Miquel dynamics for circle patterns

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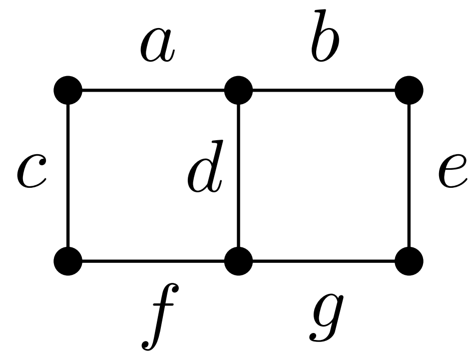
- Circle patterns form a well-studied class of objects in discrete differential geometry (discretization of conformal maps).

- Many discrete integrable systems have been discovered recently (pentagram, dimers,...).

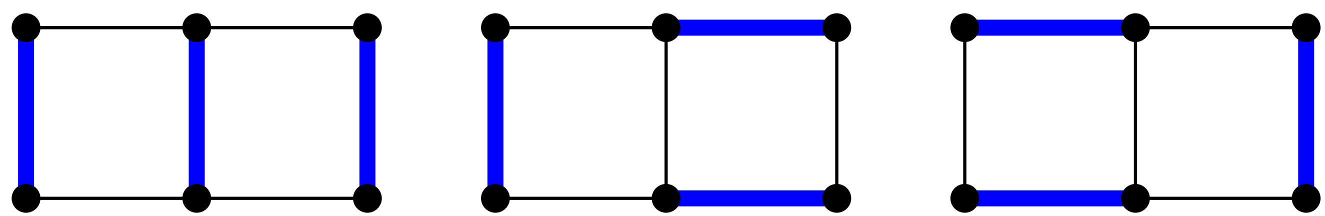
- Attempt to construct a discrete integrable system on some space of circle patterns.

- First review the Goncharov-Kenyon discrete integrable system for dimers.

The dimer model



Configurations :

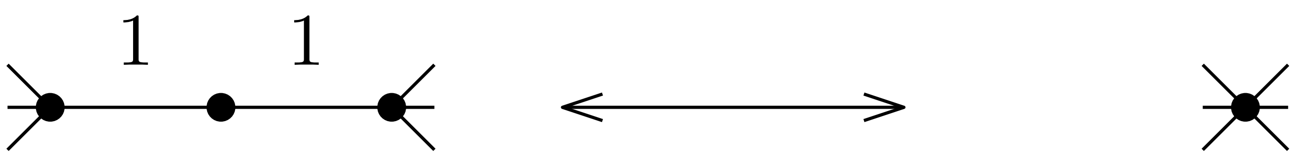


Weights : cde bcg aef

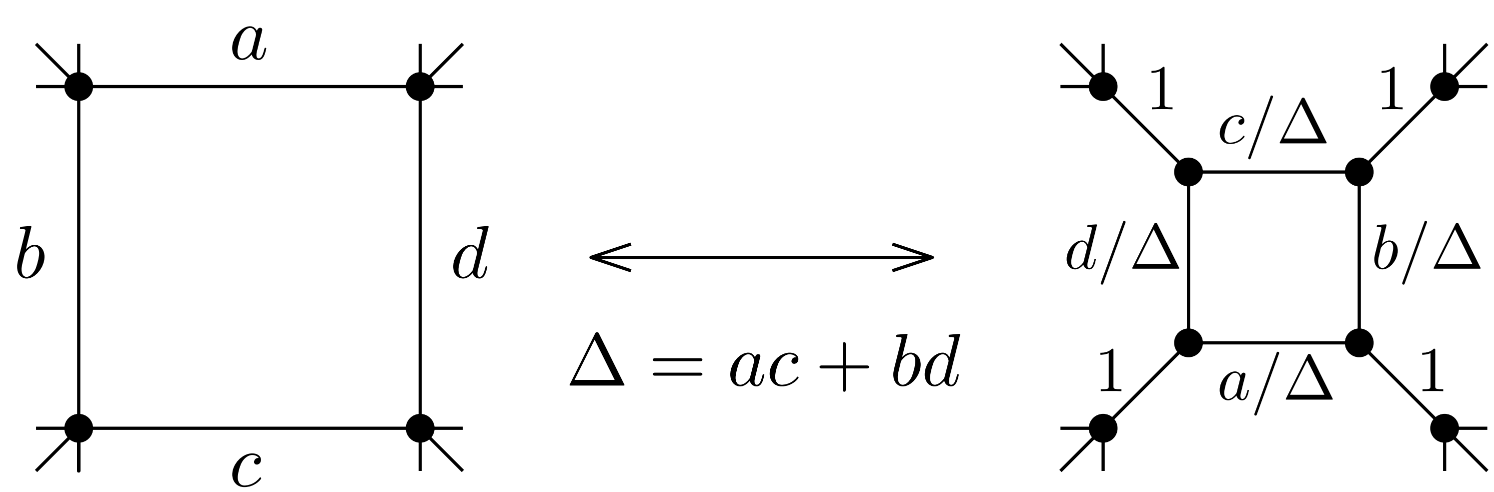
- Probabilistic model : draw a configuration at random with probability proportional to its weight. Edge correlations ?

Dimer local moves

- Contraction of degree 2 vertices :

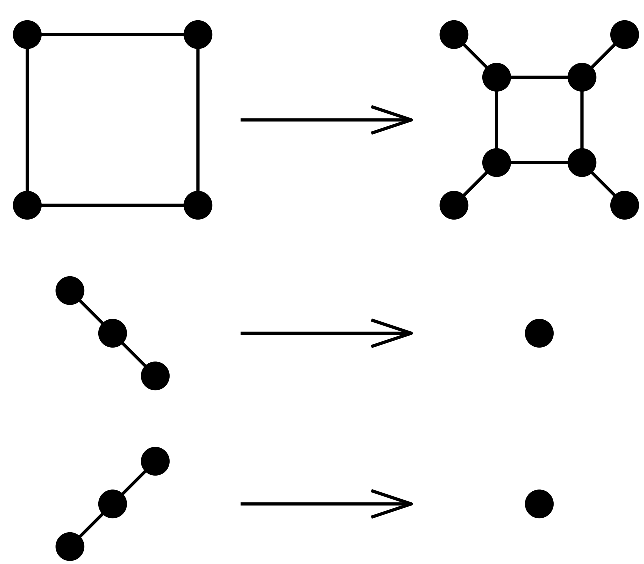


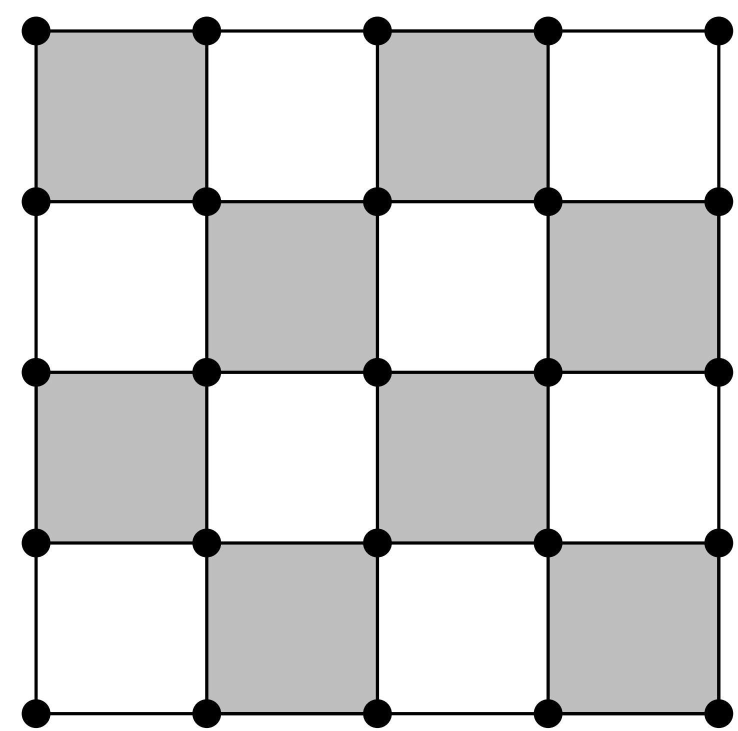
- Urban renewal :



Goncharov-Kenyon dimer dynamics

- Start with $\mathbb{Z}^2$ with some edge weights. At even (resp. odd) times, perform an urban renewal on each white (resp. black) face followed by the contraction of all the degree 2 vertices. Get $\mathbb{Z}^2$ with different edge weights.





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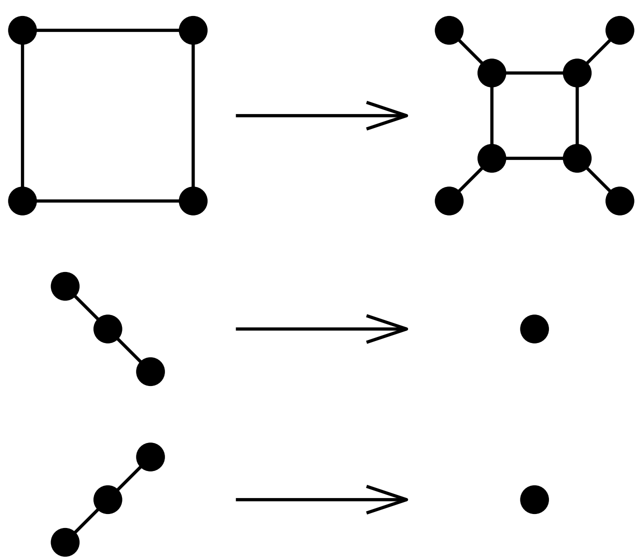

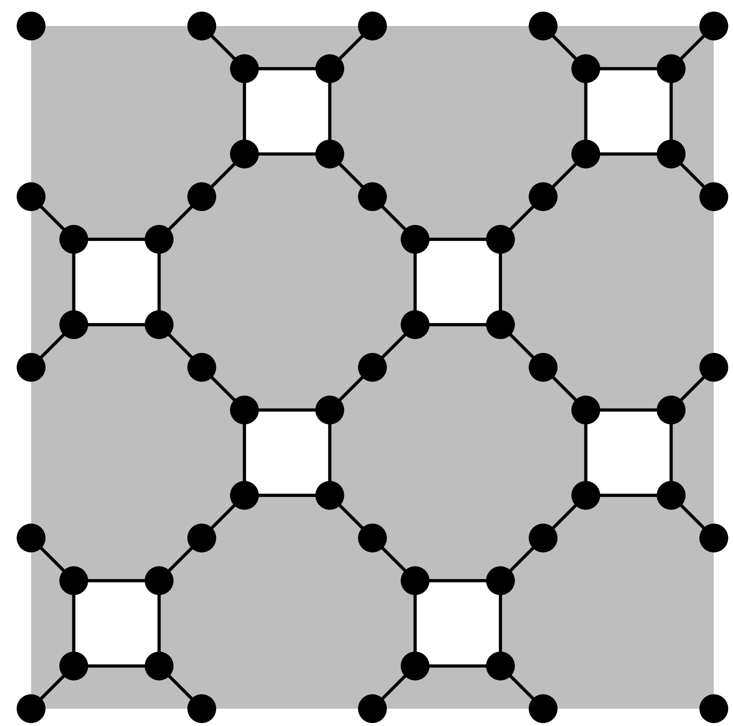

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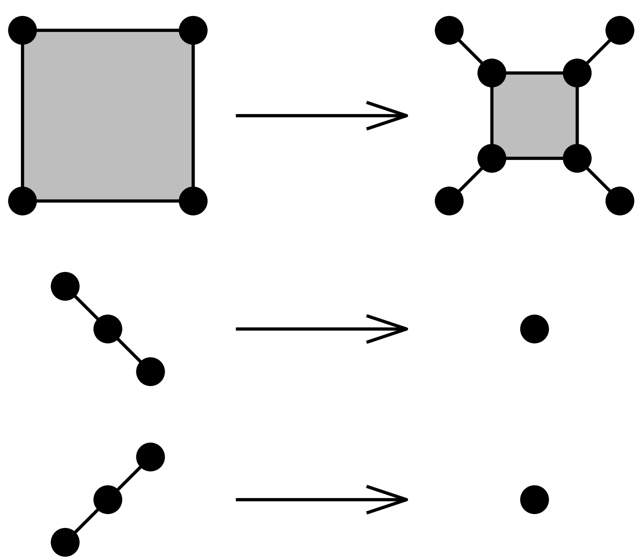

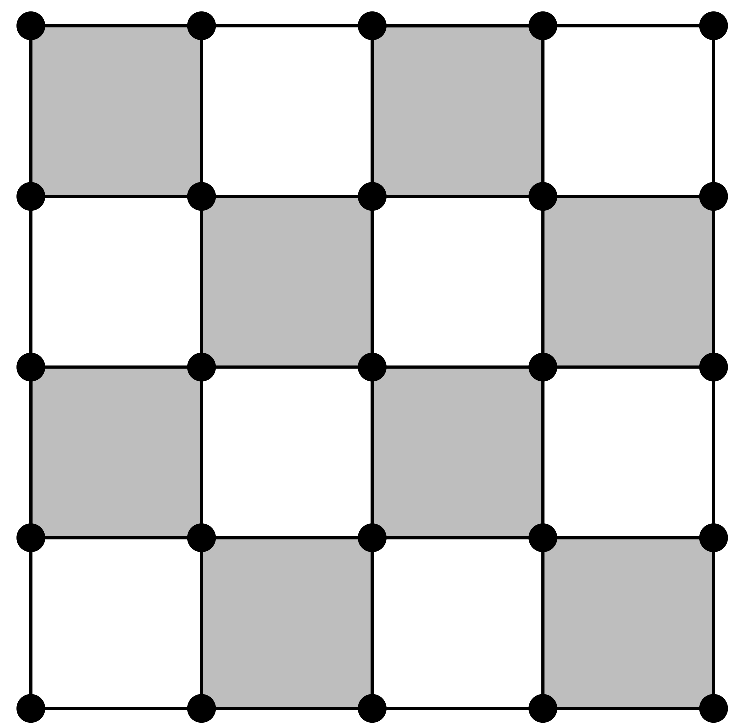

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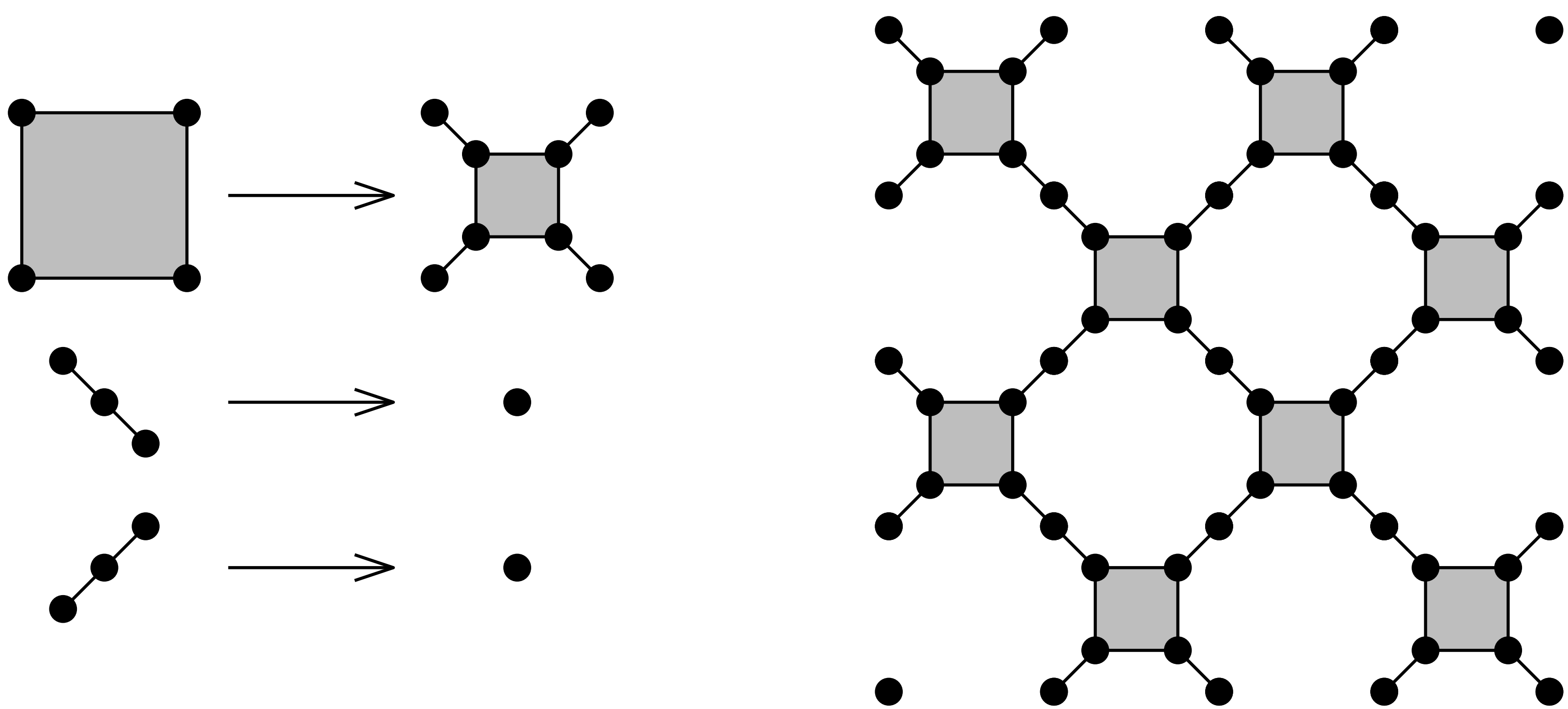

