# Dimers and circle patterns

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Joint works with:

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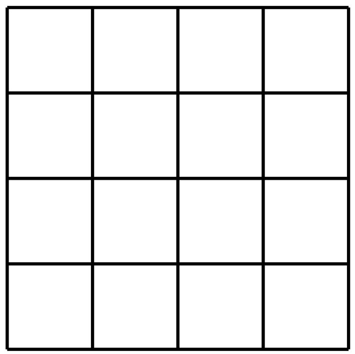
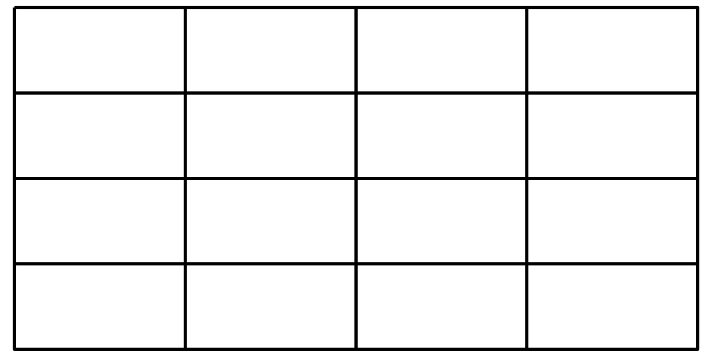
Marianna Russkikh (MIT)

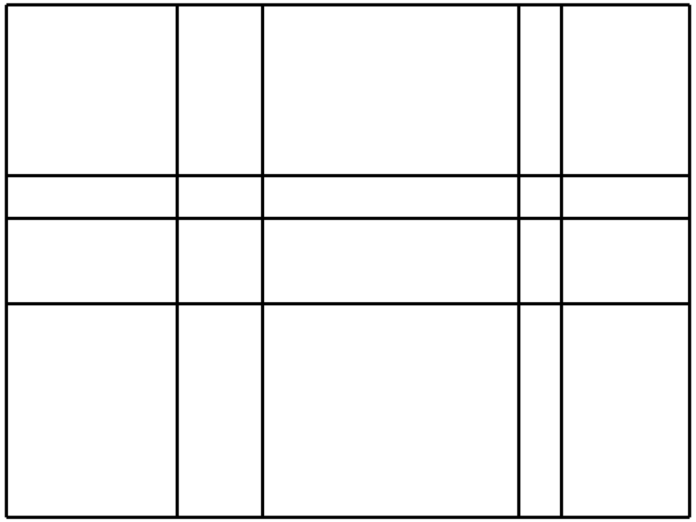
Discrete mathematical modelling seminar
The University of Tokyo, February 13 2020

# 1 Motivation: how to draw a planar graph ?

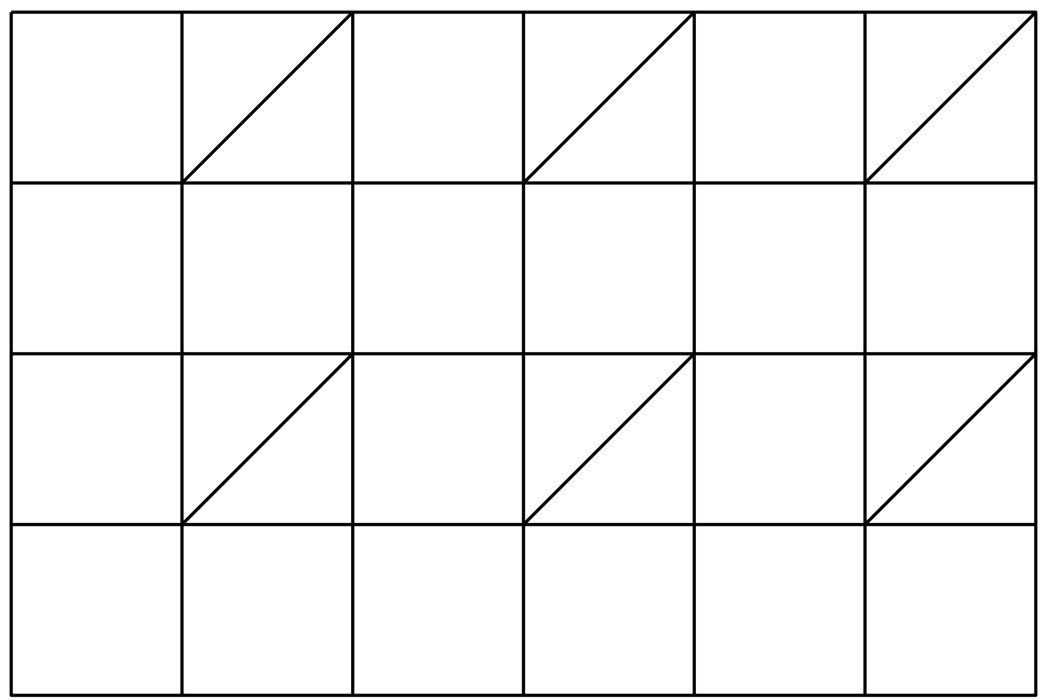
“Simple random walk on $\mathbb{Z}^2$ converges to Brownian motion.”







- More general case: edge weights, arbitrary periodic graph. Find an appropriate embedding to get the same scaling limit (universality).



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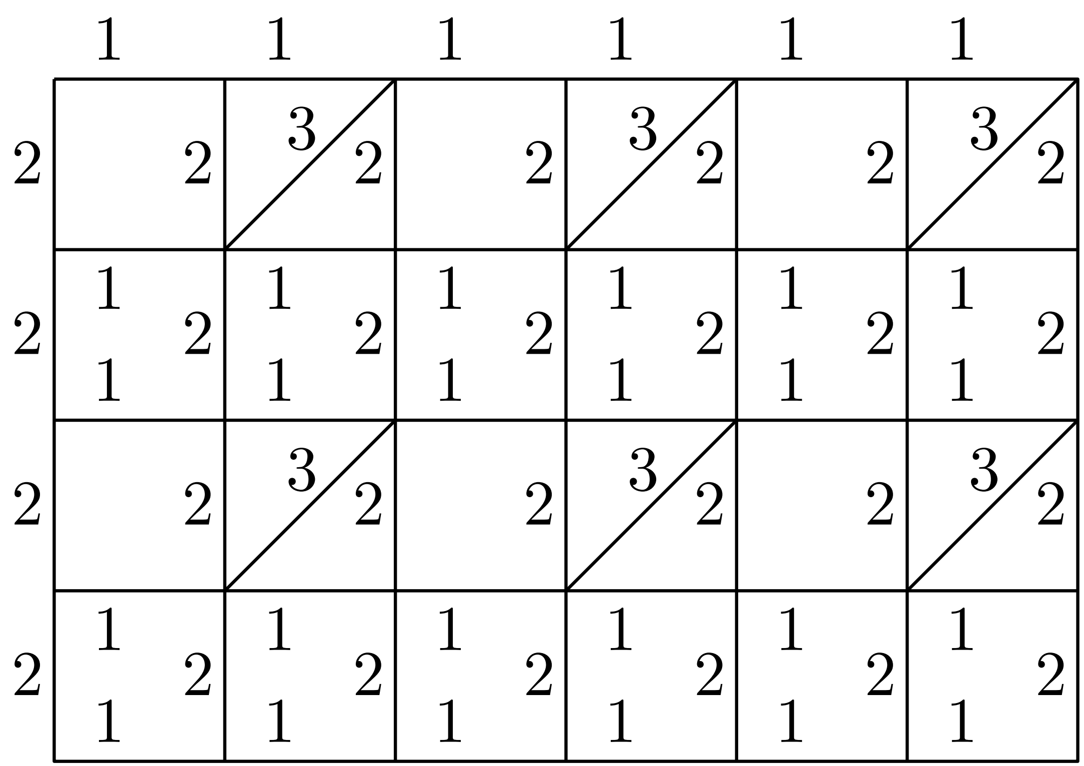
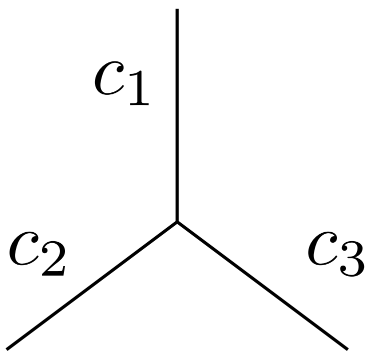
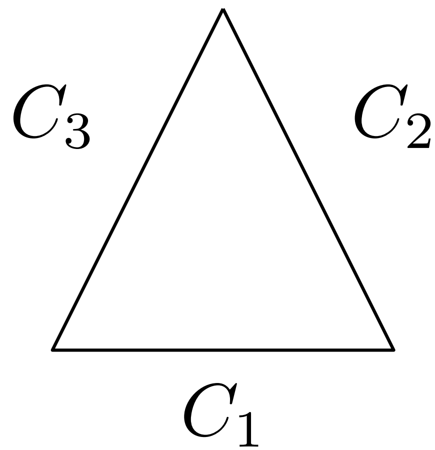


- Find a way of embedding graphs which is compatible with the local star-triangle move.




$$C_1 = \frac{c_2 c_3}{c_1 + c_2 + c_3}$$
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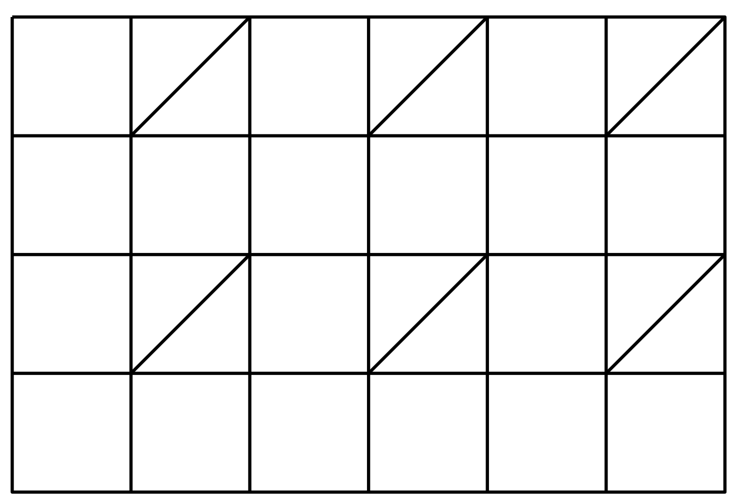
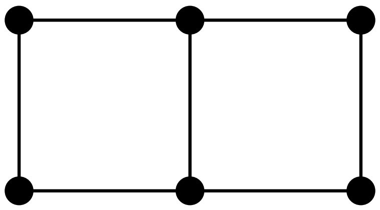


- Such an embedding giving the Brownian motion as the scaling limit and compatible with the local move exists and is called the Tutte embedding.

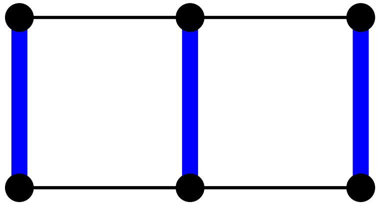
- This talk: another model of statistical mechanics (dimer model).

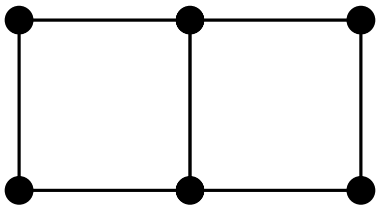
- Introduce a candidate for an embedding adapted to the dimer model.

# 2 The dimer model



- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.





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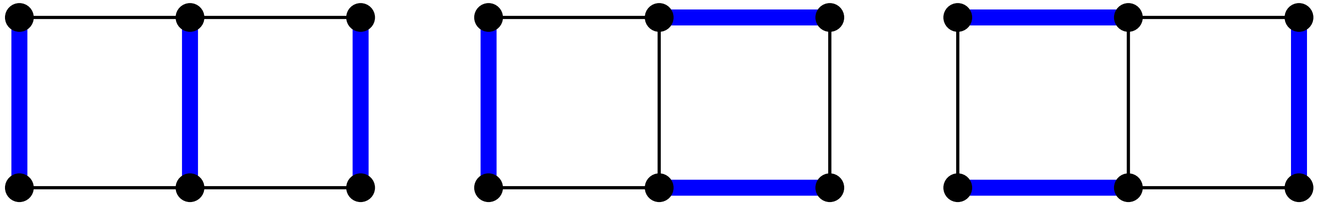
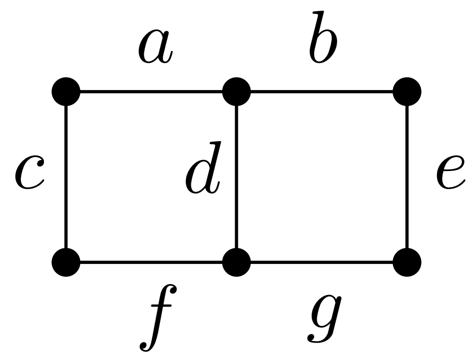
Dimer coverings:



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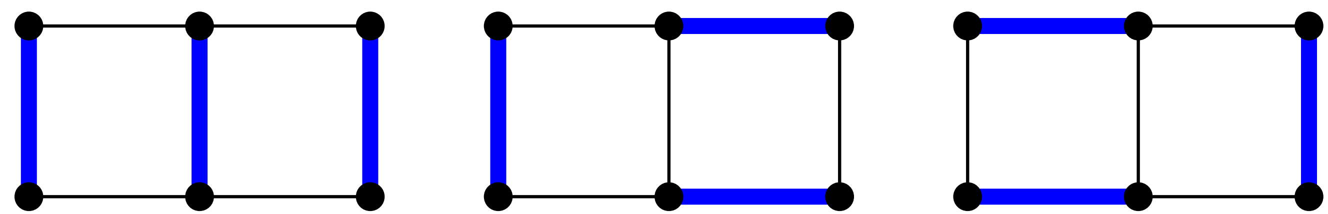
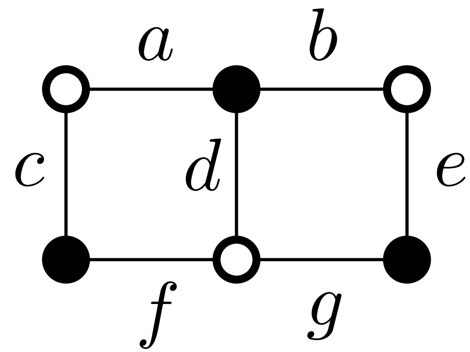


Weights: $cde$ $bcg$ $aef$

- *Boltzmann measure*: draw a dimer covering at random with probability proportional to its weight.

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *positive edge weights*.




- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.

- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge.

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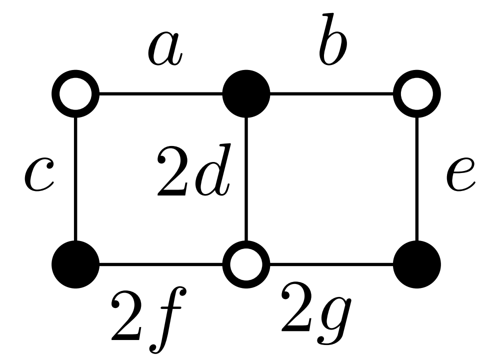


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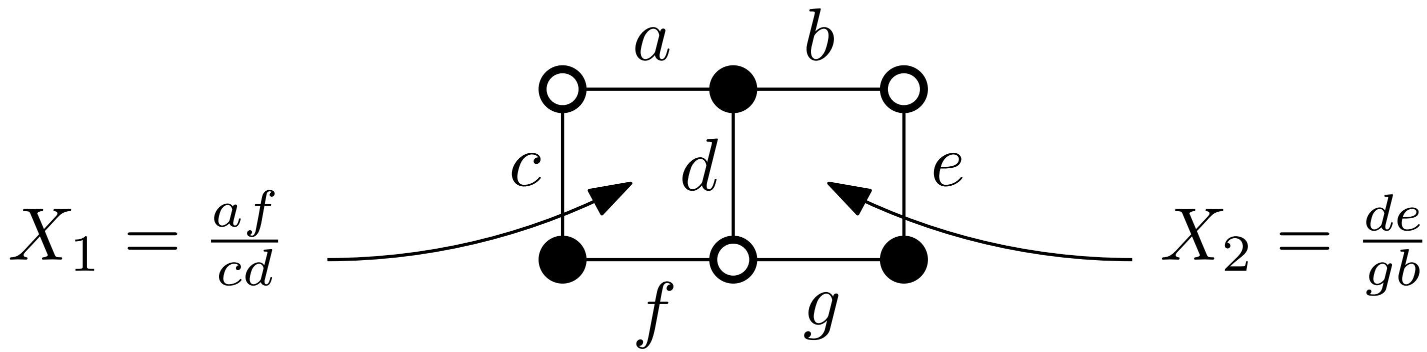


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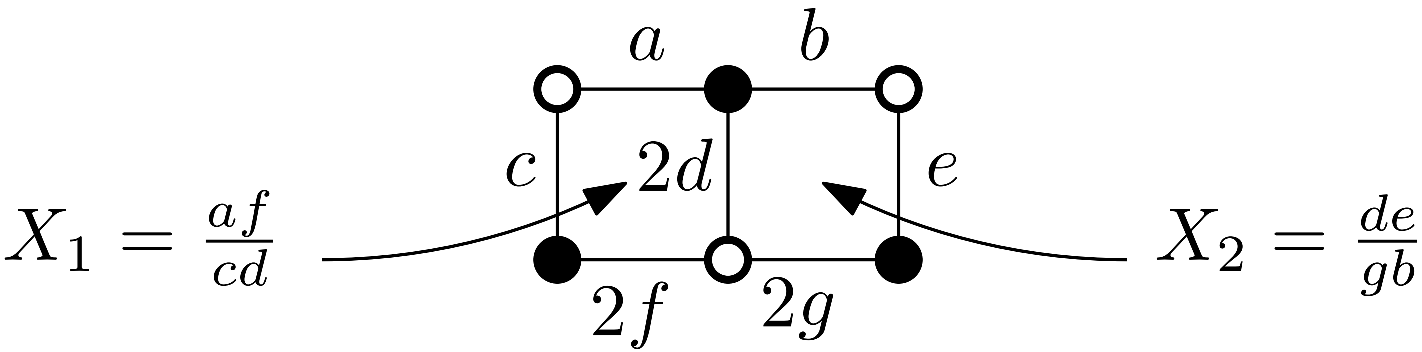


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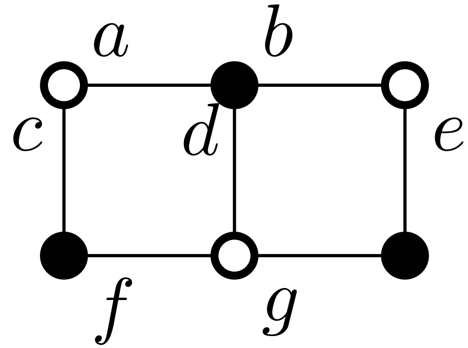
# The Kasteleyn matrix $K$




- *Kasteleyn signs*: assign a sign to each edge such that the number of minus signs around a face of degree 2 mod 4 (resp. 0 mod 4) is even (resp. odd).

- $K$: weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

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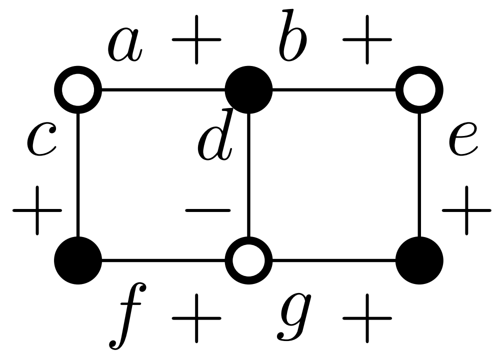


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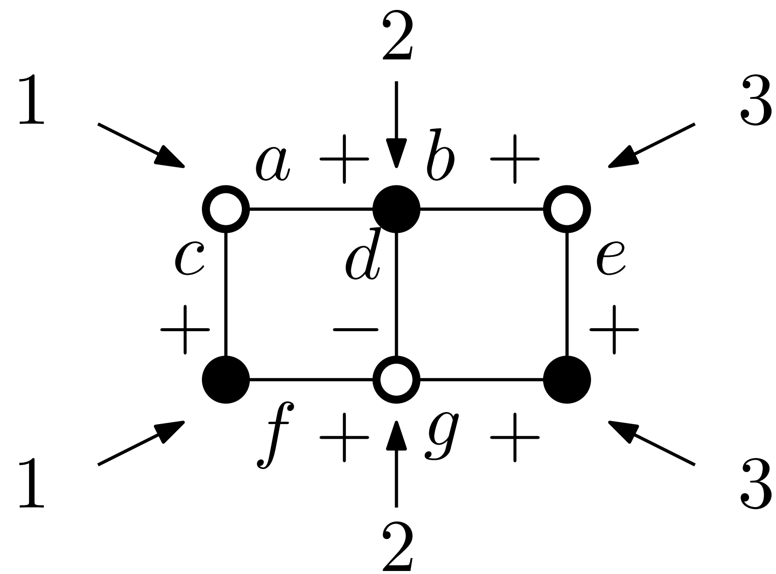


$$\begin{array}{c} \\ \circ_1 \\ \circ_2 \\ \circ_3 \end{array} \begin{array}{c} \bullet_1 \quad \bullet_2 \quad \bullet_3 \\ \begin{pmatrix} c & a & 0 \\ f & -d & g \\ 0 & b & e \end{pmatrix} \end{array}$$

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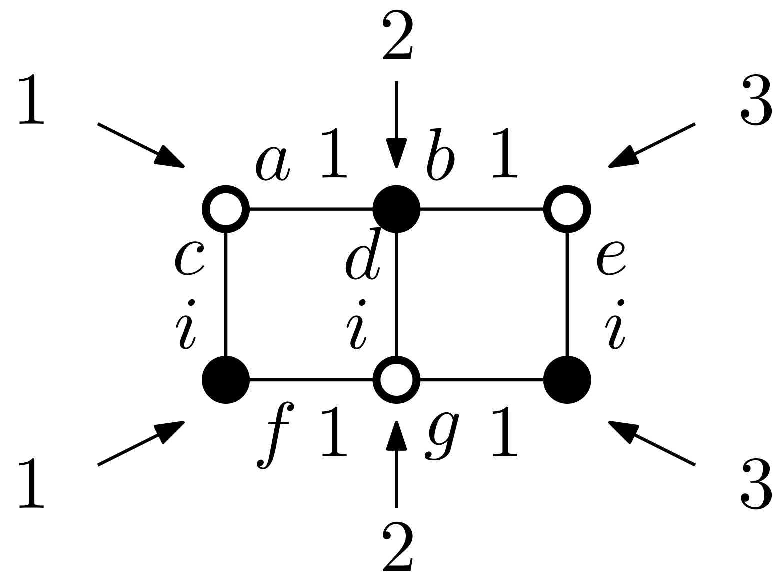


$$
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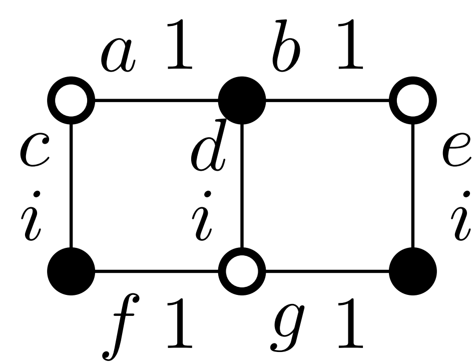
$$

- *Complex Kasteleyn signs*: assign a unit complex number to each edge such that the alternating product of these numbers around a face of degree 2 mod 4 (resp. 0 mod 4) is 1 (resp. $-1$).

- $K$: weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

- The partition function (sum of the weights of all dimer coverings) is $|\det K|$. The dimer correlations are given by minors of $K^{-1}$ (Kasteleyn, Temperley-Fisher).

- Merge the complex Kasteleyn signs with the positive edge weights to get complex edge weights (entries of $K$).



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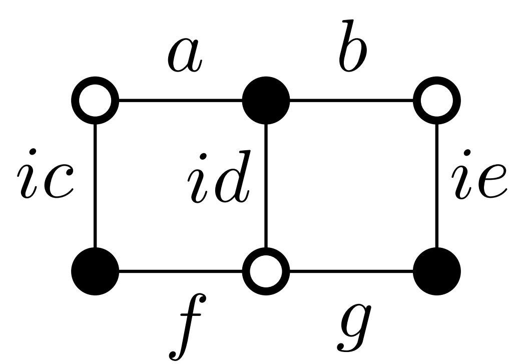

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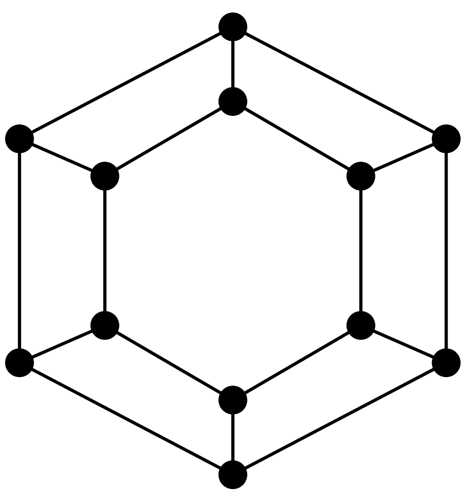
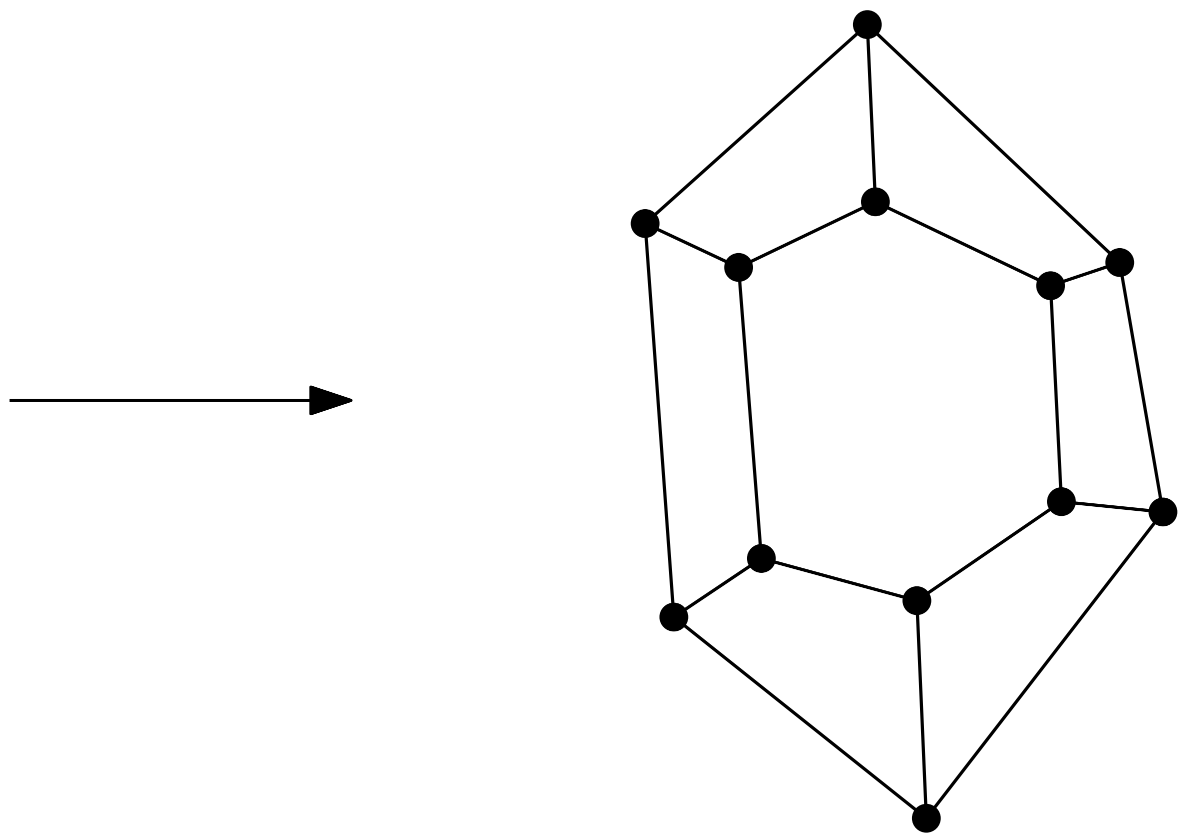
- The alternating product of complex edge weights is real positive (resp. real negative) around a face of degree 2 mod 4 (resp. 0 mod 4).

