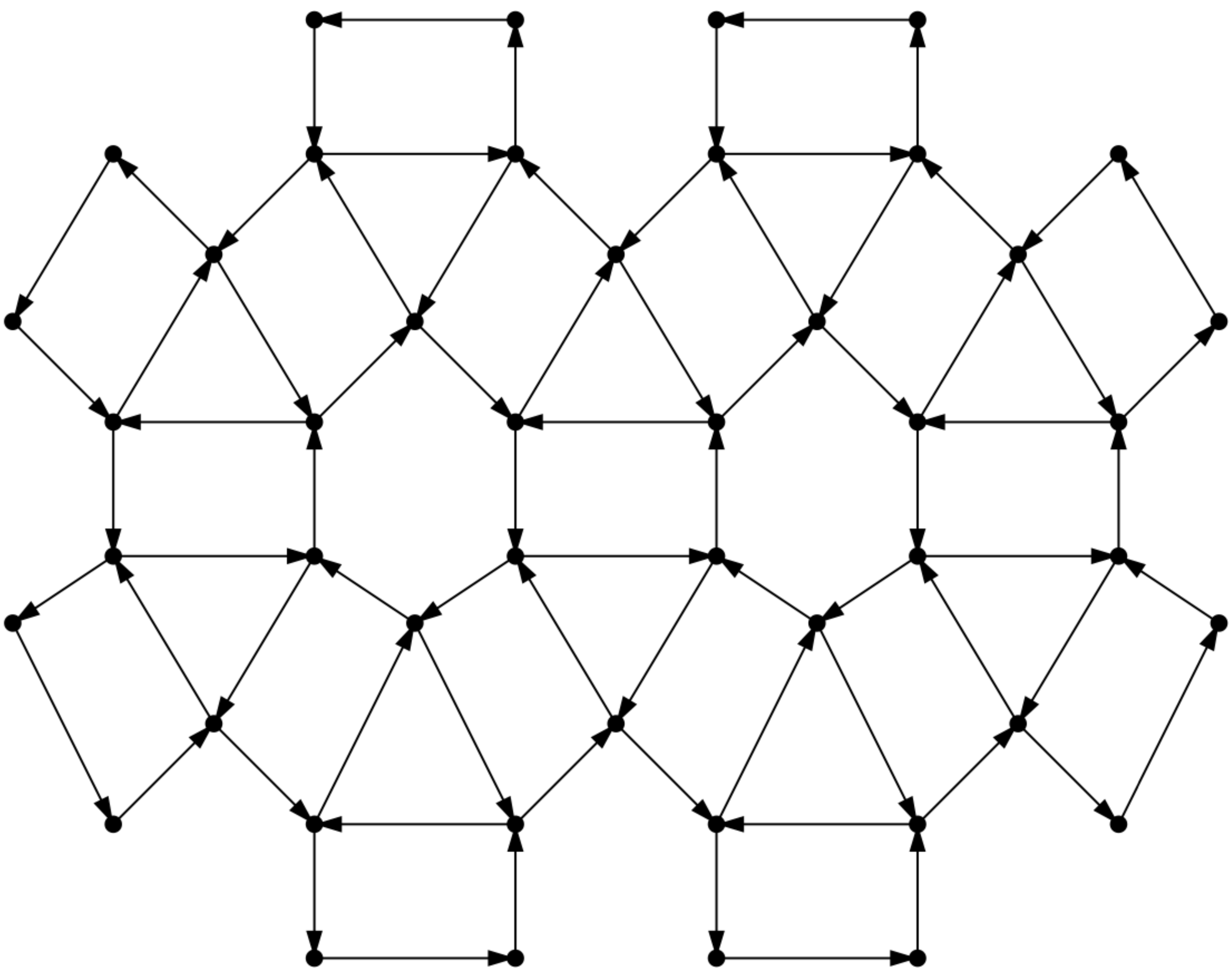
Realization as a TCD map



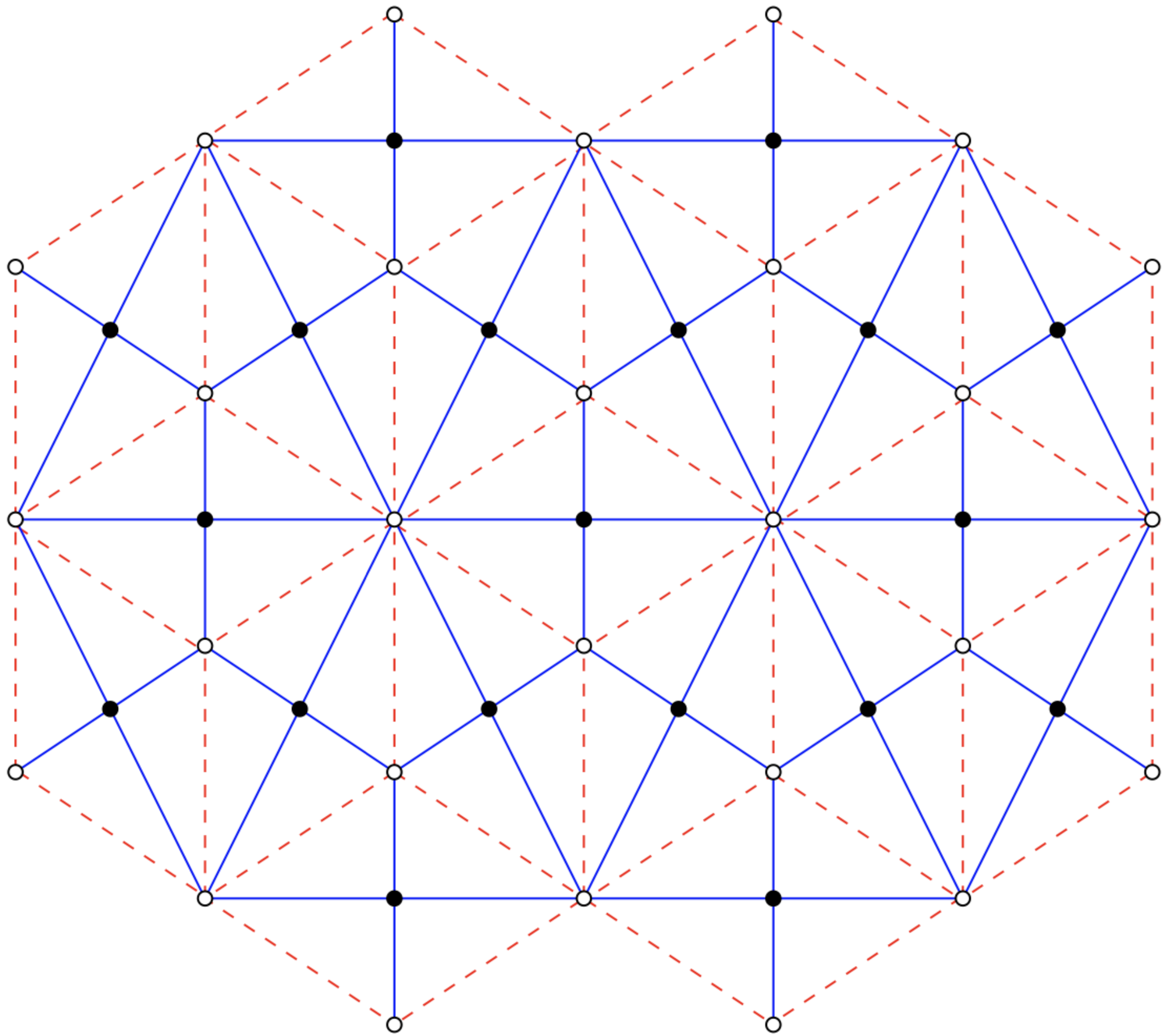
Theorem (AGPR,AGR)**.** *The cube flip producing the point associated to the new