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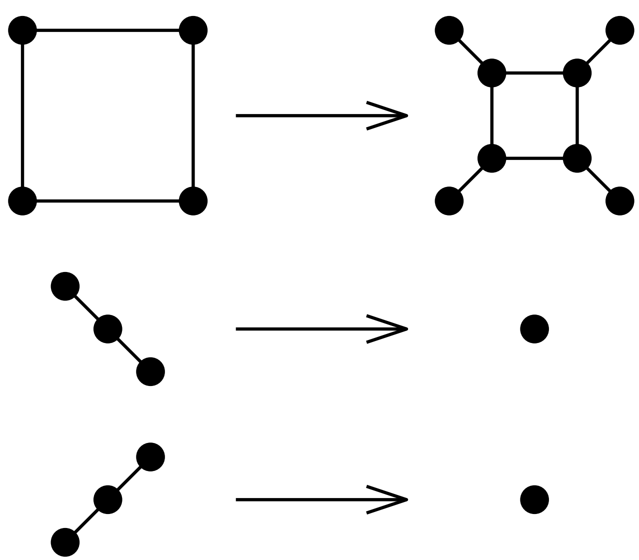
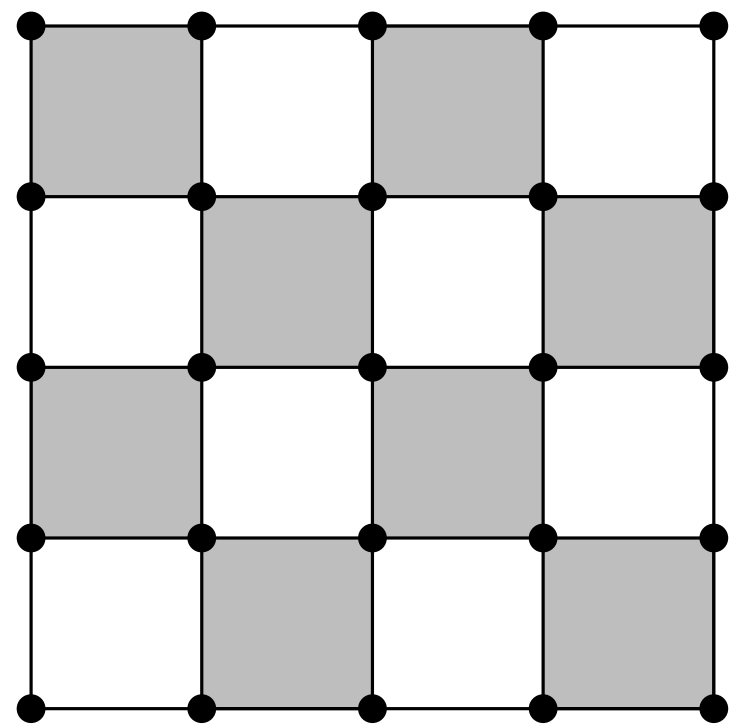
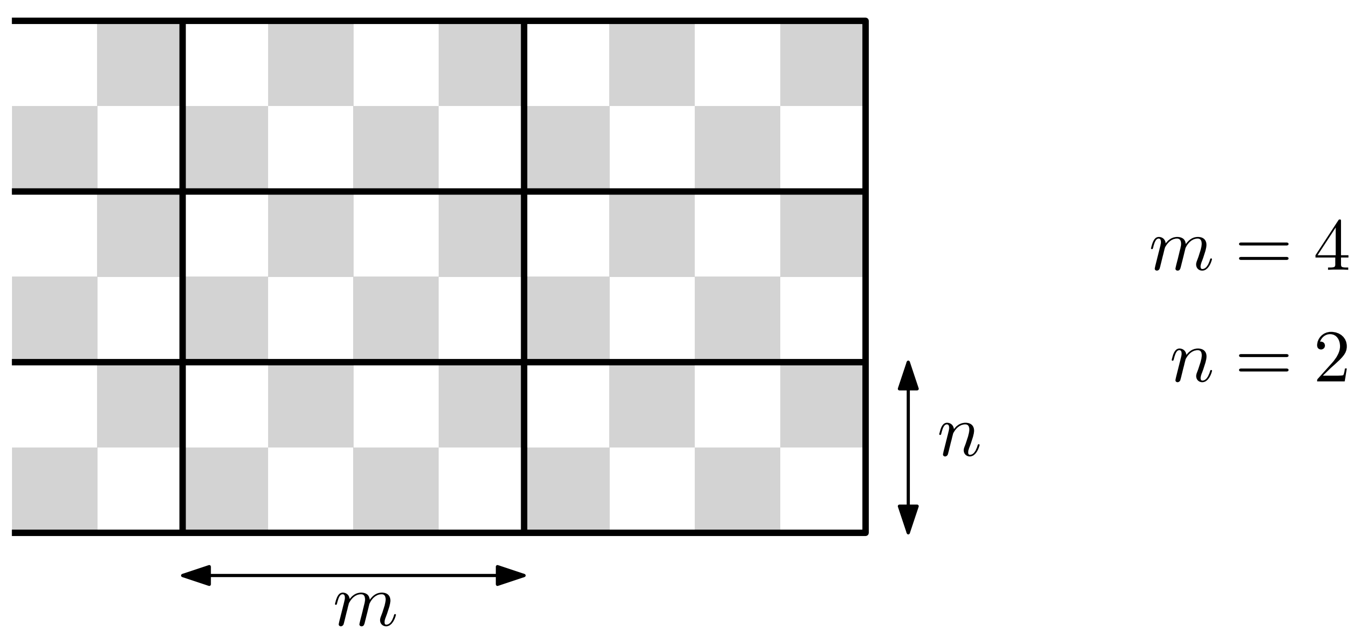




- Start with spatially biperiodic edge weights, with fundamental domain of size $m \times n$. The weights remain $m \times n$ biperiodic under the dynamics.

- It is seen as an $m \times n$ square grid graph on the torus.

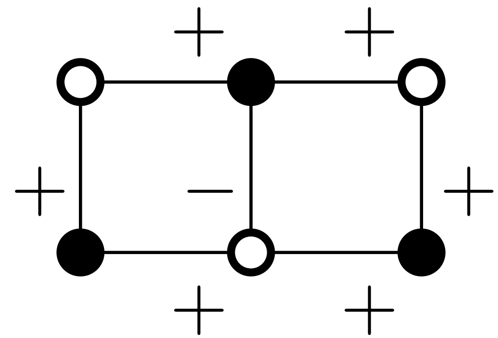


The Kasteleyn operator K

- K is a weighted signed adjacency matrix of the graph with edge weights.

- For planar graphs, the determinant of K gives the partition function (sum of the weights of all dimer configurations). The dimer correlations are given by minors of K^{-1} (Kasteleyn, Temperley, Fisher).

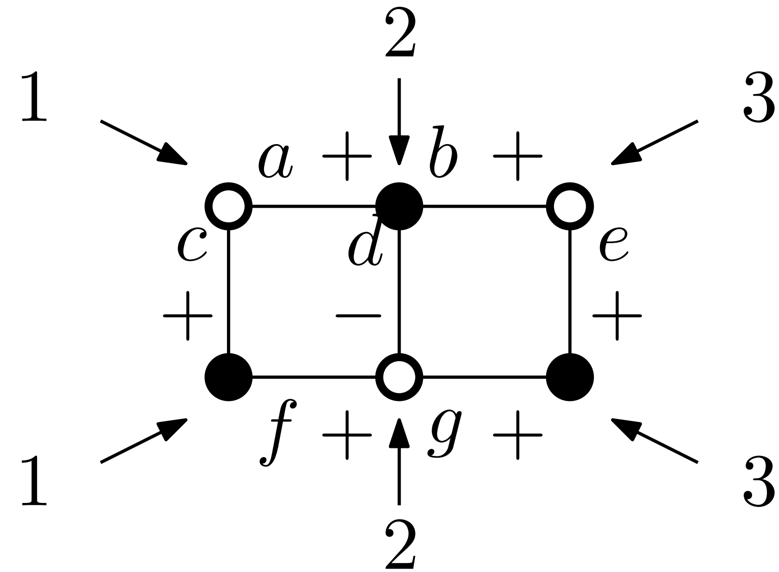
- For graphs on a torus, the Fourier transform of K will give the spectral curve.



- We shall only consider bipartite graphs, that is graphs where the vertices can be colored black and white with each edge connecting a black vertex to a white vertex.

- Kasteleyn signing : assign a sign to each edge such that the number of minus signs around a face of degree 2 mod 4 (resp. 0 mod 4) is even (resp. odd).

- K : weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

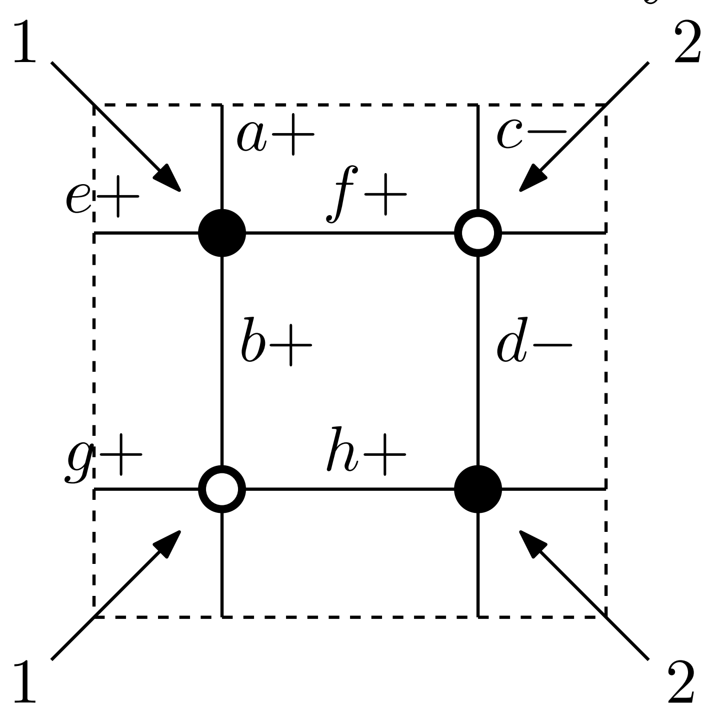


$$
\begin{array}{c} \\ \circ_1 \\ \circ_2 \\ \circ_3 \end{array}
\begin{array}{c}
\begin{array}{ccc} \bullet_1 & \bullet_2 & \bullet_3 \end{array} \\
\begin{pmatrix} c & a & 0 \\ f & -d & g \\ 0 & b & e \end{pmatrix}
\end{array}
$$

- K : weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

- Consider a bipartite graph on a torus together with two cycles generating the first homology group of the torus.

- $K(z,w)$: for an edge from a white vertex to a black vertex, the corresponding coefficient is multiplied by z, z^{-1}, w or w^{-1} if the edge crosses the right, left, top or bottom boundary of the fundamental domain.



$$K(z,w) = \begin{pmatrix} b + aw^{-1} & h + gz^{-1} \\ f + ez & -d - cw \end{pmatrix}$$

Spectral data

- Spectral curve : the set of all $(z, w) \in (\mathbb{C}^*)^2$ such that $\det K(z, w) = 0$.

- Associate to each (z, w) on the spectral curve a vector in the kernel of $K(z, w)$.

- Divisor on the spectral curve : values of (z, w) on the spectral curve for which the first coordinate of the vector vanishes.

Algebro-geometric integrability

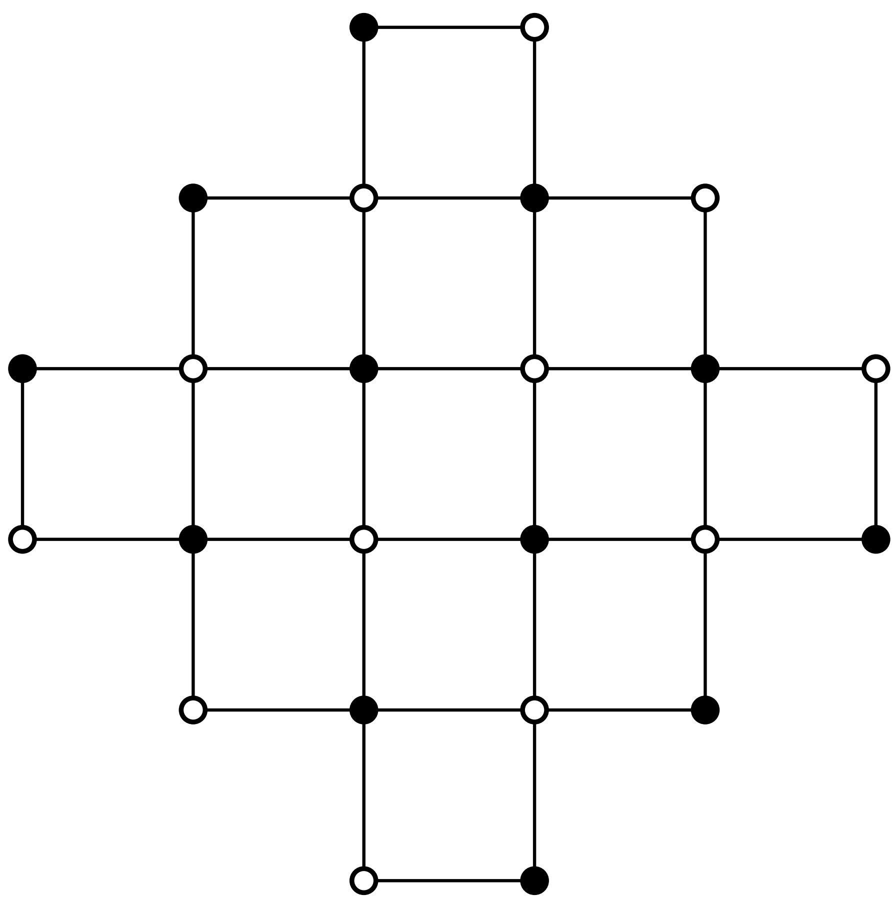
- Consider the Goncharov-Kenyon dynamics for $m \times n$ square-grid dimer model on the torus.

- The dynamics preserves the spectral curve, hence the integrals of motion are the coefficients of the polynomial $\det K(z, w)$.

- The motion of divisor corresponds to translation on the Jacobian of the spectral curve.

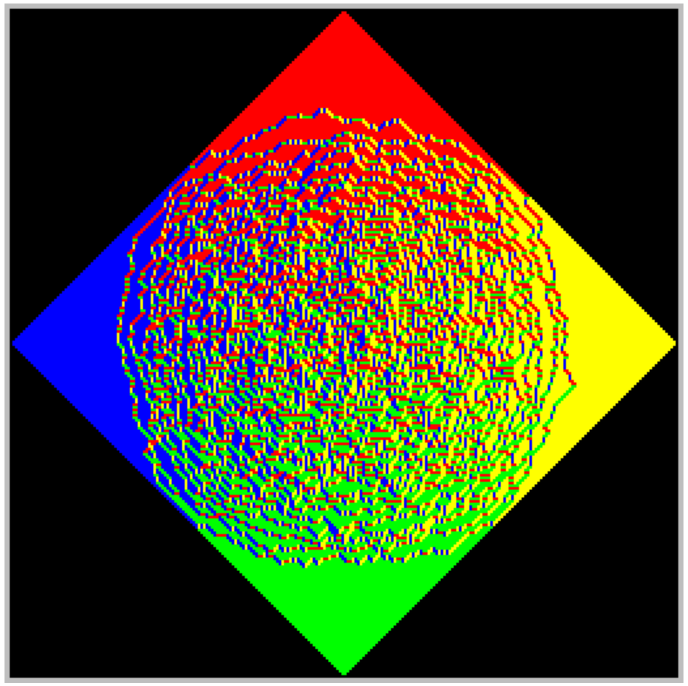
Unrelated dimer story : Aztec diamond

Aztec diamond of size 3



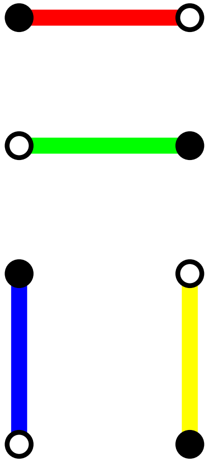
Pick a dimer configuration of the Aztec diamond of size n uniformly at random.

- Appearance of a limit shape in the limit $n \to \infty$.



picture by Cris Moore
tuvalu.santafe.edu/
~moore/aztec256.gif

color code for dimers:



- To any dimer configuration one can associate a height function, whose graph is a stepped surface.

- A random dimer configuration of the Aztec diamond is a random surface.