# 3 Circle patterns and circle centers

- *Circle pattern for $G$*: map from the vertex set of $G$ to $\mathbb{R}^2$ sending all the vertices around any bounded face to concyclic points.



$G$ planar



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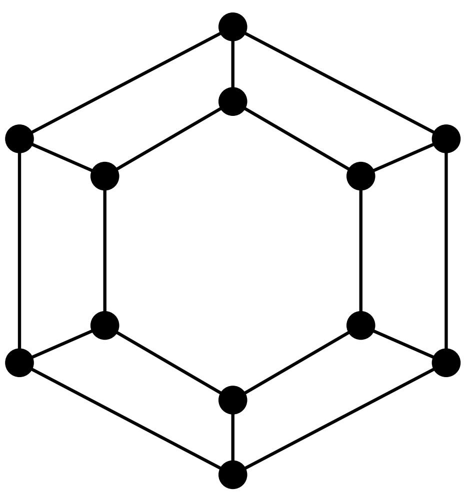

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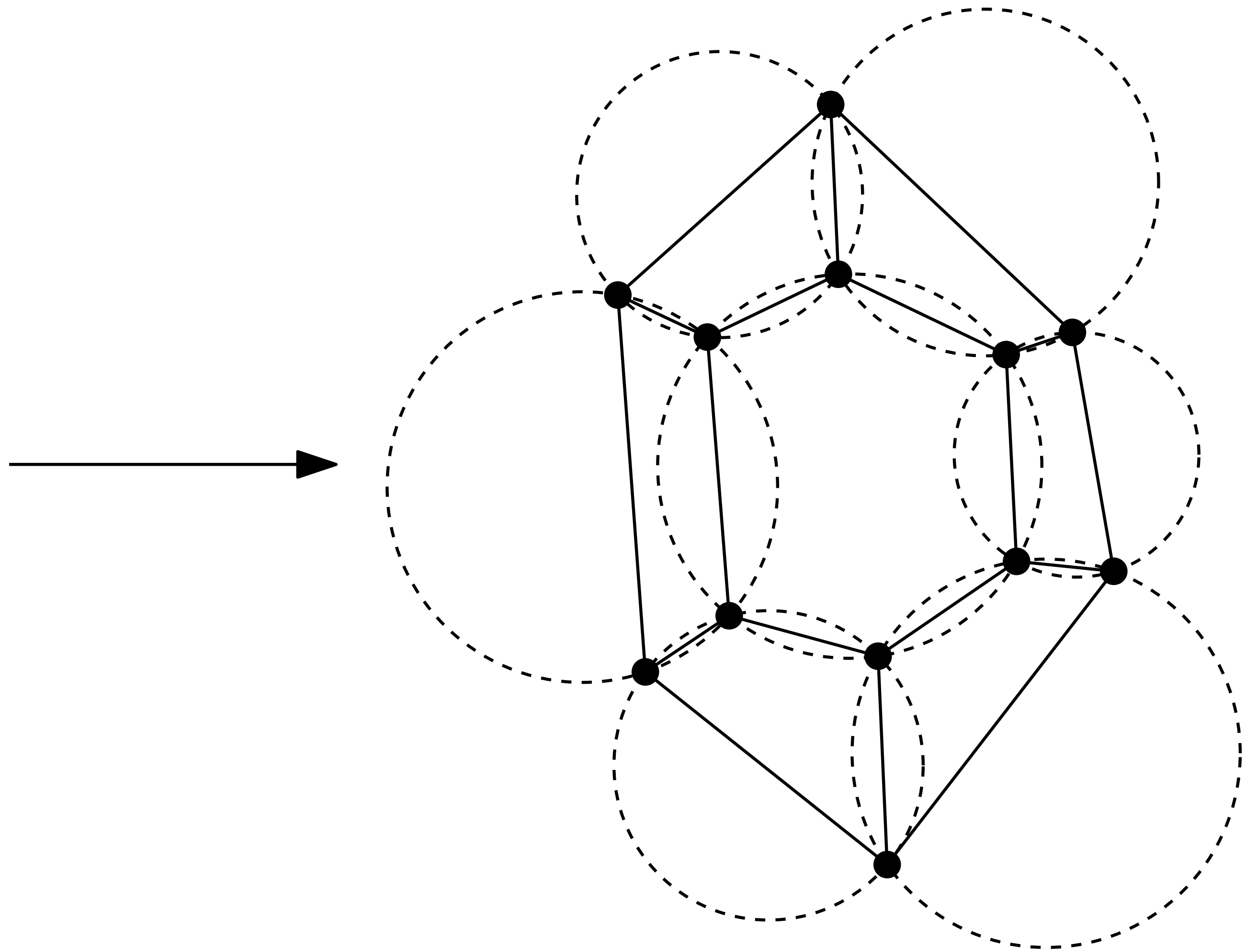

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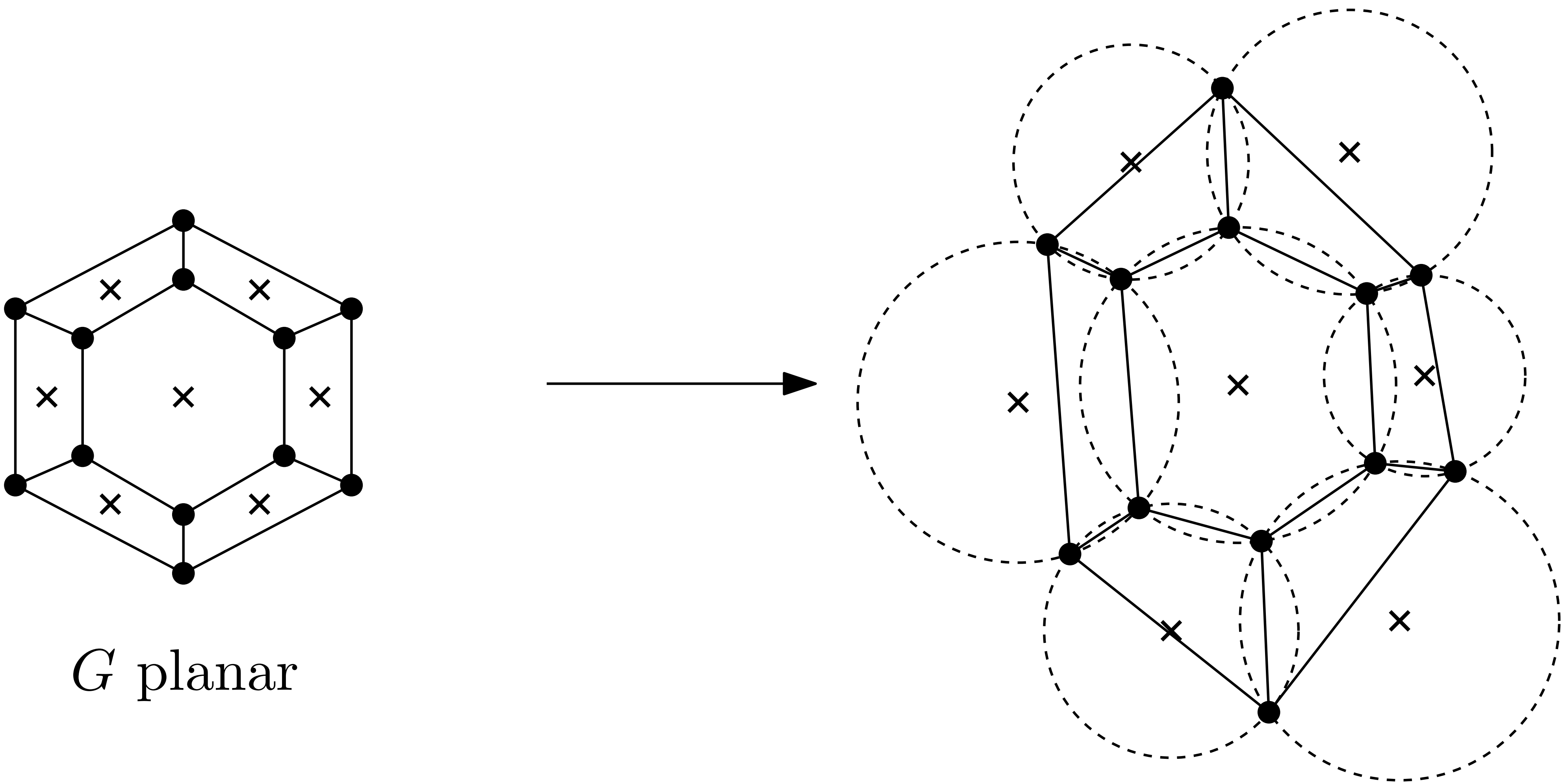

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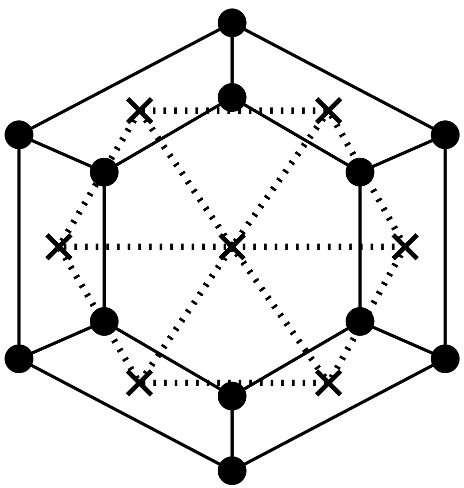

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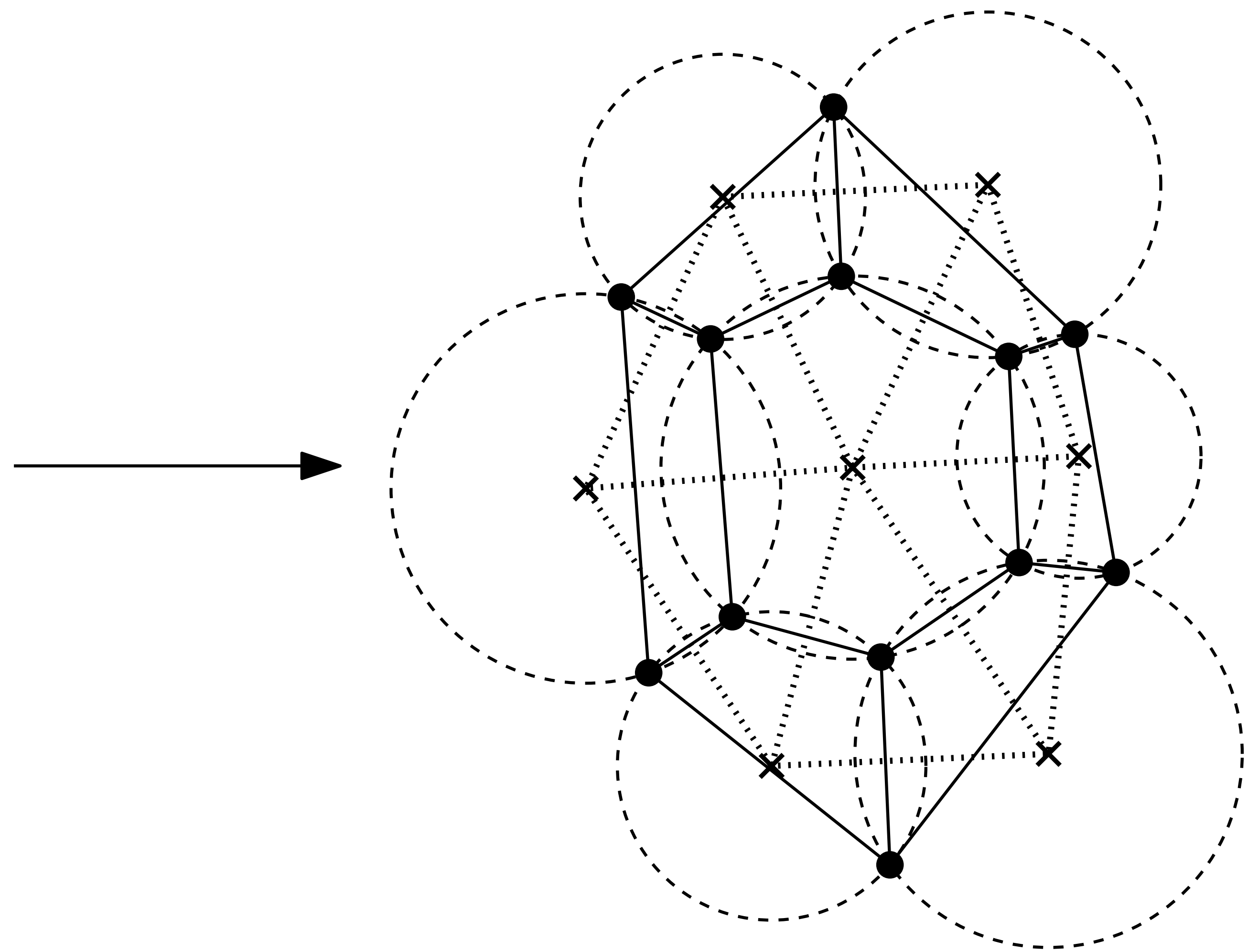
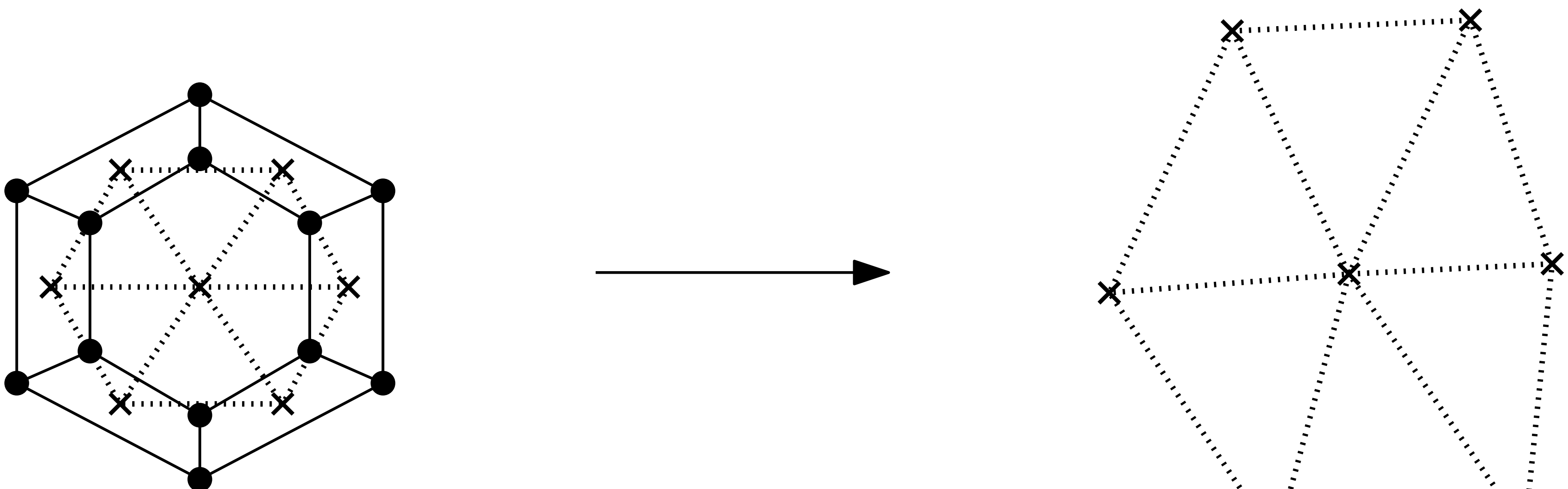


- *Circle centers for $G$*: drawing of the dual graph of $G$ arising as centers of some circle pattern for $G$.



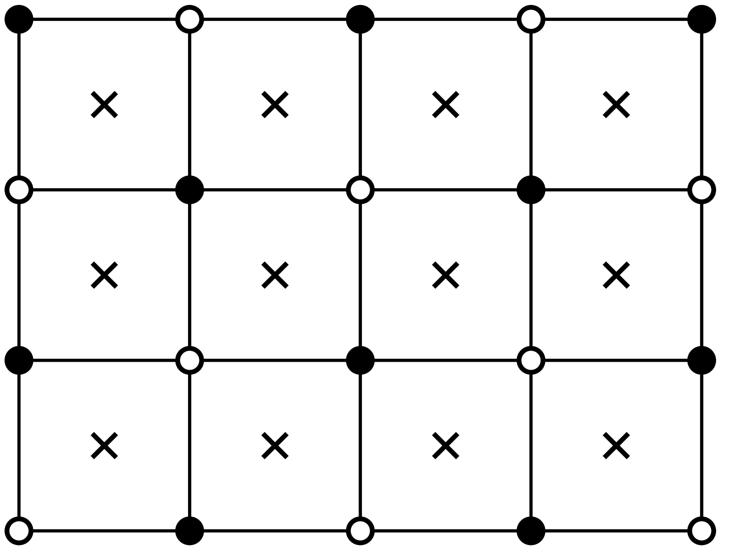
$G$ planar

- Recover the circle pattern from the circle centers ?

  How many circle patterns have the same centers ?

- Given a drawing of the dual graph of $G$, how to see if it corresponds to the centers of a circle pattern for $G$ ?

- Answers in the case when $G$ is bipartite.



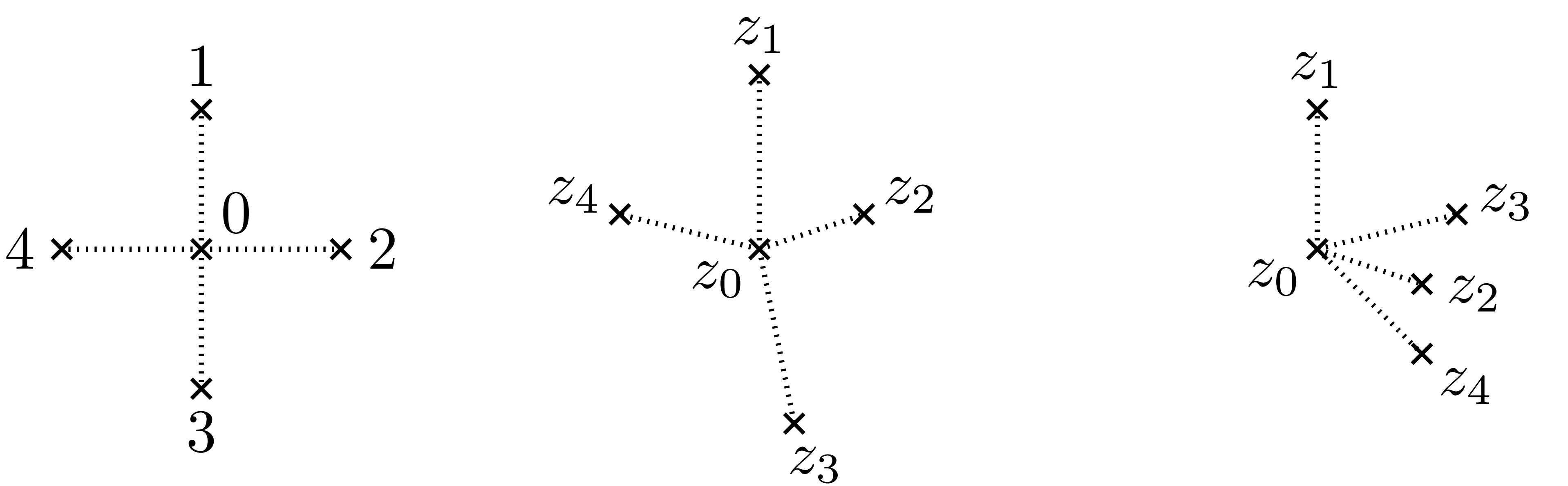
[Geogebra]

- From now on $G$ is bipartite.

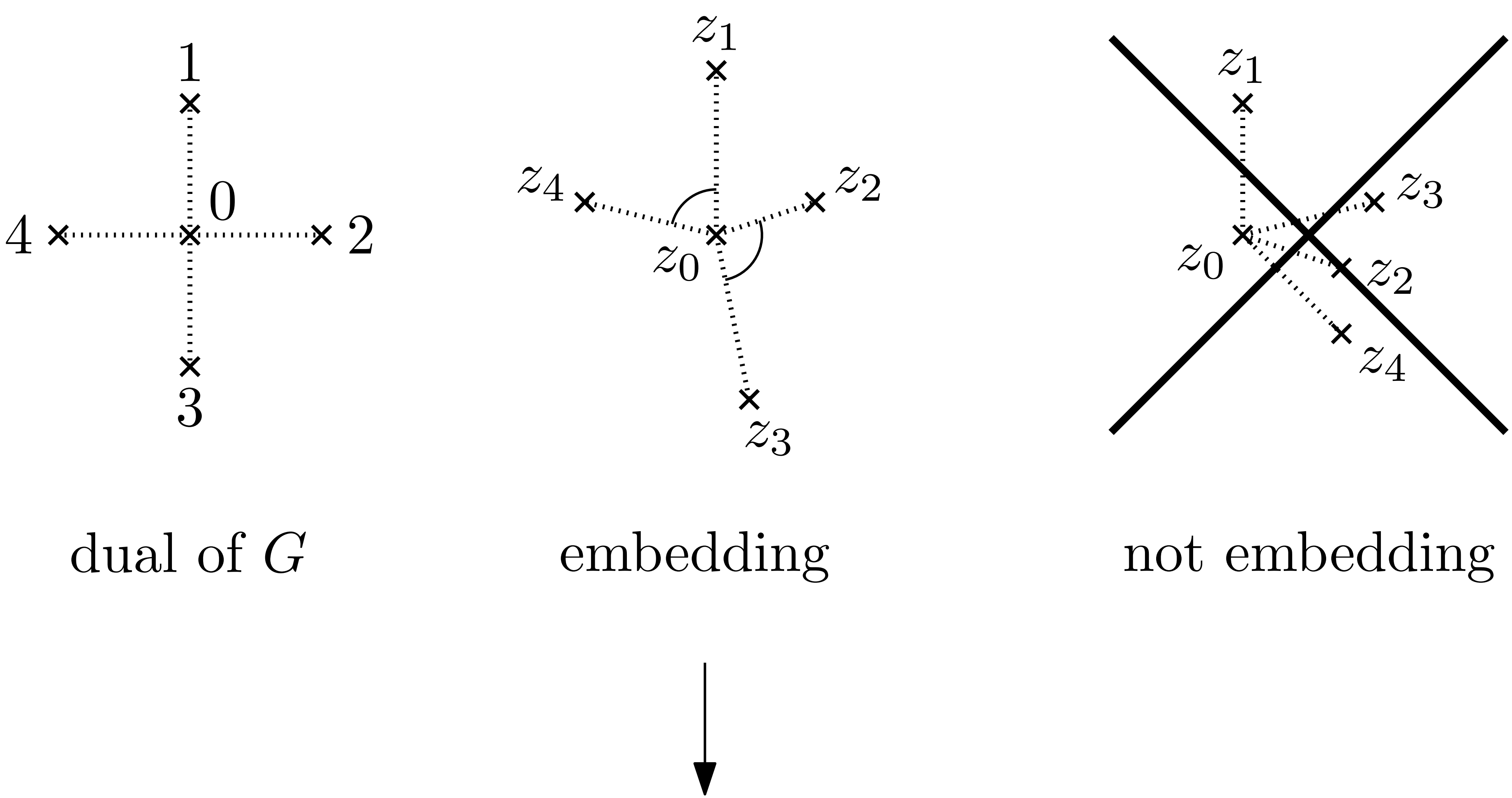
- 2-parameter family of patterns with the same centers.

- A drawing of the dual graph of $G$ corresponds to circle centers for $G$ if and only if around each dual vertex the sum of every other angle is 0 mod $\pi$.

