Dimers and circle patterns

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Joint work with:

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Part 1

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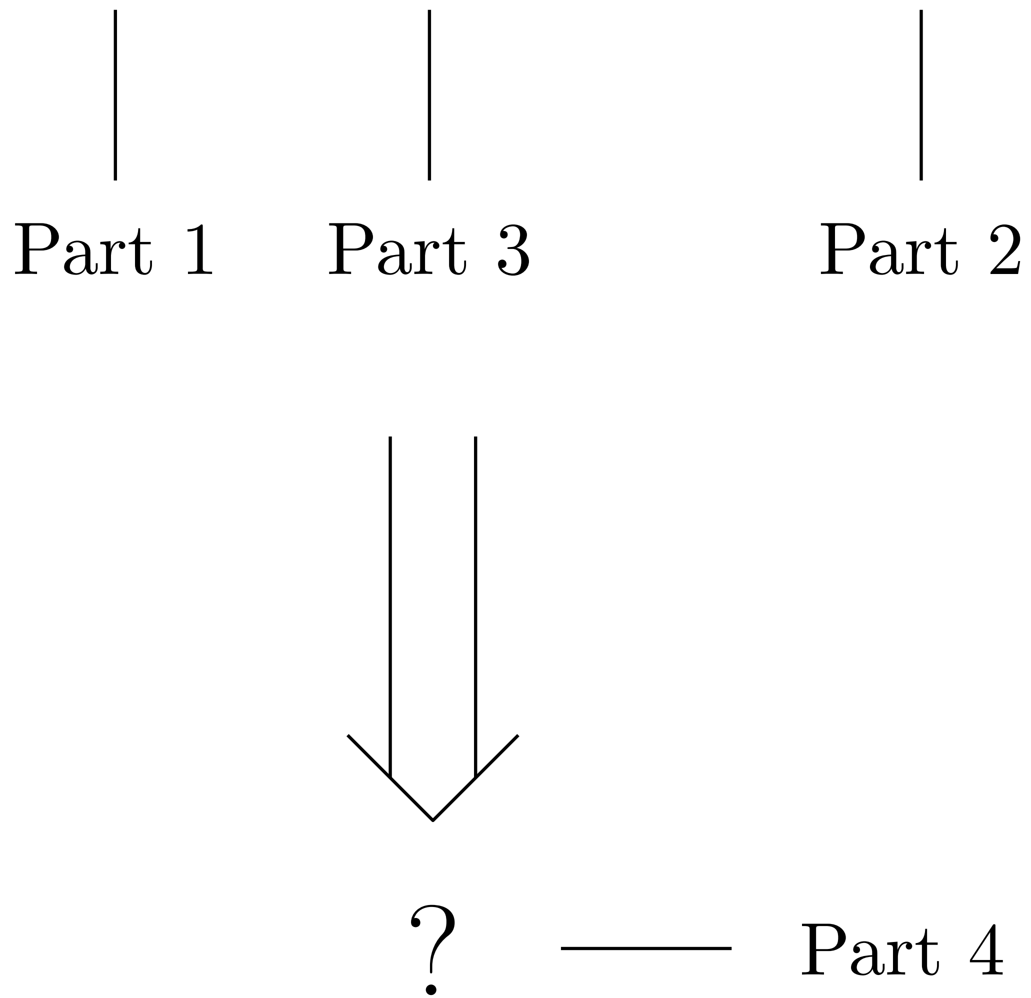
Part 1

Part 2

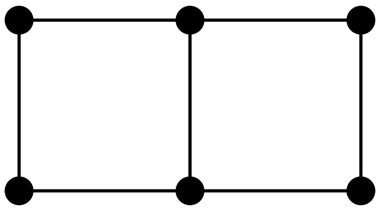
Dimers and circle patterns

Part 1 Part 3 Part 2

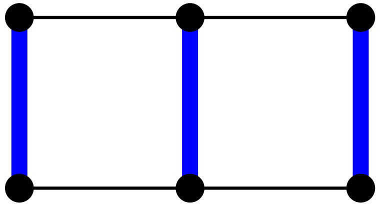
Dimers and circle patterns

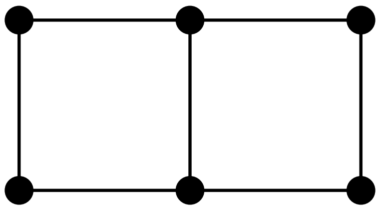


1 The dimer model



- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.





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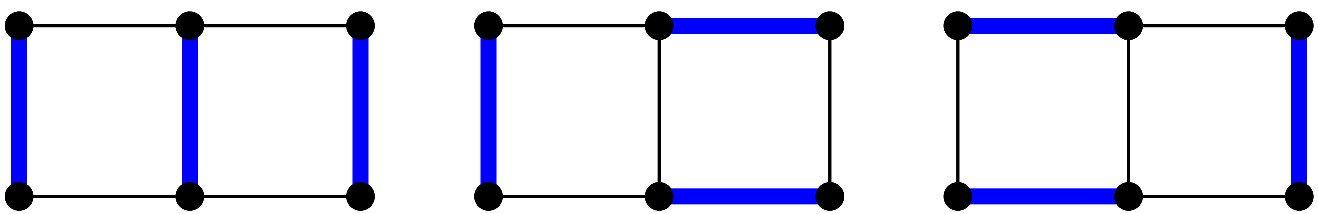
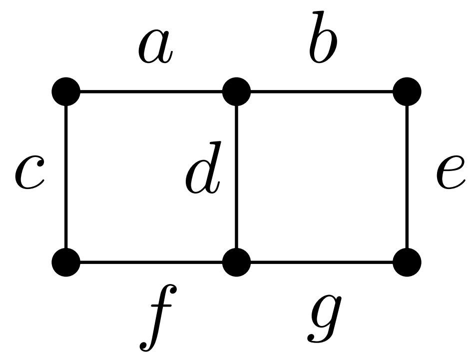
Dimer coverings: 