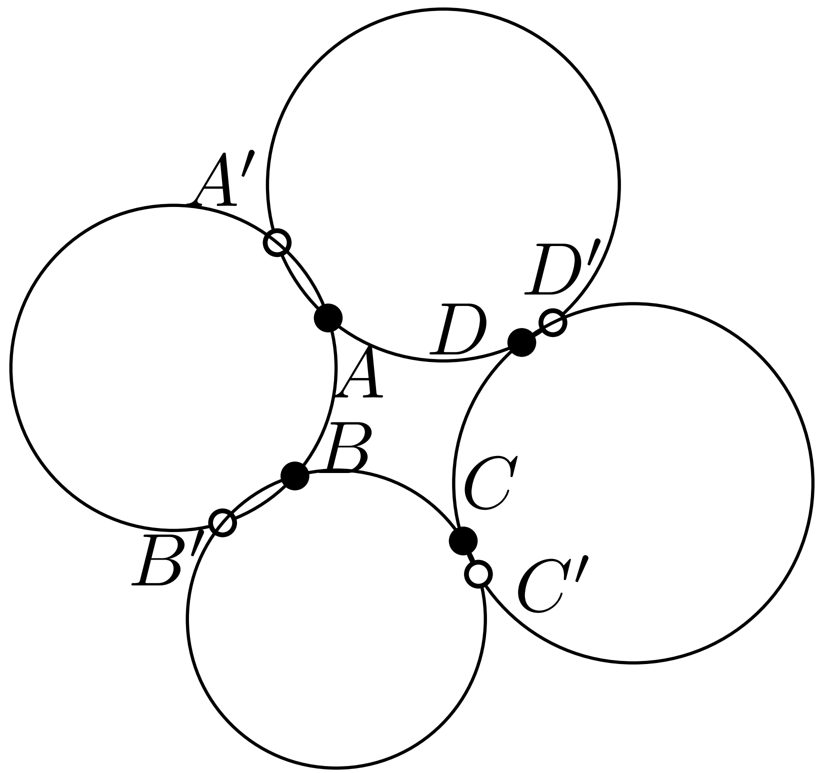
- For $n \to \infty$, this random surface concentrates around a deterministic surface called the limit shape.

- The limit shape is obtained by solving a variational problem, minimizing a certain functional with prescribed boundary conditions.

- Main topic of this talk : Miquel dynamics on circle patterns.

- Its definition by Kenyon was inspired by the Goncharov-Kenyon dimer integrable system.

Miquel's theorem



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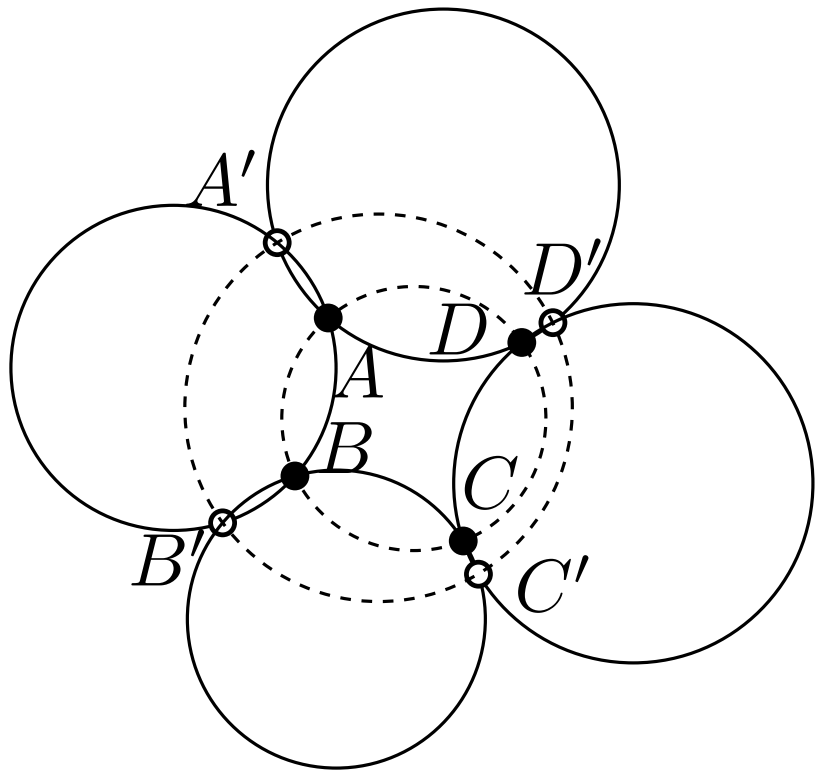


Theorem (Miquel, 1838)**.** *In this setting,*
A, B, C, D *concyclic* $\Leftrightarrow$ A', B', C', D' *concyclic.*

THÉORÈMES

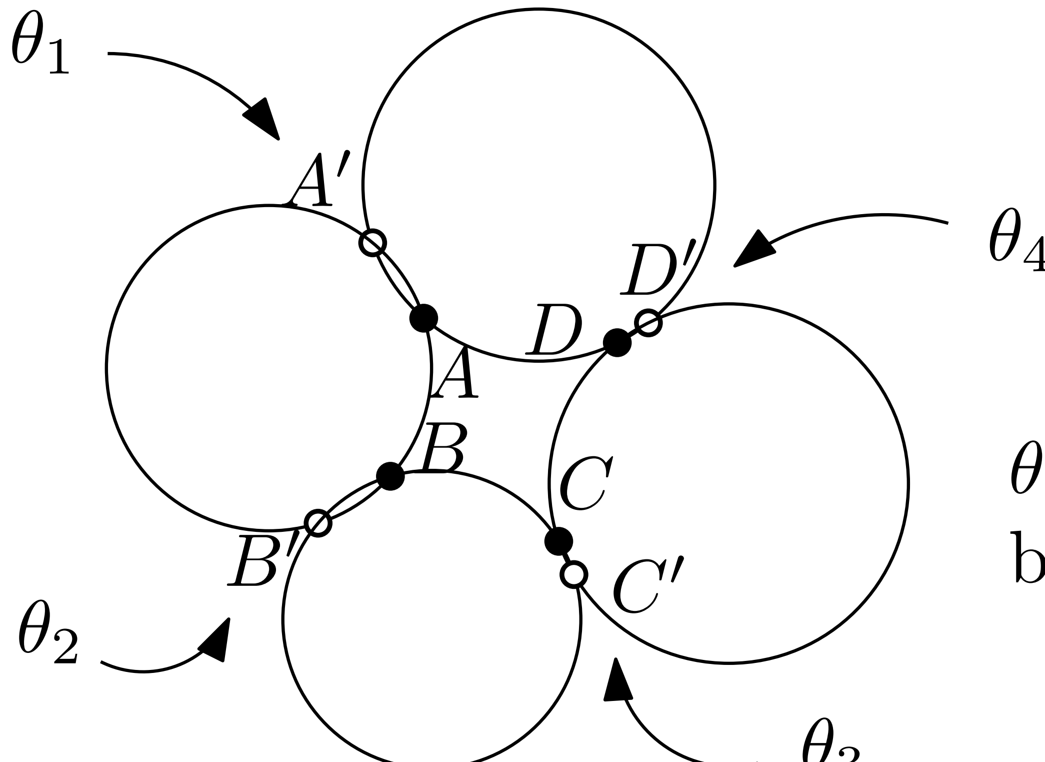
Sur les intersections des cercles et des sphères;

PAR AUG. MIQUEL.

THÉORÈME I.

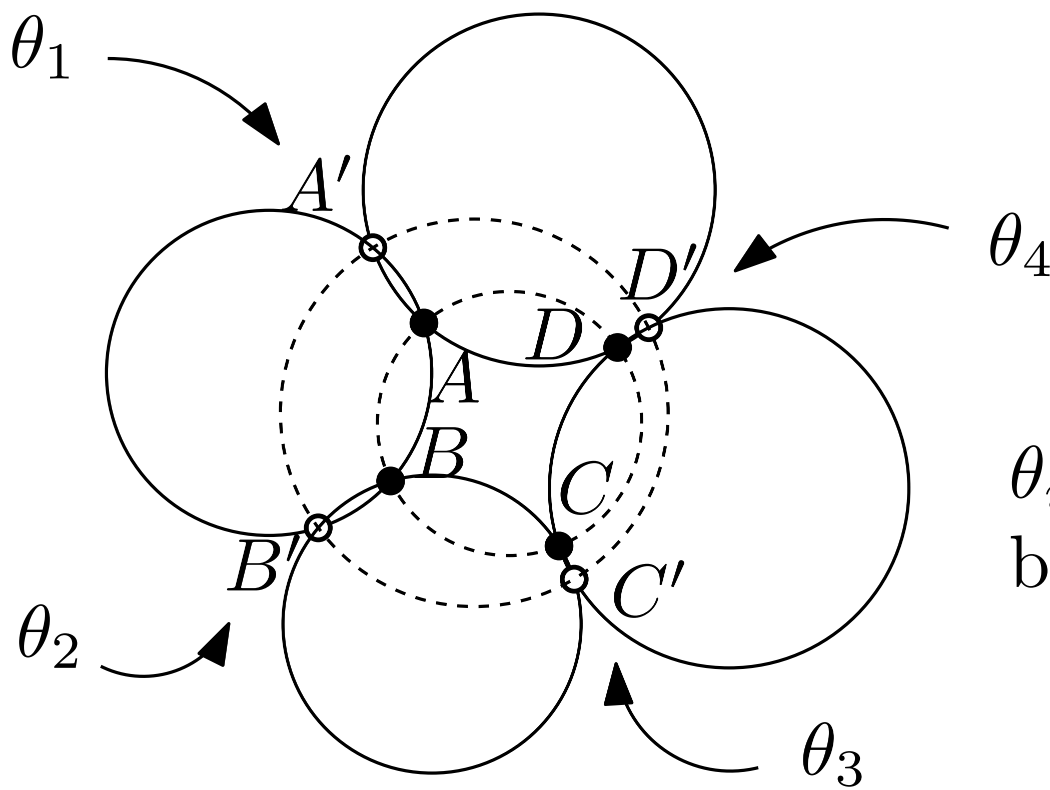
« Lorsque quatre points A, B, C, D (fig. 1, planche III) sont situés » sur une même circonférence de cercle ABCD; si par les points » consécutifs A et B, B et C, C et D, D et A, on fait passer des » circonférences de cercle, les quatre secondes intersections A′, B′, » C′, D′ des circonférences consécutives se trouveront sur une même » circonférence de cercle A′B′C′D′. »

Miquel's theorem



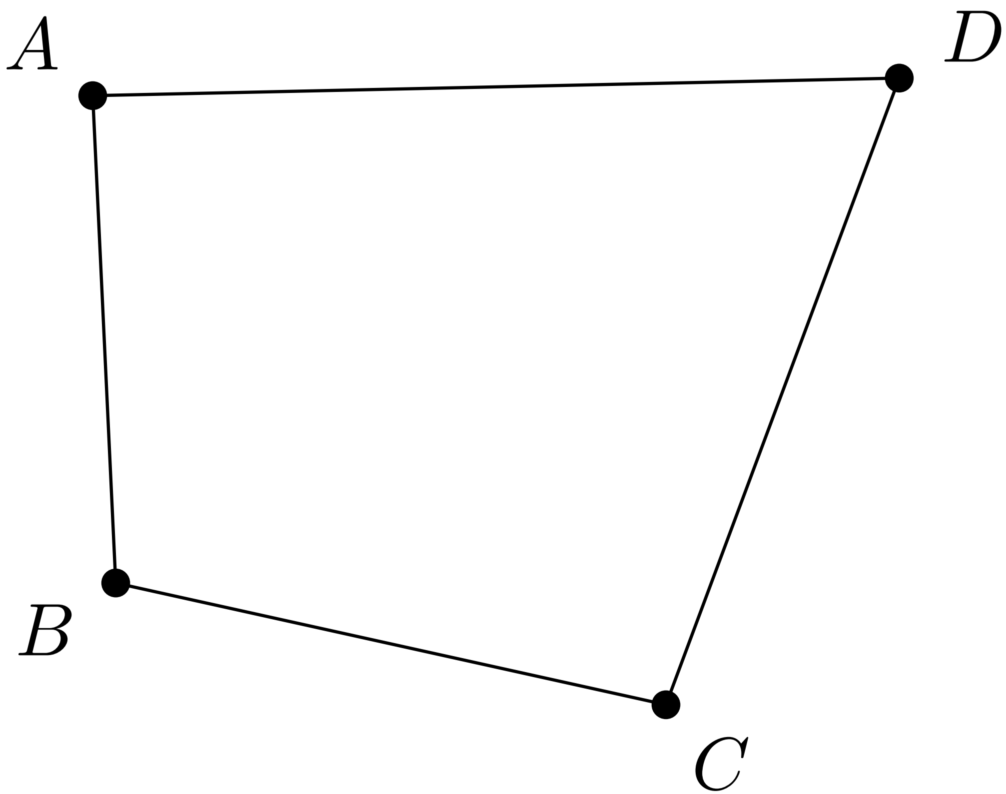
θ_i : intersection angle between two circles

Miquel's theorem

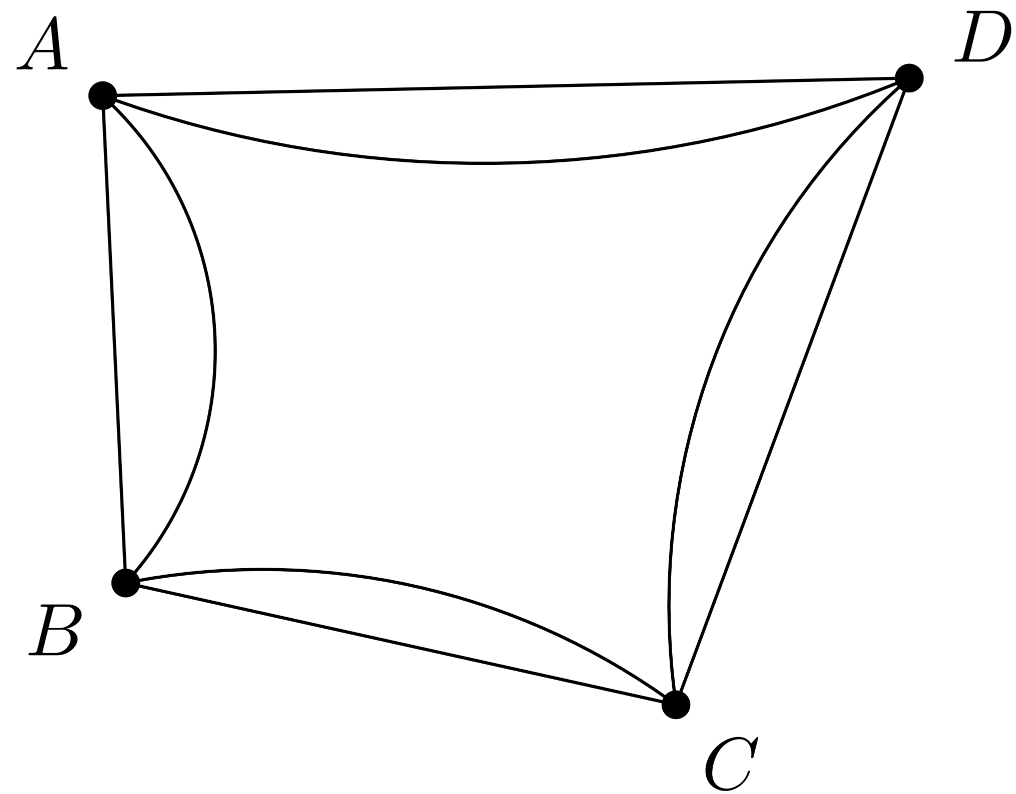


θ_i : intersection angle between two circles

Theorem (R., 2017). $\theta_1 + \theta_3 = \theta_2 + \theta_4 \Leftrightarrow$ A, B, C, D *concyclic* $\Leftrightarrow$ A', B', C', D' *concyclic.*



$$ABCD \text{ concyclic} \Leftrightarrow \hat{A} + \hat{C} = \hat{B} + \hat{D}$$



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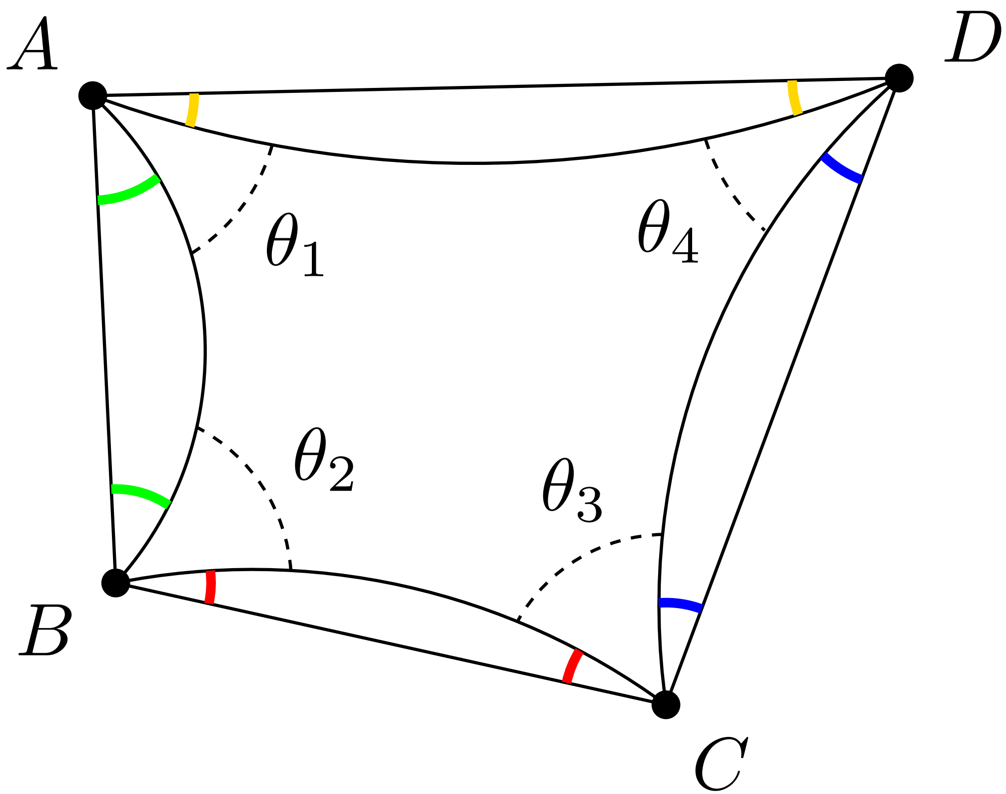