- An *embedding* of the dual graph of $G$ corresponds to circle centers for $G$ if and only if around each dual vertex the sum of every other angle is equal to $\pi$.




dual of $G$ embedding not embedding




$$\arg \frac{z_4 - z_0}{z_1 - z_0} + \arg \frac{z_2 - z_0}{z_3 - z_0} = \pi \Leftrightarrow \frac{(z_2 - z_0)(z_4 - z_0)}{(z_1 - z_0)(z_3 - z_0)} \in \mathbb{R}_{<0}$$

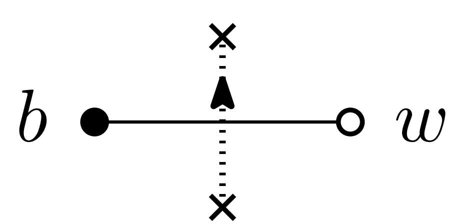
# 4 Dimer models and circle centers

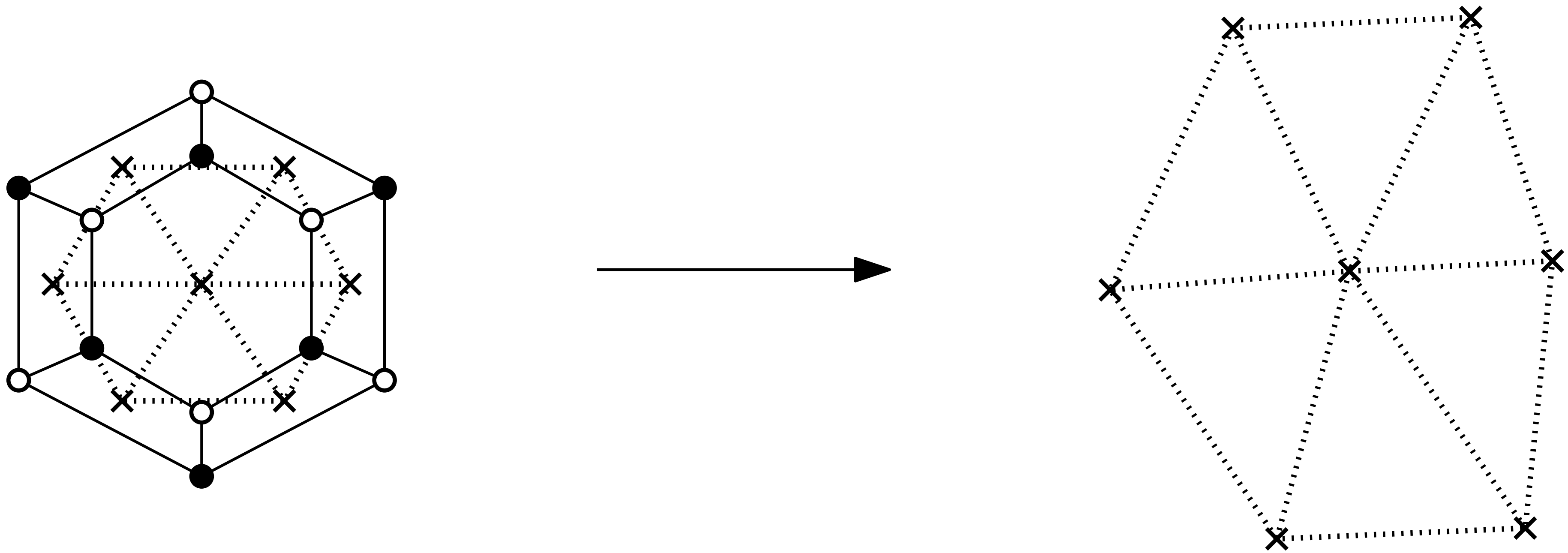
# From circle centers to dimer weights

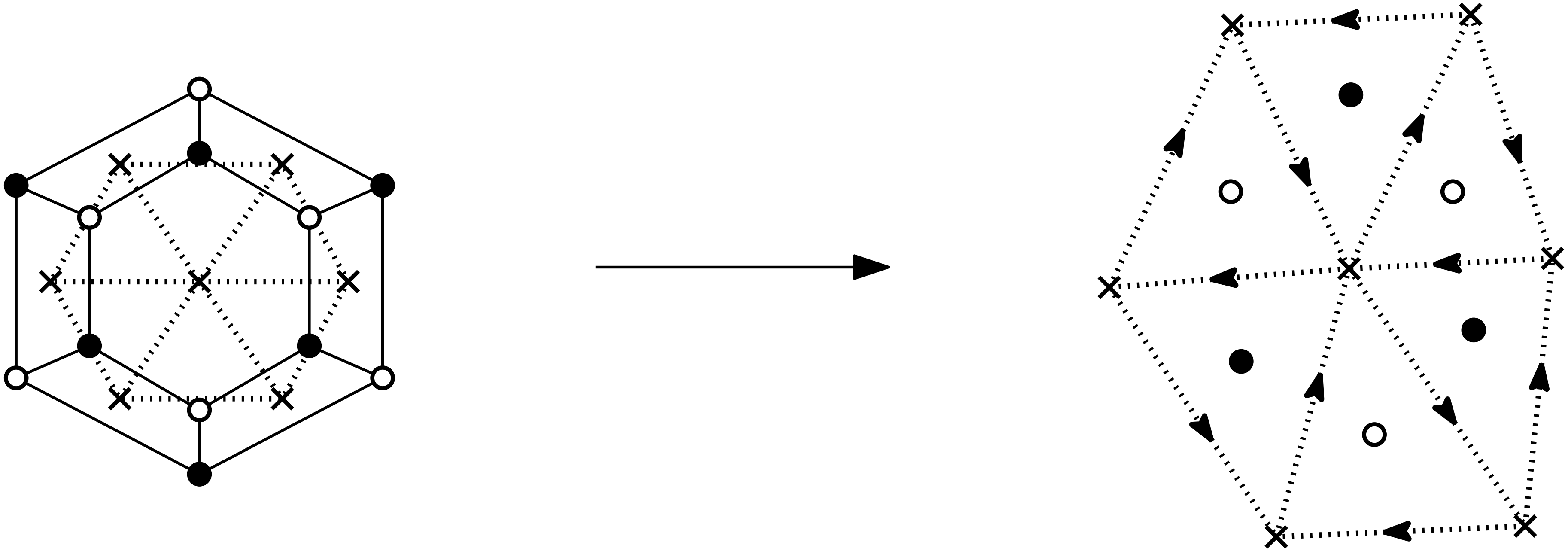
- Fix $G$ a planar unweighted bipartite graph. Start with an embedding of the dual of $G$ as circle centers (a.k.a. t-embedding for $G$).

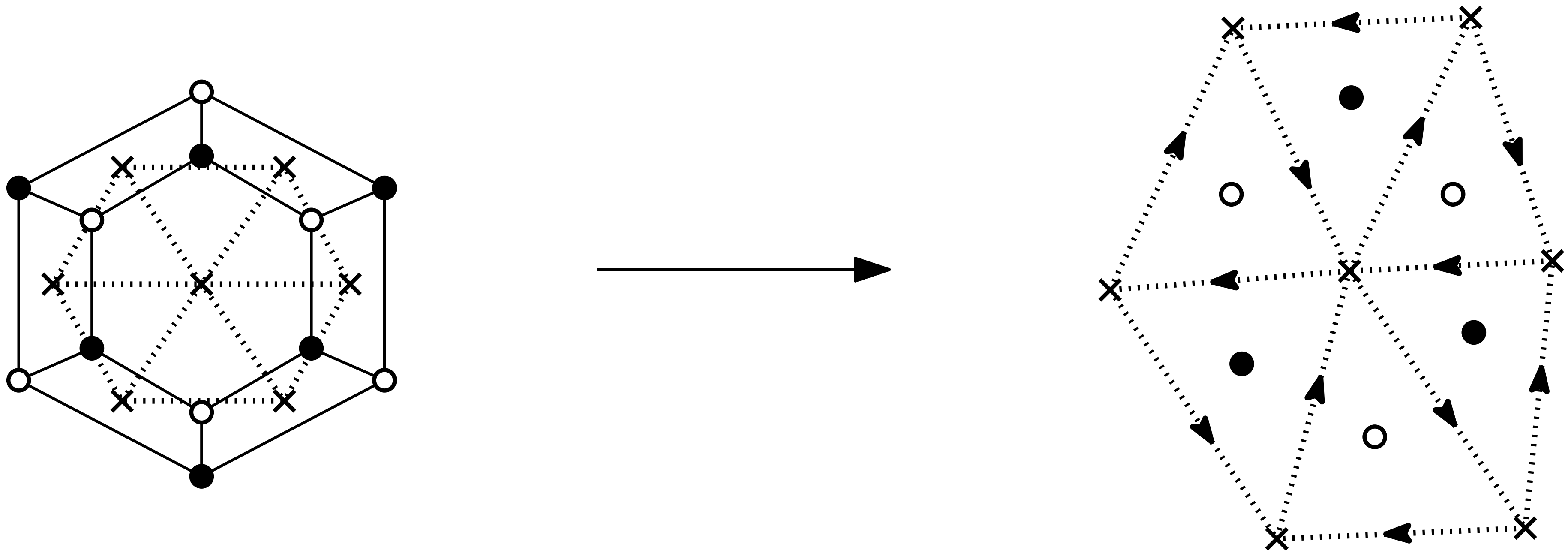
- Construct complex edge weights for $G$ associated to that embedding which satisfy the Kasteleyn condition.

- For an edge in $G$ between $b$ and $w$, the weight is the vector (complex number) of its corresponding dual edge, oriented so that $b$ lies to its left.

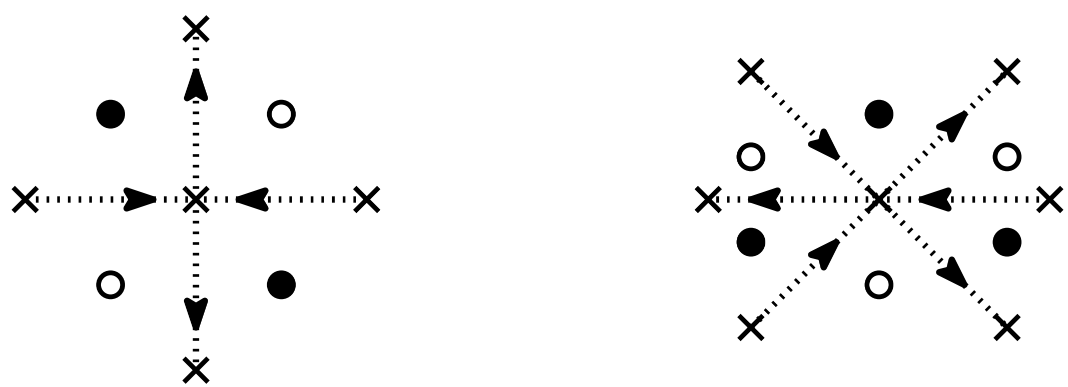


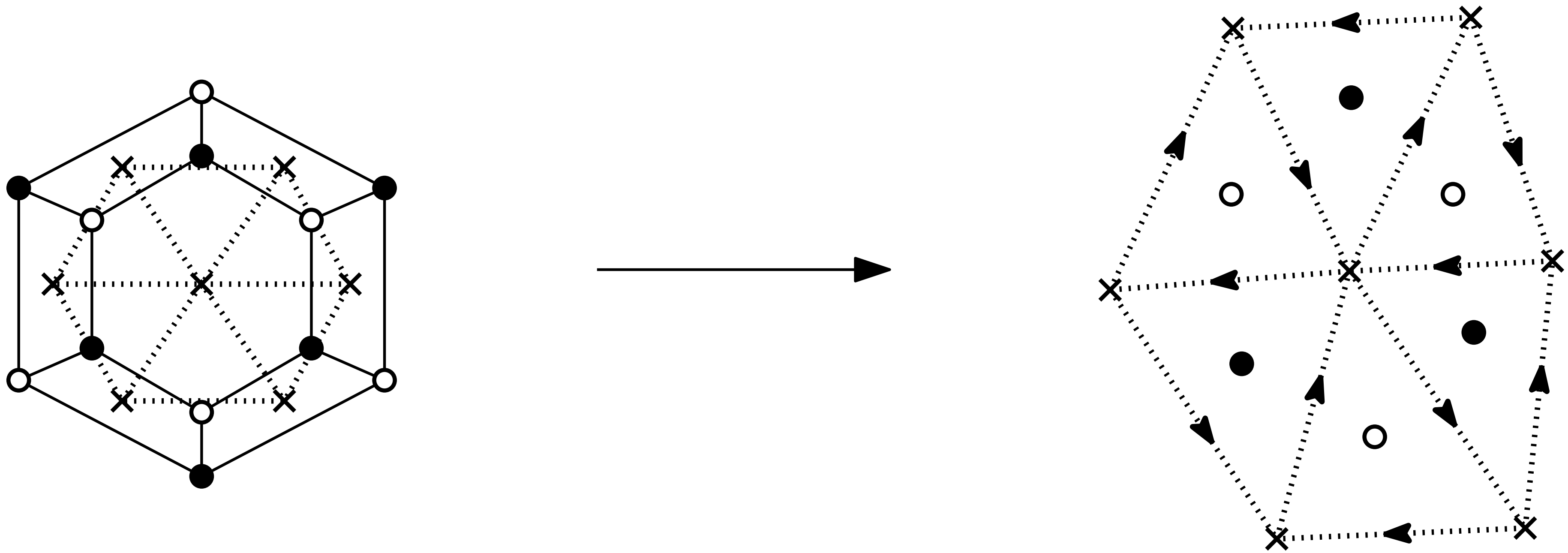






- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree 2 mod 4 (resp. 0 mod 4) is positive (resp. negative).





- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree 2 mod 4 (resp. 0 mod 4) is positive (resp. negative).

- Around every vertex, the sum of the complex edge weights is zero, i.e. the edge weights have zero divergence.

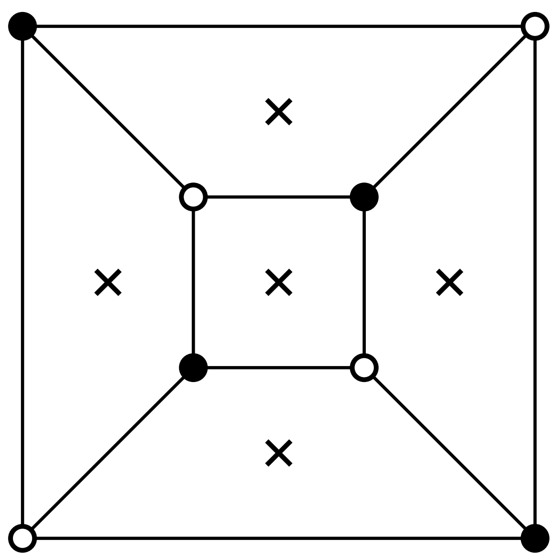
For a bipartite graph, the geometric local condition
  "being centers of a circle pattern with embedded dual"
implies the local condition
  "being Kasteleyn edge weights with zero divergence"
(Kenyon-Lam-R.-Russkikh, 2018)

- The fact that circle center embeddings satisfy the Kasteleyn condition was also observed by Affolter (2018).
- Positive edge weights are obtained from circle centers as distances between adjacent centers.
- Generalizes the construction from the isoradial case (Kenyon 2002).

# From dimer weights to circle centers

- Given a bipartite graph with positive edge weights, find gauge equivalent weights coming from circle centers.

→ *Coulomb gauge*



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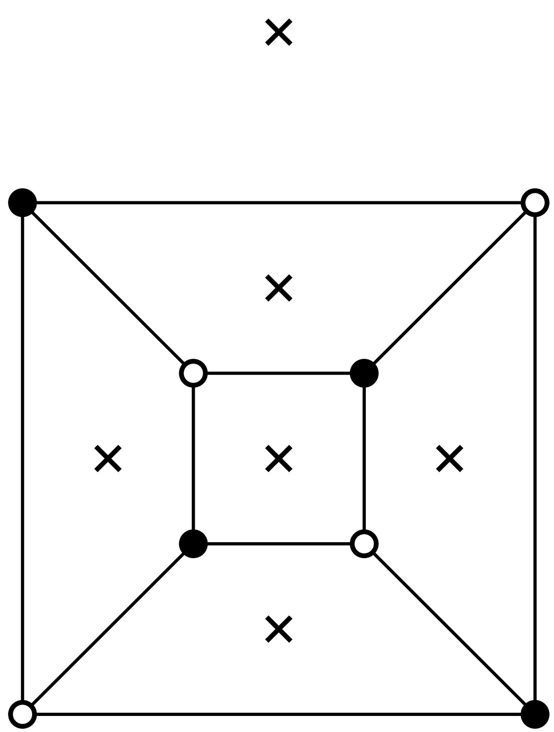

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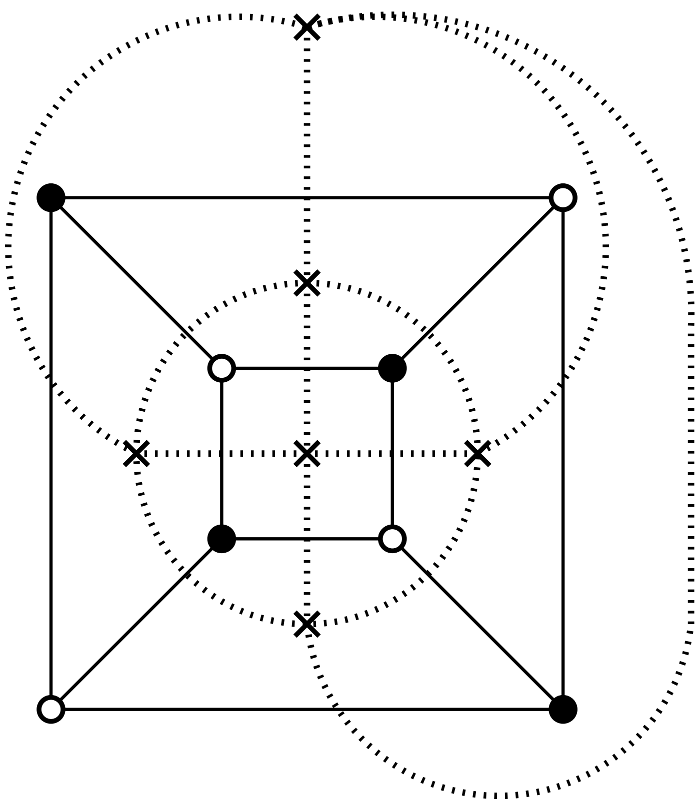

dual

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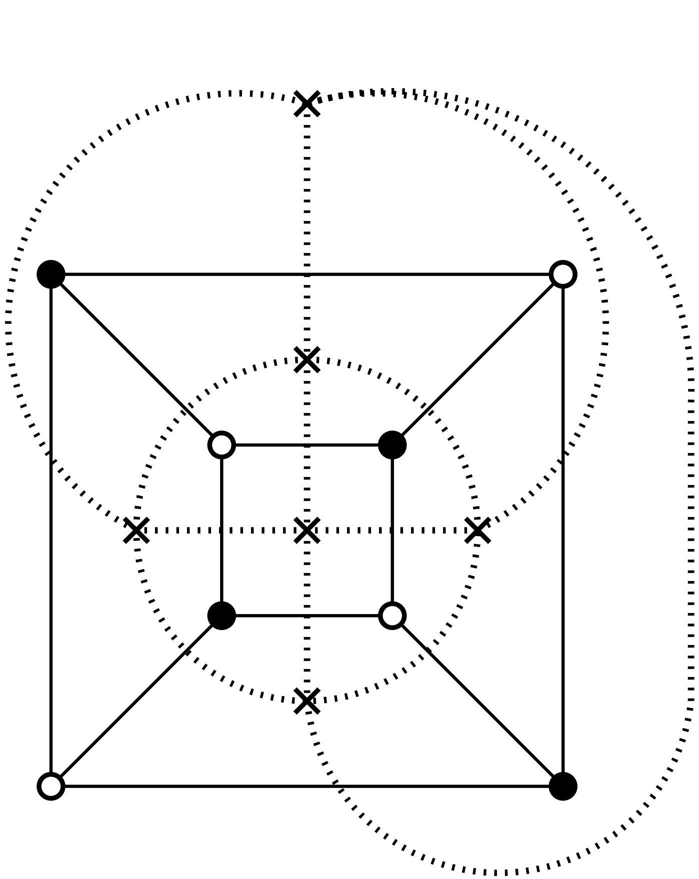

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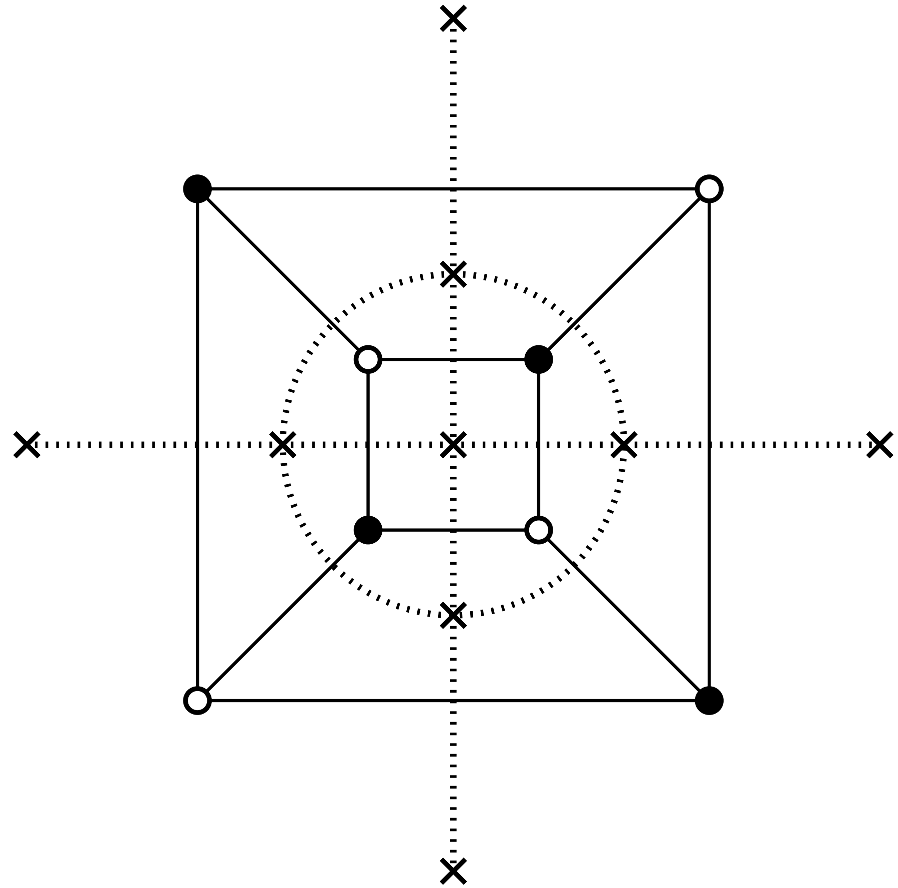

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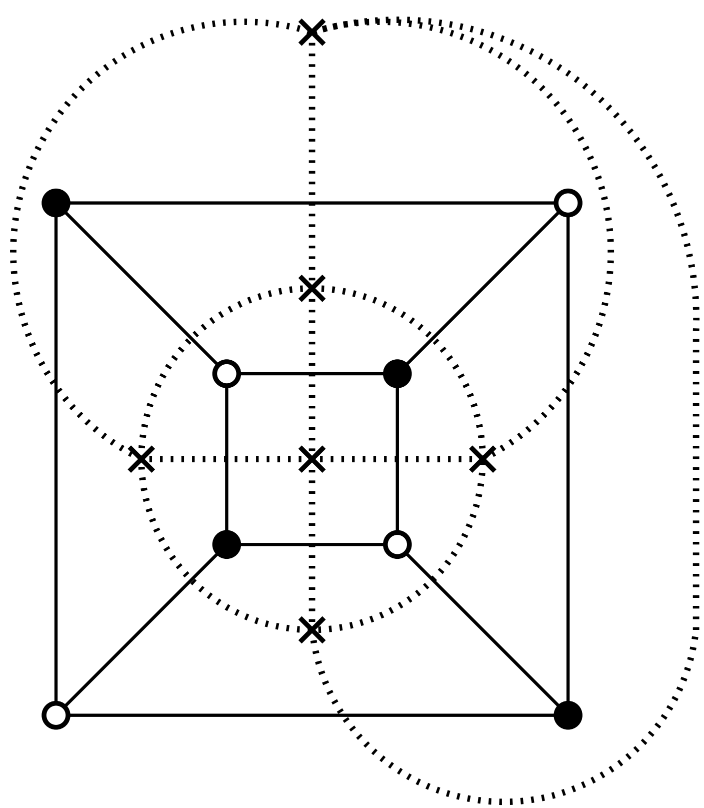
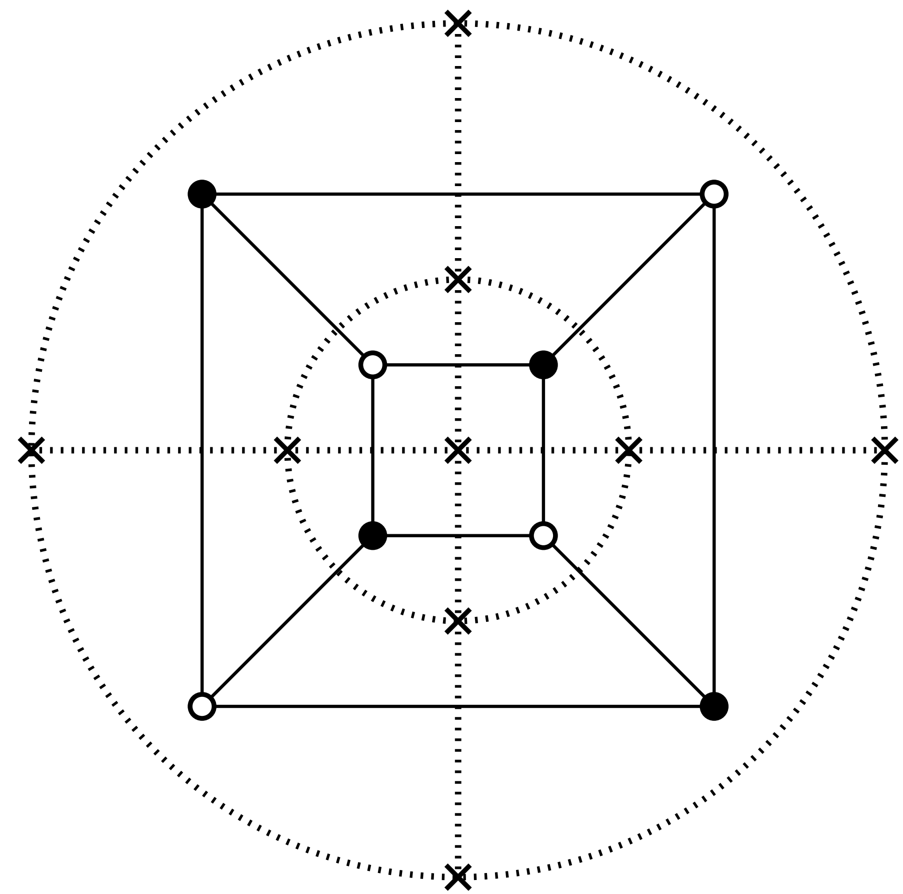
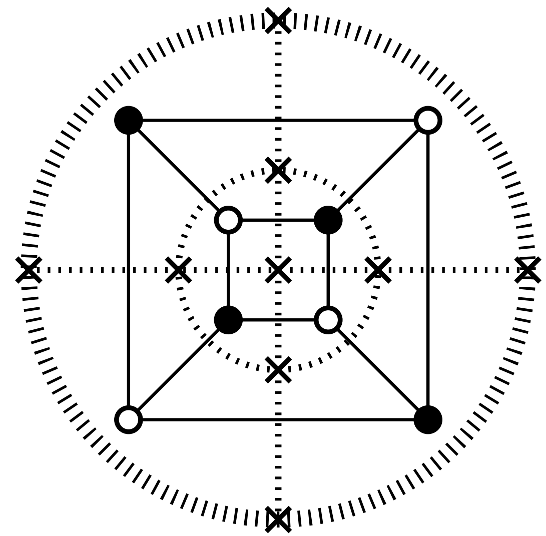


dual



augmented dual



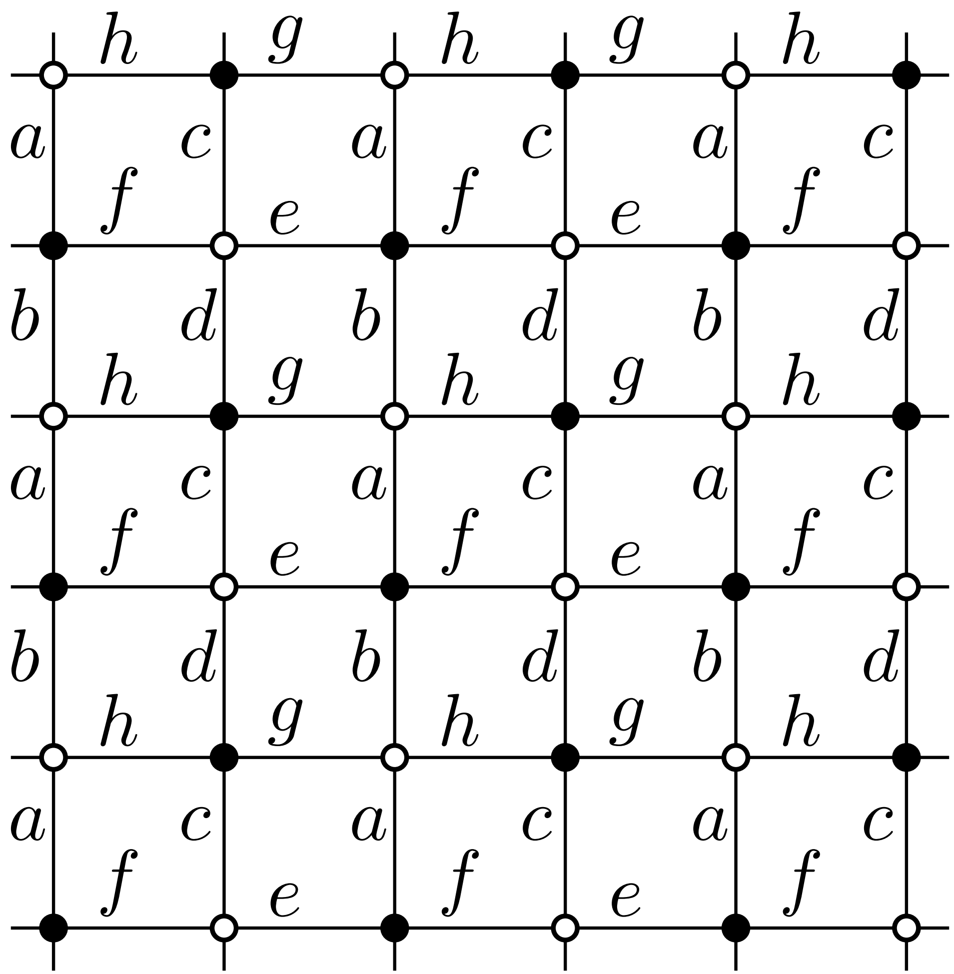
**Theorem** (Kenyon-Lam-R.-Russkikh 2018). *Let $G$ be a planar bipartite weighted graph with outer face of degree 4. Fix a convex quadrilateral $P$.*
*There are two circle center embeddings of the augmented dual of $G$ which produce weights that are gauge equivalent to the original weights and such that the four outer dual vertices are mapped to the vertices of $P$.*