cube vertex arises as a composition of seven resplits and four spider moves.*



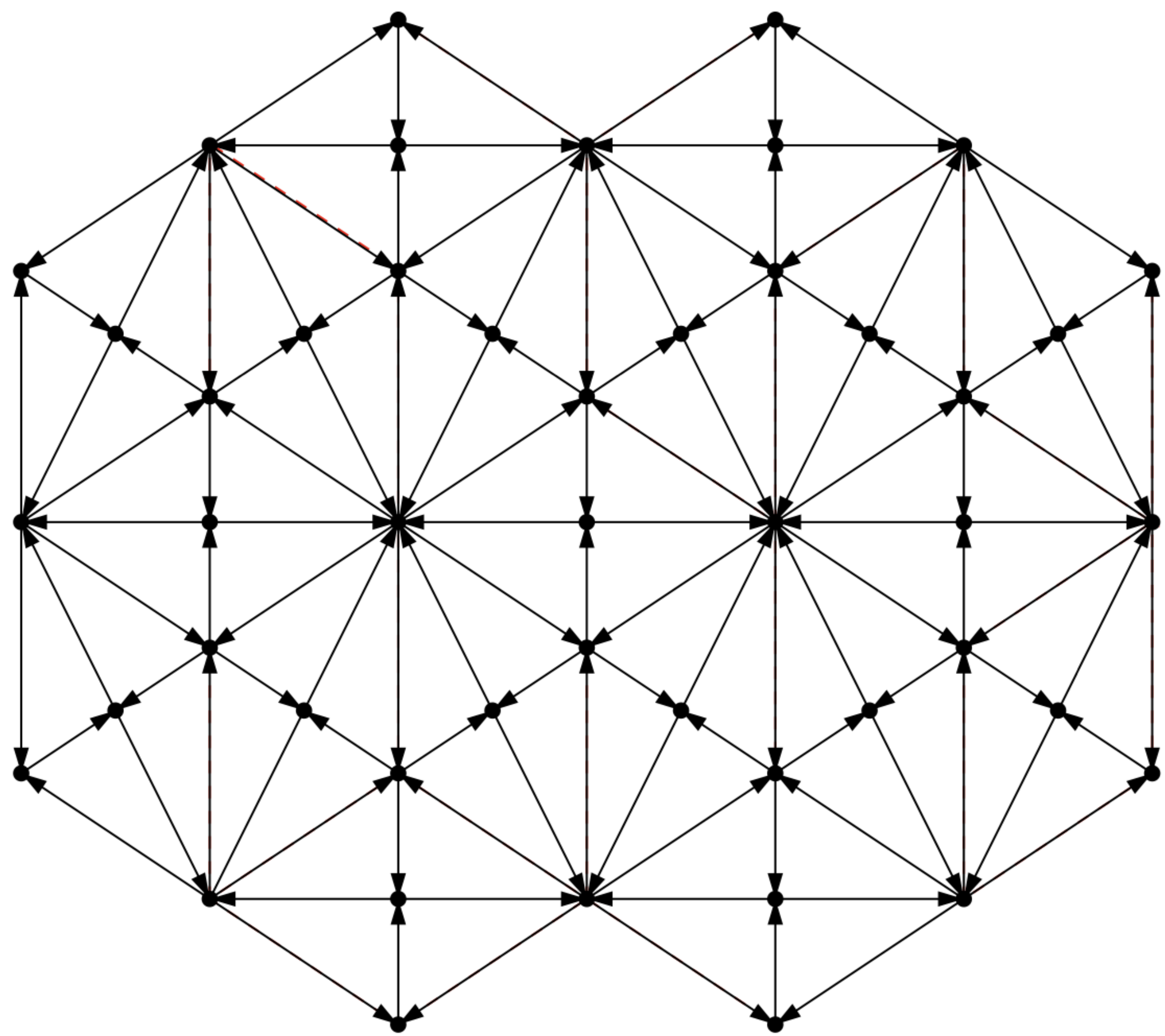
A large portion of the projective quiver for $\mathbb{Z}^3$ Q-nets