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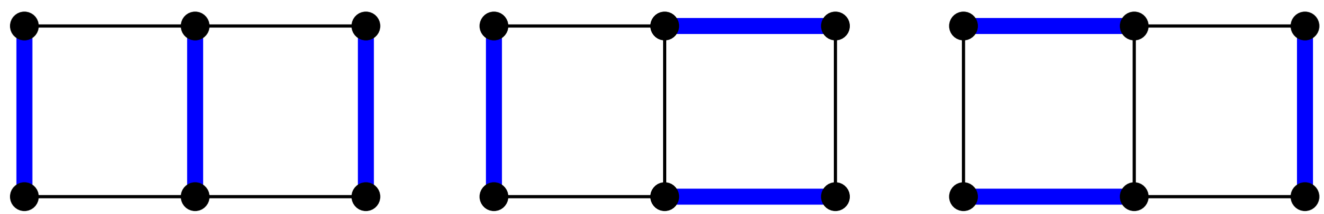
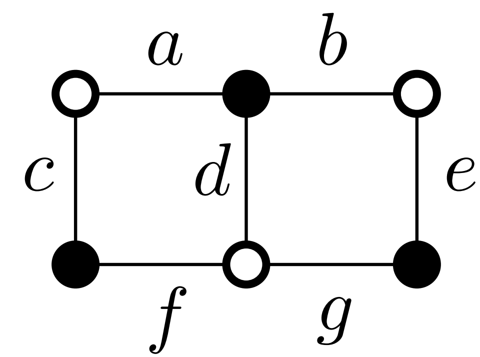
Dimer coverings:



Weights: cde bcg aef

- *Boltzmann measure*: draw a dimer covering at random with probability proportional to its weight.

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *positive edge weights*.



- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.

- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge.

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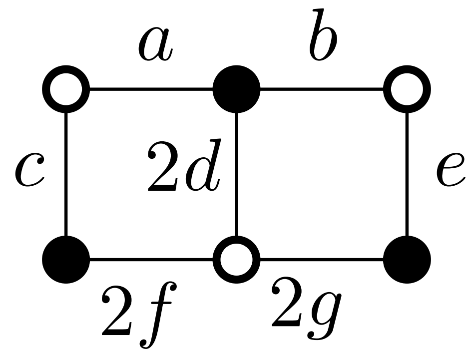

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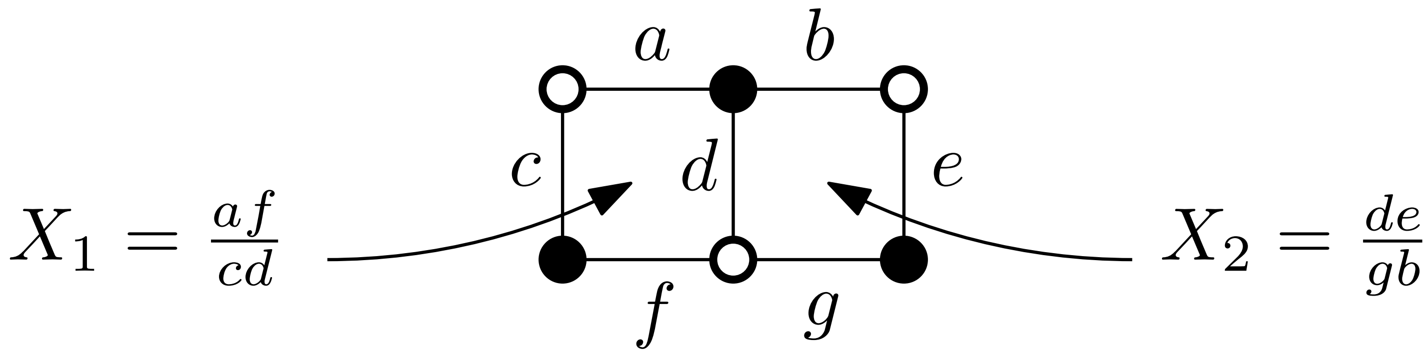