- Given
  - an unweighted bipartite planar graph $G$ with boundary of length 4
  - a convex quadrilateral (boundary condition)

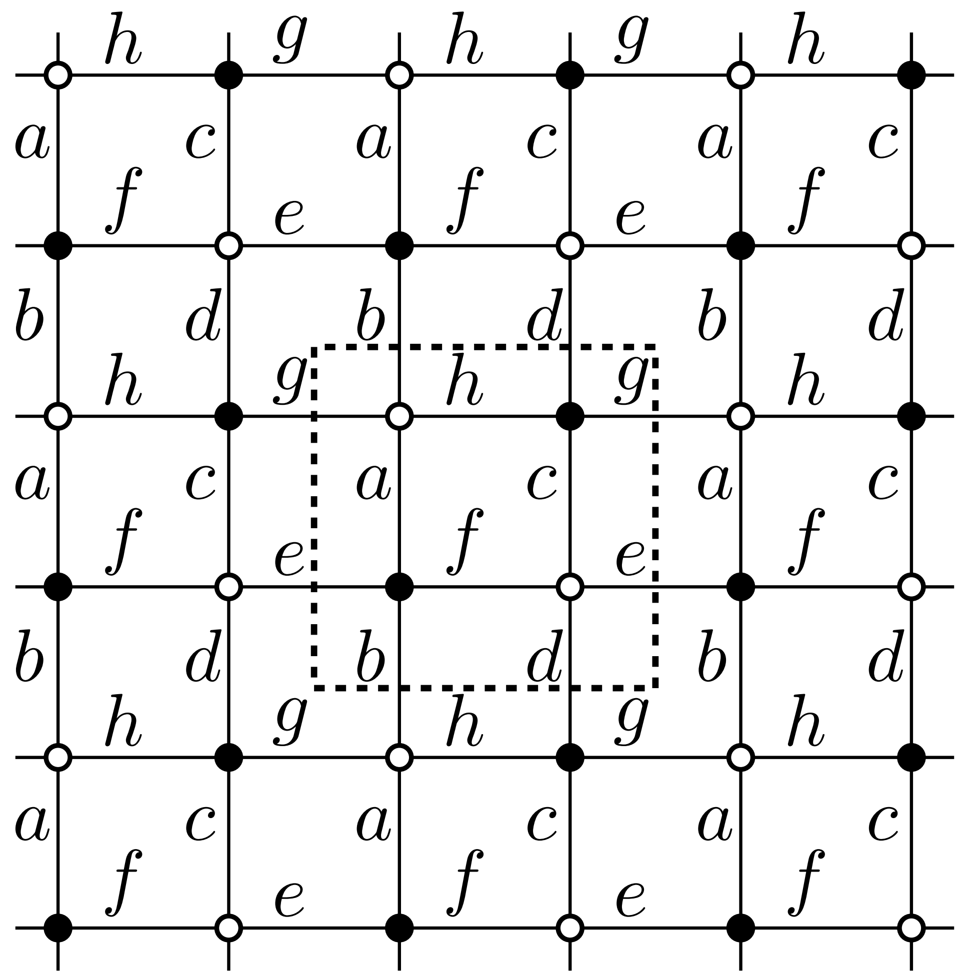
  there is a 2-to-1 correspondence between embeddings of the augmented dual of $G$ as circle centers and dimer Boltzmann measures on $G$.

- Expected to hold in some form for other boundary lengths.

- Other setting: infinite planar bipartite graphs, periodic in two directions with edge weights also periodic.



- Other setting: infinite planar bipartite graphs, periodic in two directions with edge weights also periodic.



- Let $G$ be an infinite periodic weighted graph.

- *Gibbs measure*: probability measure on the dimer coverings of $G$, whose restriction to finite subgraphs are Boltzmann measures induced by the edge weights.

- *Ergodic Gibbs measure*: not a convex combination of other Gibbs measures.

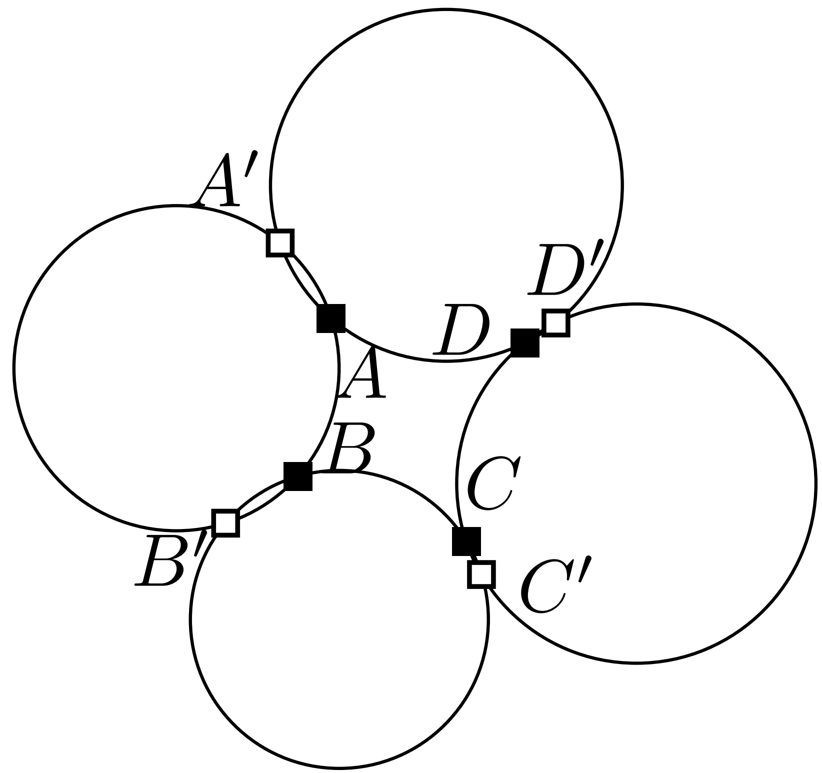
- *Liquid*: correlations decay polynomially.

**Theorem** (Kenyon-Lam-R.-Russkikh 2018). *Let $G$ be an infinite weighted bipartite graph, periodic in two directions. Periodic circle center embeddings of the dual of $G$ producing edge weights that are gauge equivalent to the original ones are in bijection with liquid ergodic Gibbs measures on $G$.*

- In both the finite and the infinite case, the construction of a circle center embedding associated with a weighted planar graph $G$ depends globally (not locally) on $G$.

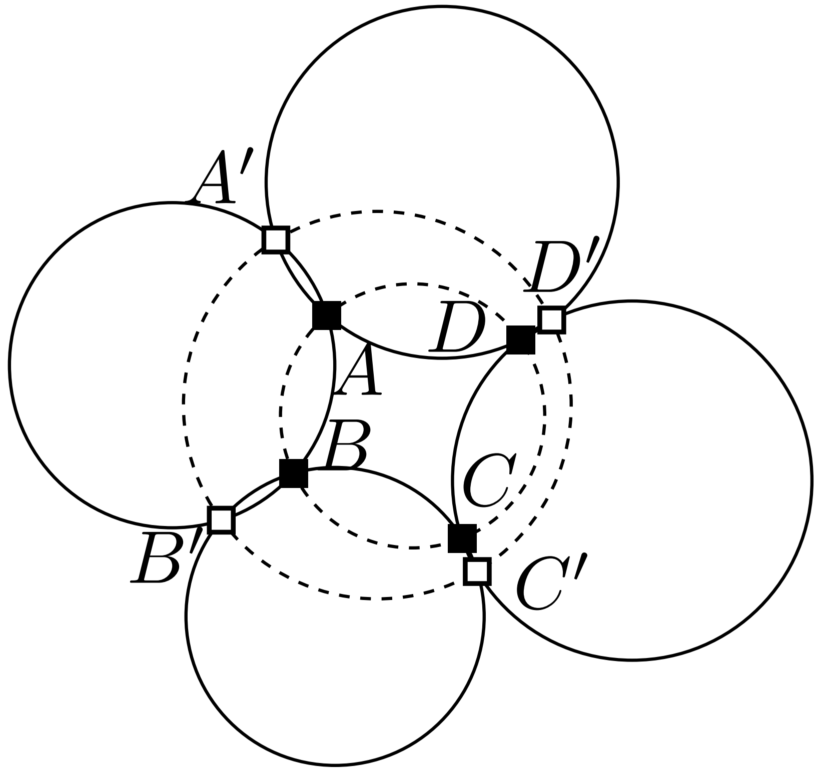
# 5 Local moves and scaling limits

# Miquel's theorem




**Theorem** (Miquel, 1838)**.** *In this setting,*
$A, B, C, D$ *concyclic* $\Leftrightarrow$ $A', B', C', D'$ *concyclic.*

# Miquel's theorem

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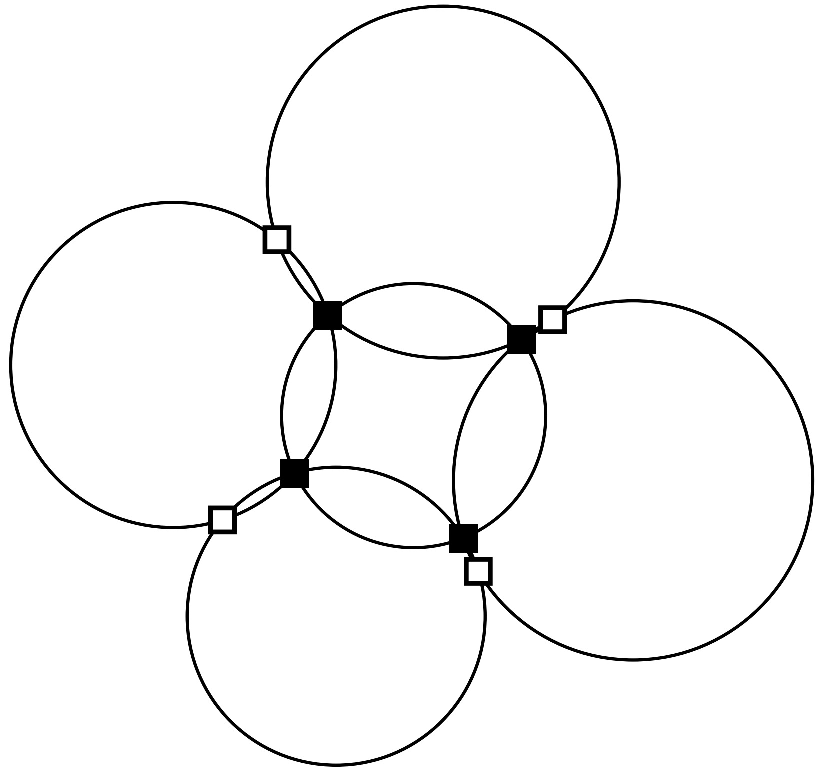

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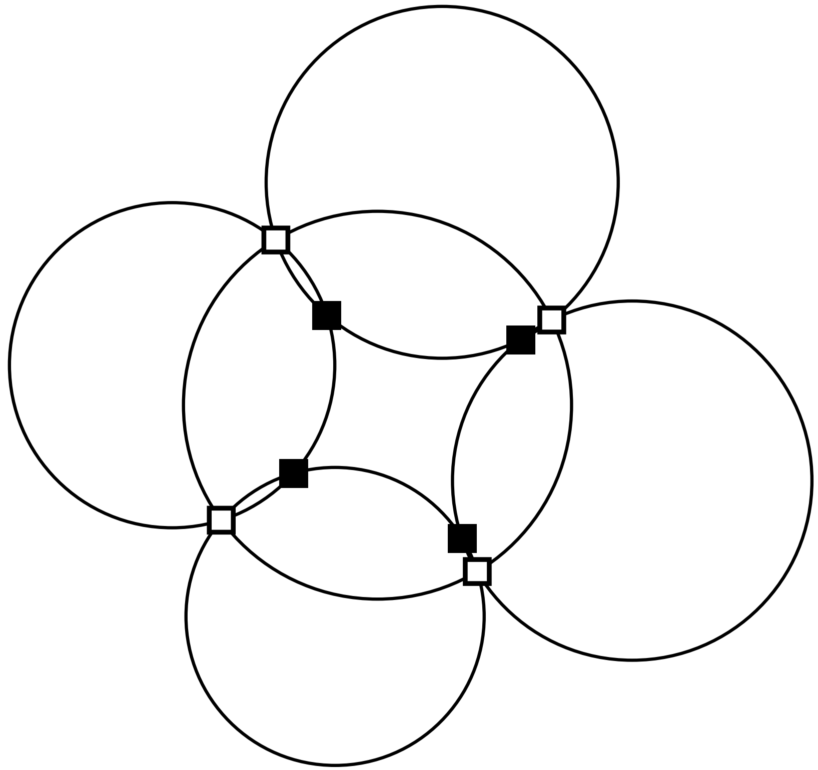

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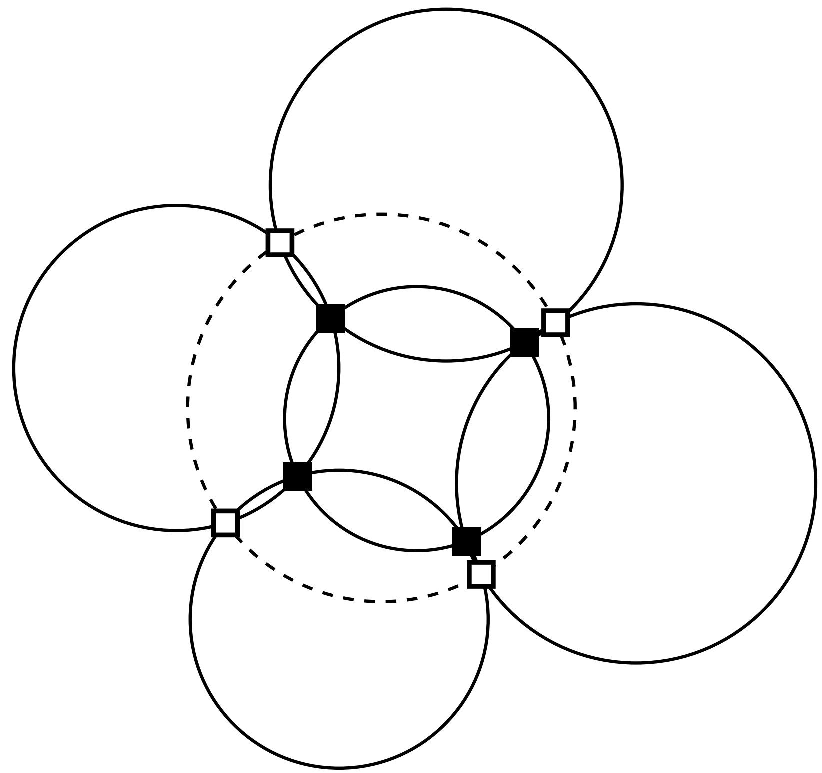
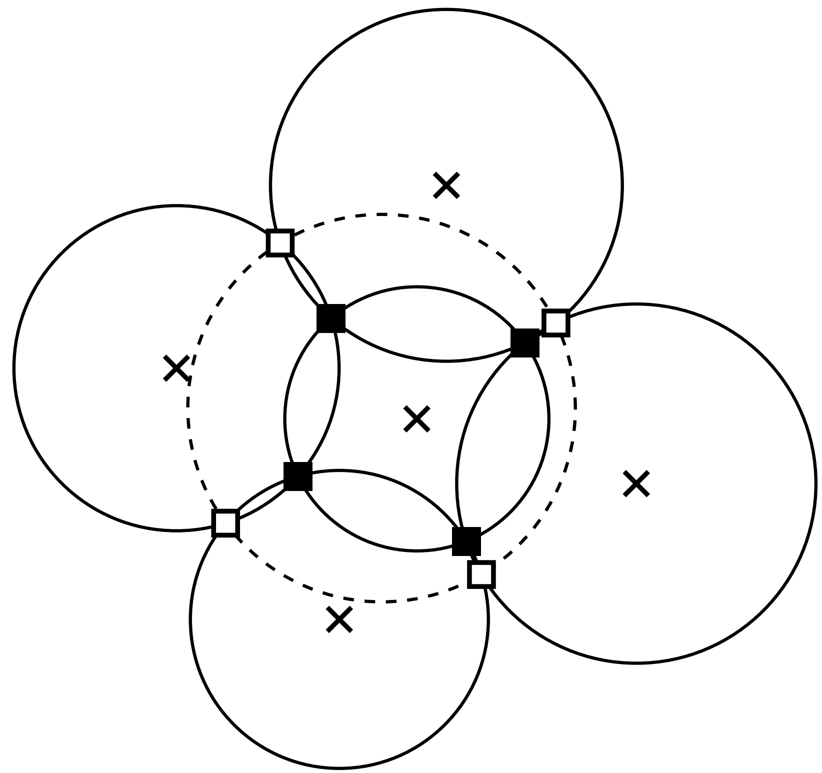
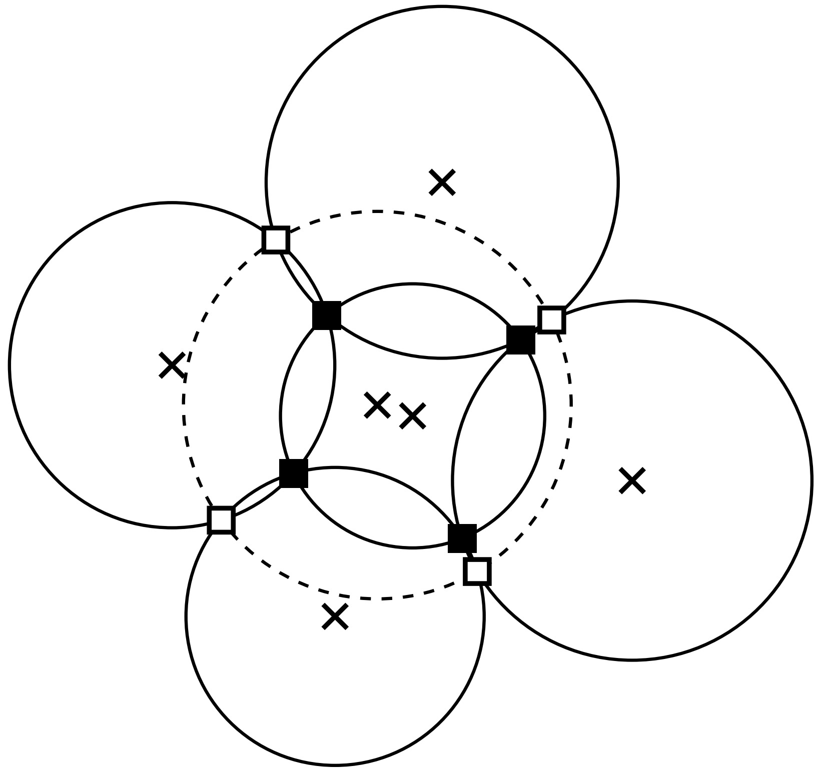
# Miquel's theorem revisited



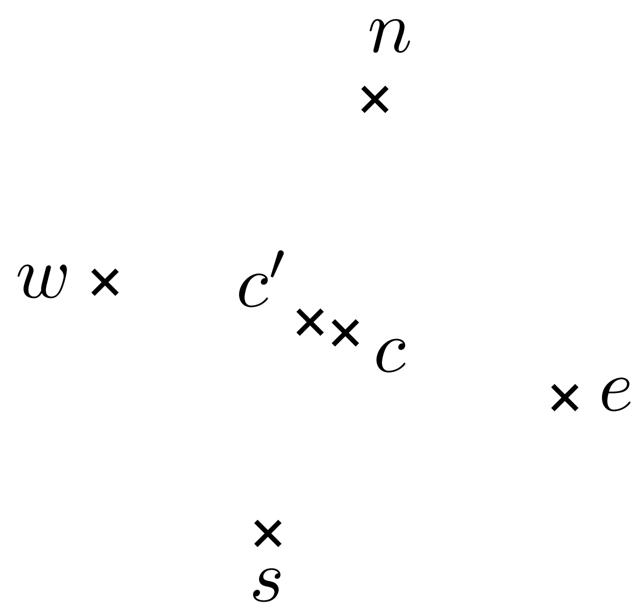
# Miquel's theorem revisited



# Miquel's theorem revisited



# Miquel's theorem revisited



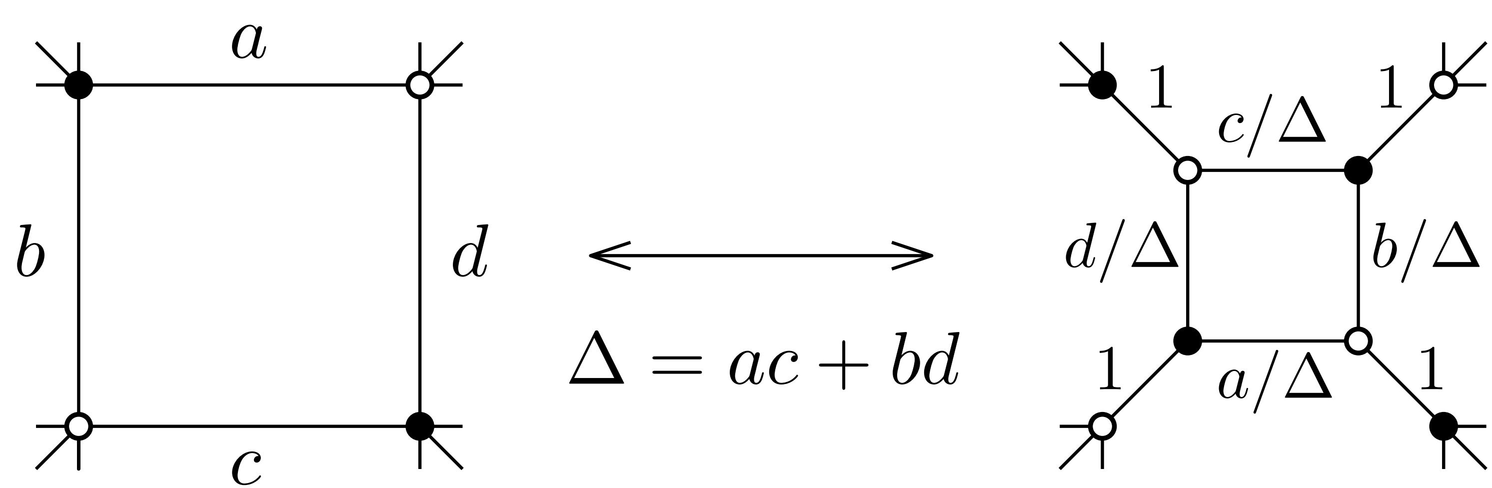

$n, w, s, e, c, c' \in \mathbb{C}$

**Theorem** (Affolter 2018, Kenyon-Lam-R.-Russkikh 2018).

$$\frac{(c-w)(s-c')(e-n)}{(w-s)(c'-e)(n-c)} = -1$$

Discrete Schwarzian KP equation

# Urban renewal



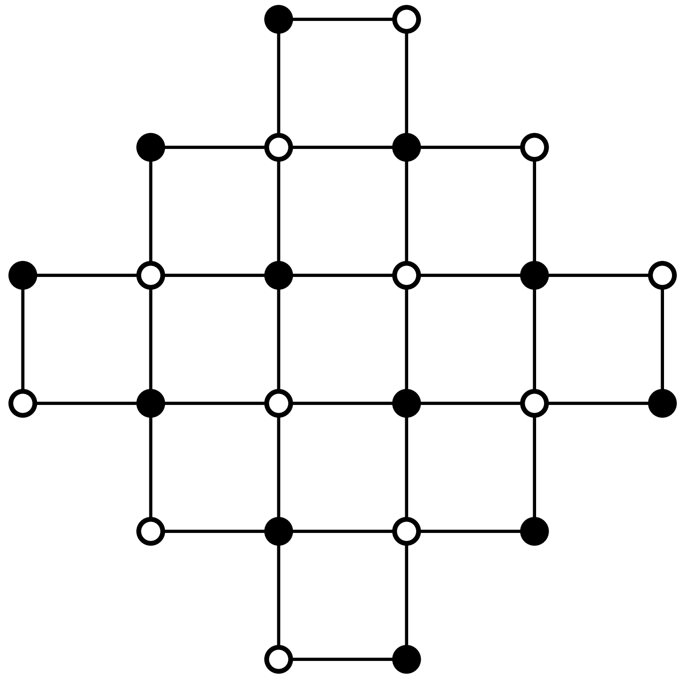

**Theorem** (Affolter 2018, Kenyon-Lam-R.-Russkikh, 2018)**.** *The Miquel move for circle centers corresponds to the urban renewal for dimer models.*