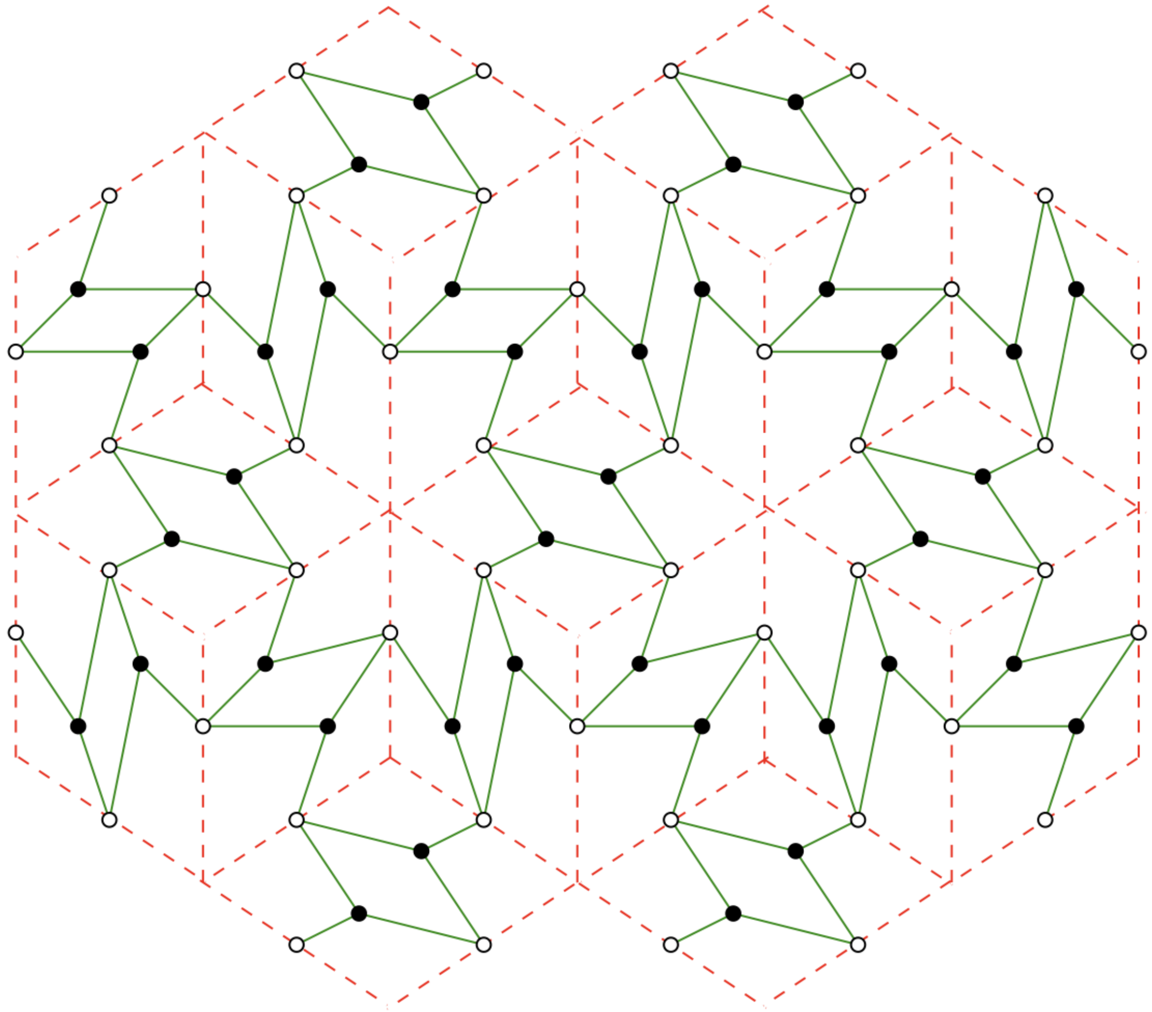
Taking its dual graph, we recognize the bipartite graph associated with spanning trees on the hexagonal lattice.