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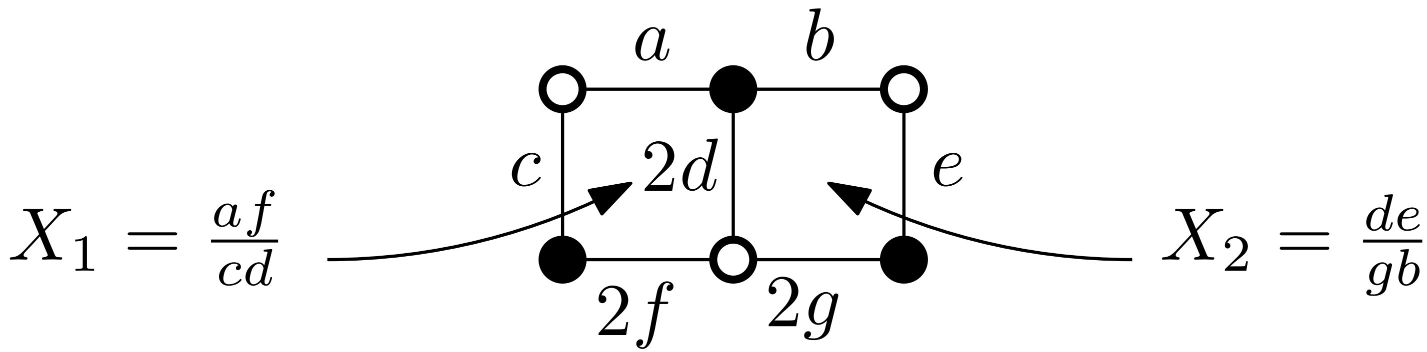

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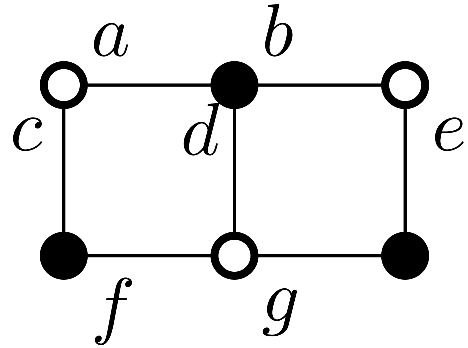
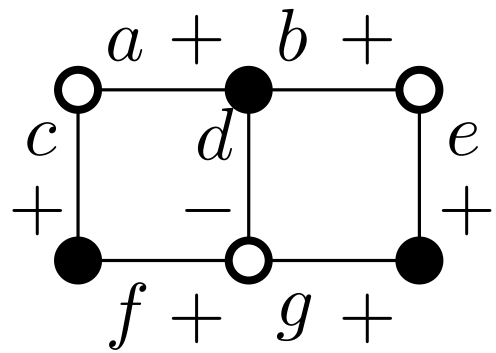
The Kasteleyn matrix K



- *Kasteleyn signs*: assign a sign to each edge such that the number of minus signs around a face of degree 2 mod 4 (resp. 0 mod 4) is even (resp. odd).

- K: weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

The Kasteleyn matrix K



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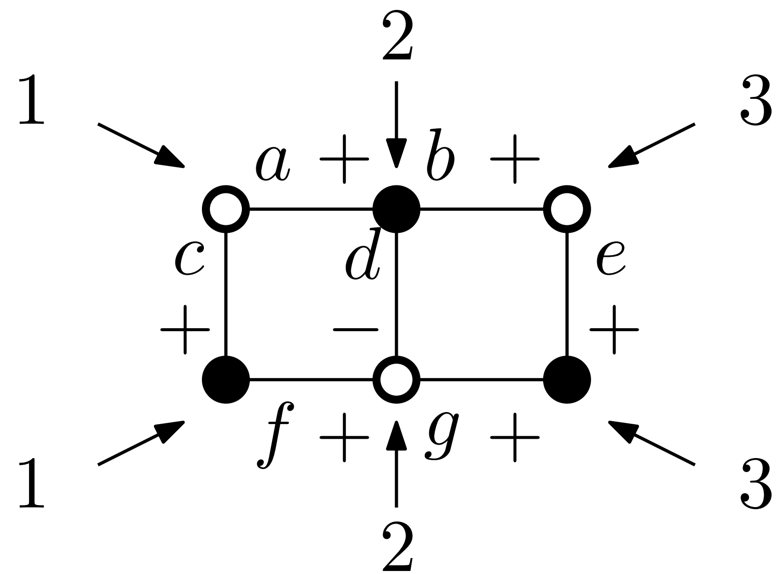

$$\begin{array}{c|ccc} & \bullet_1 & \bullet_2 & \bullet_3 \\ \hline \circ_1 & c & a & 0 \\ \circ_2 & f & -d & g \\ \circ_3 & 0 & b & e \end{array}$$

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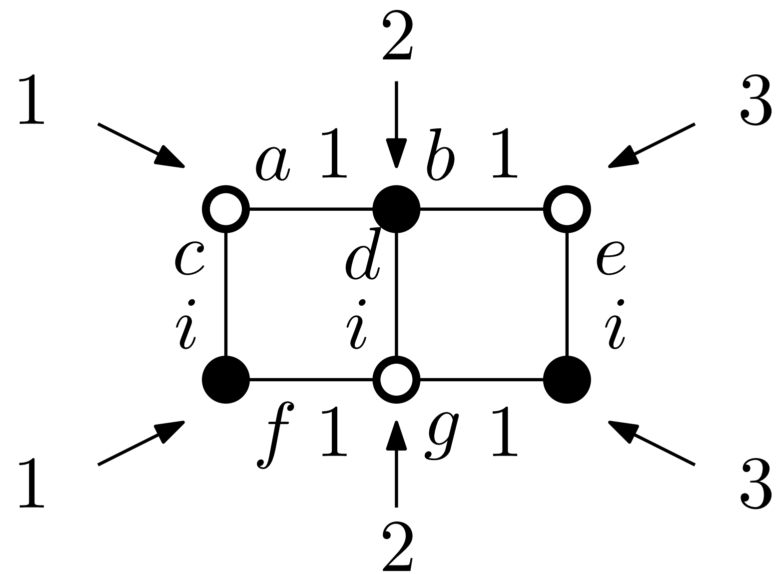