Relation to cluster algebras and integrable systems

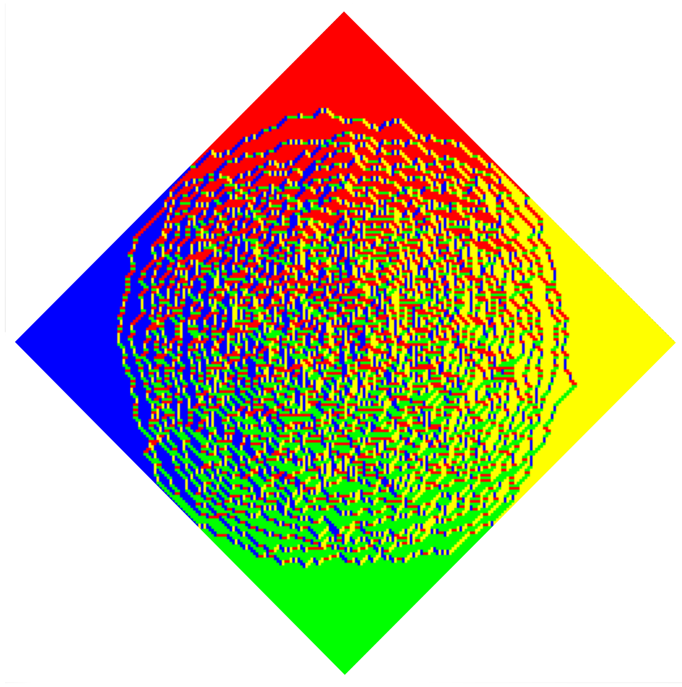
$$\Downarrow$$

More about this in Monday's talk

# The Aztec diamond

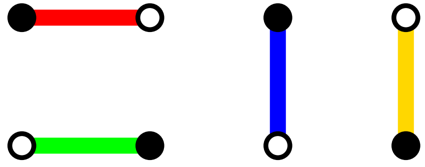


Aztec diamond of size 3



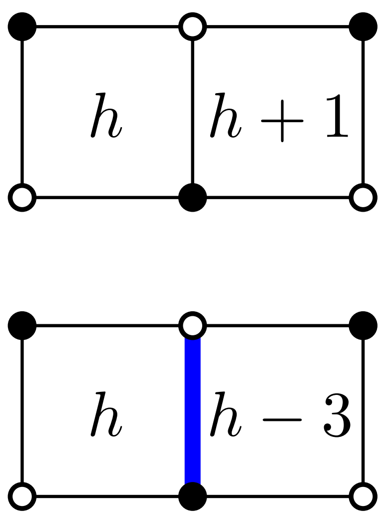
picture by Cris Moore

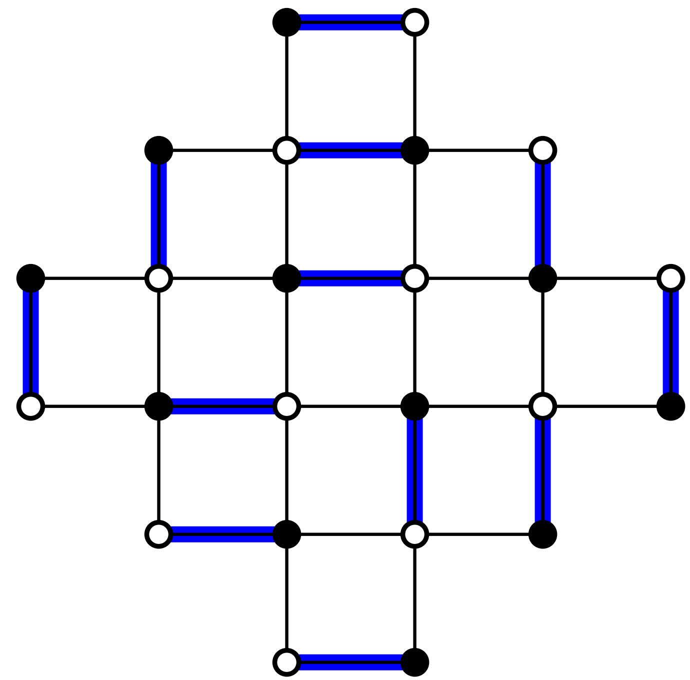
tuvalu.santafe.edu/
~moore/aztec256.gif



# The height function

- A dimer covering of a bipartite graph can be seen as a stepped surface, by associating to each face an integer with the following rules.



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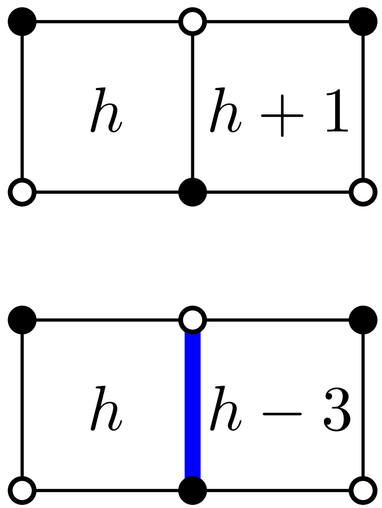


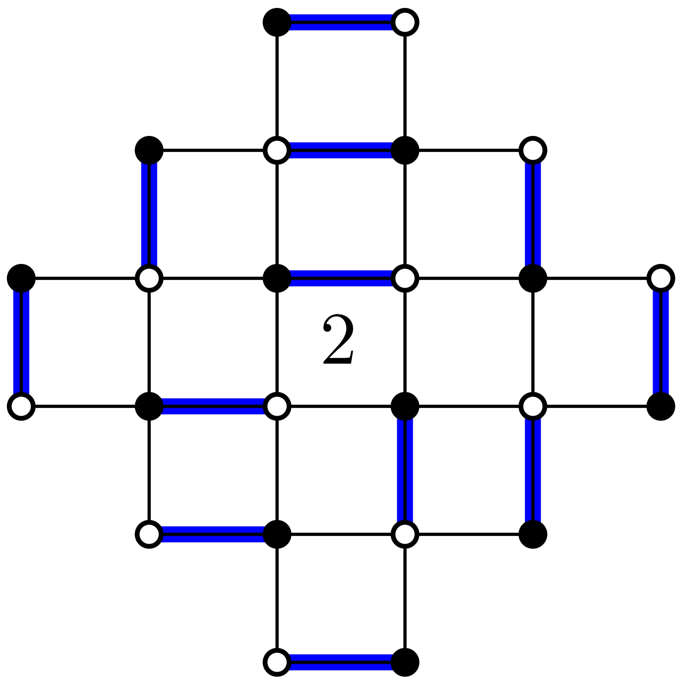

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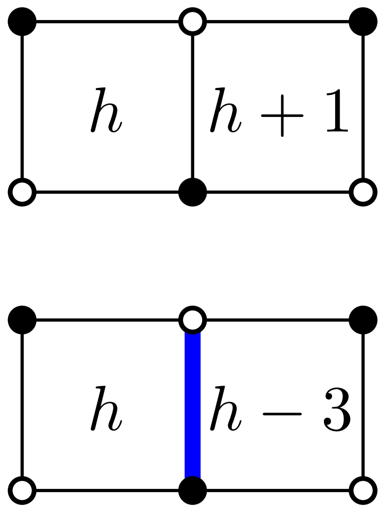


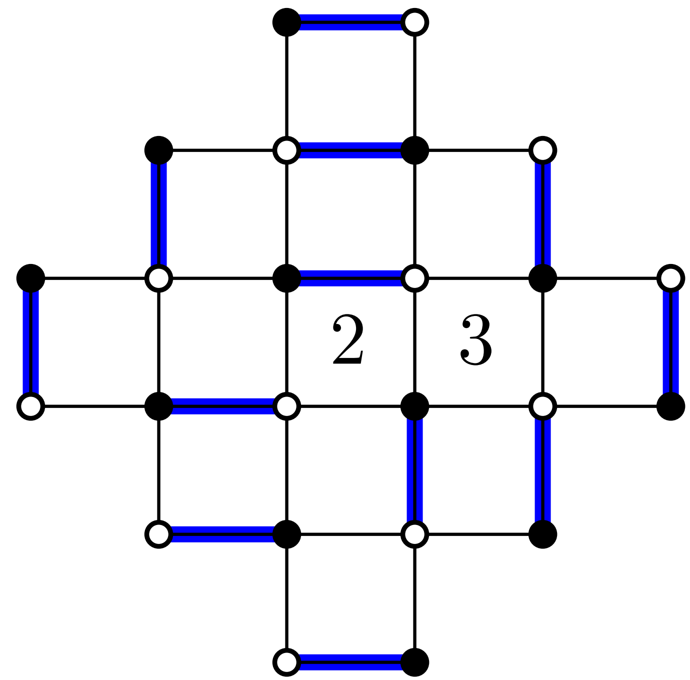

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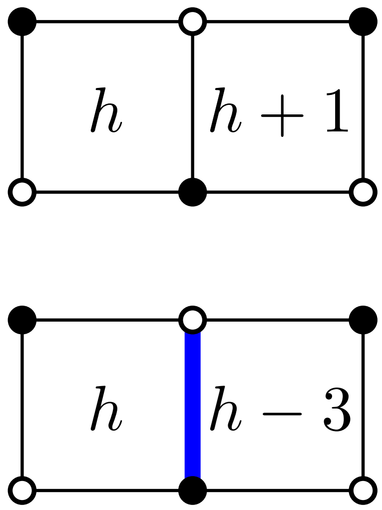


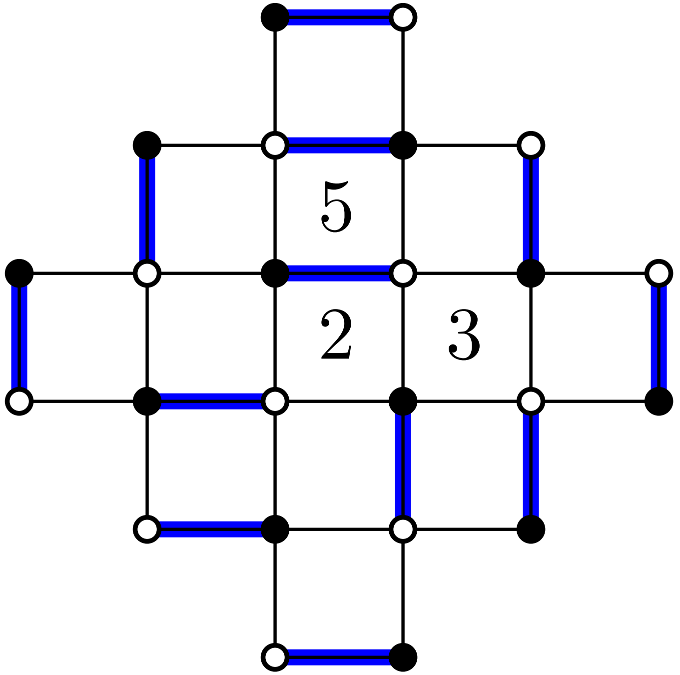

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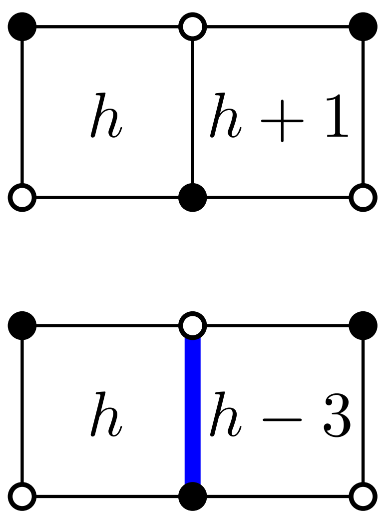


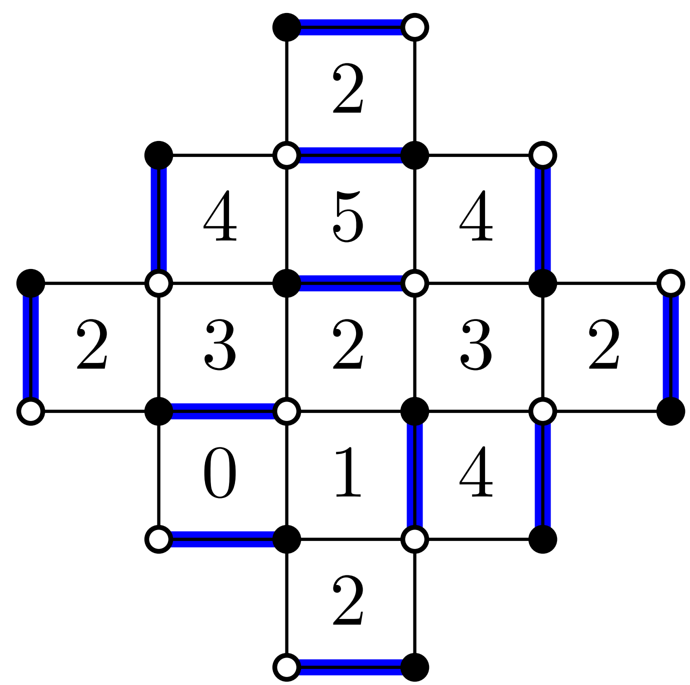

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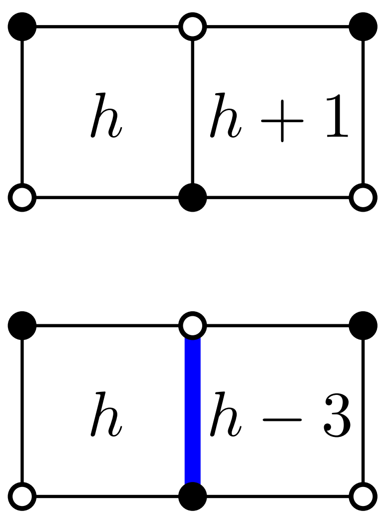
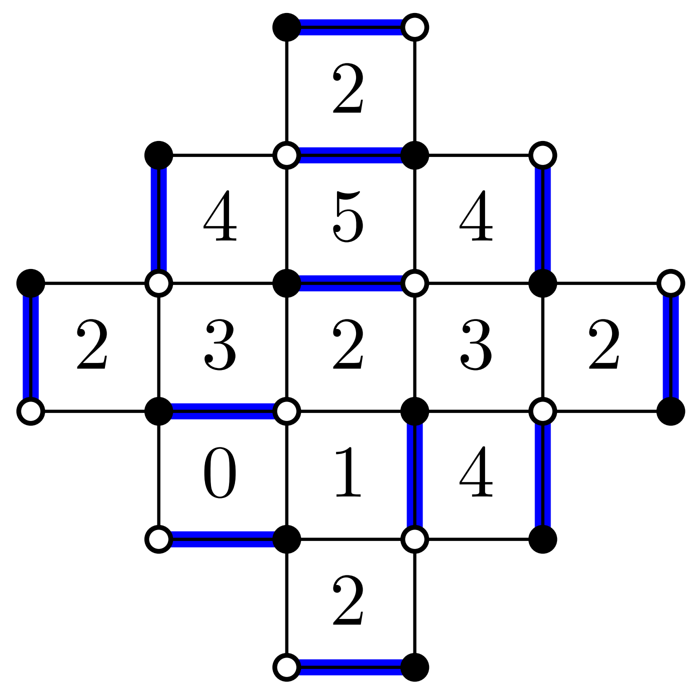
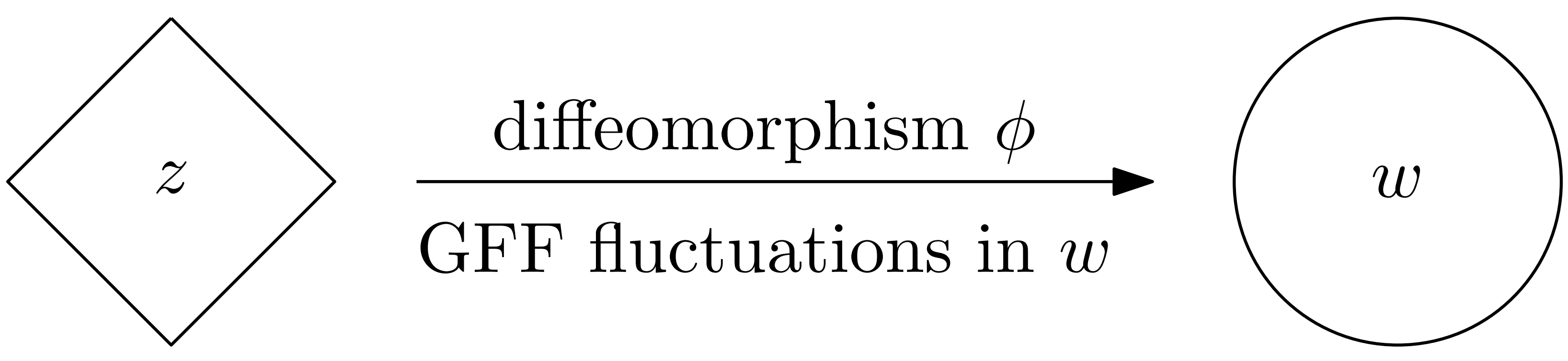



- The random stepped surface concentrates around a deterministic surface called the limit shape.

# The Gaussian Free Field (GFF)

- Real-valued Gaussian process defined on a domain in the plane. Like a Brownian bridge with 2d time.
- Conformally invariant object arising as a scaling limit of many natural random processes (universality).
- Fluctuations of the stepped surface around the limit shape are given by the GFF... for a conformal structure different from the one given by Euclidean metric !



- These are well-established classical results:

  Jockush-Propp-Shor 1998

  Cohn-Kenyon-Propp 2001

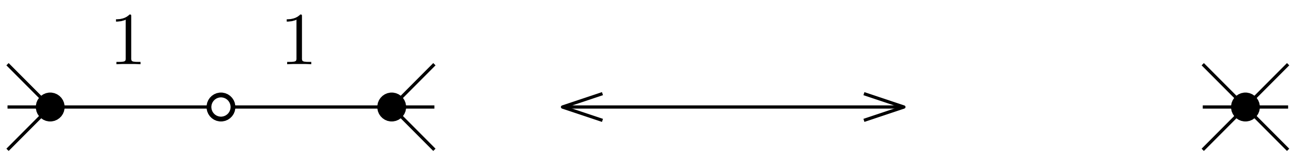
  Chhita-Johansson-Young 2015

  Bufetov-Gorin 2018, ...

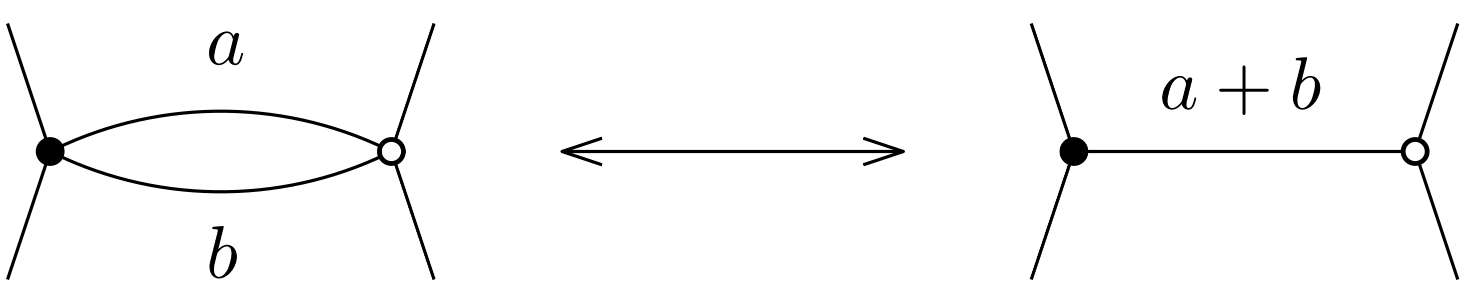
- Do they naturally arise from circle center embeddings ?

# Two more dimer local moves

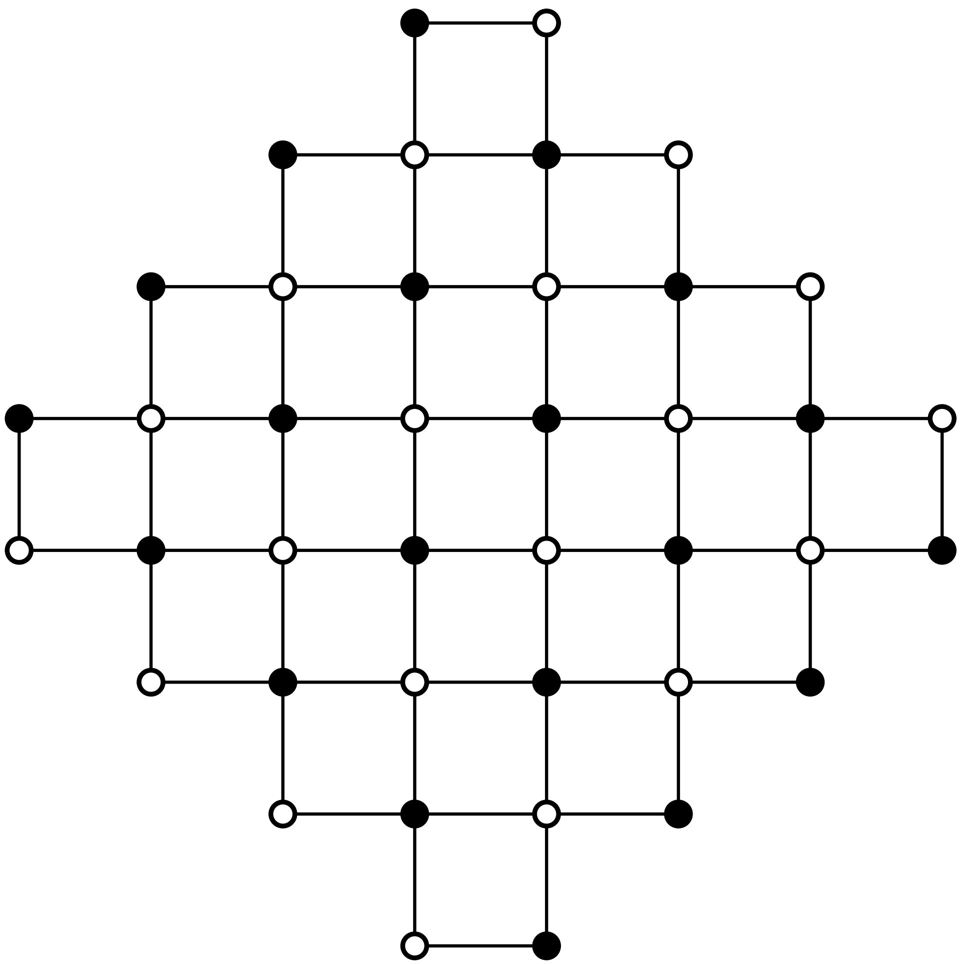
- Other local moves preserving the probabilistic model.
- Contraction of degree 2 vertices:




- Merging of parallel edges:



- After contraction of degree 2 vertices and merging of parallel edges, the outer face has degree 4.