A large portion of the affine quiver for $\mathbb{Z}^3$ Q-nets



The dual graph of the affine quiver for $\mathbb{Z}^3$ Q-nets
(NOT the VRC for $\mathbb{Z}^3$ Q-nets !)



Dubédat bipartite graph for Ising on the hexagonal lattice

3D circular nets

- Assume the Cauchy data of a $\mathbb{Z}^3$ Q-net is such that any four points around a quad are concyclic and not just coplanar. Then this property is preserved after doing cube flips (Miquel, ca. 1850). This is called a 3D circular net.

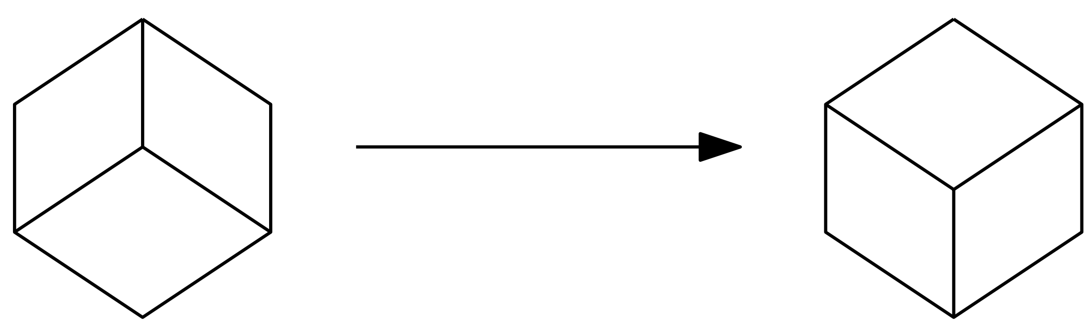
Theorem (AGR). *We recover the Poisson bracket (and probably the quantization) for 3D circular nets of Bazhanov-Mangazeev-Sergeev via the affine cluster structure for $\mathbb{Z}^3$ Q-nets.*

Darboux maps

- Fix $n \geq 2$. A ($\mathbb{Z}^3$) Darboux map is a map from the edges of $\mathbb{Z}^3$ to $\mathbb{C}P^n$ such that the images of any 4 points around a 2-cell are colinear.

- The dynamics consists in propagating some Cauchy data starting from a stepped surface.

Cube flip:



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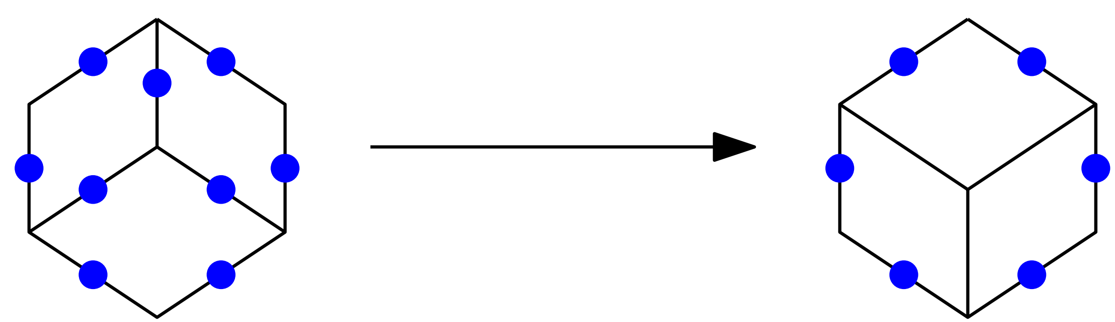