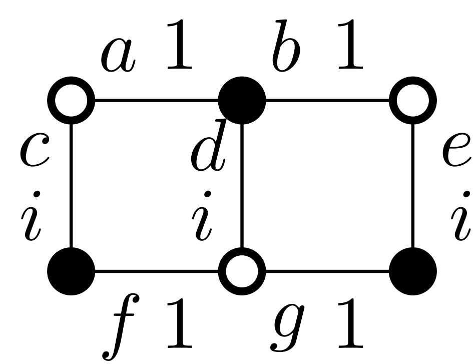
$$\begin{array}{c} \\ \circ_1 \\ \circ_2 \\ \circ_3 \end{array} \begin{array}{c} \bullet_1 \quad \bullet_2 \quad \bullet_3 \\ \begin{pmatrix} ic & a & 0 \\ f & id & g \\ 0 & b & ie \end{pmatrix} \end{array}$$

- *Complex Kasteleyn signs*: assign a unit complex number to each edge such that the alternating product of these numbers around a face of degree 2 mod 4 (resp. 0 mod 4) is 1 (resp. -1).

- K: weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

- The partition function (sum of the weights of all dimer coverings) is $|\det K|$. The dimer correlations are given by minors of K^{-1} (Kasteleyn, Temperley-Fisher).

- Merge the complex Kasteleyn signs with the positive edge weights to get complex edge weights (entries of K).



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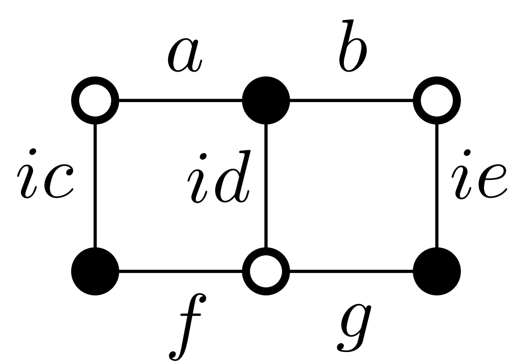

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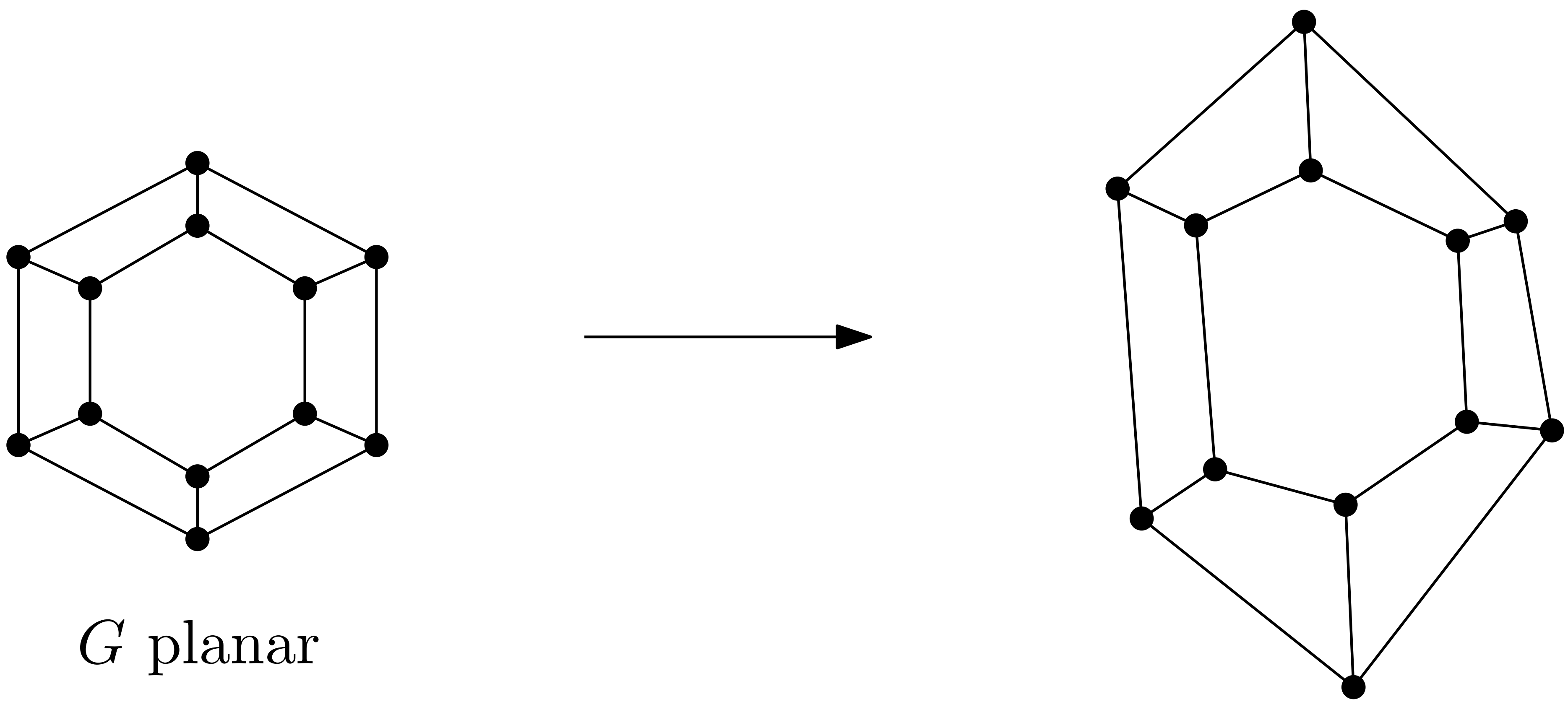
- The alternating product of complex edge weights is real positive (resp. real negative) around a face of degree 2 mod 4 (resp. 0 mod 4).