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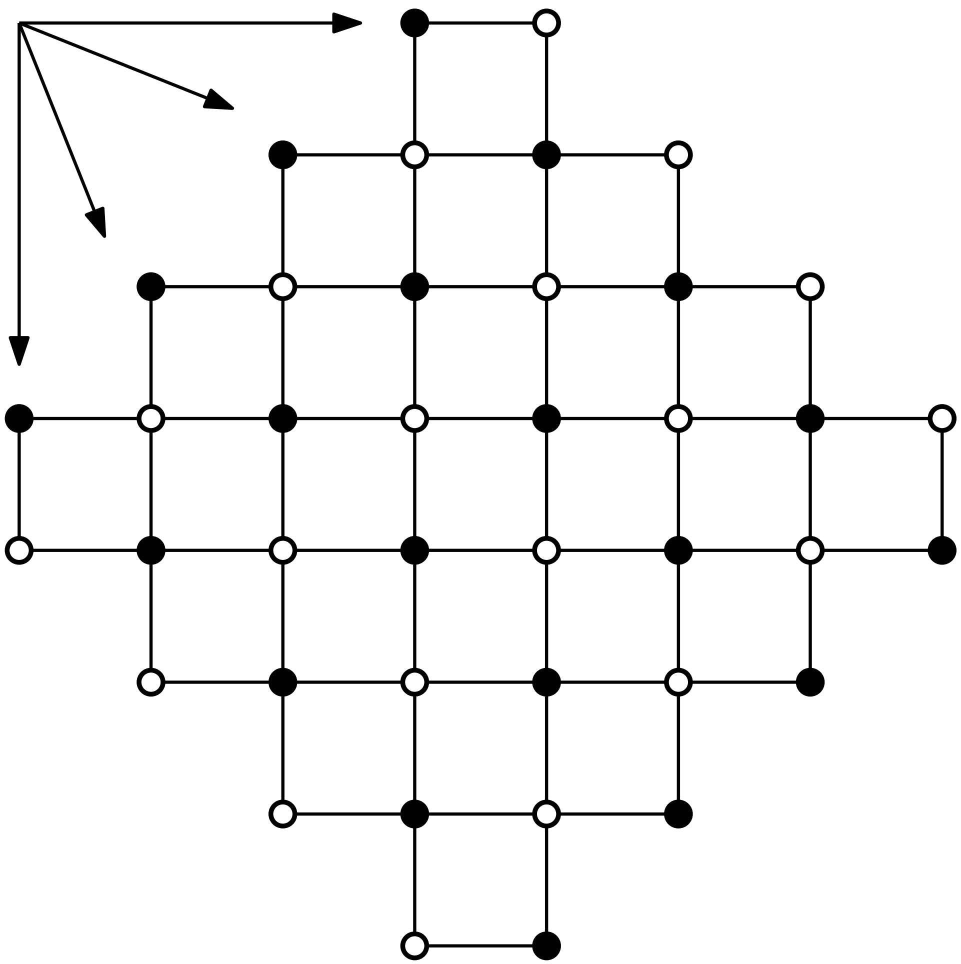

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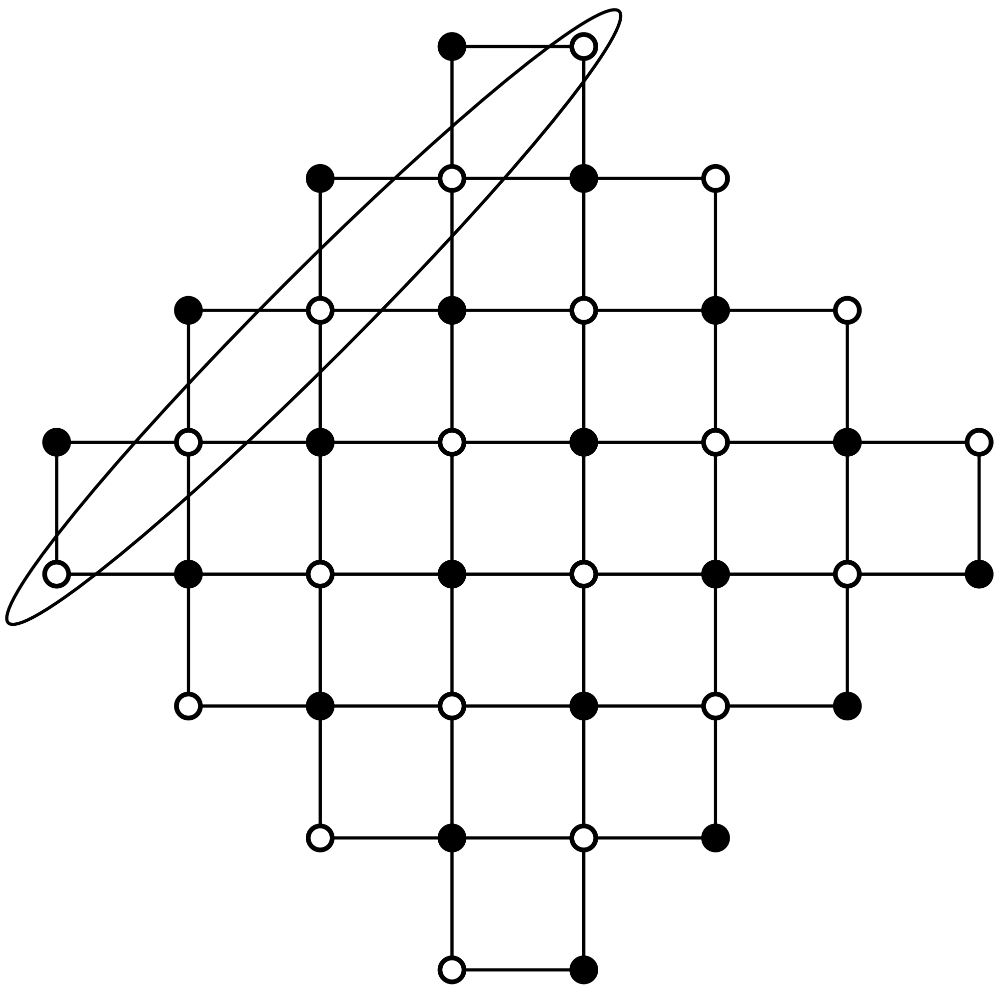
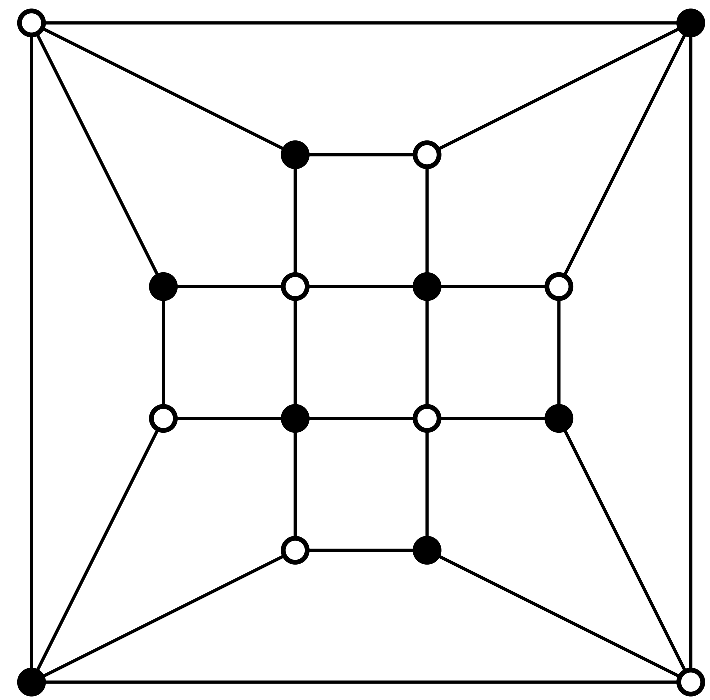
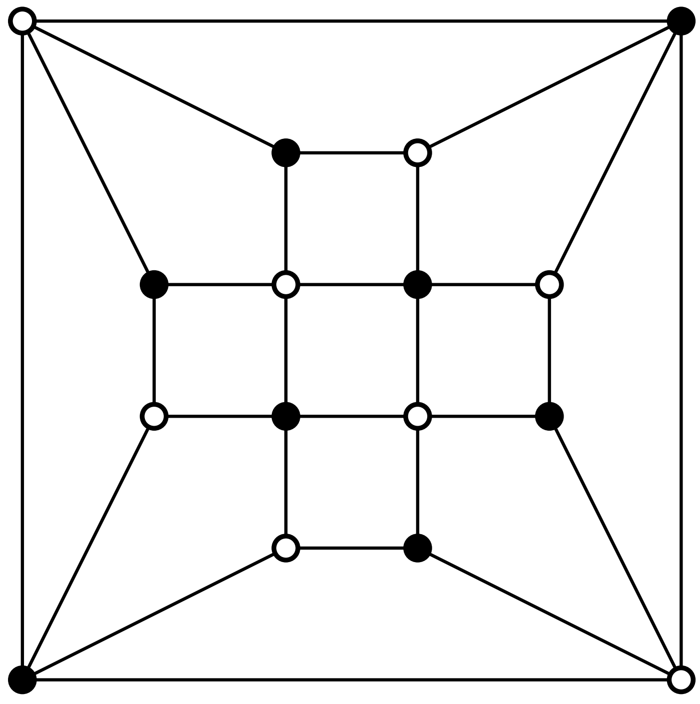
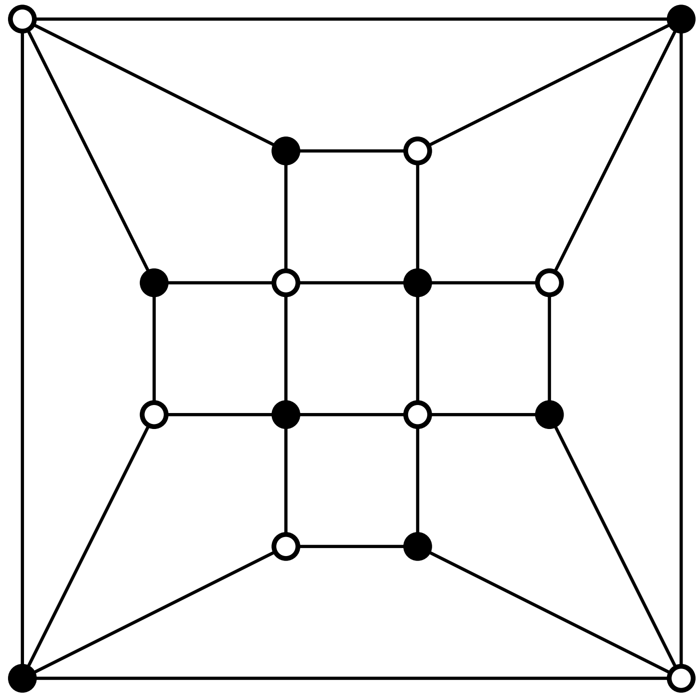
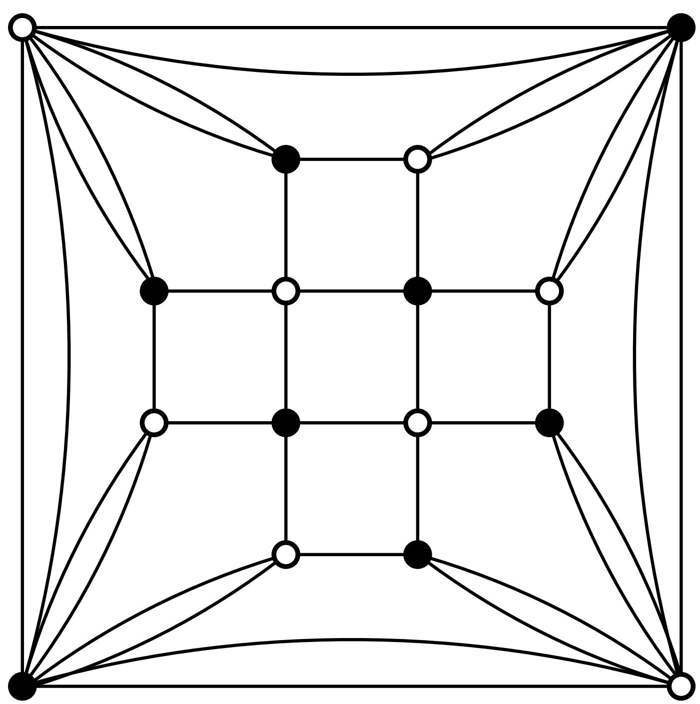
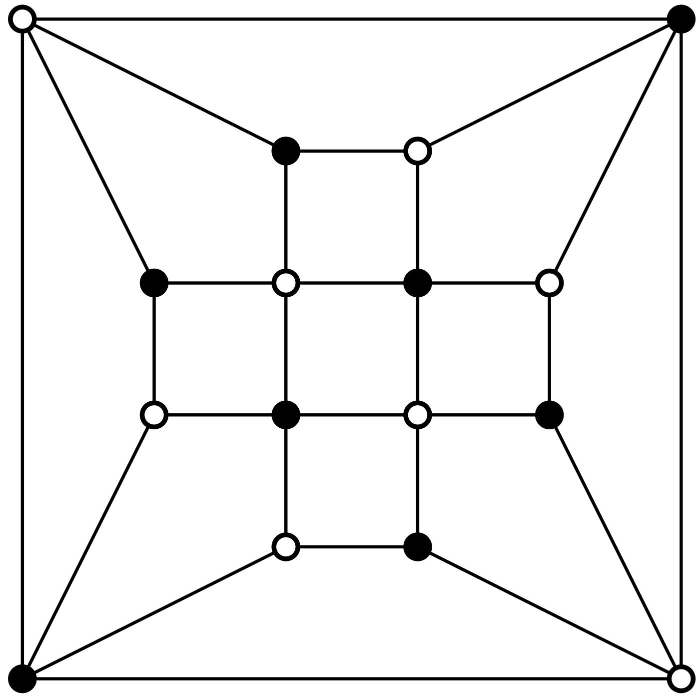