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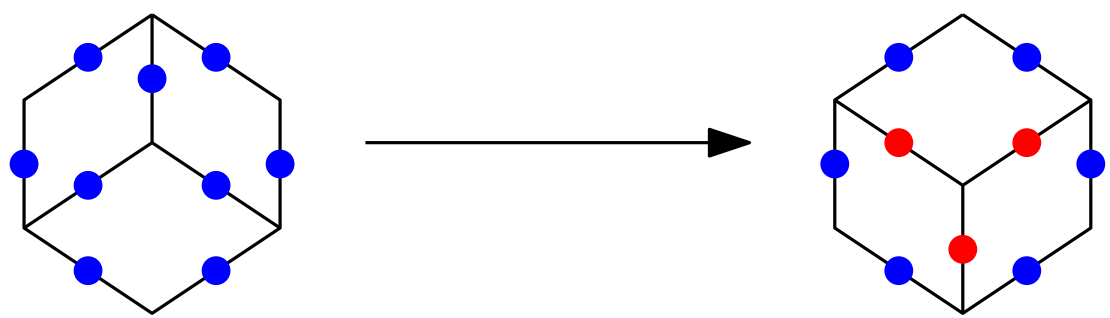
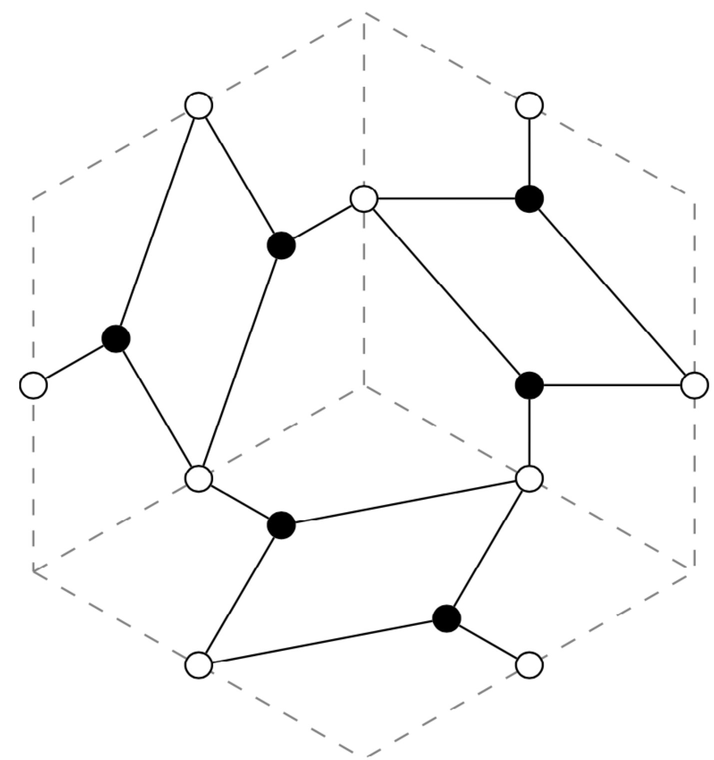
Cube flip:



AGPR,AGR: The TCD map for Darboux maps is obtained by gluing pieces like this one.



- Its projective quiver is the Ising quiver and its affine quiver is the spanning tree quiver.
- In terms of TCDs, a cube flip is realized by seven spider moves and four resplits.

Lines complexes

- Fix $n \geq 4$. A ($\mathbb{Z}^3$) line complex is a map from the 2-cells of $\mathbb{Z}^3$ to $\mathbb{C}P^n$ such that the images of any 6 points associated with 6 2-cells around a cube are colinear.

- Again a local propagation rule, described in terms of 2-2 moves for TCDs.

- AGR: both the projective and the affine quivers for line complexes are the spanning tree quiver.

Lines complexes

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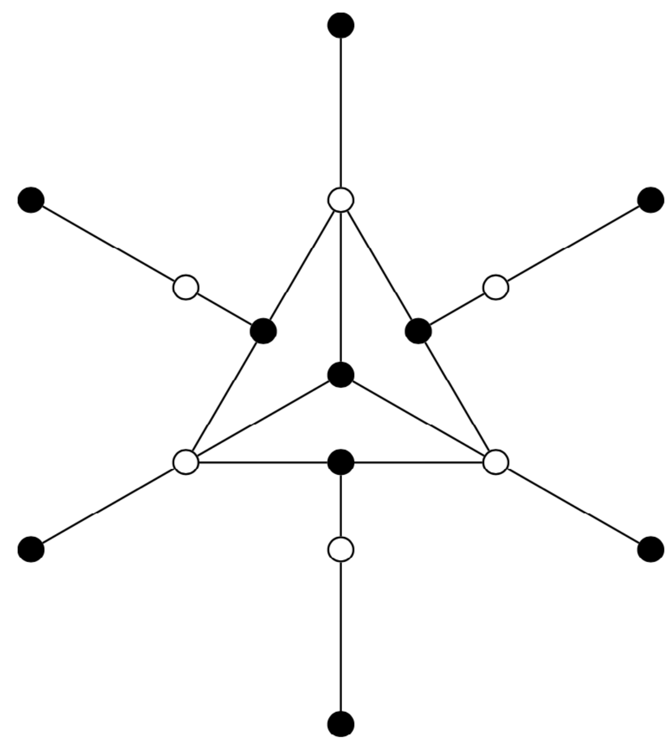
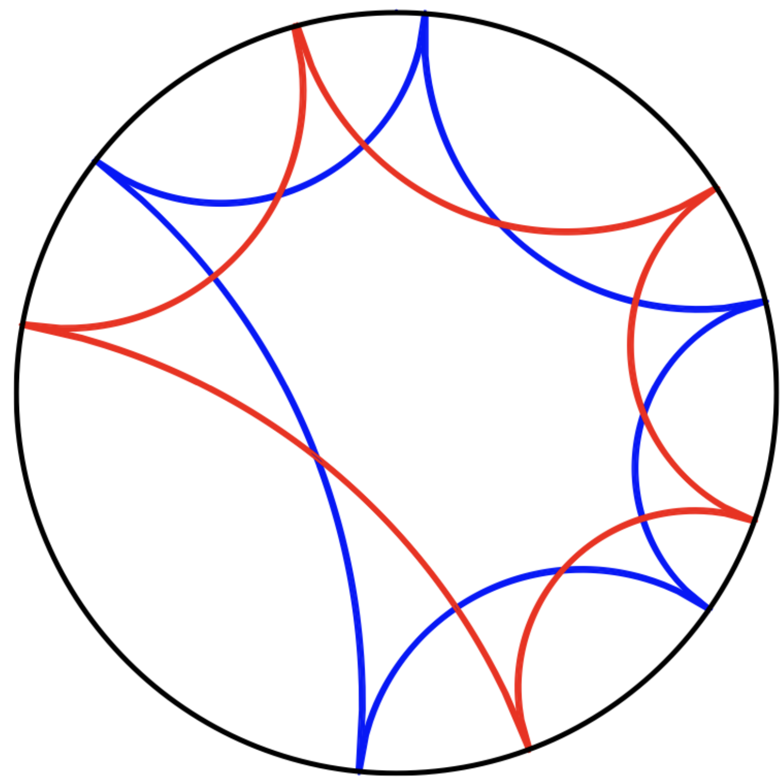
TCD for line complexes