2 Circle patterns and circle centers

- *Circle pattern for G*: map from the vertex set of G to $\mathbb{R}^2$ sending all the vertices around any bounded face to concyclic points.



G planar

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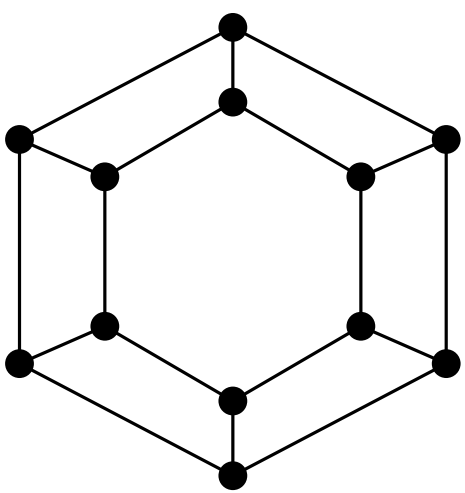

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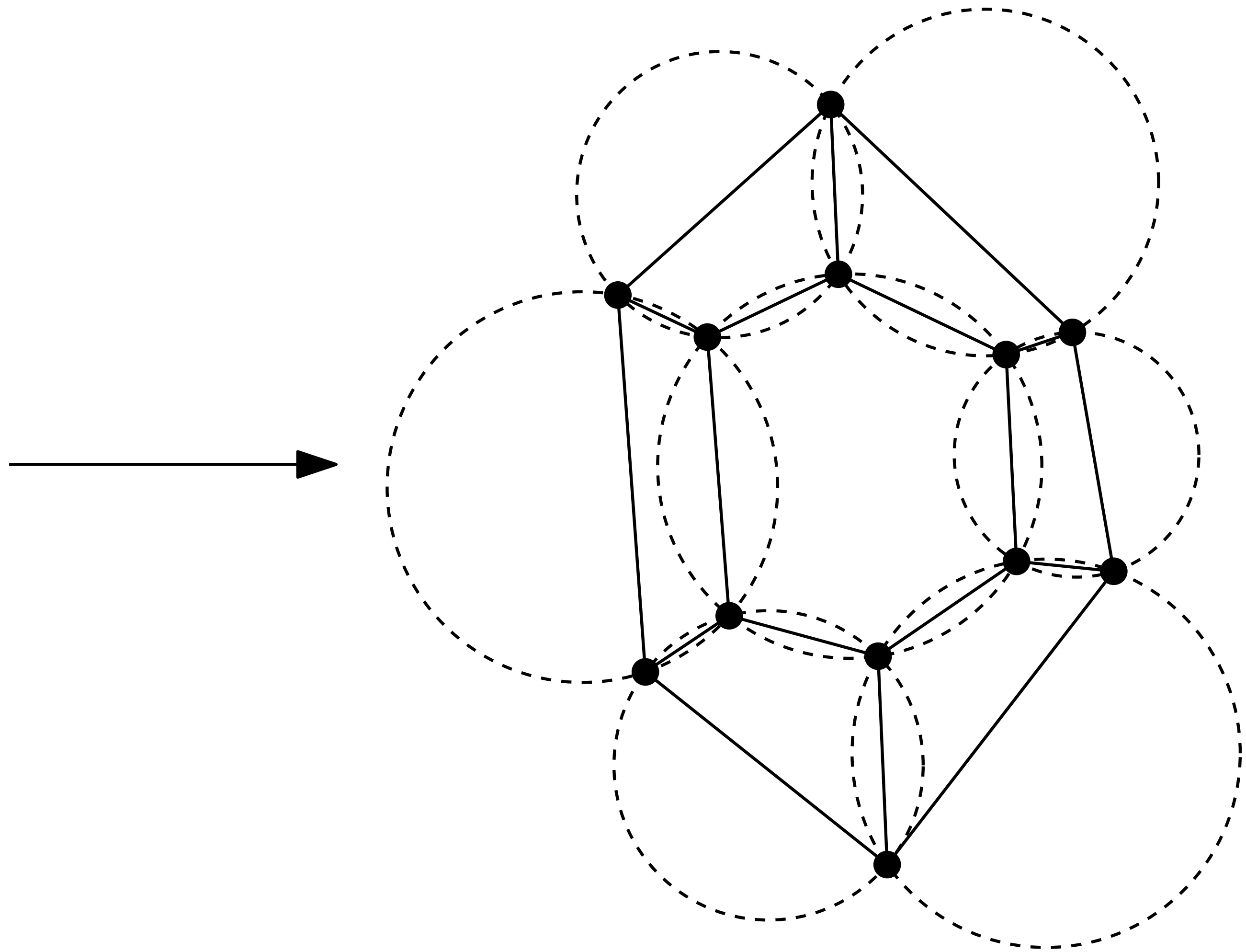