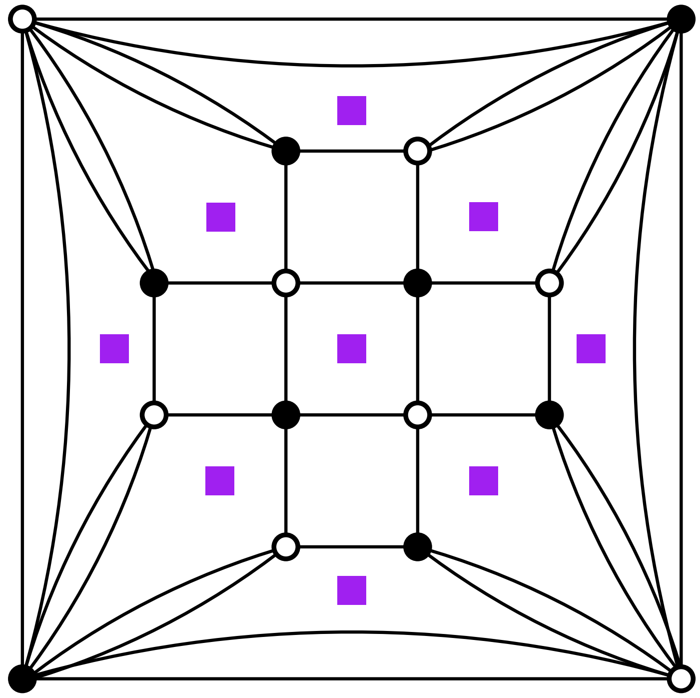


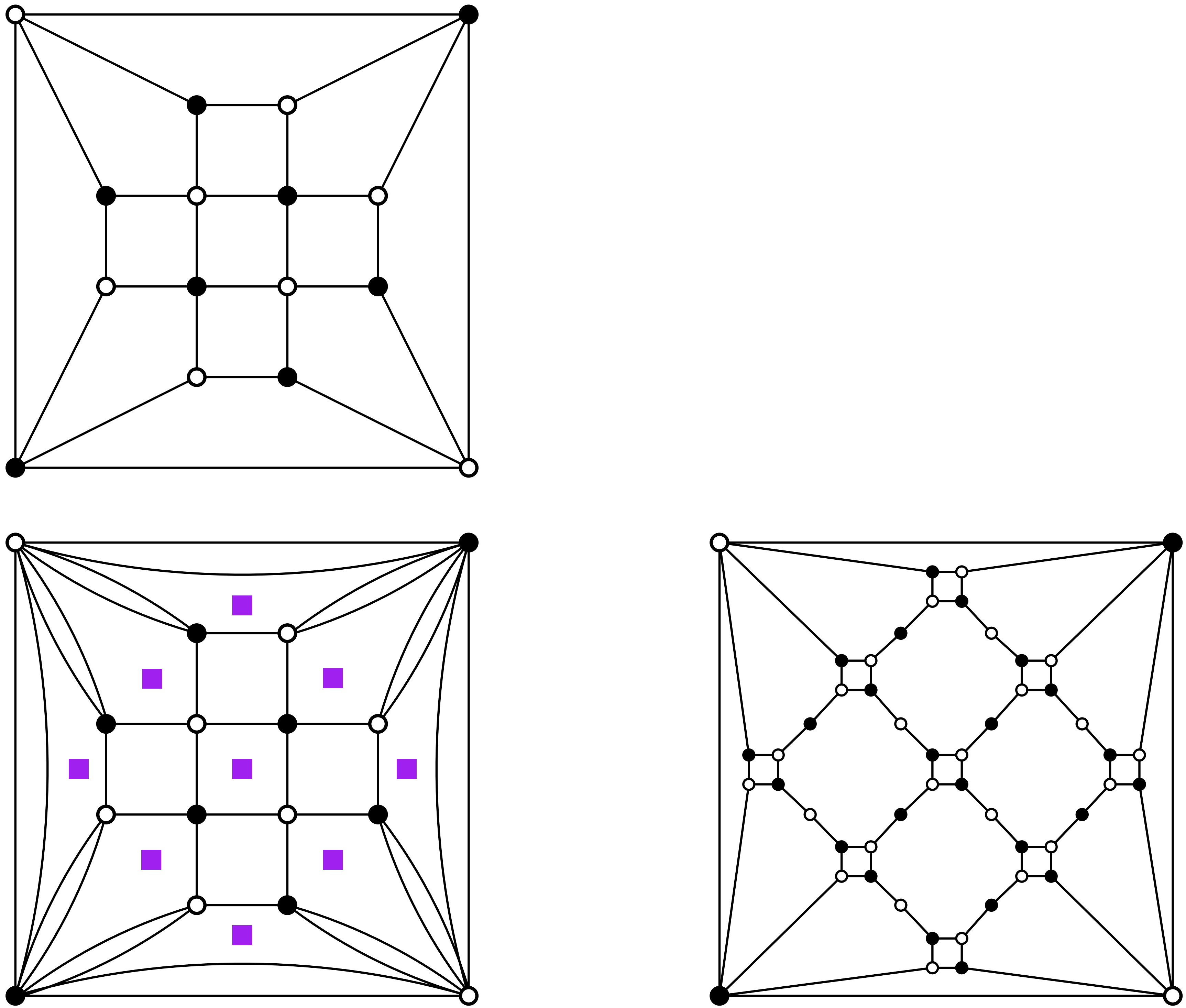


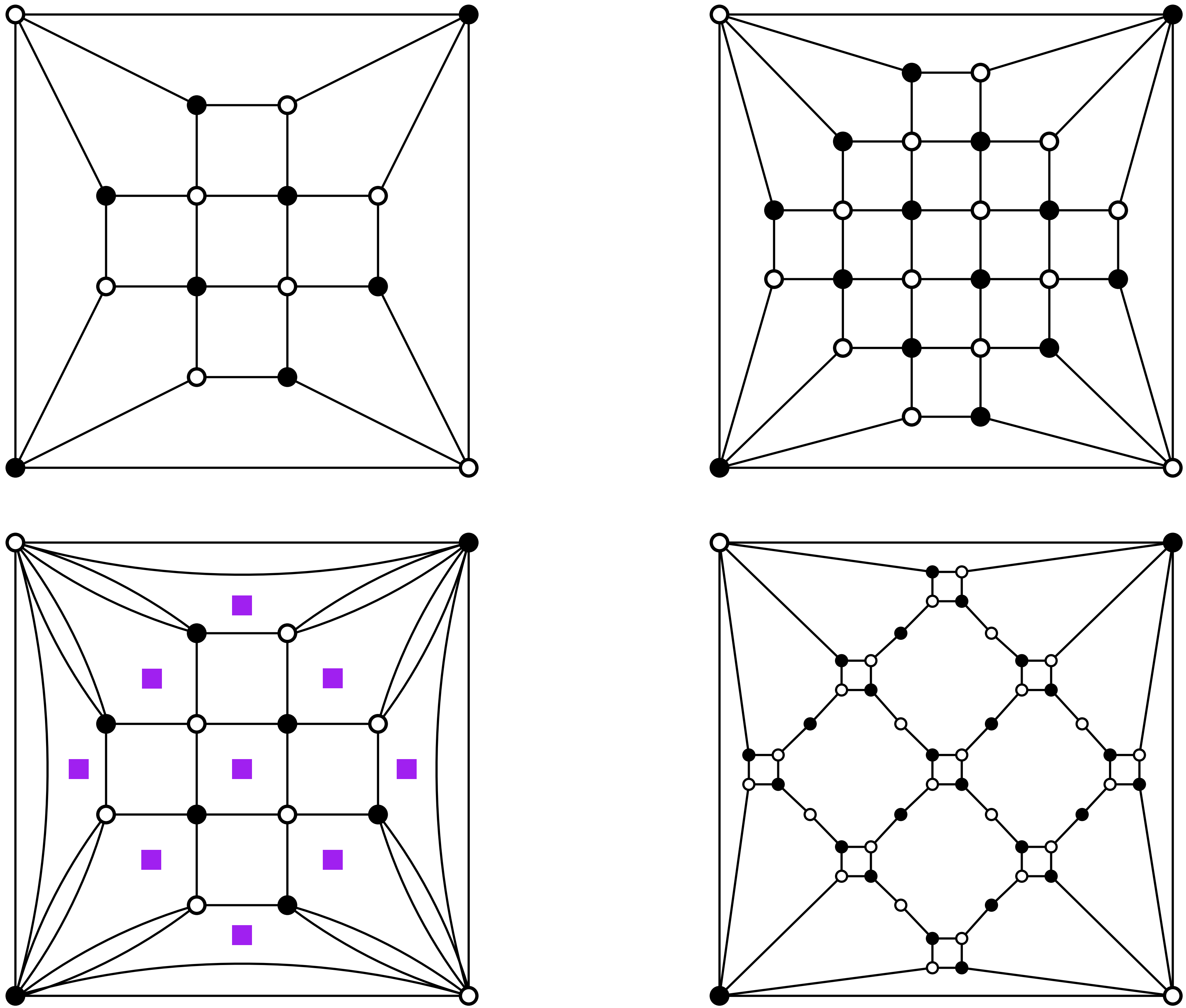


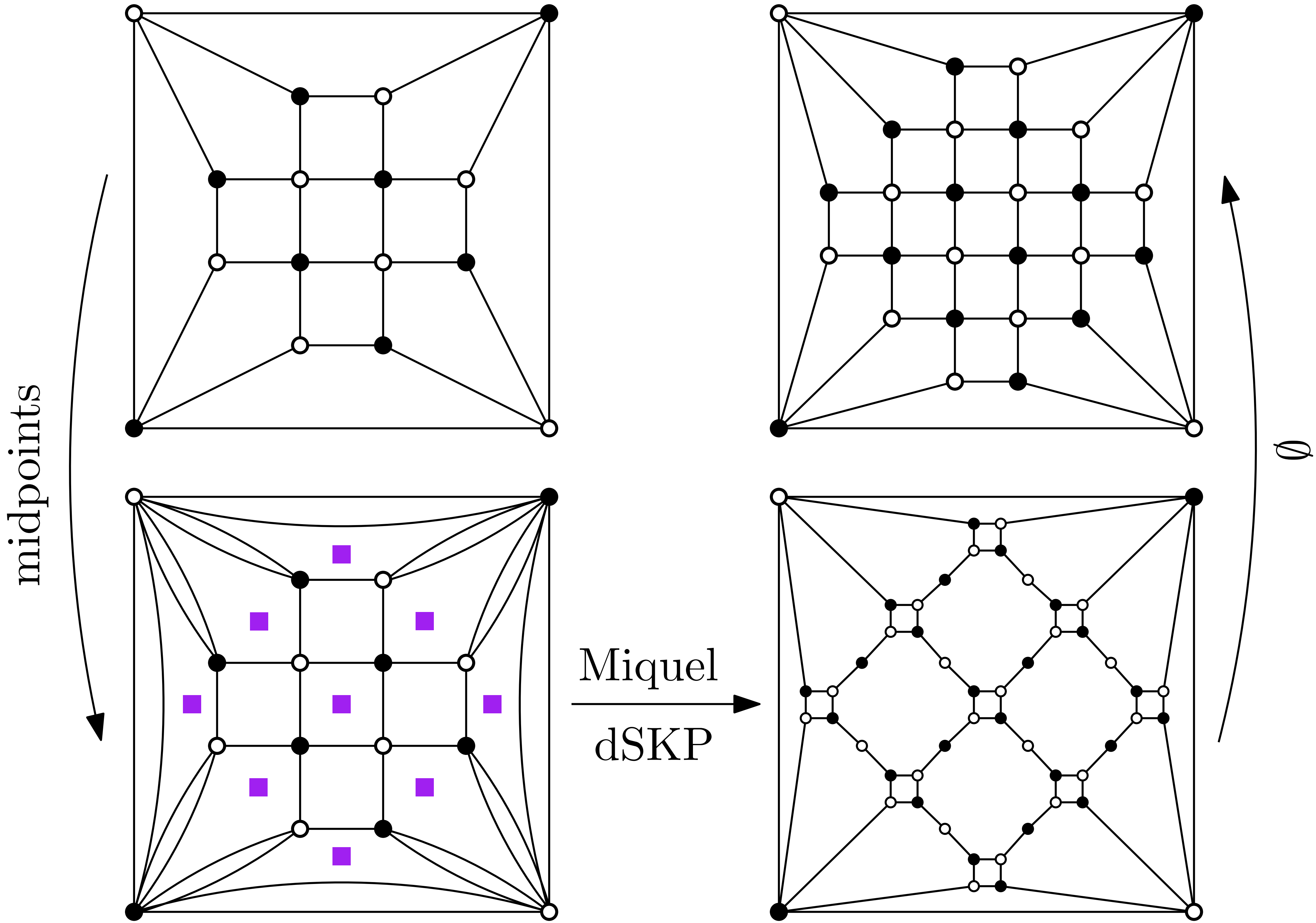




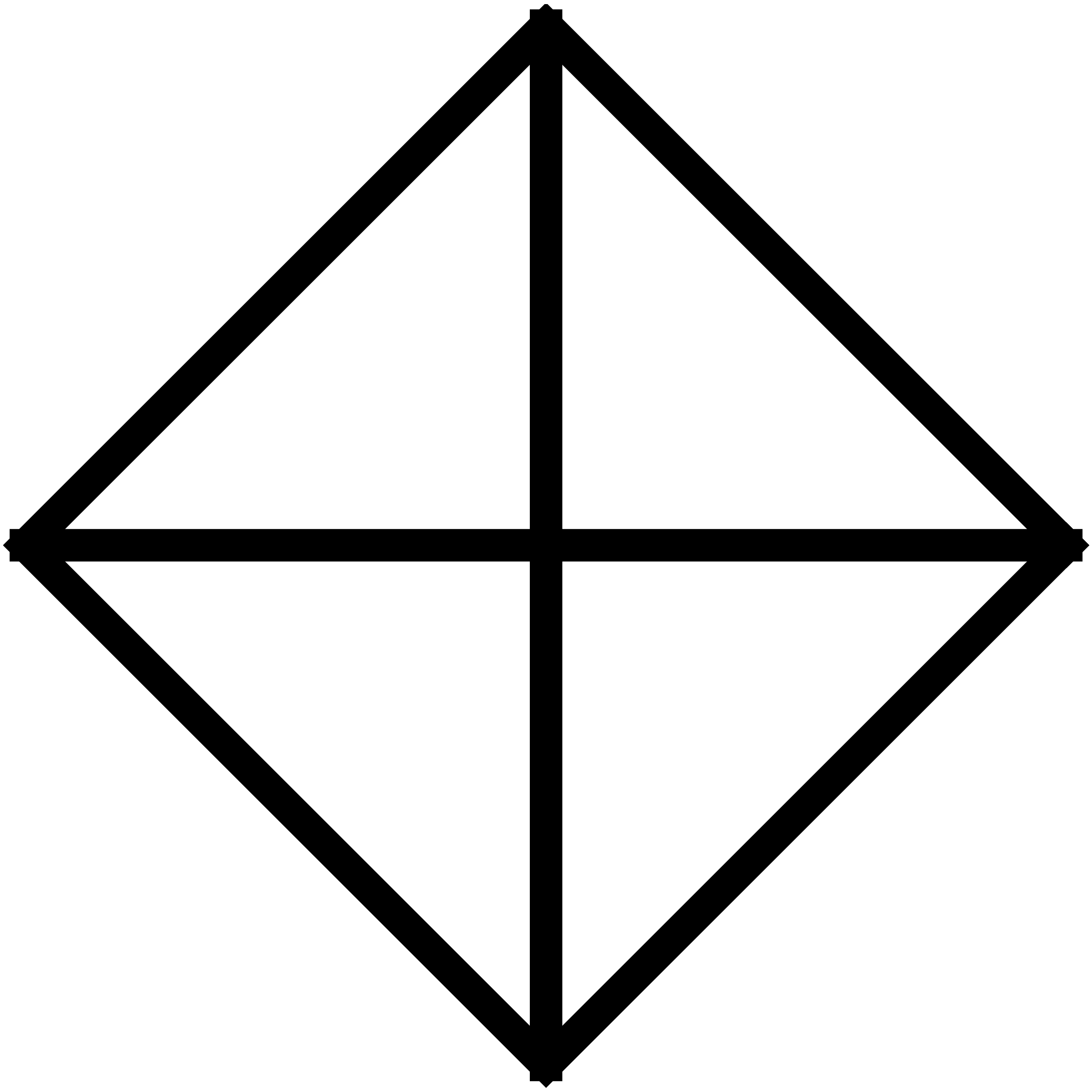




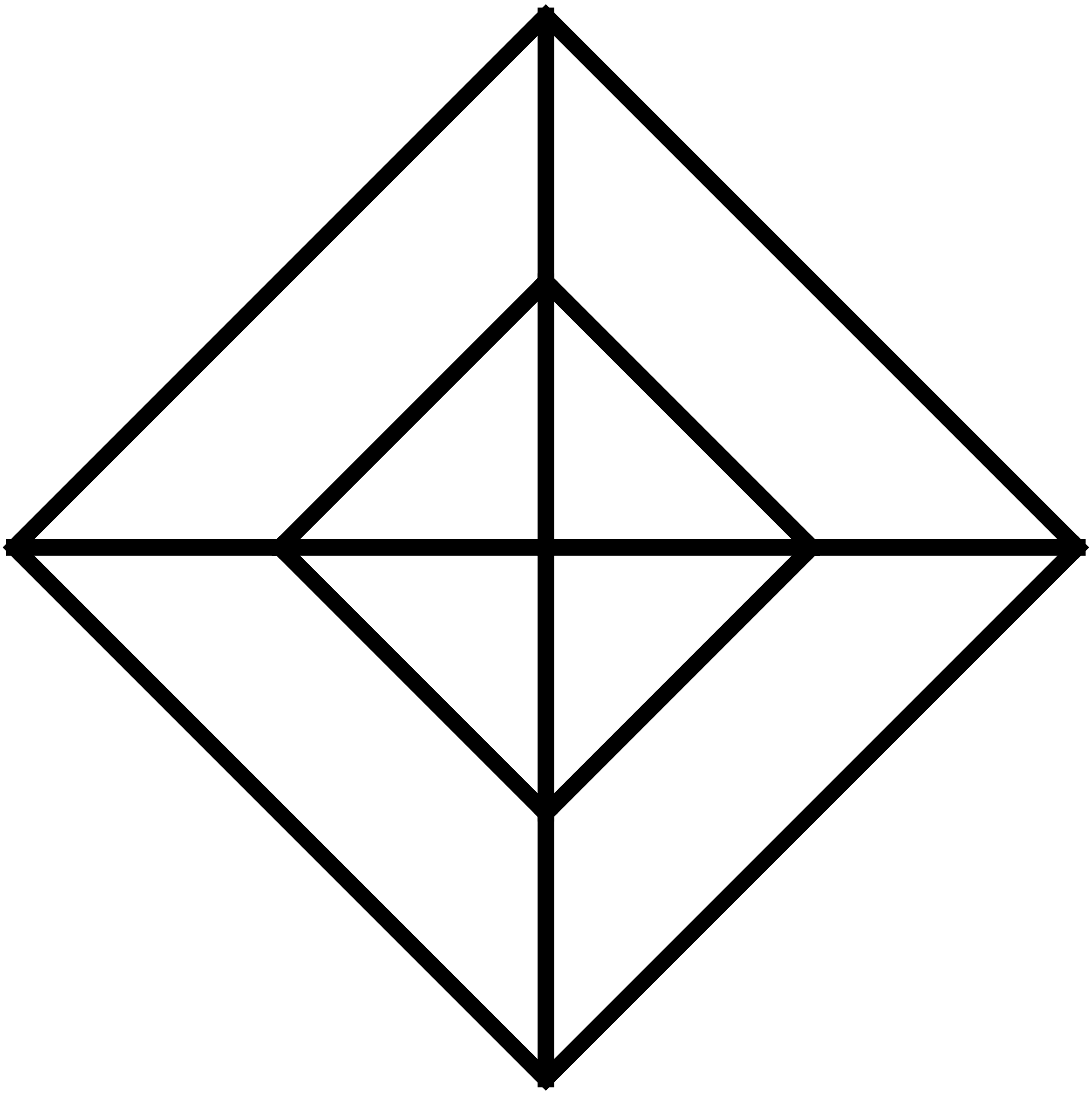




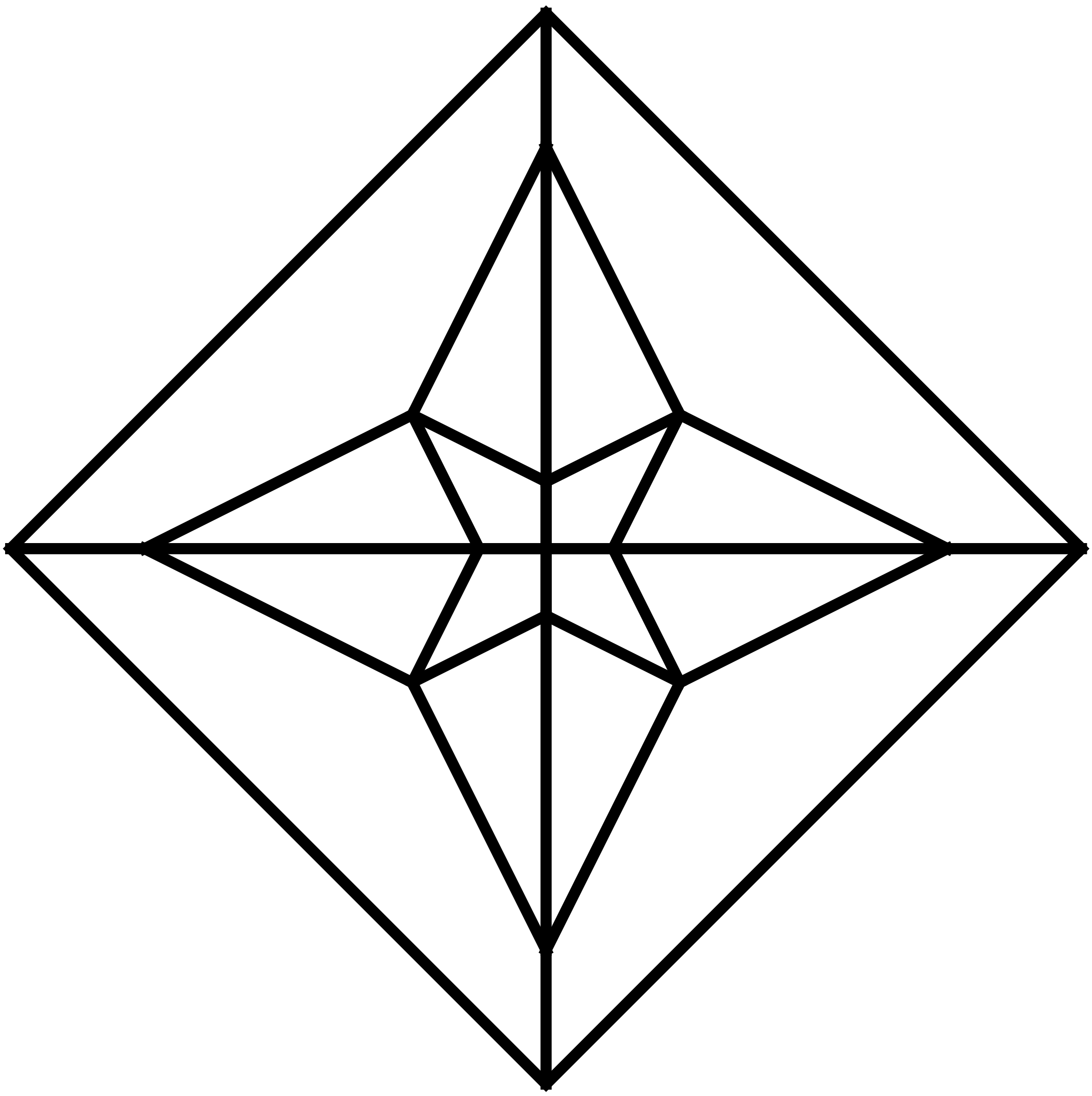
midpoints
Miquel
dSKP
∅



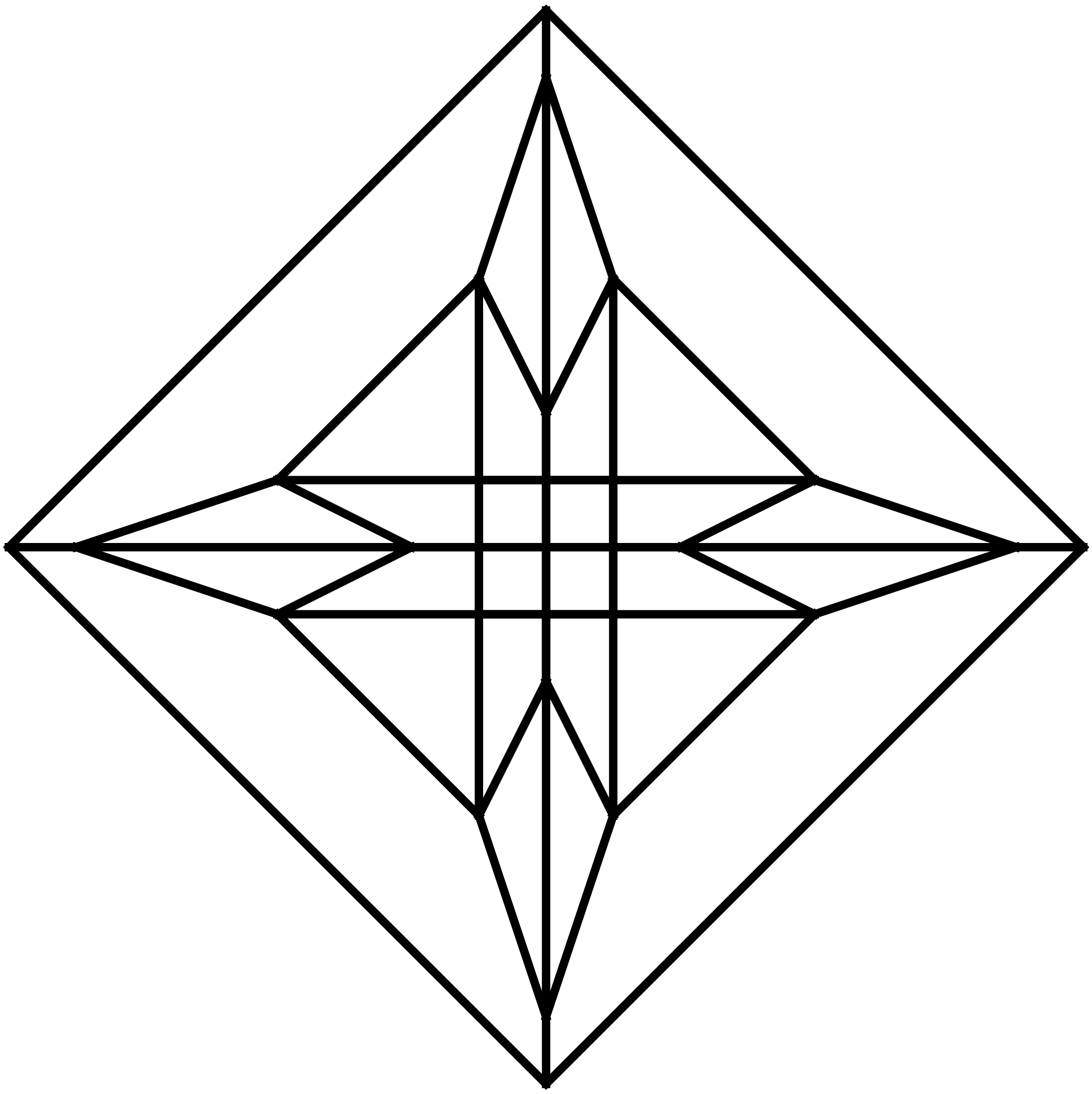
Size 2



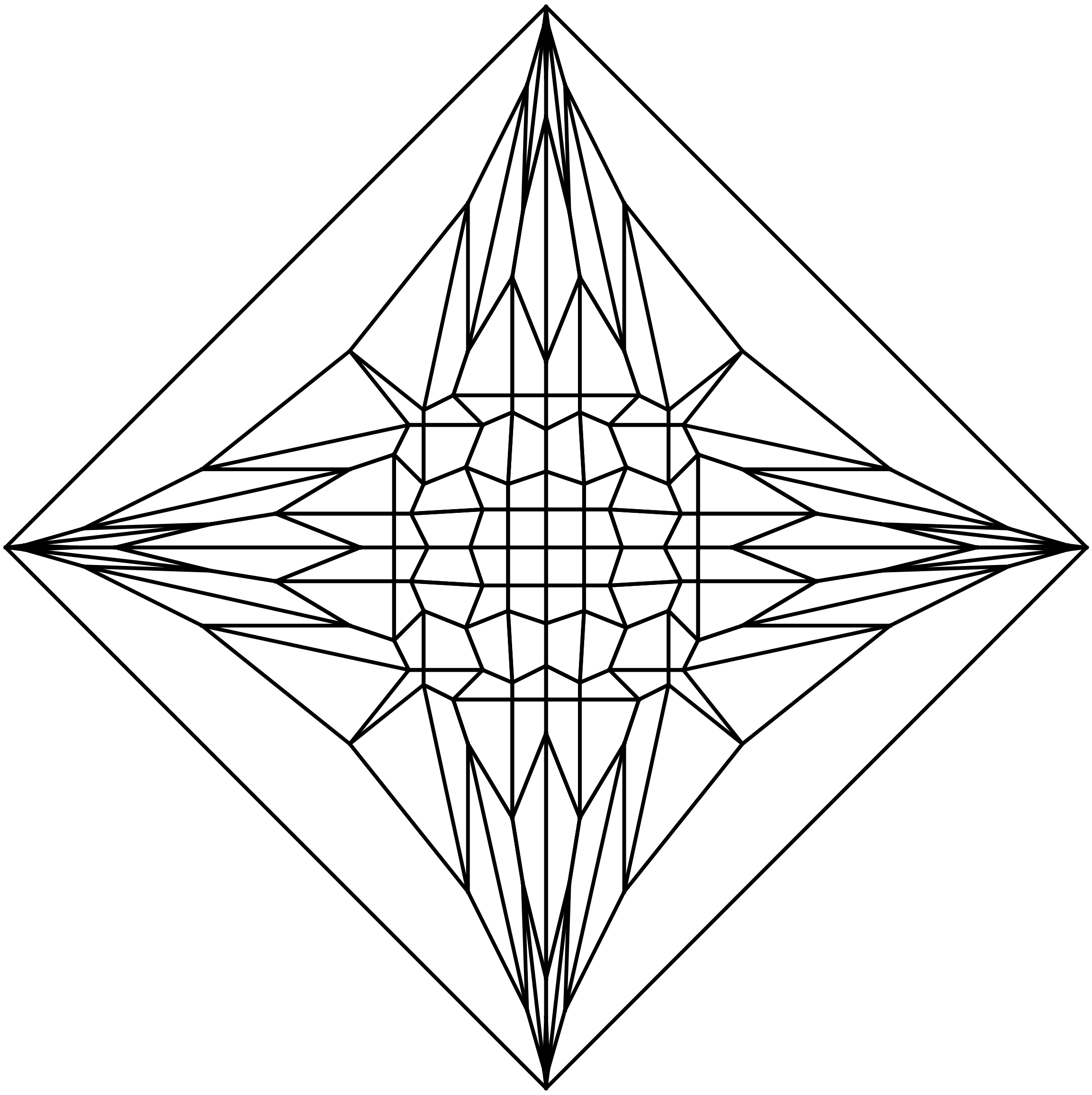
Size 3



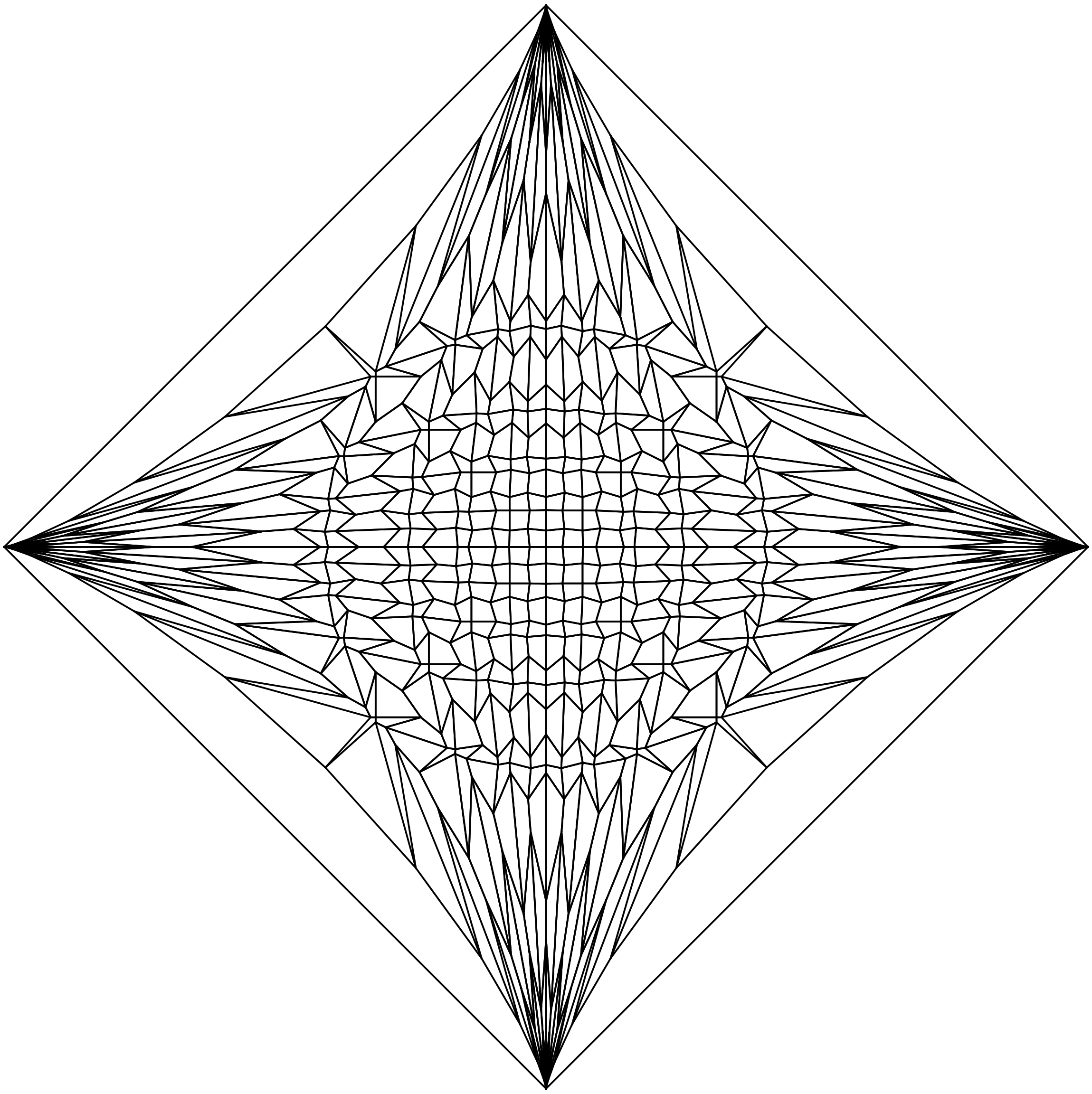
Size 4



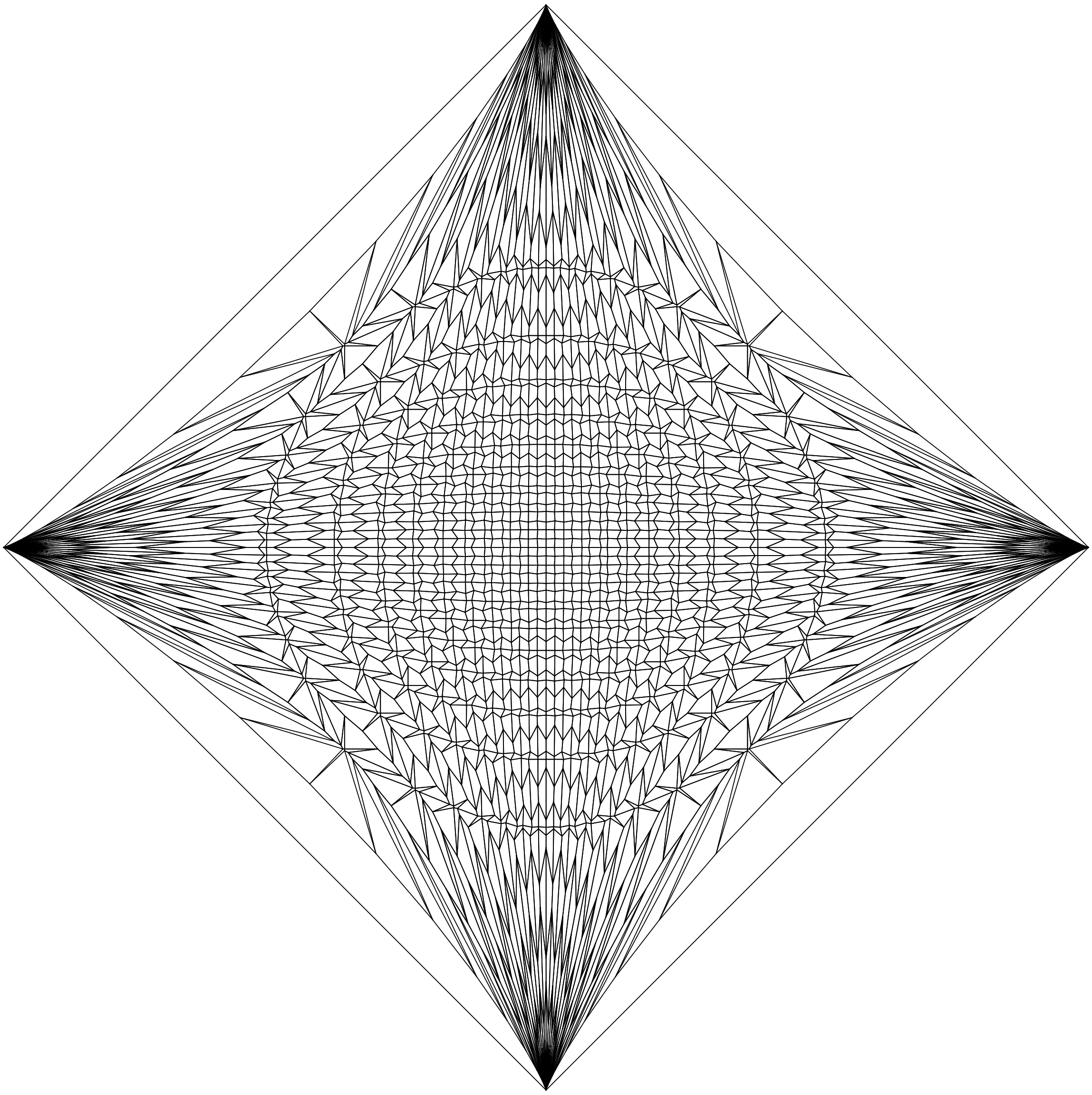
Size 5



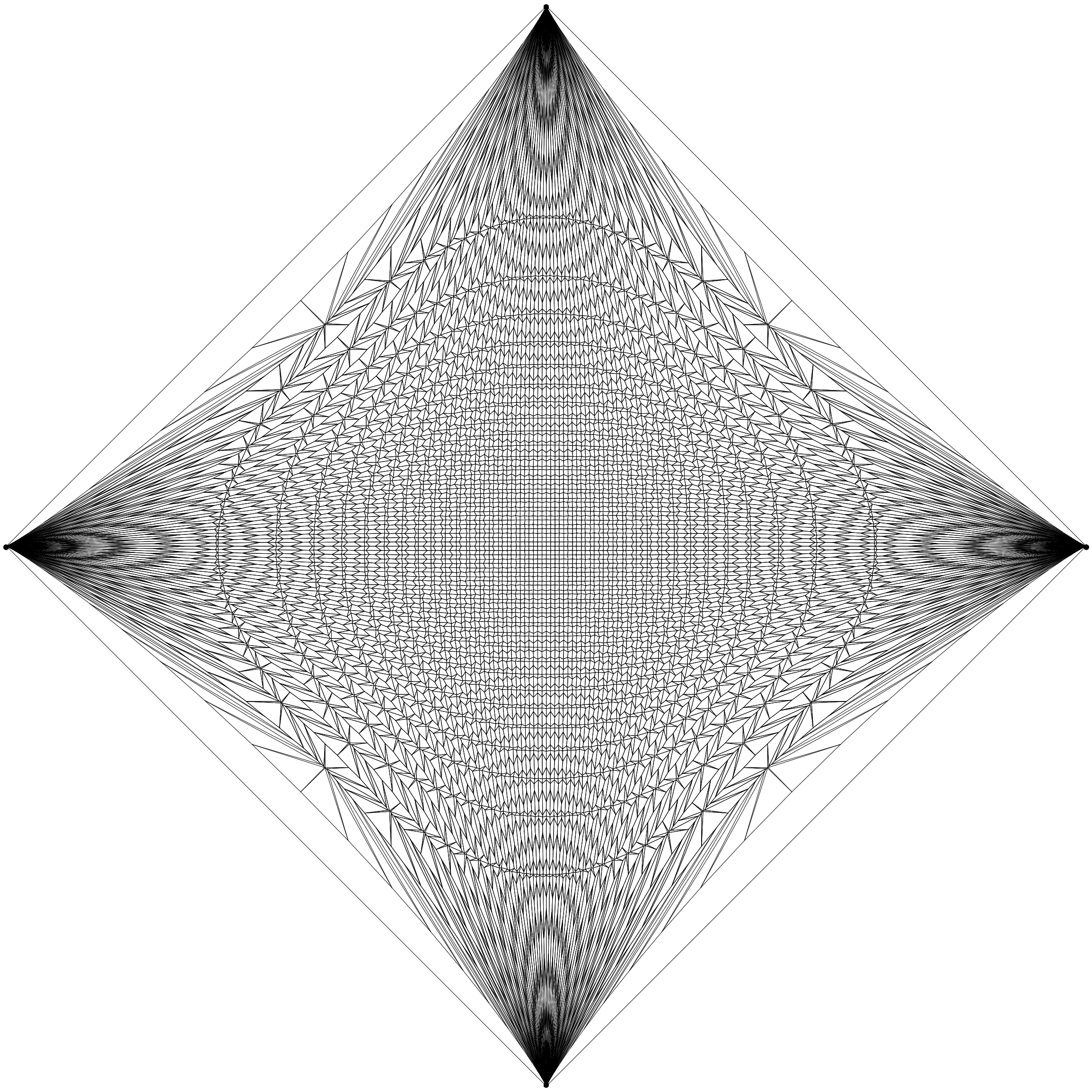
Size 10



Size 20

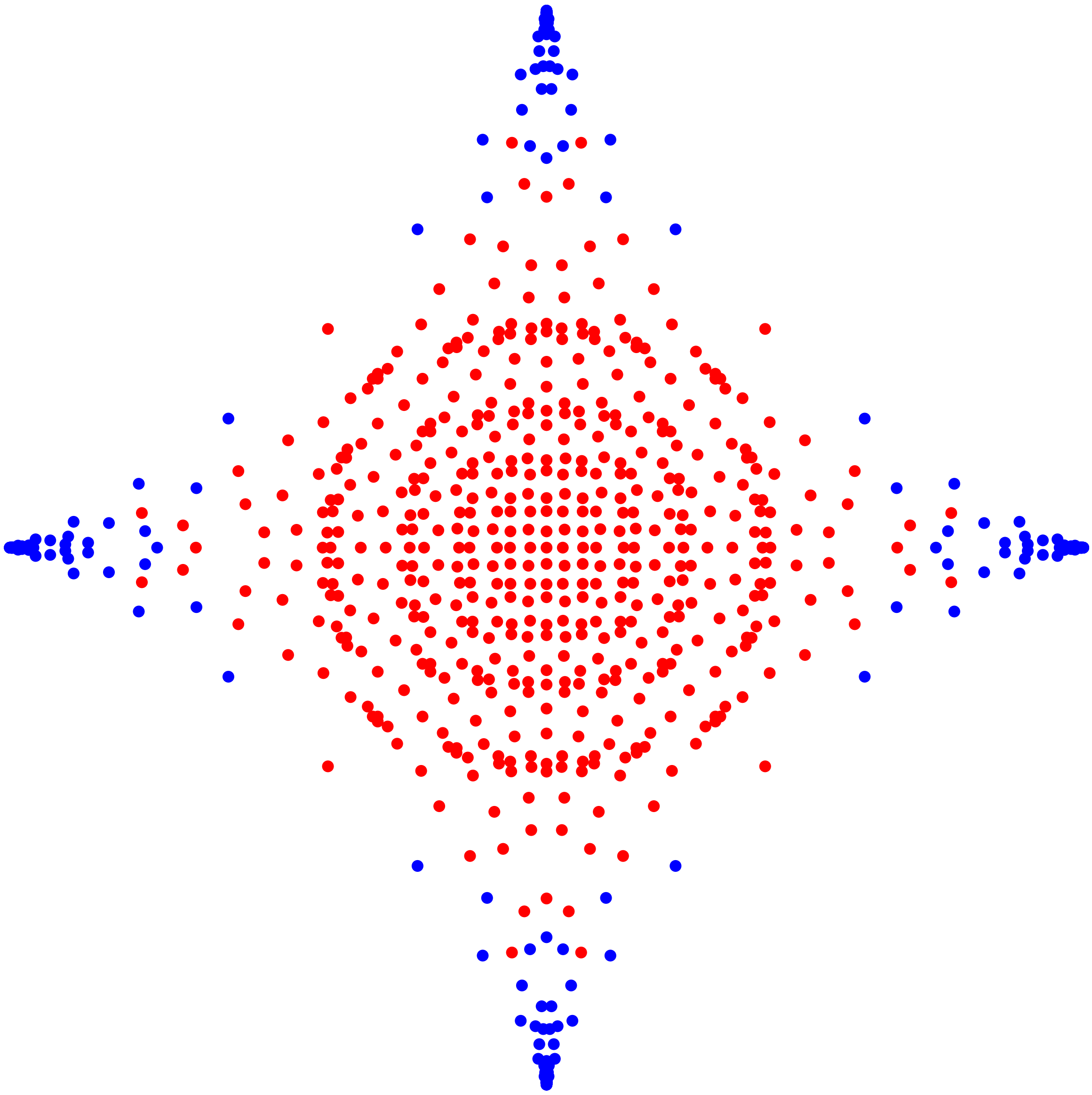


Size 40



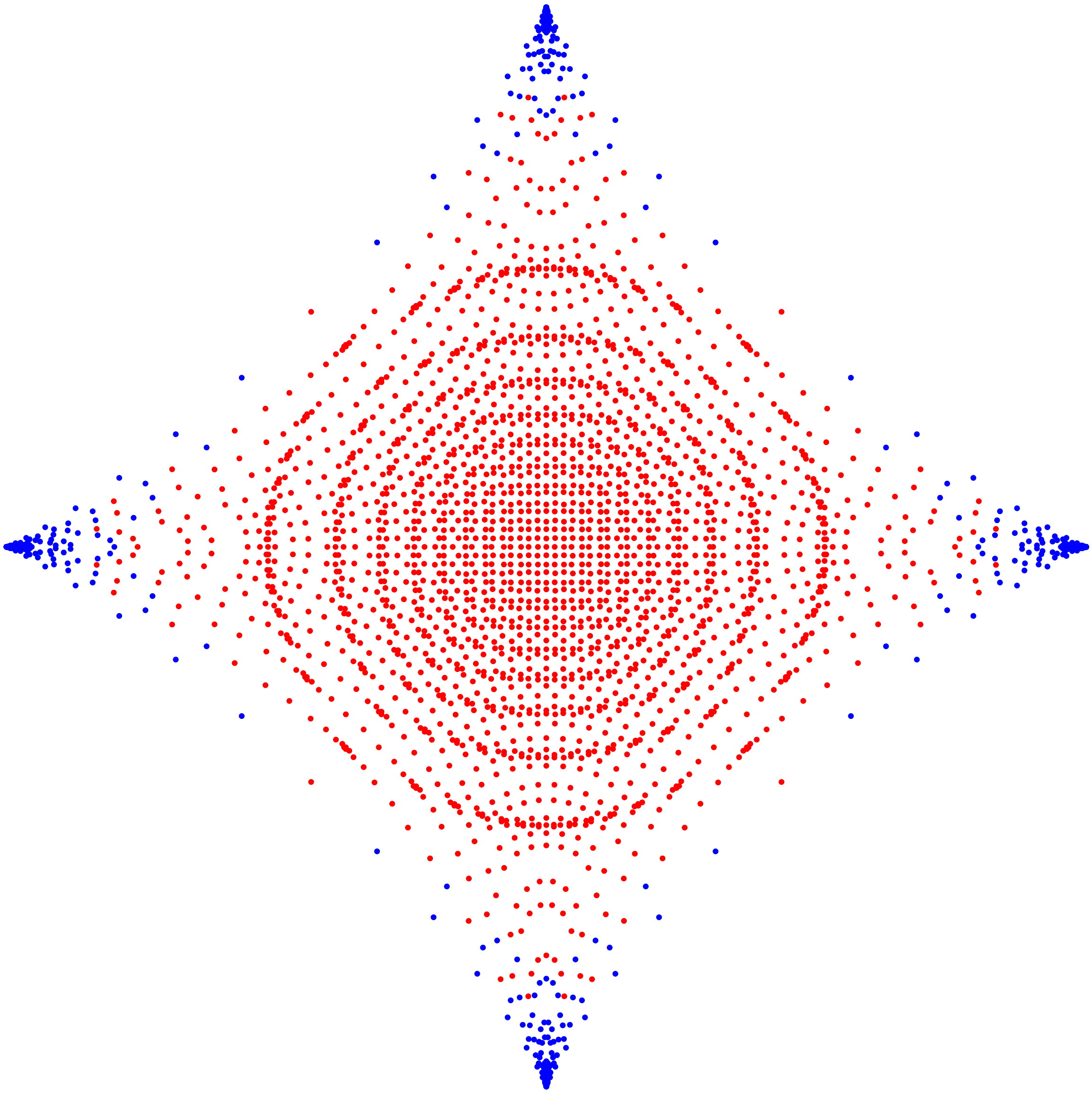
Size 80

The image of a dual vertex inside (resp. outside) the arctic circle is red (resp. blue).



# Size 20

The image of a dual vertex inside (resp. outside) the arctic circle is red (resp. blue).



Size 40

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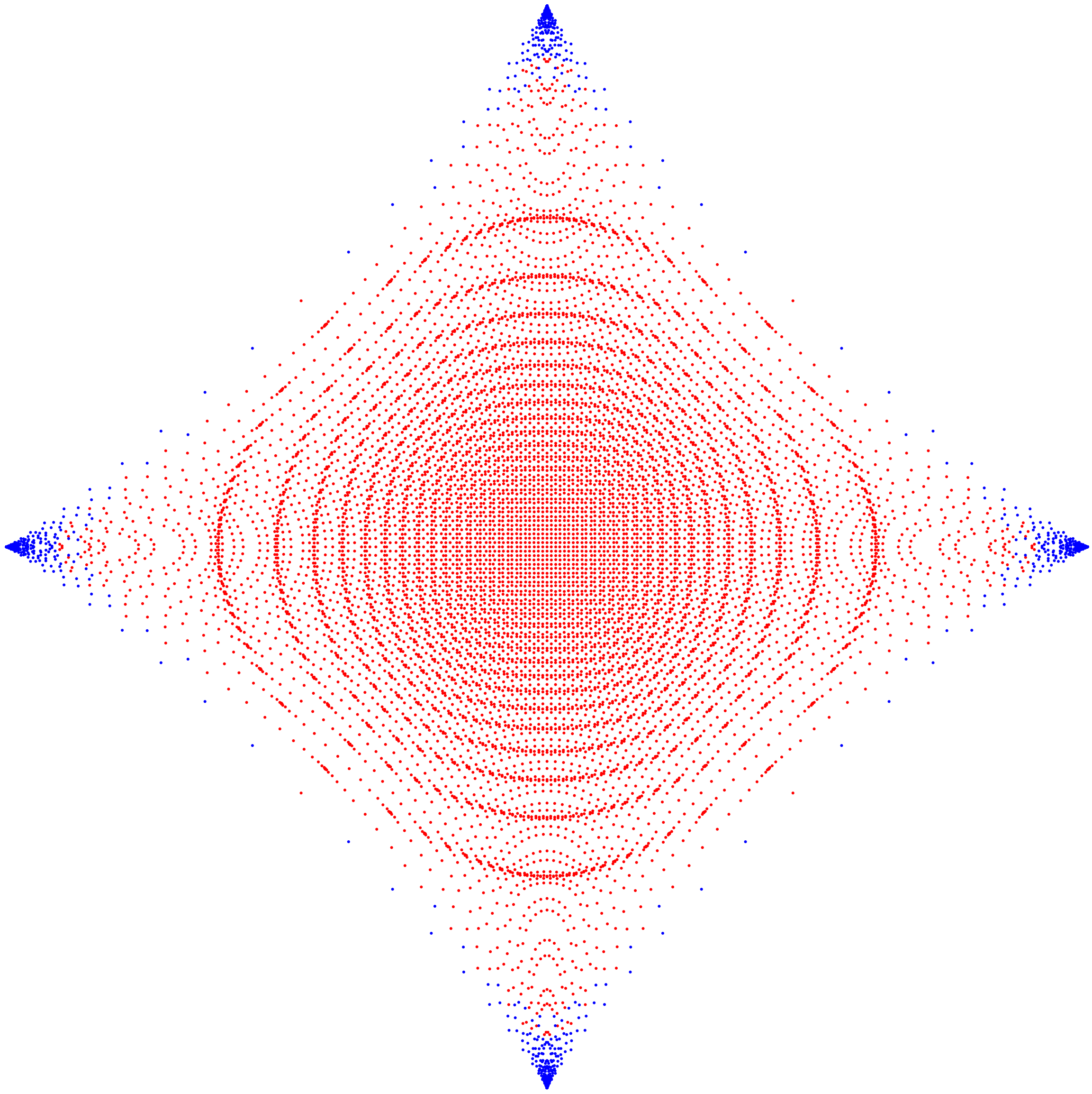
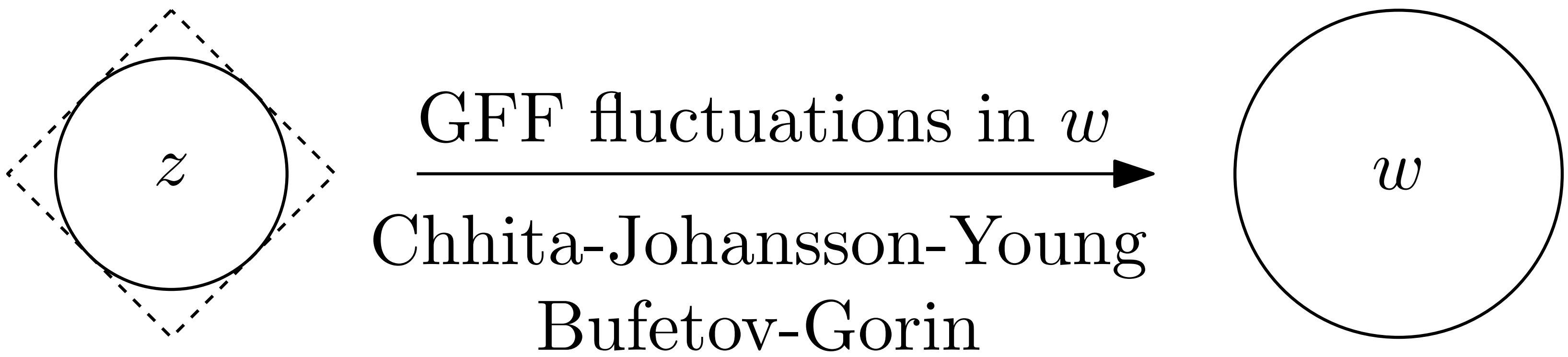


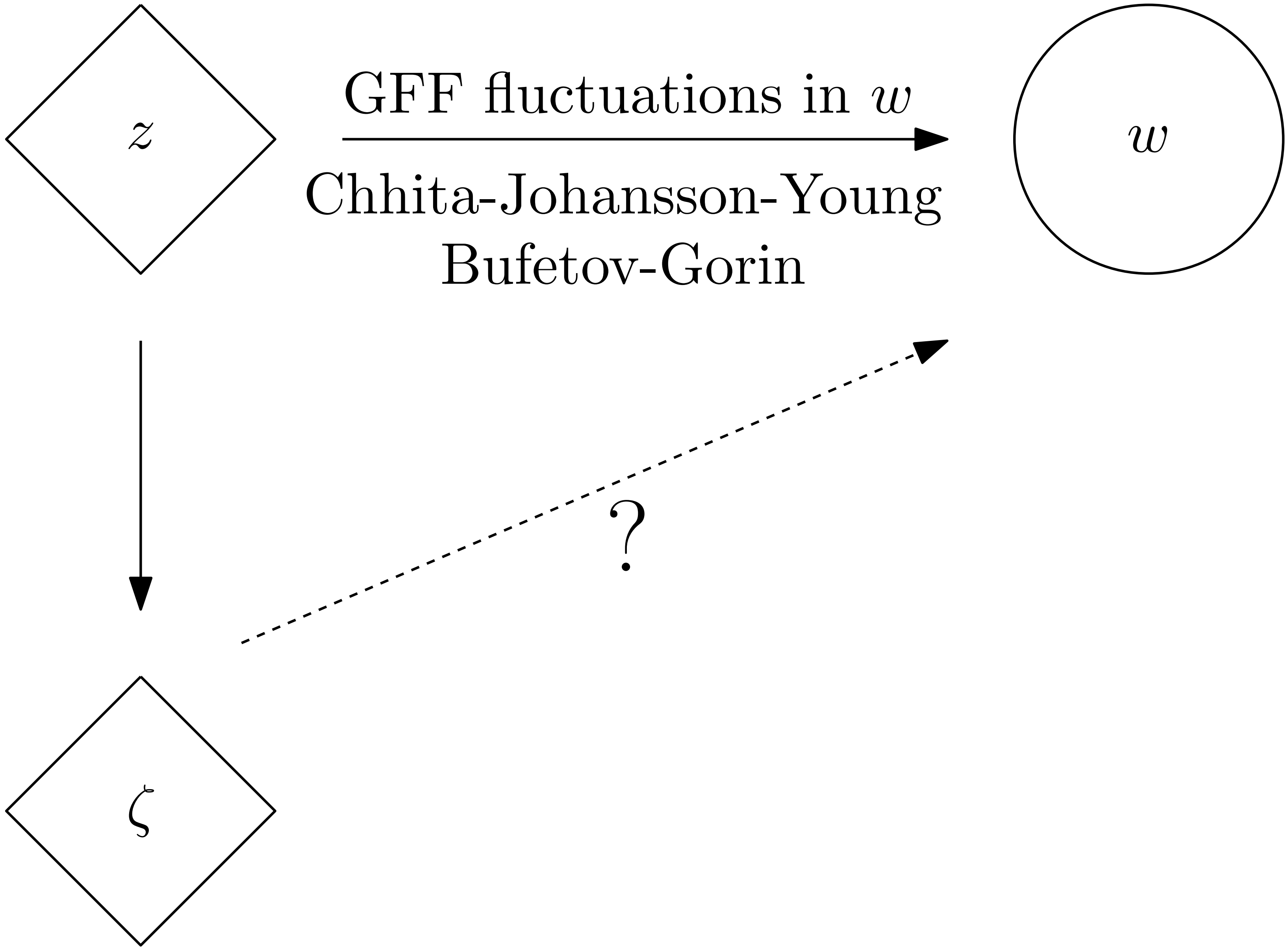
# Size 80

- Expect convergence to a continuous map $z \mapsto \zeta$ from the unit square to itself.

- Each frozen region is collapsed to a vertex of the square.

- The map $\zeta$ does not directly give the right conformal structure to describe the GFF fluctuations.


z
GFF fluctuations in w
Chhita-Johansson-Young
Bufetov-Gorin
w


z
GFF fluctuations in w
w
Chhita-Johansson-Young
Bufetov-Gorin
?
ζ

# The 折り紙 map

- Start with a circle center embedding of the dual graph (black and white faces).

- In that picture, reverse the orientation of all the white faces and glue black and white faces together with the same adjacency relations as in the circle center embedding. This is the *origami map*.

- Equivalent definition: fold the circle center embedding along each edge.