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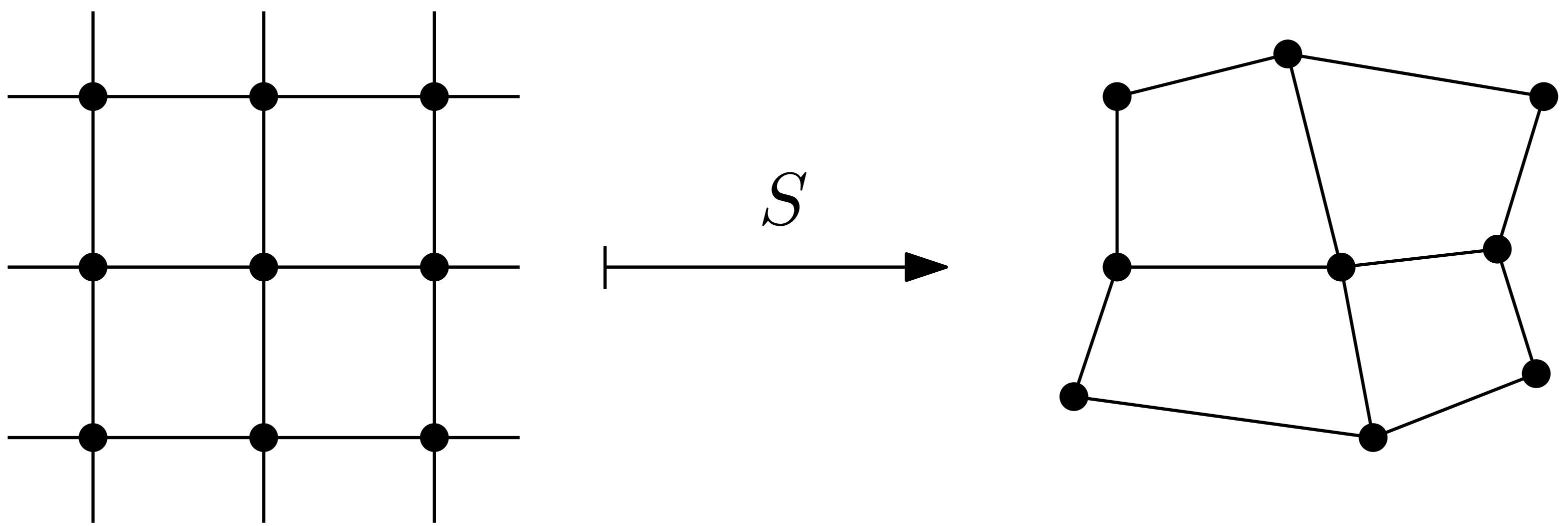
$$\hat{A} = \theta_1 +) +) \qquad \hat{B} = \theta_2 +) +)$$

$$\hat{C} = \theta_3 +) +) \qquad \hat{D} = \theta_4 +) +)$$

Square grid circle patterns

- A square grid circle pattern (SGCP) is a map $S : \mathbb{Z}^2 \to \mathbb{R}^2$ such that any four vertices around a face of $\mathbb{Z}^2$ get mapped to four concyclic points.



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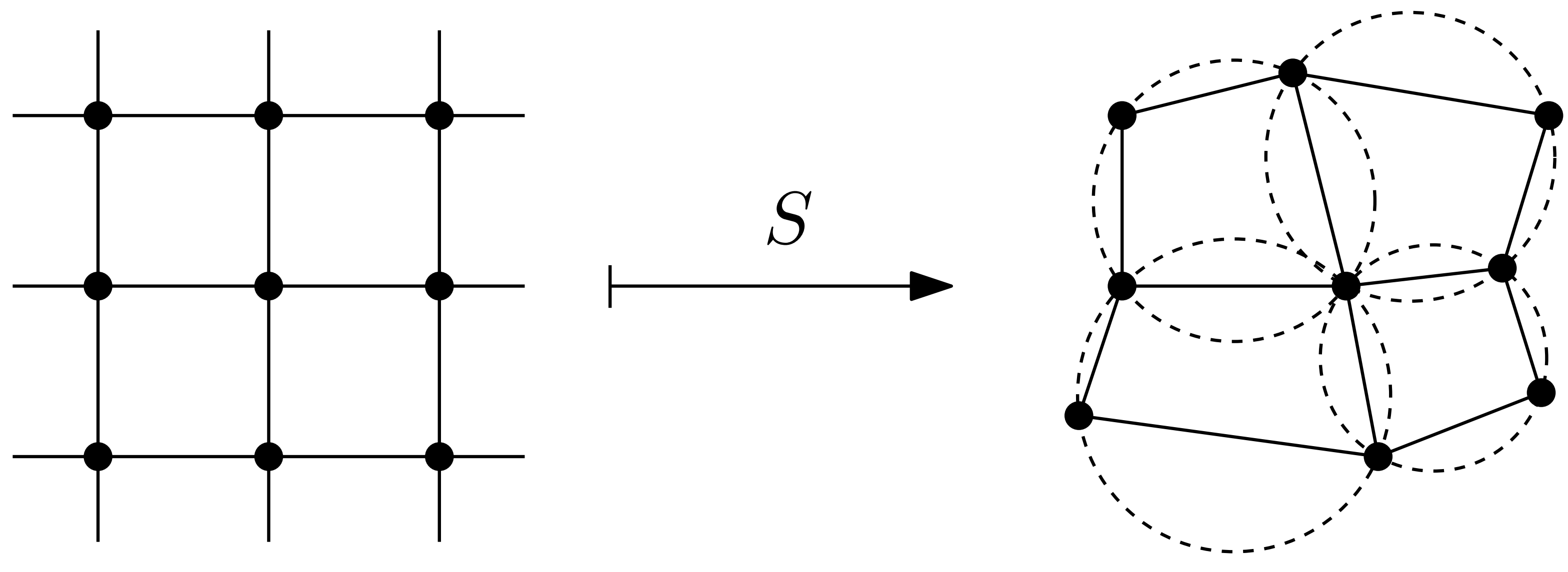
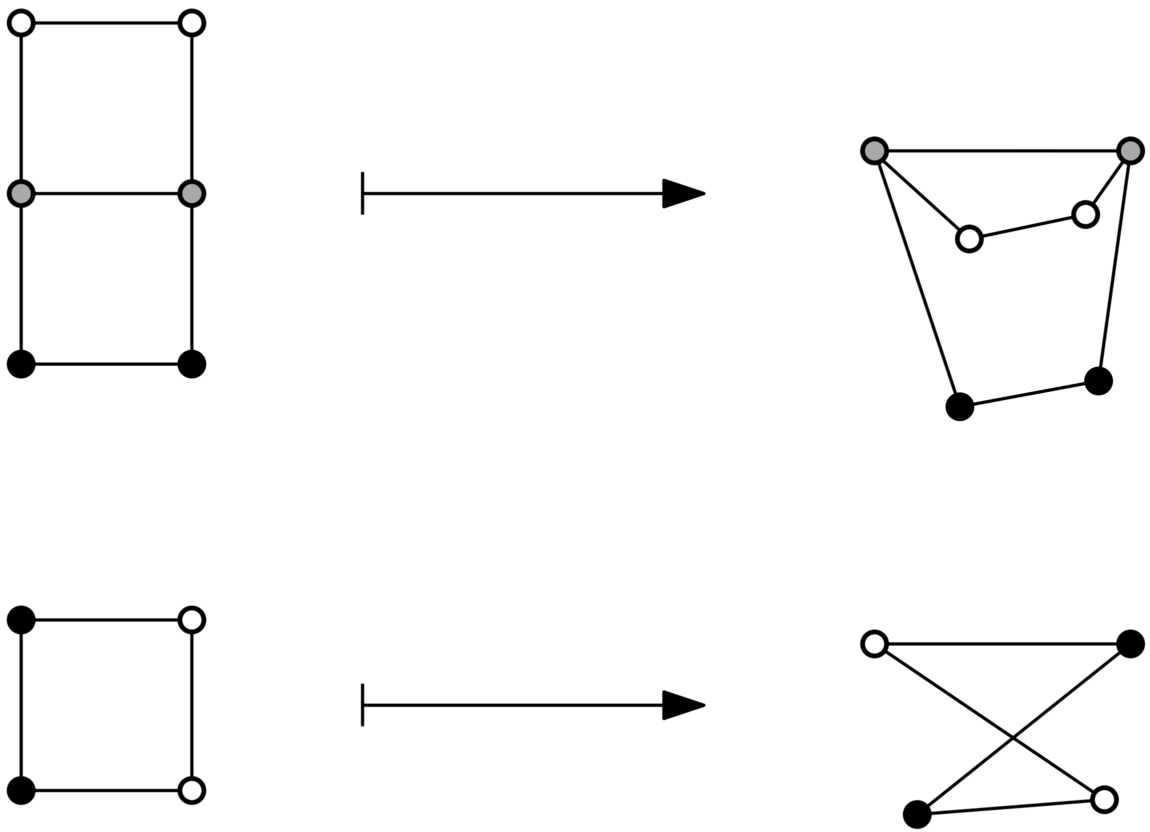


- Folds and non-convex quadrilaterals are allowed.



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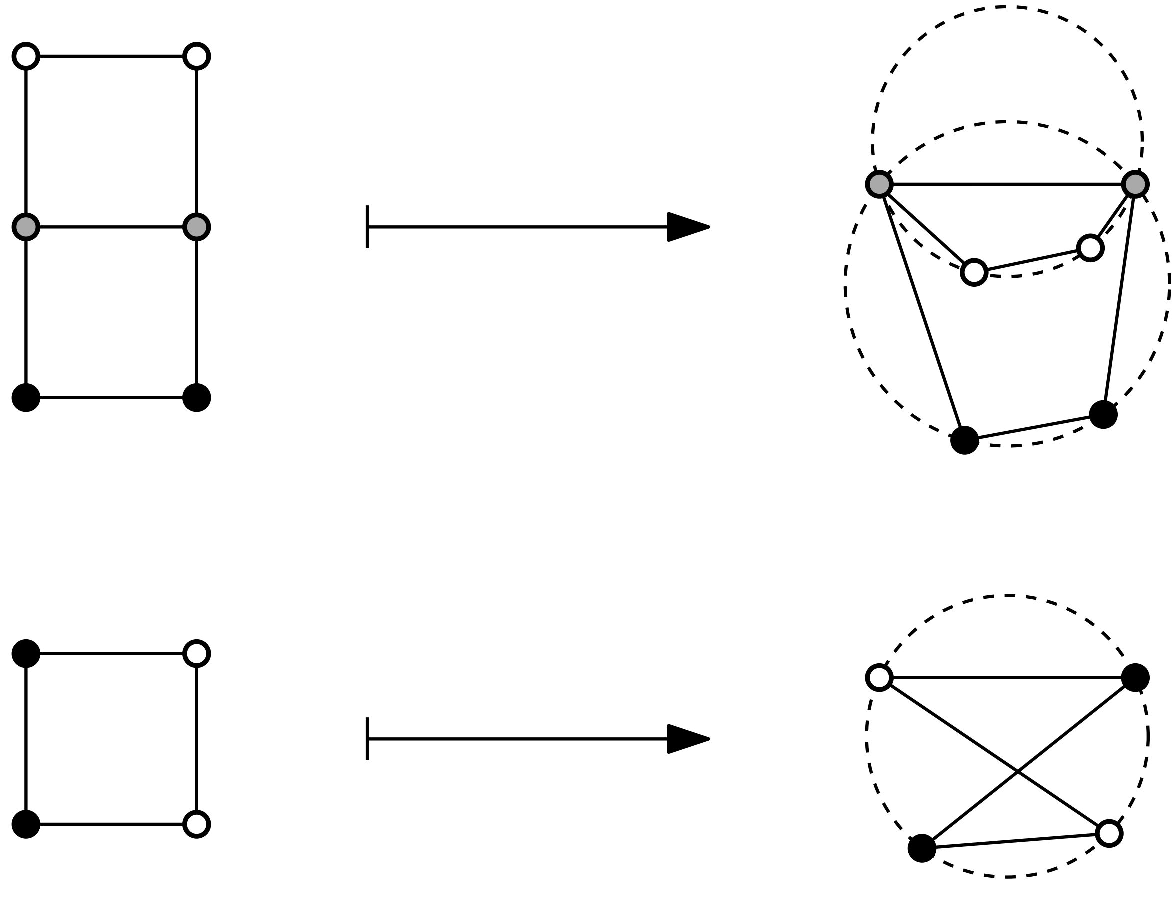
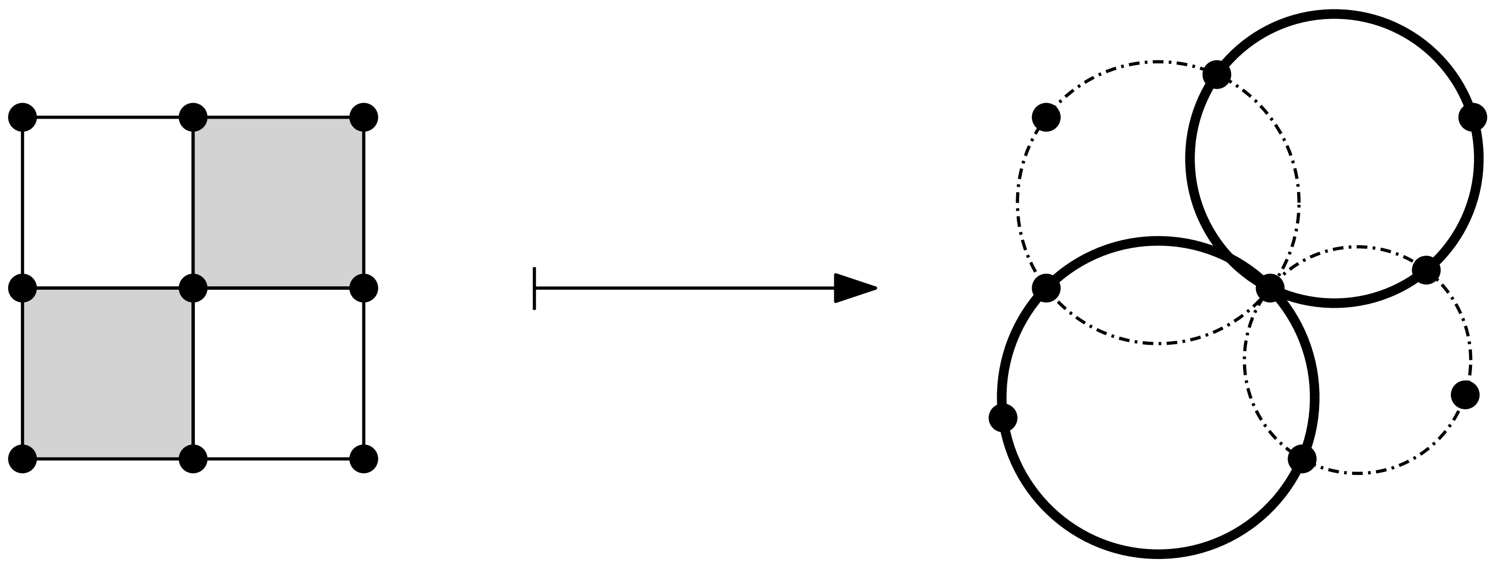


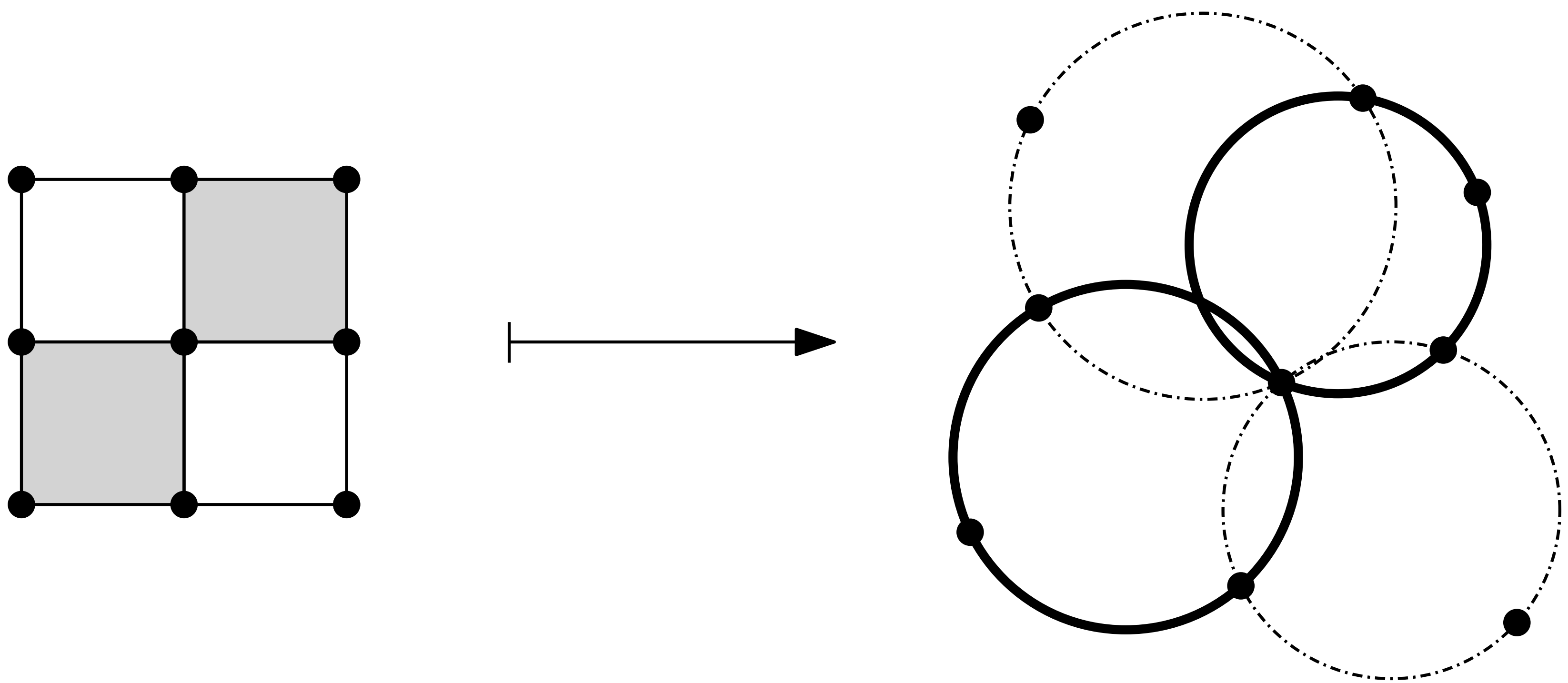
Miquel dynamics

- Checkerboard coloring of the faces of $\mathbb{Z}^2$: black and white circles.