Cross-ratio dynamics

Given an ideal n-gon P_0 in the hyperbolic plane there are two n-gons P_{-1} and P_1 with sides pairwise orthogonal to those of P_0.

Work in progress with Niklas Affolter and Terrence George: cluster structure for this dynamics



Picture from the Arnold-Fuchs-Izmestiev-Tabachnikov paper

3 Operations on TCD maps and more cluster structures

Projective vs affine cluster structures

- Given a planar quiver Q with alternating orientations in/out around each vertex, construct a bipartite graph G such that Q is the projective/affine quiver of G.

- For a projective quiver, one can reconstruct G_p up to resplits.

- For an affine quiver, one can reconstruct G_a up to spider moves.

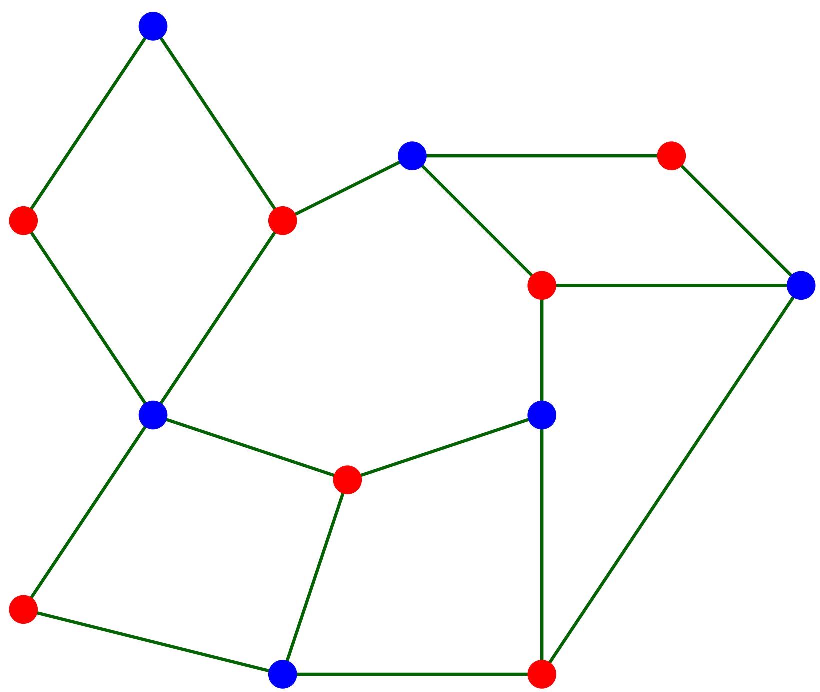
- Combinatorial relation between G_p and G_a? Geometric relation between TCD maps associated with G_p and G_a?