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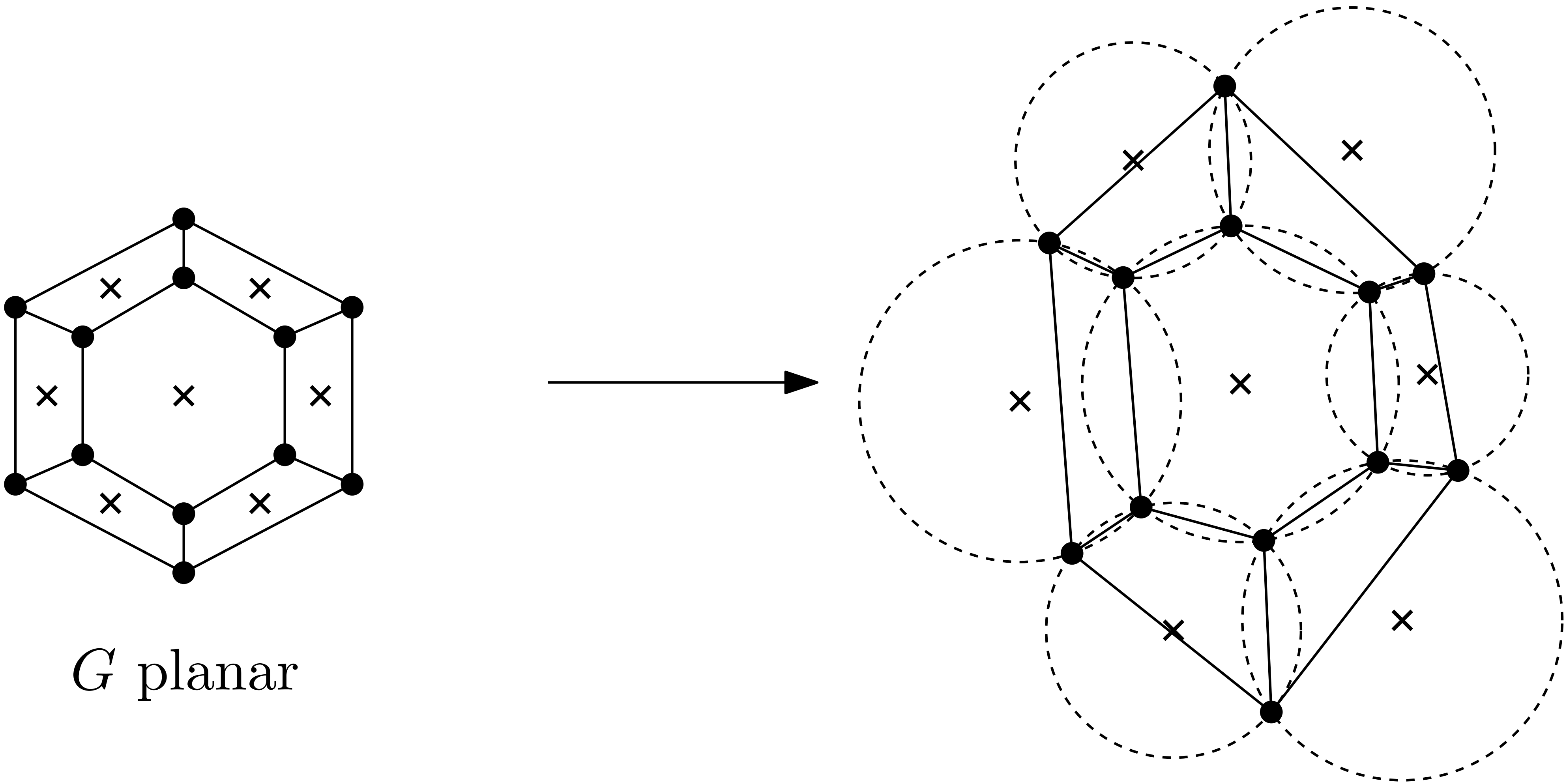