- Define two maps from the set of SGCPs to itself, black mutation μ_B and white mutation μ_W.

- Black mutation μ_B : each vertex gets moved to the other intersection point of the two white circles it belongs to. All the vertices move simultaneously.



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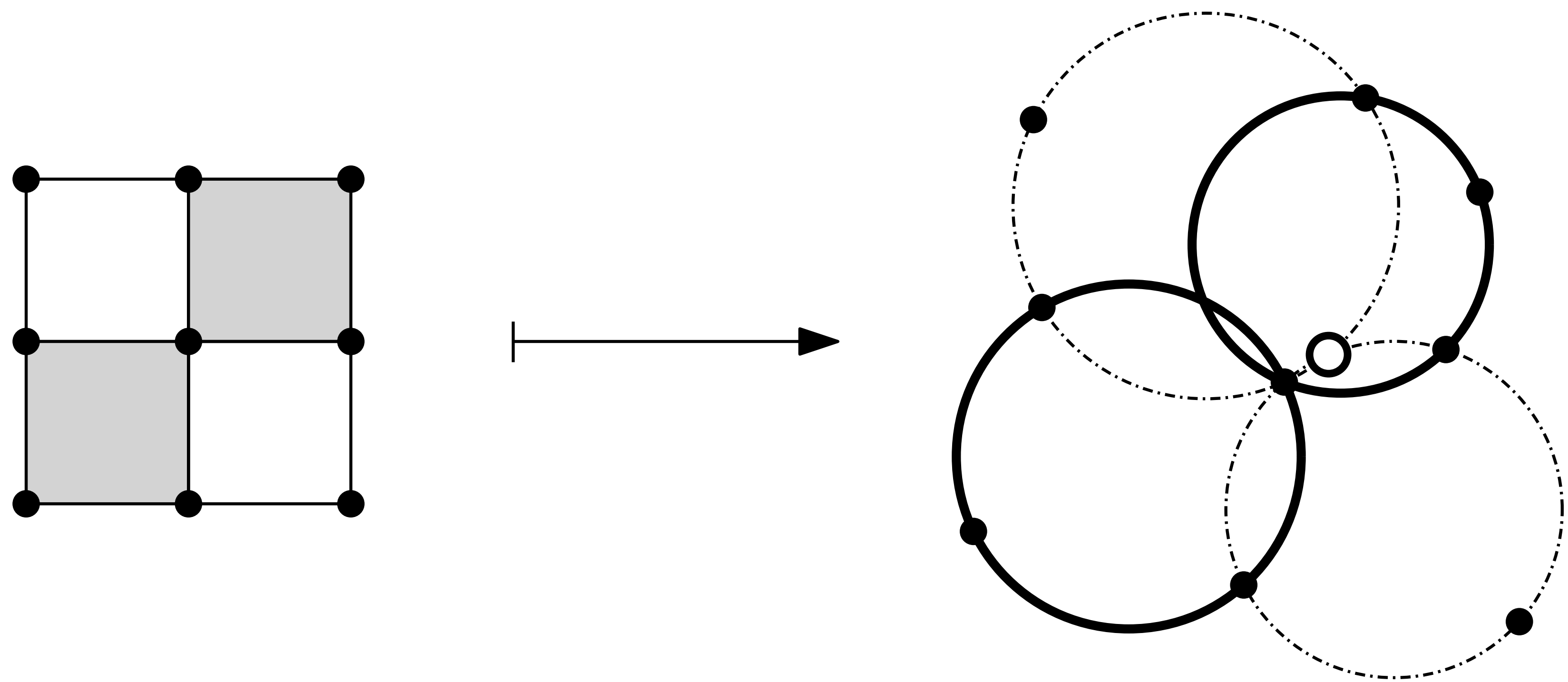

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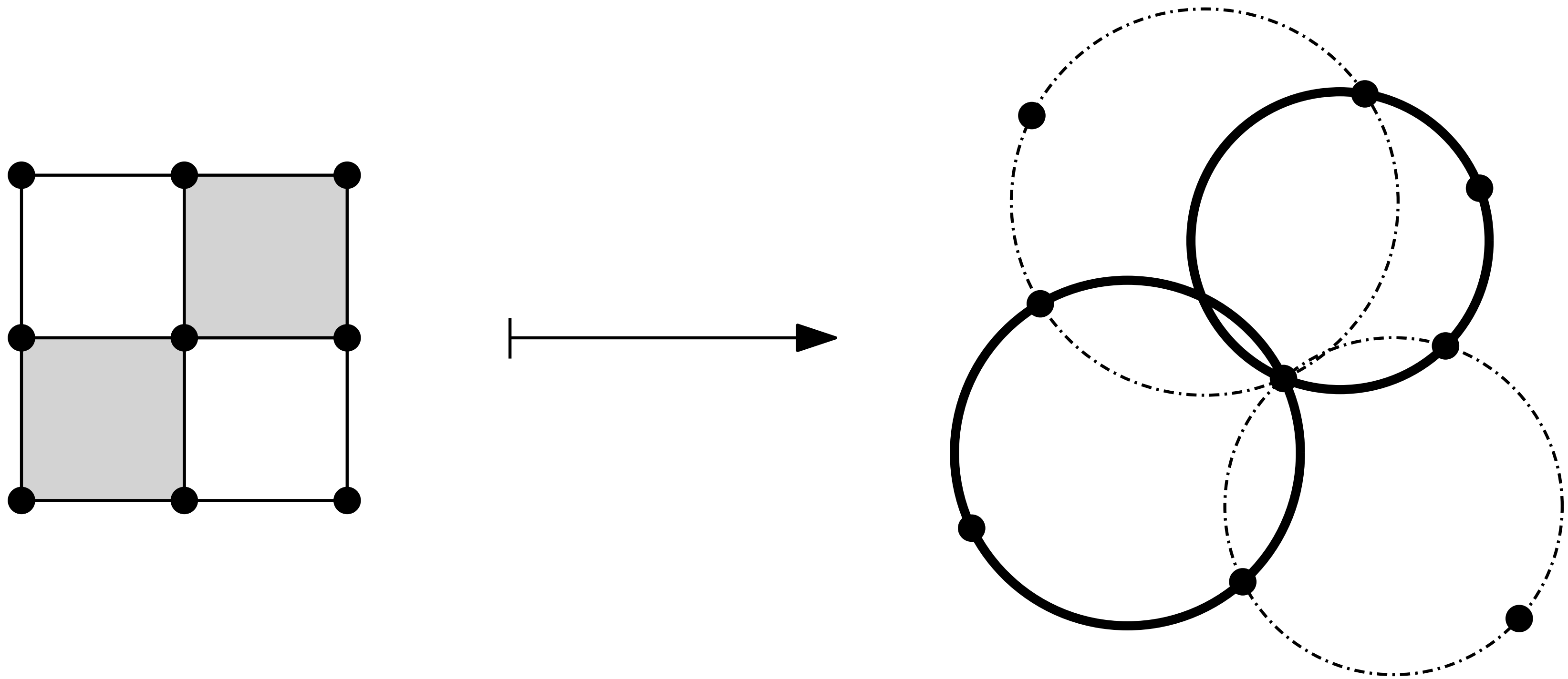

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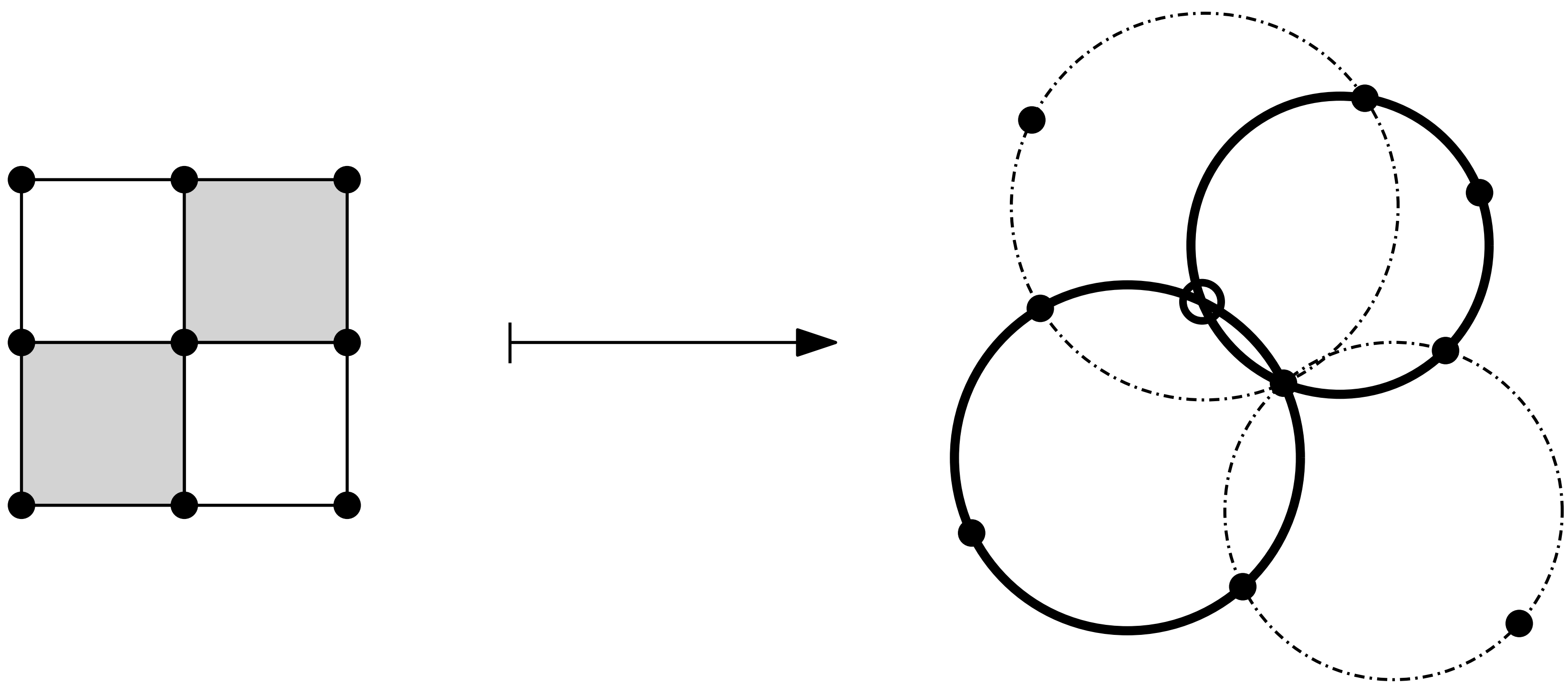

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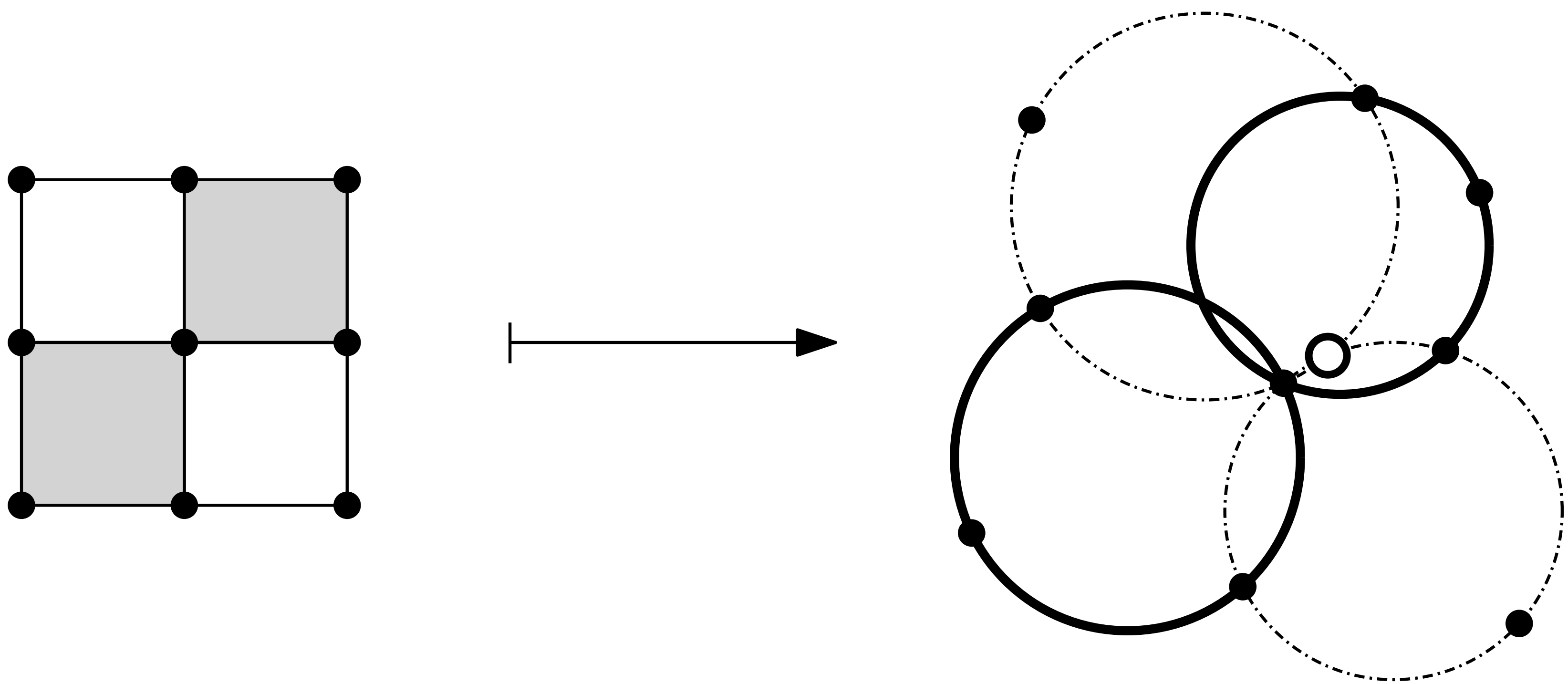
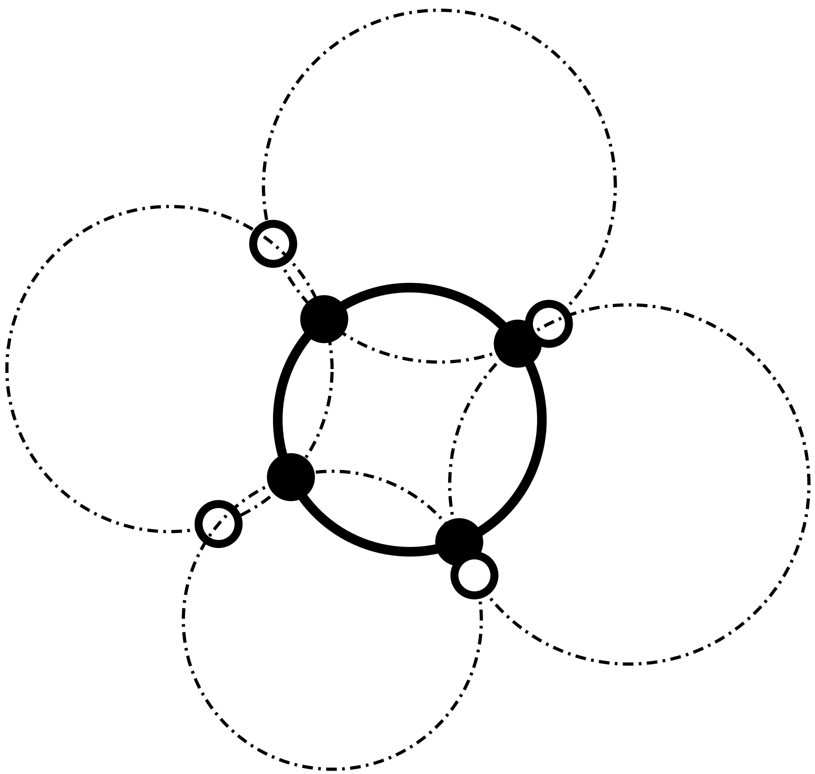