Tutte's trinity

“black” vertex

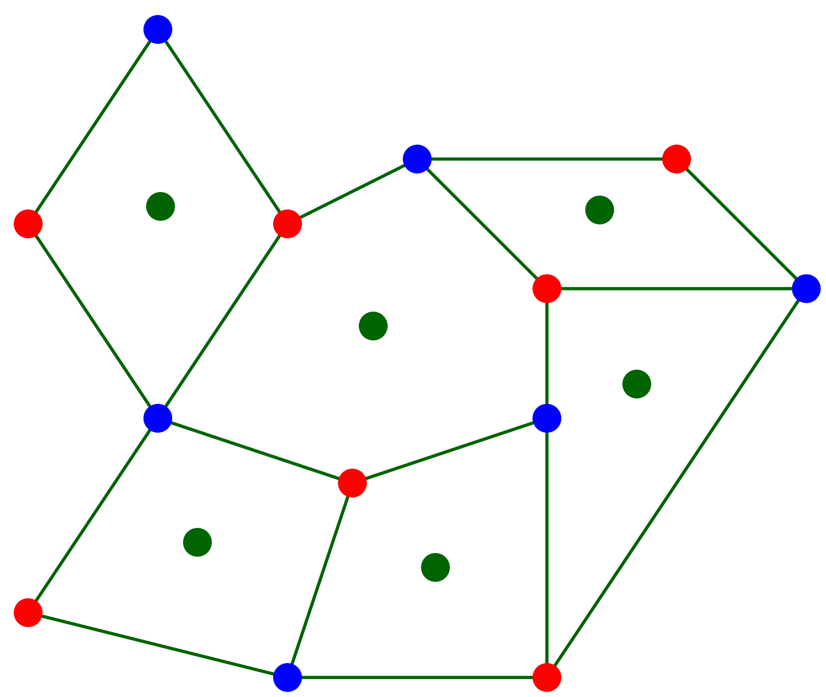
“white” vertex

edge



Tutte's trinity

- "black" vertex
- "white" vertex
- dual vertex
- edge



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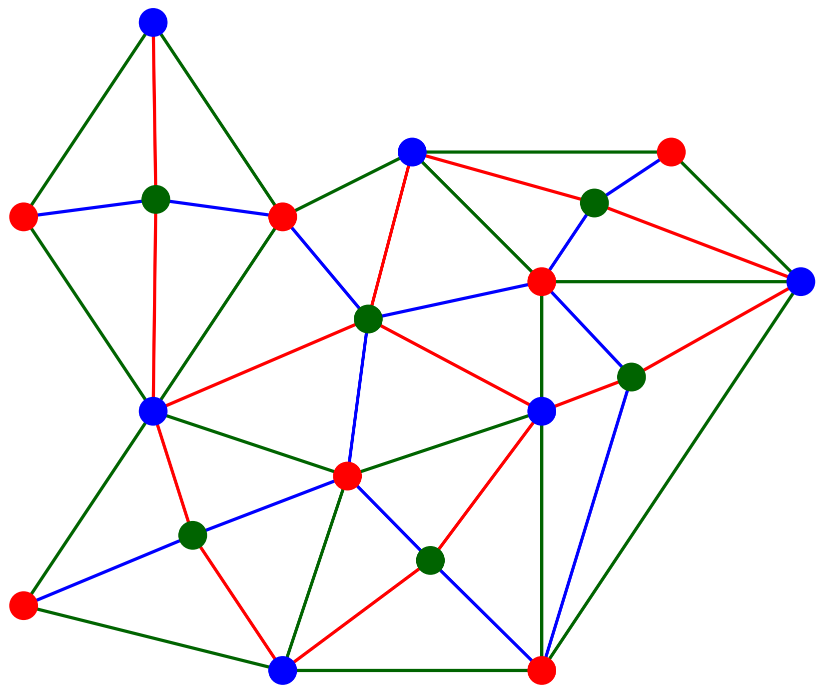