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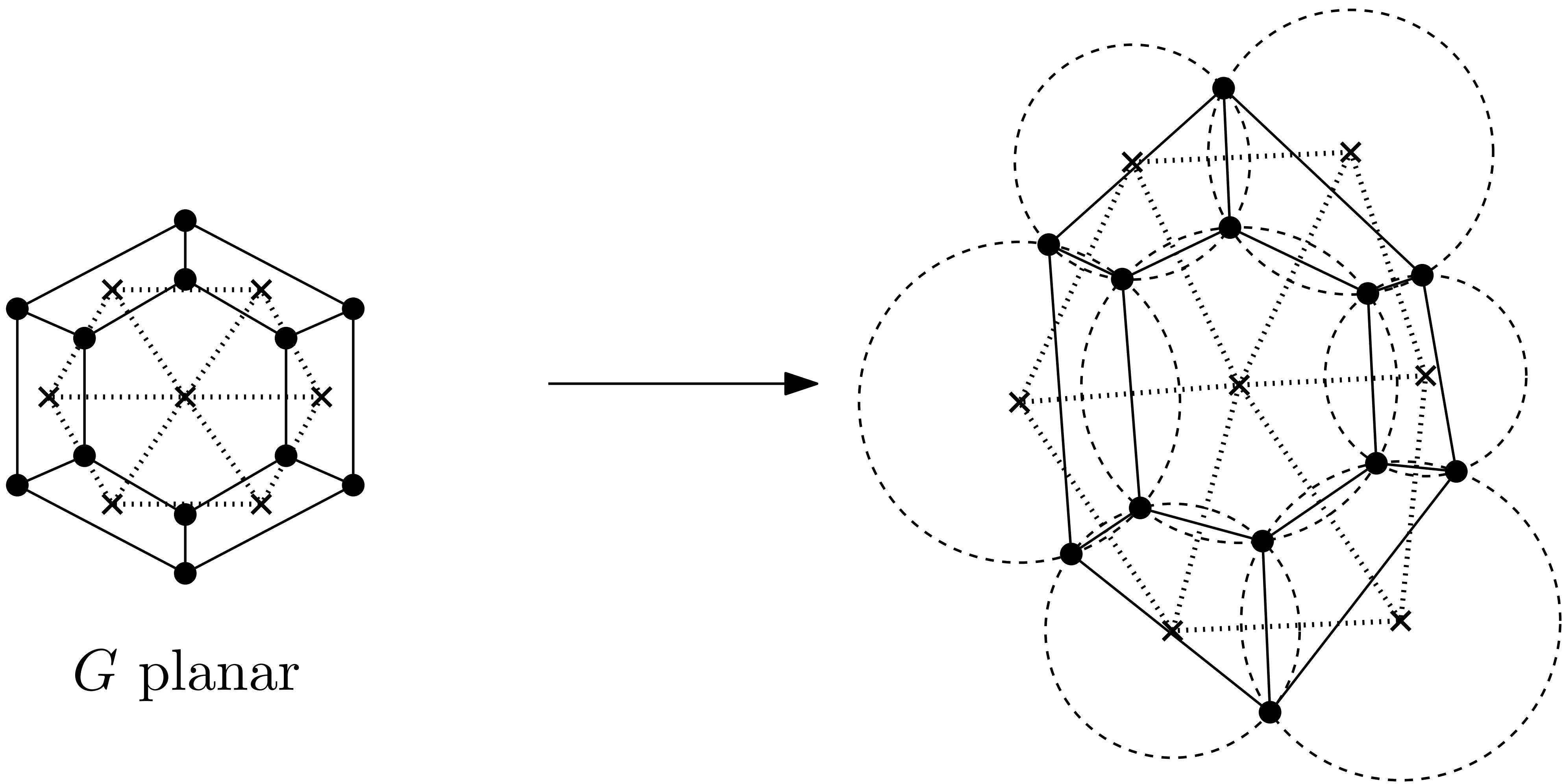
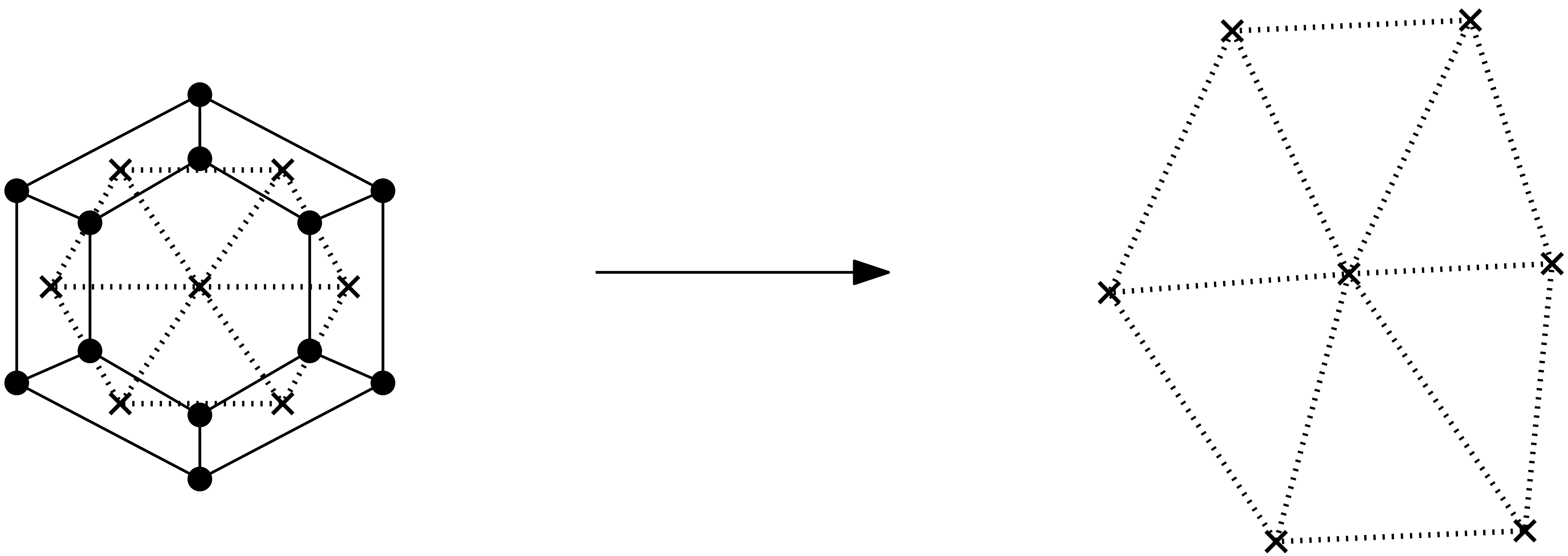


G planar

- *Circle centers for G*: drawing of the dual graph of G arising as centers of some circle pattern for G.