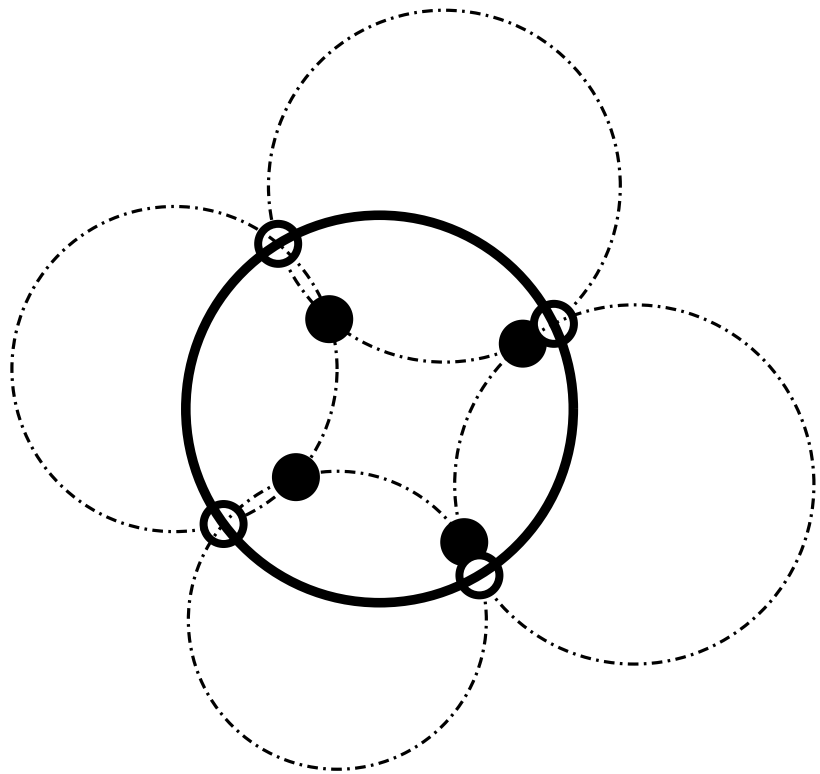
- Why does μ_B produce an SGCP ?



- The maps μ_B and μ_W are involutions.

- Miquel dynamics : discrete-time dynamics obtained by alternating between μ_B and μ_W.

- Invented by Richard Kenyon.



Miquel's theorem !

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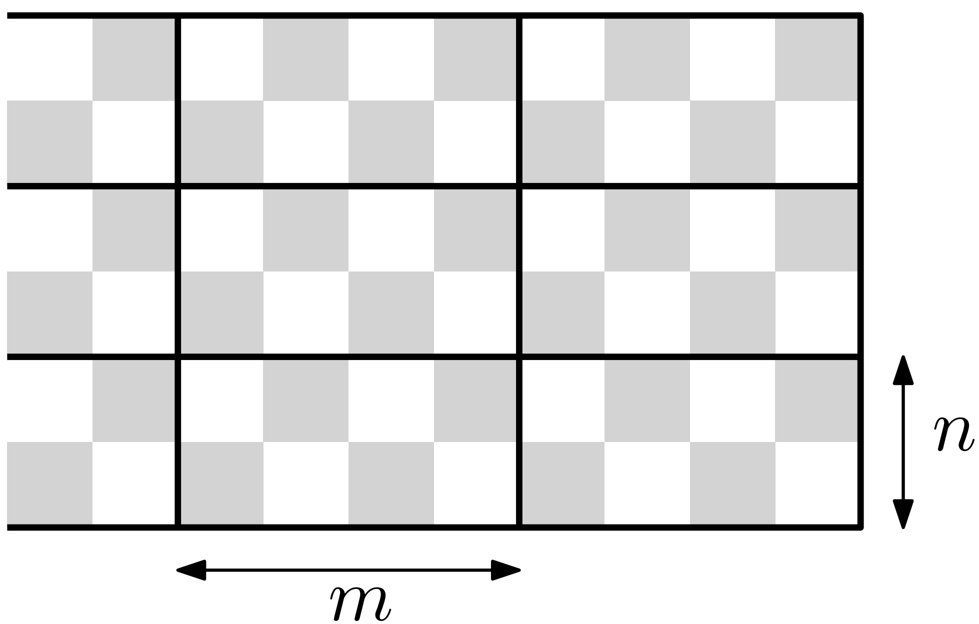
Biperiodic SGCPs

- An SGCP S is spatially biperiodic if there exist m, n integers and $\vec{u}, \vec{v} \in \mathbb{R}^2$ such that for all $(x, y) \in \mathbb{R}^2$,

$$S(x+m, y) = S(x, y) + \vec{u}$$
$$S(x, y+n) = S(x, y) + \vec{v}$$



$m = 4$

$n = 2$

- The vector $\vec{u}$ (resp. $\vec{v}$) is called the monodromy in the direction $(m, 0)$ (resp. $(0, n)$).

- A biperiodic SGCP is mapped by Miquel dynamics to another biperiodic SGCP with the same periods and monodromies.

- This reduces the problem to a finite-dimensional one.

- A biperiodic circle pattern in the plane projects down to a circle pattern on a flat torus.

[Mathematica]

Dimers vs circle patterns

- The limit shape in the dimer model is a deterministic surface which minimizes some surface tension with prescribed boundary conditions (Cohn-Kenyon-Propp).

- For circle patterns, one can find the radii knowing the intersection angles by solving a variational principle. The functional minimized is similar to the one occurring for dimers (Rivin, Bobenko-Springborn).

- Miquel dynamics mimics the Goncharov-Kenyon dimer discrete integrable system. Can it give us a direct connection between dimers and circle patterns ?

Miquel dynamics property wish list

Dimension of the space

Coordinates on that space

"Many" independent conserved quantities

Spectral curve

Identify black and white mutation as
cluster algebra mutations

Compatible Poisson bracket

Miquel dynamics property wish list

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Spectral curve	$\sim$
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Compatible Poisson bracket	$\times$

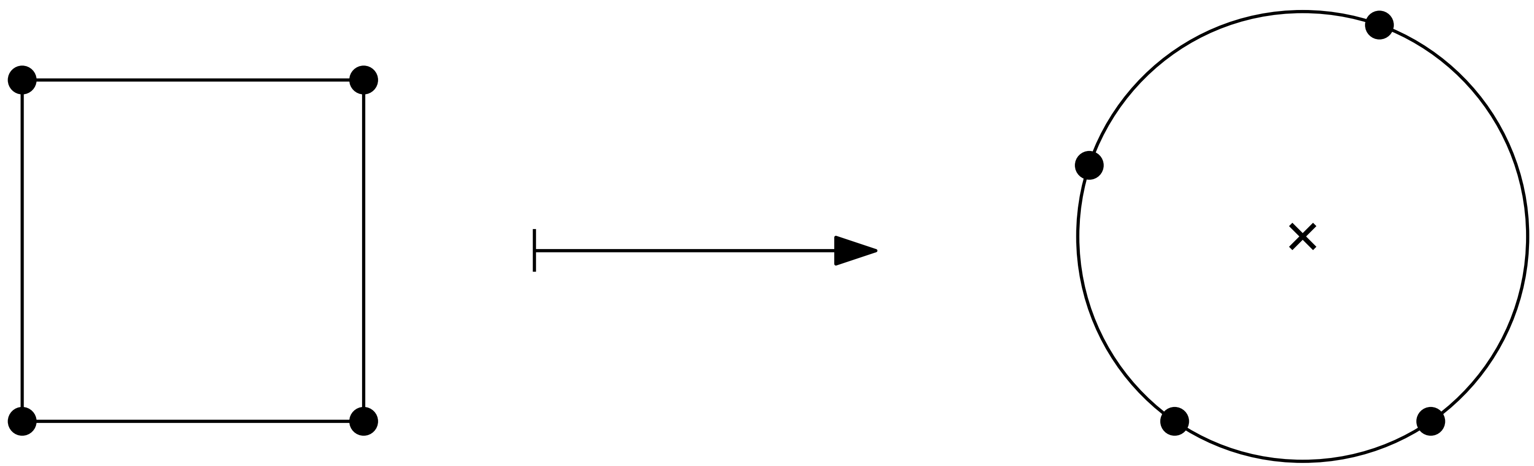
What space of circle patterns ?

- Fix m and n in $\mathbb{Z}_+$ and consider the space $\mathcal{M}_{m,n}$ of SGCPs that have both $(m, 0)$ and $(0, n)$ as a period, considered up to similarity.

- SGCPs whose faces form a cell decomposition of the torus (no folds, no non-convex quads) are an open subset of $\mathcal{M}_{m,n}$.

- Bobenko-Springborn (2004) : this subspace of cell-decomposition SGCPs has dimension $mn + 1$.

Coordinates for $\mathcal{M}_{m,n}$



- Four ϕ variables in each of the mn faces of a fundamental domain.
- These variables must satisfy some relations.

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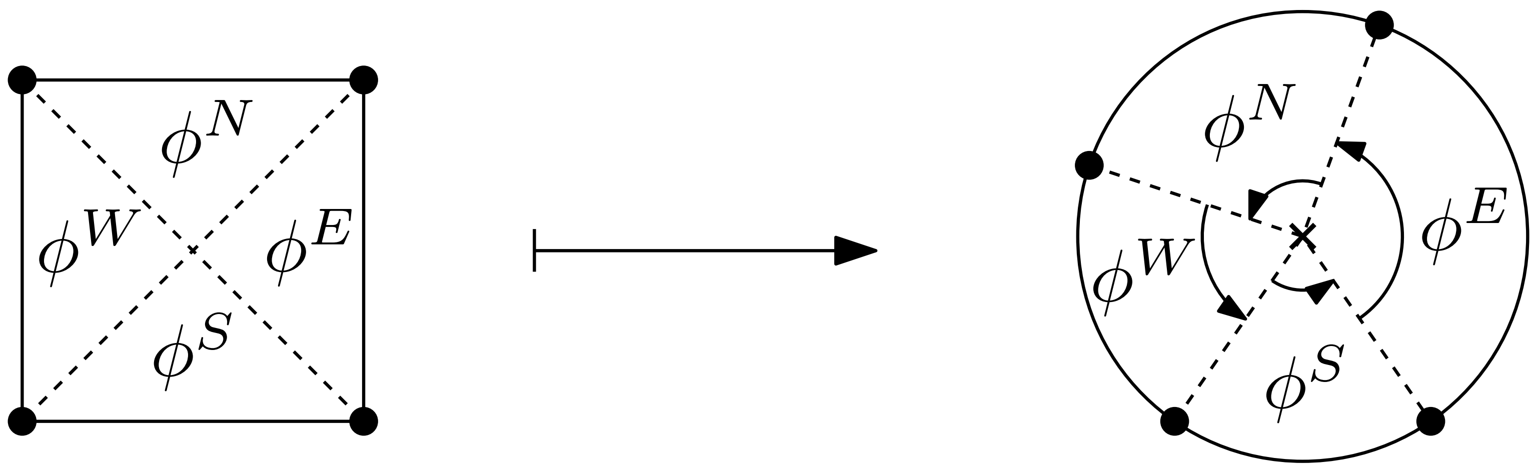
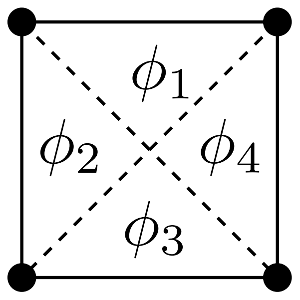


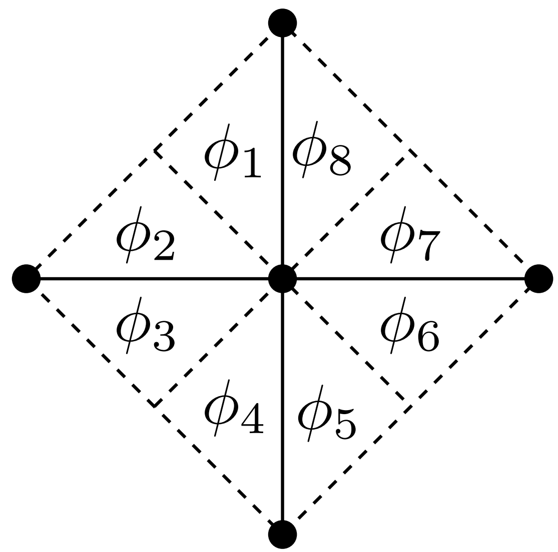
- Four ϕ variables in each of the mn faces of a fundamental domain.

- These variables must satisfy some relations.

- Flatness at each face and vertex.
- Consistency of radii around a vertex.



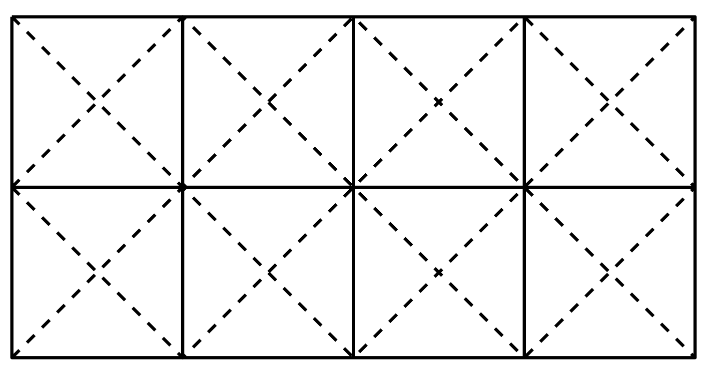
$$\sum_{i=1}^{4} \phi_i = 2\pi$$



$$\sum_{i=1}^{8} \phi_i = 4\pi$$

$$\frac{\sin\frac{\phi_1}{2}\sin\frac{\phi_3}{2}\sin\frac{\phi_5}{2}\sin\frac{\phi_7}{2}}{\sin\frac{\phi_2}{2}\sin\frac{\phi_4}{2}\sin\frac{\phi_6}{2}\sin\frac{\phi_8}{2}} = 1$$

- Flatness at each face and vertex.
- Consistency of radii around a vertex.
- Global relations across the torus, expressing the consistency of radii and the parallelism of edges.



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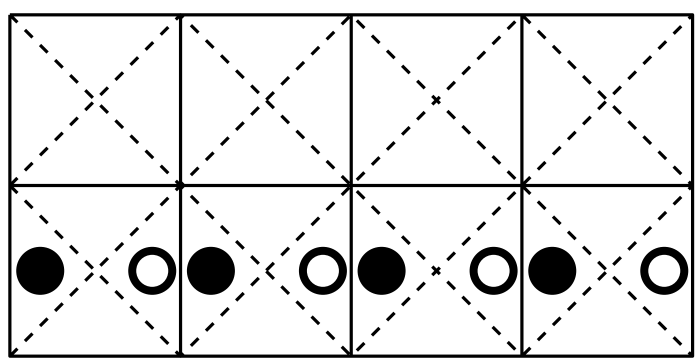

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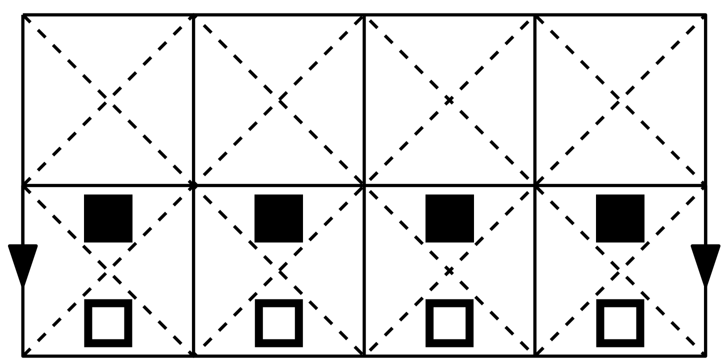

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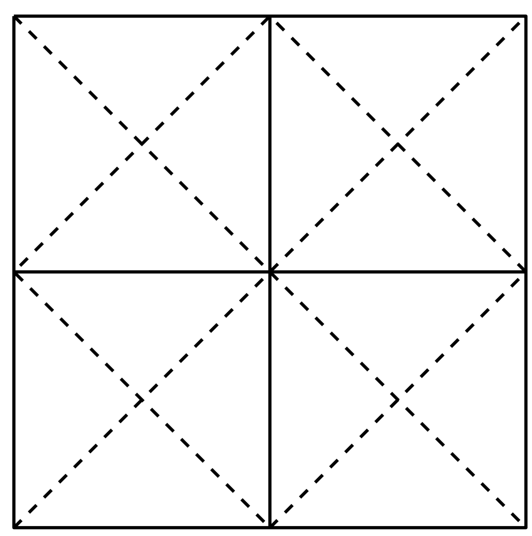
$$\prod_{\bullet} \sin \frac{\phi}{2} = \prod_{\circ} \sin \frac{\phi}{2}$$

$$\sum_{\blacksquare} \phi = \sum_{\square} \phi$$

Theorem (R., 2017)**.** *The* $4mn$ ϕ *variables satisfying the above relations coordinatize* $\mathcal{M}_{m,n}$*.*

We easily reconstruct the SGCP from such ϕ variables.