- Chelkak-Laslier-Russkikh provide a general perspective for general bipartite dimer models.

- In the continuum limit, the graph of the origami map as a function of the circle center embedding is a minimal surface in Lorentz space, whose canonical conformal structure describes the GFF fluctuations.

- Requires technical assumptions which, though natural, are not known to be verified in any explicit case.

- Chelkak-R. (2020+): in the Aztec diamond case, this indeed holds, with an explicit minimal surface (assuming the convergence of the circle center embeddings).

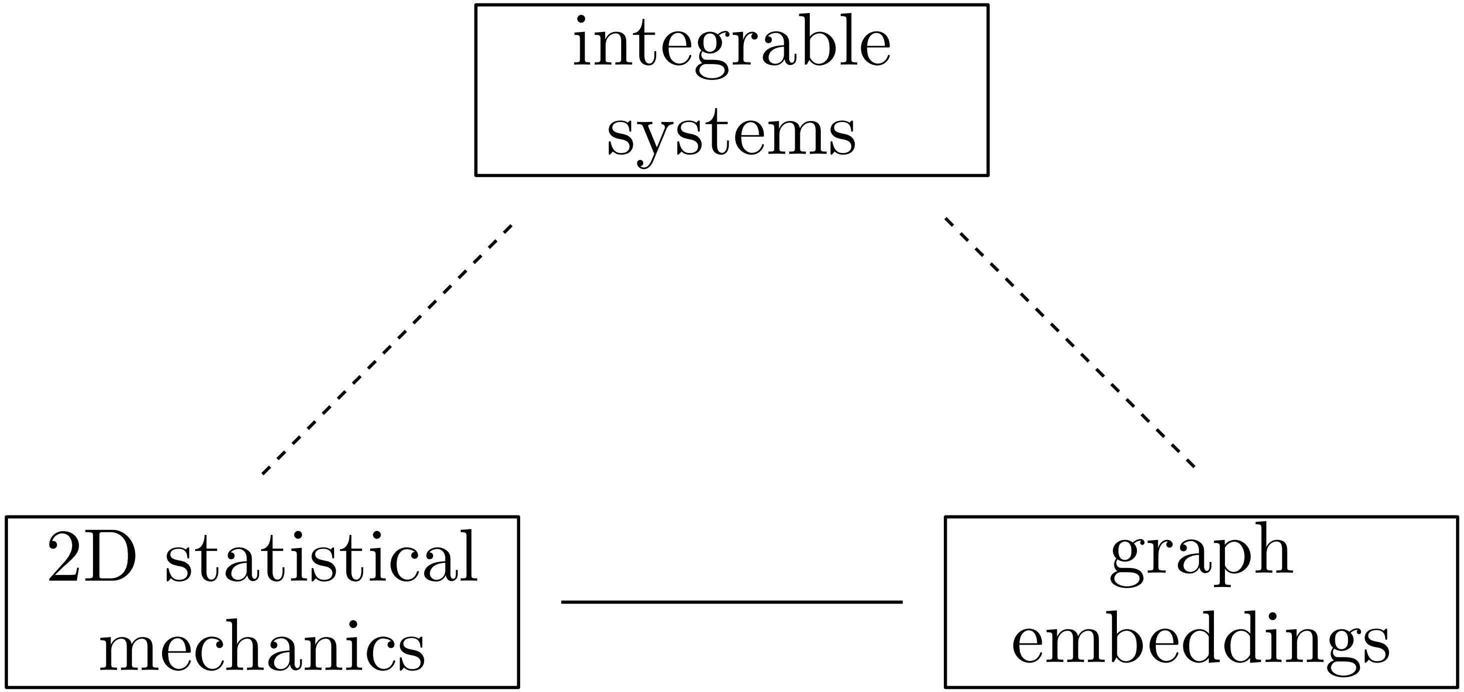
# Conclusion

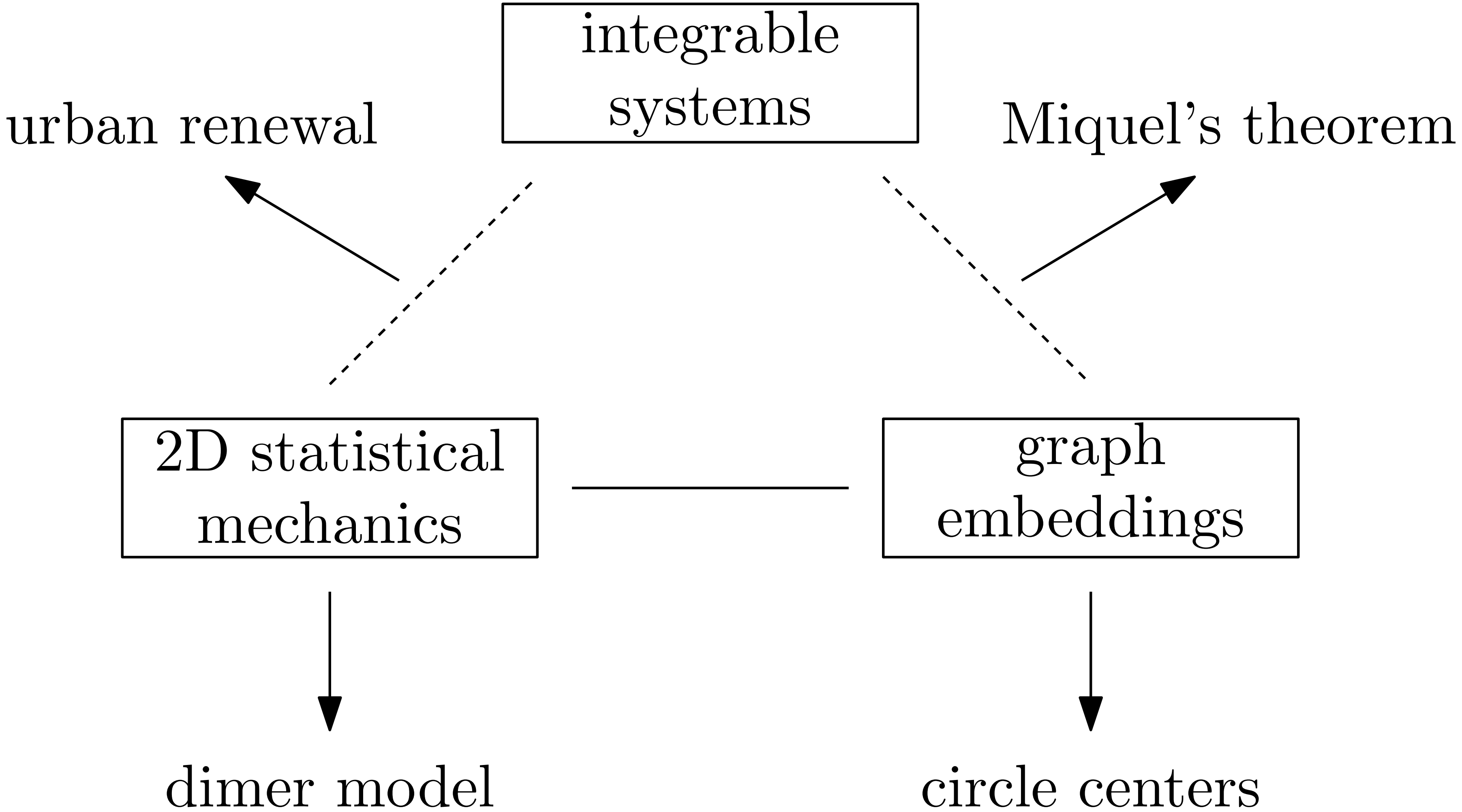
- Circle center embeddings (a.k.a. t-embeddings) for bipartite graphs are a particular choice of gauge.

- The angle condition for circle centers implies the Kasteleyn condition.

- Correspondence between embeddings as circle centers and Boltzmann/Gibbs measures for the dimer model for planar graphs with outer face of degree 4 and graphs periodic in two directions.

- Expected to provide the right geometric setting to study the scaling limit of the bipartite dimer model.


integrable systems
2D statistical mechanics
graph embeddings


integrable systems
urban renewal
Miquel's theorem
2D statistical mechanics
graph embeddings
dimer model
circle centers

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N. Affolter, Miquel dynamics, Clifford lattices and the dimer model, arXiv:1808.04227 (2018).

D. Chelkak and S. Ramassamy, Fluctuations in the Aztec diamond via a Lorentz-minimal surface, in preparation.

D. Chelkak, B. Laslier and M. Russkikh, Dimer model and holomorphic functions on t-embeddings of planar graphs, arXiv:2001.11871 (2020).

D. Chelkak, B. Laslier and M. Russkikh, Bipartite dimer model: p-embeddings and Lorentz-minimal surfaces, in preparation.

# THANK YOU !