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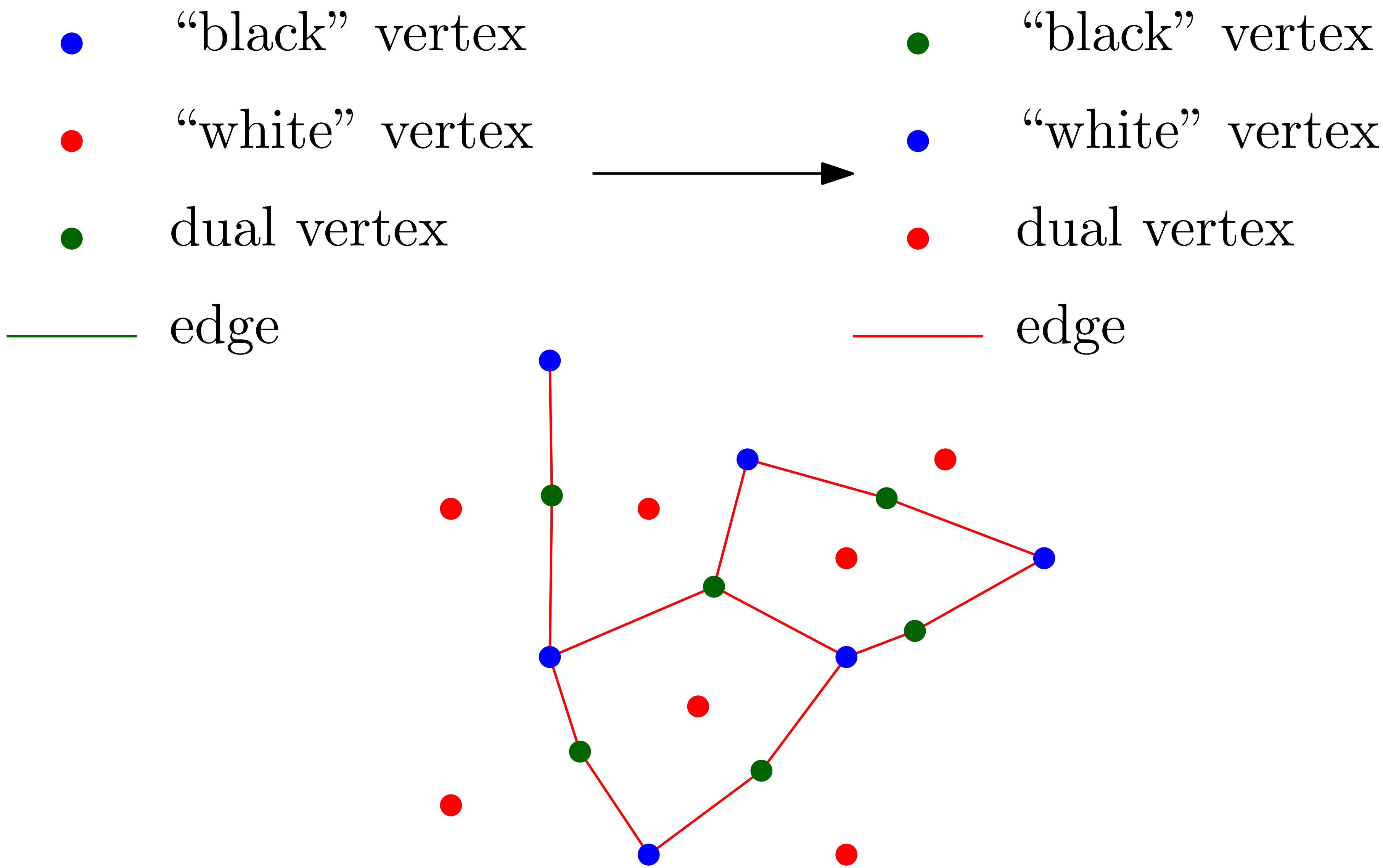
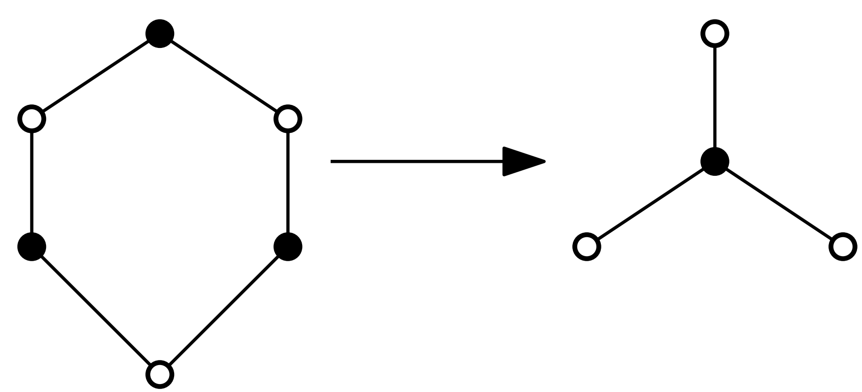


Section of a TCD map

- Let T be a TCD map to $\mathbb{C}P^n$ and let E be a hyperplane of $\mathbb{C}P^n$. Denote by L_b the line at black vertex b.

- The section $\sigma_E(T)$ is the TCD map obtained by "rotating the colors" and placing at each black vertex b of T the point $L_b \cap E$.

Faces of T induce relations among points of $\sigma_E(T)$.



Theorem (AGR). *The affine cluster structure of T relative to E is equal to the projective cluster structure of $\sigma_E(T)$ (quivers and variables coincide).*

- Iterated sections are well-defined. If H and H' are two hyperplanes of $\mathbb{C}P^n$, one can unambiguously define $\sigma_{H\cap H'}(T)$.

- Associated to a single TCD map to $\mathbb{C}P^n$, we obtain $n+1$ cluster structures.

$$\begin{array}{r} \mathcal{T}_4 \\ \sigma_{E_3}\downarrow \\ \mathcal{T}_3 \\ \sigma_{E_2}\downarrow \\ \mathcal{T}_2 \\ \sigma_{E_1}\downarrow \\ \mathcal{T}_1 \end{array}$$

- A section of a $\mathbb{Z}^2$ Q-net is a $\mathbb{Z}^2$ Q-net.
- A section of a $\mathbb{Z}^3$ Q-net is a Darboux map.
- A section of a Darboux map is a line complex.
- A section of a line complex is a $\mathbb{Z}^3$ Q-net.

Projective duality

- Another operation on flags of TCD maps, produces a dual flag.

- All the affine cluster structures of the dual flag are related to those of the primal flag by reverting quiver arrows and inverting variables.

- In total, $n+2$ cluster structures associated with a TCD map to $\mathbb{C}P^n$.

$$
\begin{array}{ccc}
\mathcal{T}_4 & \qquad & \mathcal{T}_4^* \\
\sigma_{E_3}\downarrow & & \downarrow \sigma_{E_3^*} \\
\mathcal{T}_3 & & \mathcal{T}_3^* \\
\sigma_{E_2}\downarrow & & \downarrow \sigma_{E_2^*} \\
\mathcal{T}_2 & & \mathcal{T}_2^* \\
\sigma_{E_1}\downarrow & & \downarrow \sigma_{E_1^*} \\
\mathcal{T}_1 & & \mathcal{T}_1^*
\end{array}
$$

THANK YOU !