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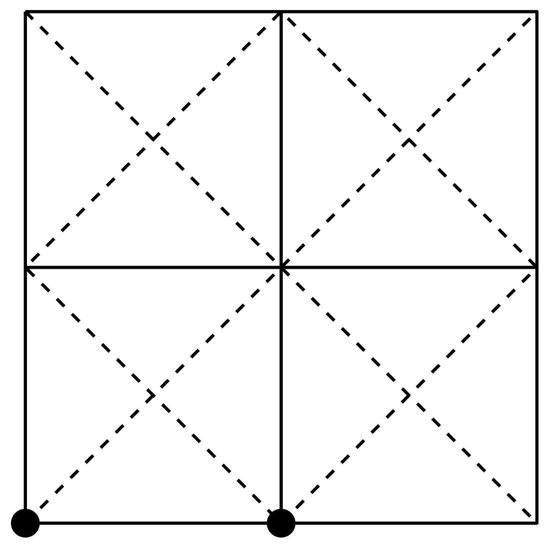

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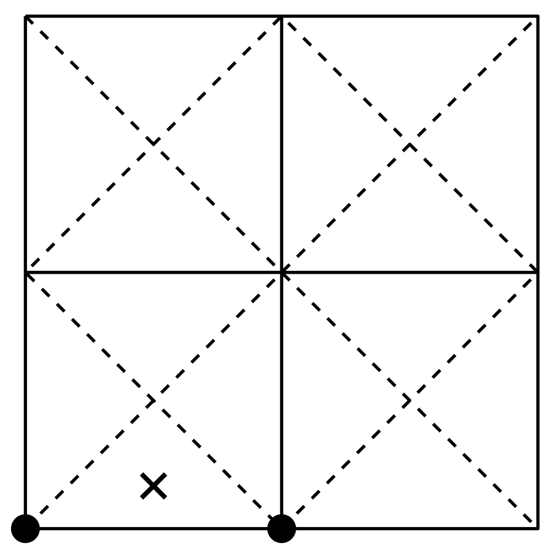

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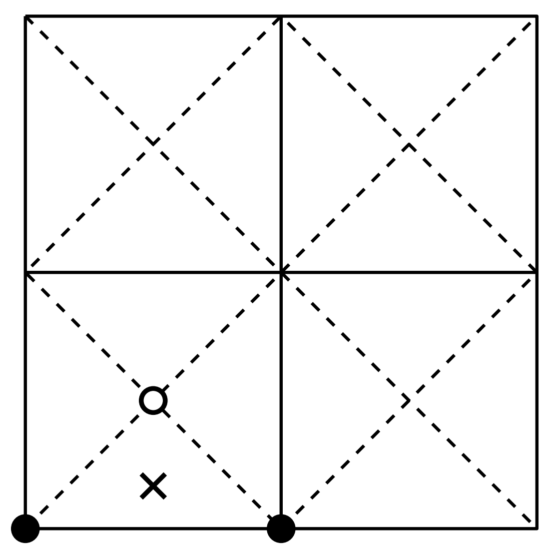

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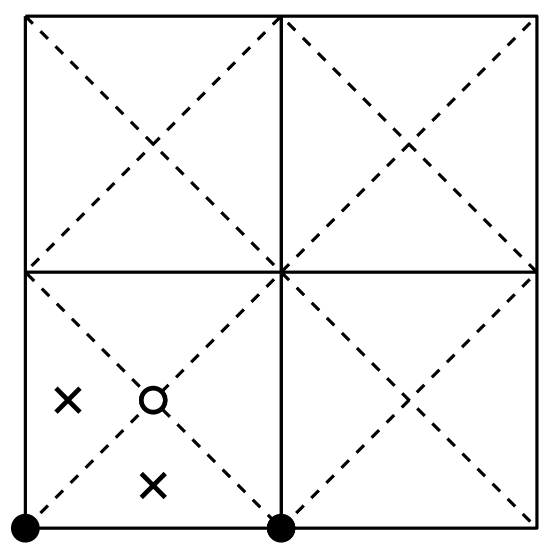

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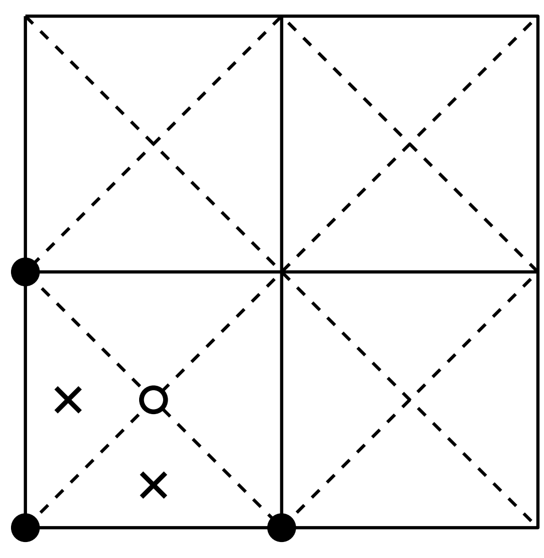

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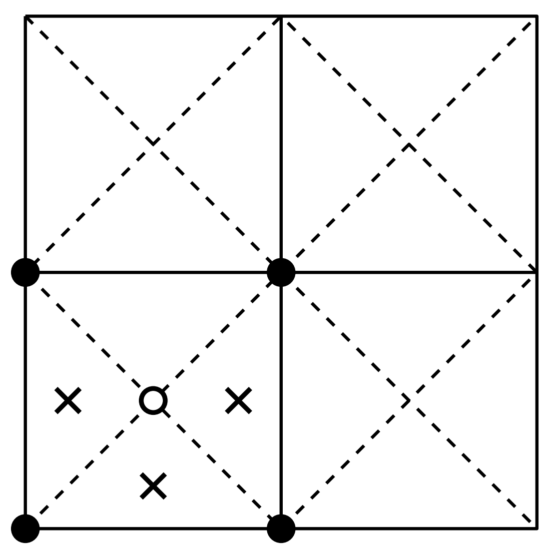

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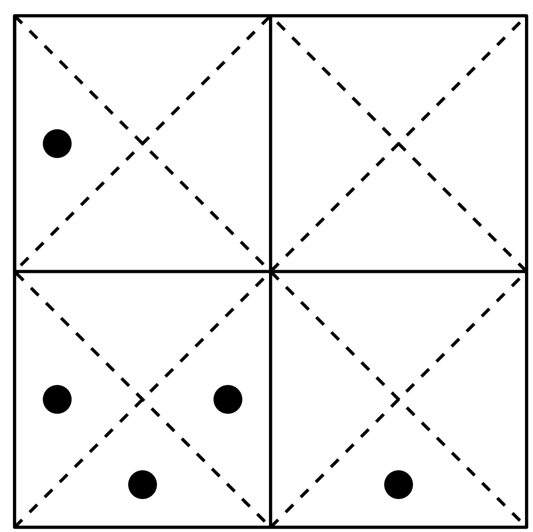

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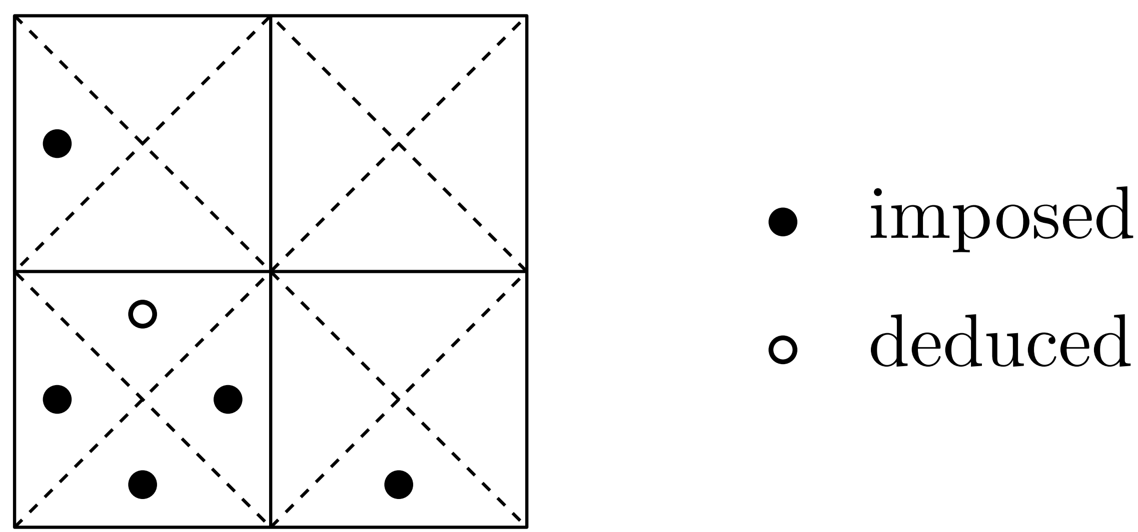

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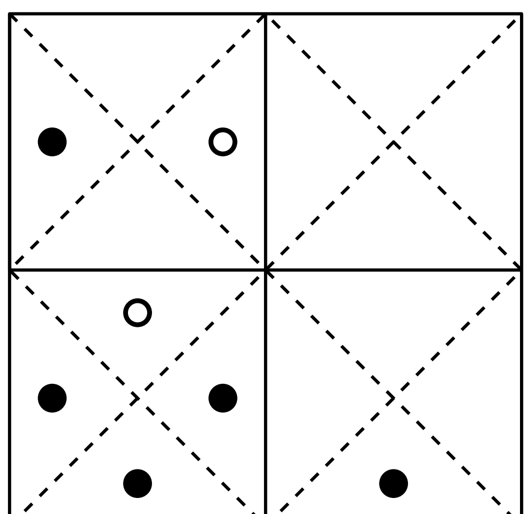

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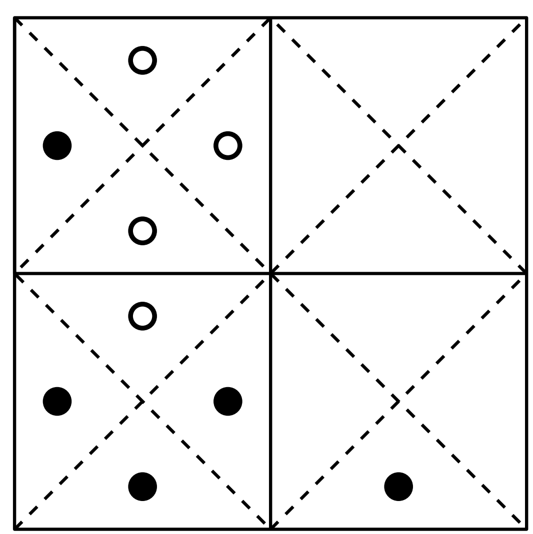

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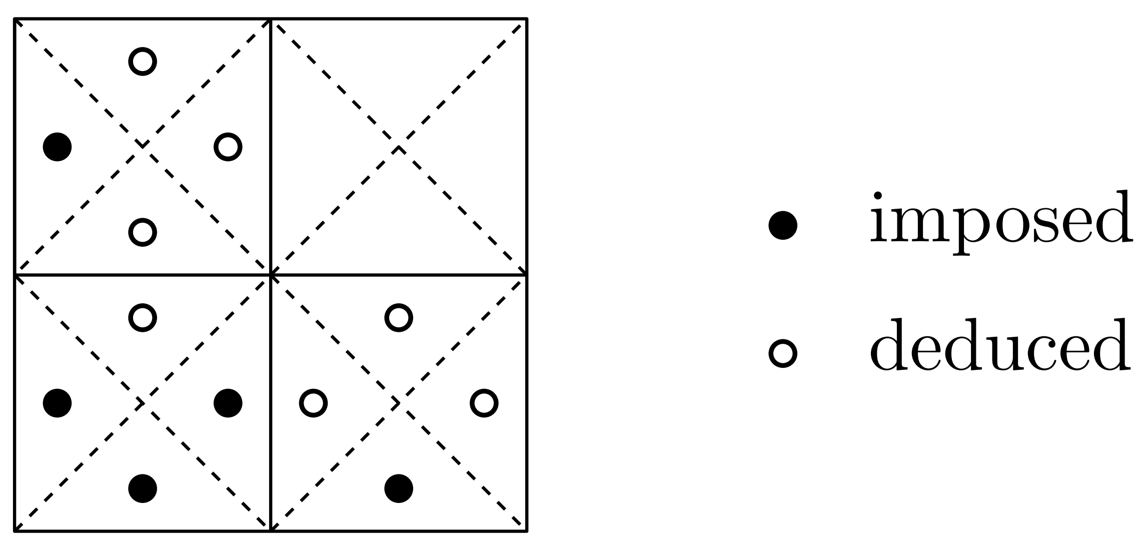

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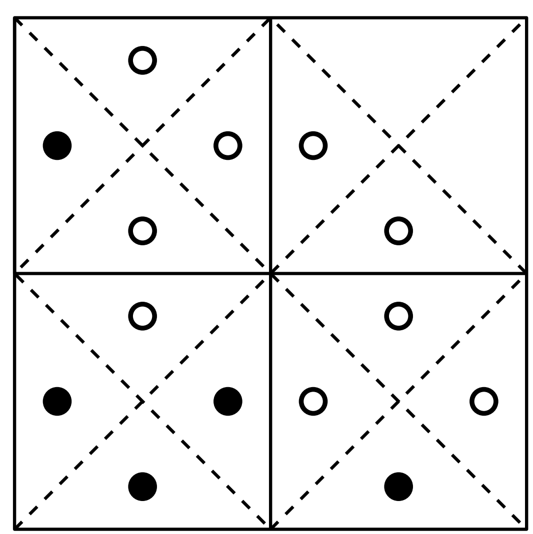

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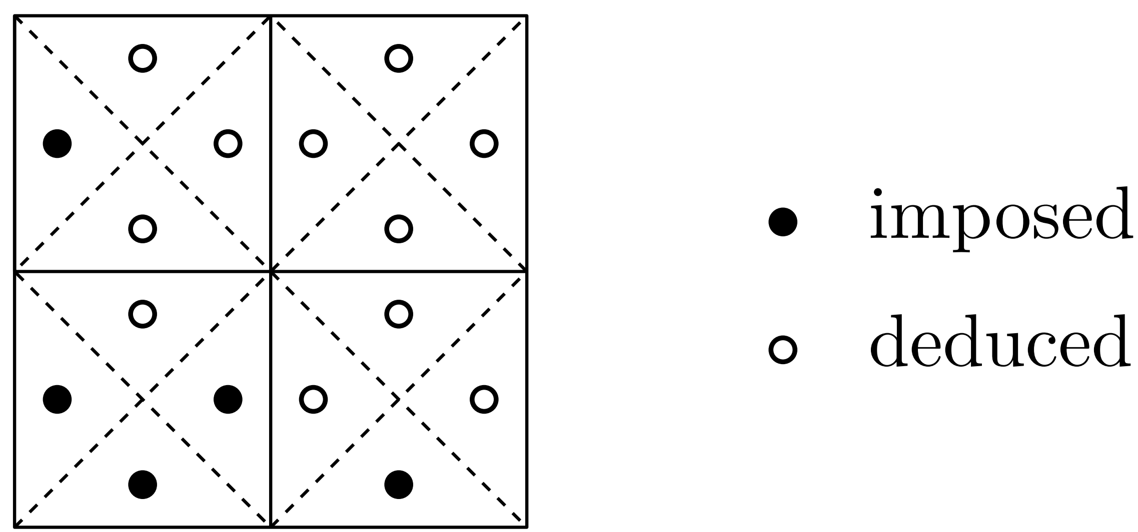
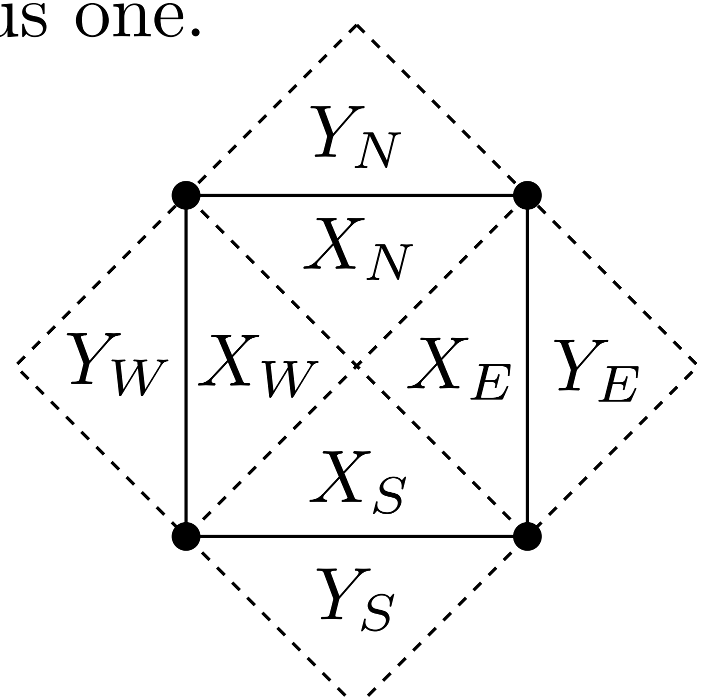


Local recurrence formulas

- Replace the ϕ variables by $X = e^{i\phi}$, complex numbers of modulus one.



central face is black

- How does black mutation act on the X_i's and Y_i's ?

$$Y'_N = Y_N \frac{\left(1 - \frac{\left(1-X_W^{-1}\right)\left(1-Y_N^{-1}\right)}{\left(1-Y_W\right)\left(1-X_N\right)}\right)\left(1 - \frac{\left(1-X_E^{-1}\right)\left(1-Y_N^{-1}\right)}{\left(1-Y_E\right)\left(1-X_N\right)}\right)}{\left(1 - \frac{\left(1-X_W\right)\left(1-Y_N\right)}{\left(1-Y_W^{-1}\right)\left(1-X_N^{-1}\right)}\right)\left(1 - \frac{\left(1-X_E\right)\left(1-Y_N\right)}{\left(1-Y_E^{-1}\right)\left(1-X_N^{-1}\right)}\right)}$$

$$X'_N = \frac{1 - \frac{\left(1-X_N^{-1}\right)\left(1-Y_W^{-1}\right)\left(1-Y_N'^{-1}\right)}{\left(1-Y_N\right)\left(1-X_W\right)\left(1-Y'_W\right)}}{1 - \frac{\left(1-X_N\right)\left(1-Y_W\right)\left(1-Y'_N\right)}{\left(1-Y_N^{-1}\right)\left(1-X_W^{-1}\right)\left(1-Y_W'^{-1}\right)}}$$

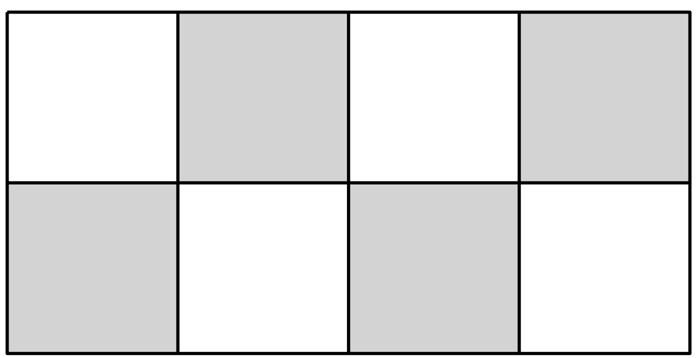
- Reminiscent of the mutation of ratios of cluster variables in cluster algebras.

Conserved quantities

- The pair of monodromy vectors $(\vec{u}, \vec{v})$ up to similarity (two real conserved quantities).

- Signed sums of intersection angles along loops on the torus.

- Draw dual graph to the square grid on the torus and orient the horizontal (resp. vertical) dual edges from white dual vertices to black dual vertices (resp. from black dual vertices to white dual vertices).



$m = 4$

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- For any directed loop l drawn on the dual graph, define $\gamma(l) = \sum_{e \in l} \pm\theta_e$.

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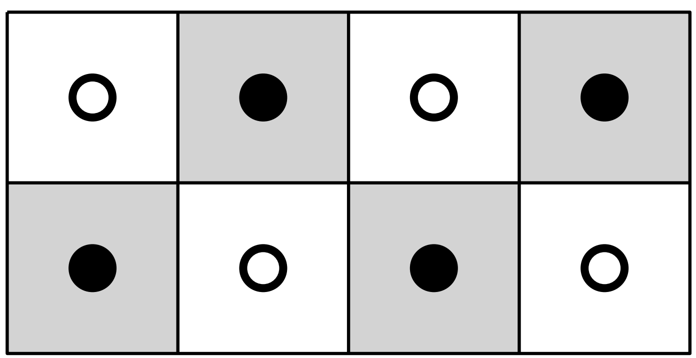

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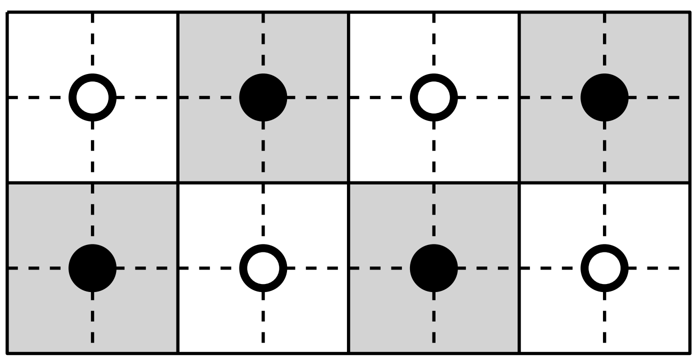

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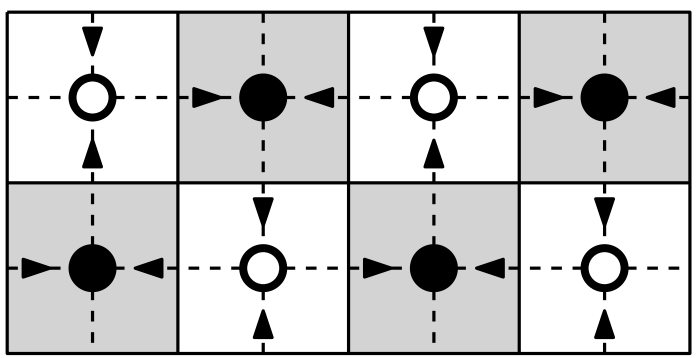

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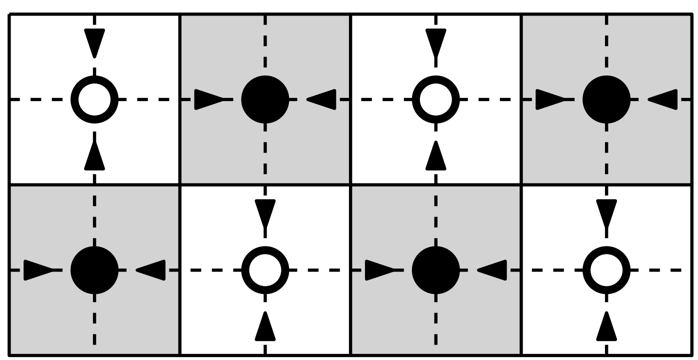

$m = 4$

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- For any directed loop l drawn on the dual graph, define $\gamma(l) = \sum_{e \in l} \pm \theta_e$.

  - θ_e: intersection angle associated with e
  - $\pm$: minus if traversing e in the wrong way

- $\gamma(l)$ only depends on the homology class of l on the torus. We can upgrade γ to be a group homomorphism from $H_1(\mathbb{T}, \mathbb{Z})$ to $\mathbb{R}/(2\pi\mathbb{Z})$.

Theorem (R., 2017)**.** *Black mutation and white mutation change γ to $-\gamma$.*

- Provides only two independent conserved quantities.

Isoradial patterns

- An SGCP is called isoradial if all the circles have a common radius.

Theorem (R., 2017). *Isoradial patterns are periodic points in $\mathcal{M}_{m,n}$, with a common period depending on m and n.*

- When $(m, n) = (2, 1)$ or $(m, n) = (4, 1)$, every pattern is isoradial.

- The isoradial Miquel dynamics coincides with the isoradial dimer dynamics.

Isoradial Miquel vs Goncharov-Kenyon

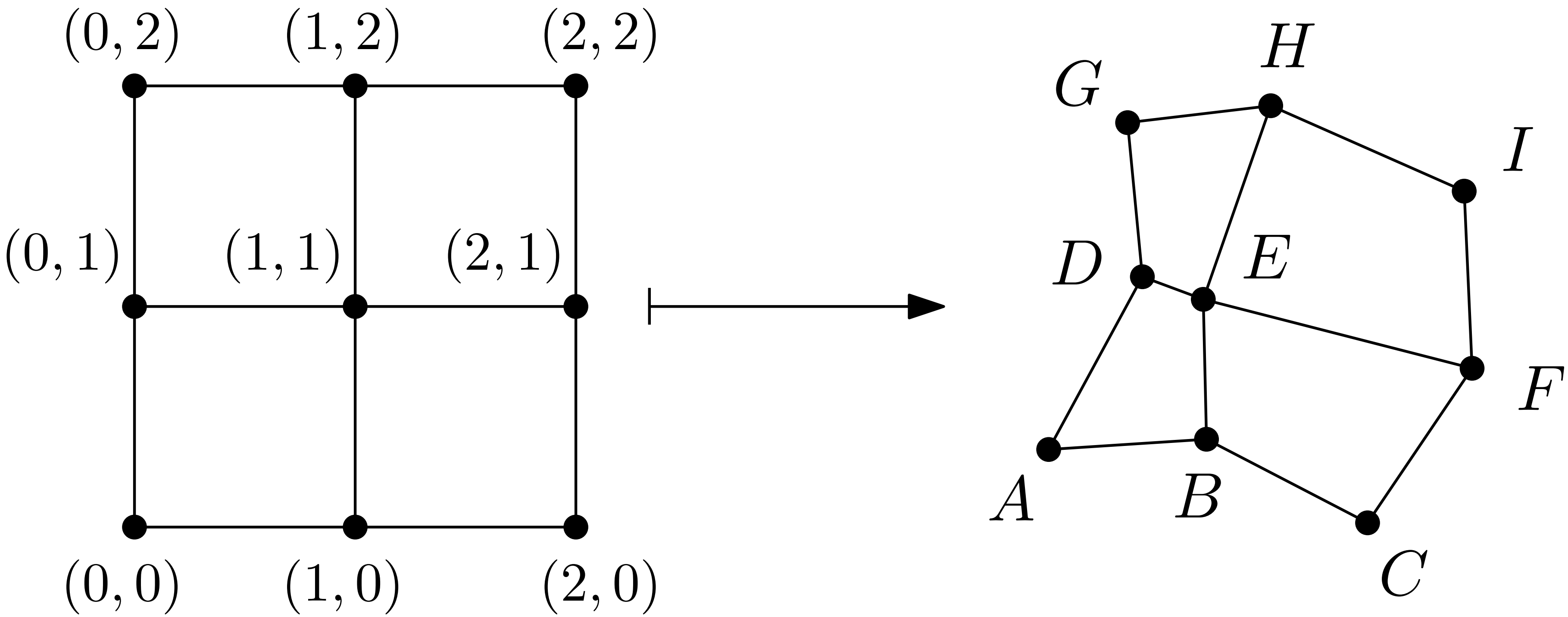
Isoradial SGCP with intersection angles θ_e on each edge e. → Square grid with weights $\sin \frac{\theta_e}{2}$ on each edge e.

↓ Black/white mutation

↓ Black/white G-K map

Isoradial SGCP with intersection angles θ'_e on each edge e. → Square grid with weights $\sin \frac{\theta'_e}{2}$ on each edge e.

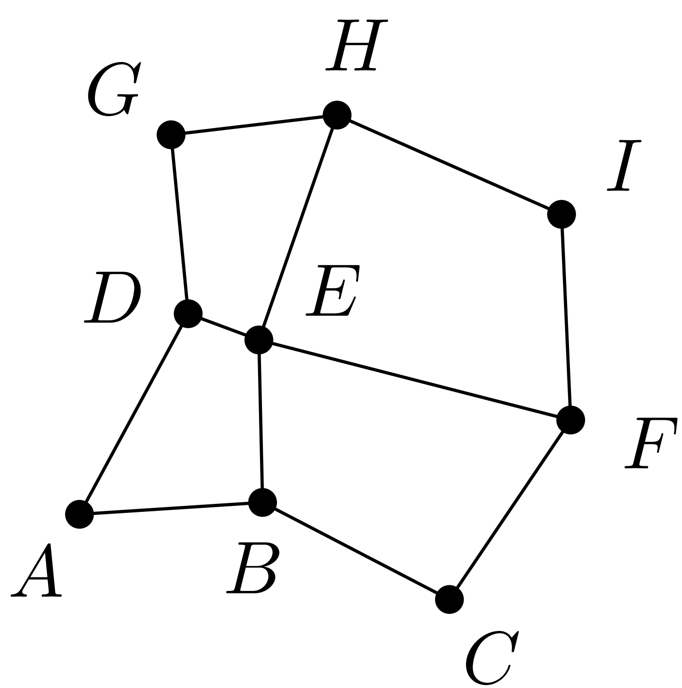
The 2×2 case



- Construction of a 2×2 SGCP : pick B, D, F and H freely, pick E to be any point on the equilateral hyperbola through B, D, F, H and extend it to a biperiodic SGCP with monodromies $\vec{u} = \overrightarrow{DF}$ and $\vec{v} = \overrightarrow{BH}$.

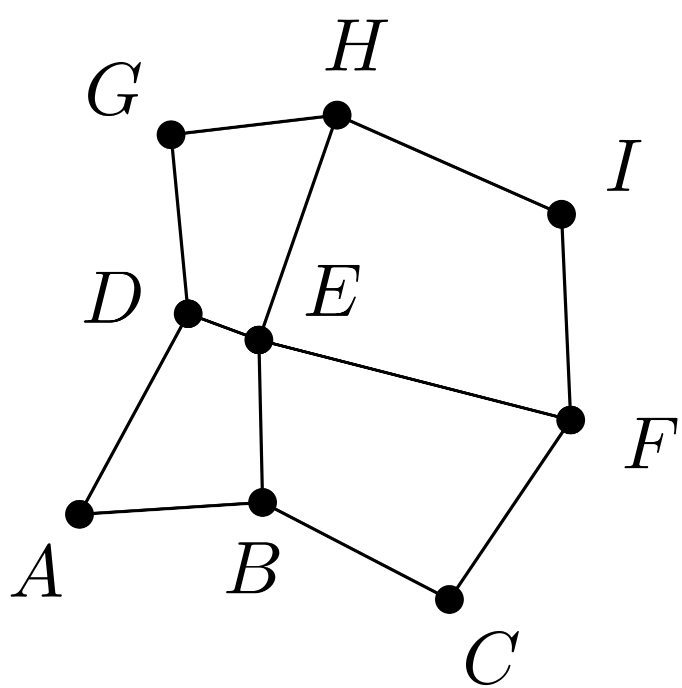
[Geogebra]

- Absolute motion : iterating Miquel dynamics, all the points usually drift to infinity.

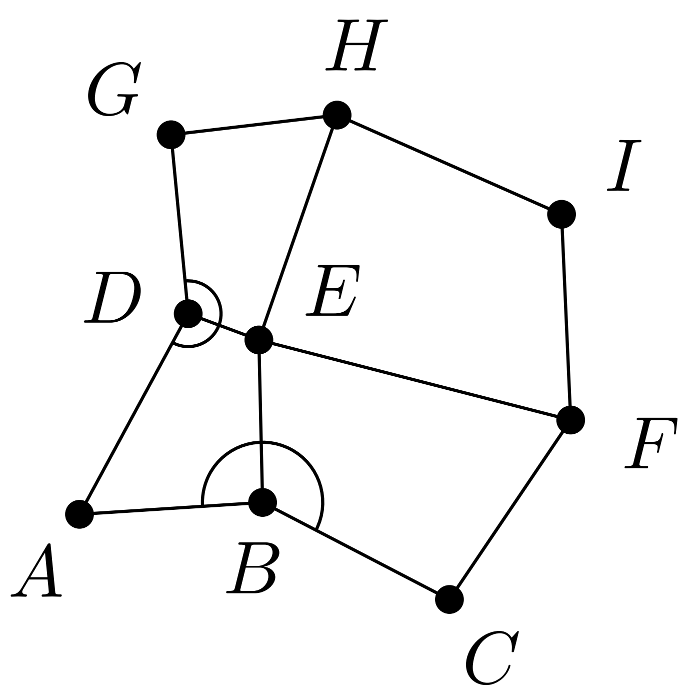


- Relative motion : apply black or white mutation and translate to bring A back to its original position.

- A, C, G and I are fixed.



- Relative motion : apply black or white mutation and translate to bring A back to its original position.
- A, C, G and I are fixed.
- B, D, F and H move along arcs of circles.



Quartic curve for E

- Given a 2×2 SGCP, elementary construction of three points Ω, P, P' (cf Geogebra).

Theorem (R., 2017). *E moves along the quartic curve $\mathcal{Q}$ defined as the set of the points M satisfying*

$$PM^2 P'M^2 - \lambda \Omega M^2 = k,$$

where λ and k are chosen such that the curve goes through A, C, G and I.

[Geogebra]

then

[Mathematica]

Binodal quartic curves

- Taking coordinates centered at Ω such that P is on the horizontal axis, $\mathcal{Q}$ has an equation of the form

$$(X^2 + Y^2)^2 + aX^2 + bY^2 + c = 0$$

- As a curve in $\mathbb{CP}^2$, the quartic $\mathcal{Q}$ has two nodes, the circular points at infinity $(1 : i : 0)$ and $(1 : -i : 0)$, hence has geometric genus 1 and its normalization $\widehat{\mathcal{Q}}$ is an elliptic curve.

- For any binodal quartic curve $\mathcal{C}$ with nodes P_1 and P_2, the group law on $\widehat{\mathcal{C}}$ can be defined using conics going through P_1, P_2 and a fixed base point $P_0 \in \mathcal{C}$: the other three intersection points of the conic with $\mathcal{C}$ are declared to have zero sum.

Theorem (Glutsyuk-R., 2018). *Denote by E'_w (resp. E'_b) the renormalized position of E after white (resp. black) mutation. Then Miquel dynamics is translation on $\widehat{\mathcal{Q}}$:*

$$E'_w = -E - 2A$$
$$E'_b = -E - 2C$$

[Geogebra]
cubic then quartic

(then the end)