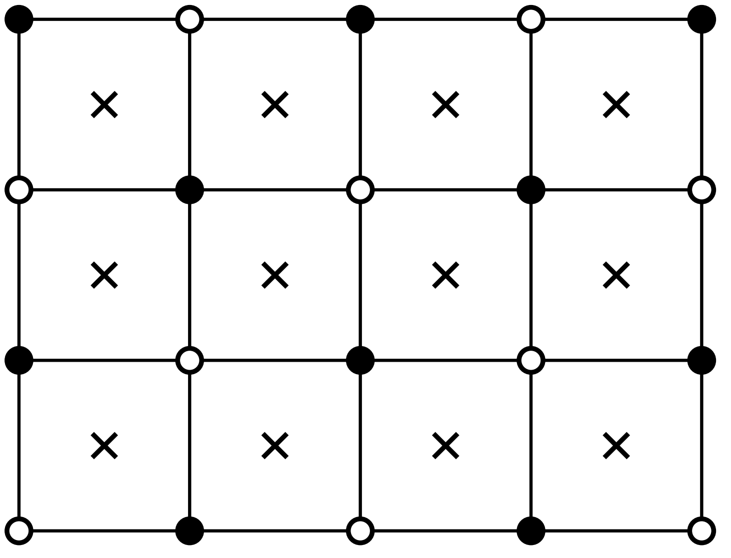
G planar

- Recover the circle pattern from the circle centers ?

  How many circle patterns have the same centers ?

- Given a drawing of the dual graph of G, how to see if it corresponds to the centers of a circle pattern for G ?

- Answers in the case when G is bipartite.



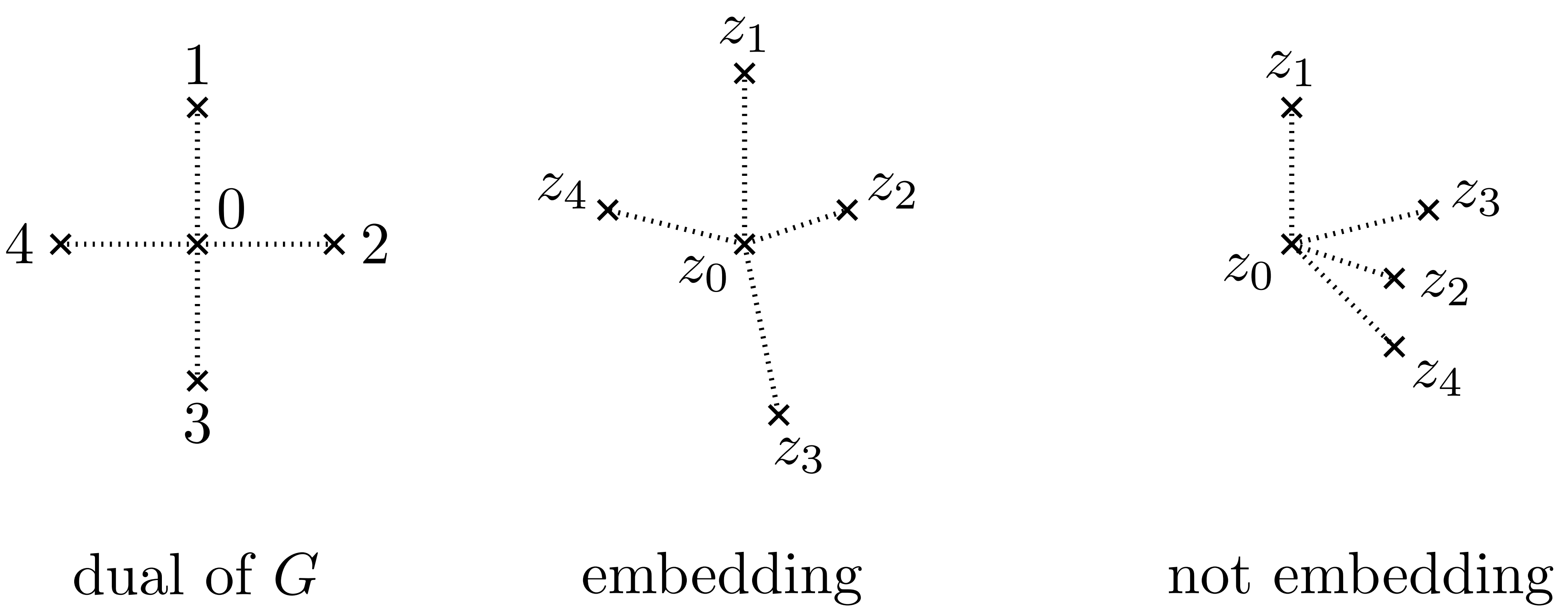
[Geogebra]

- From now on G is bipartite.

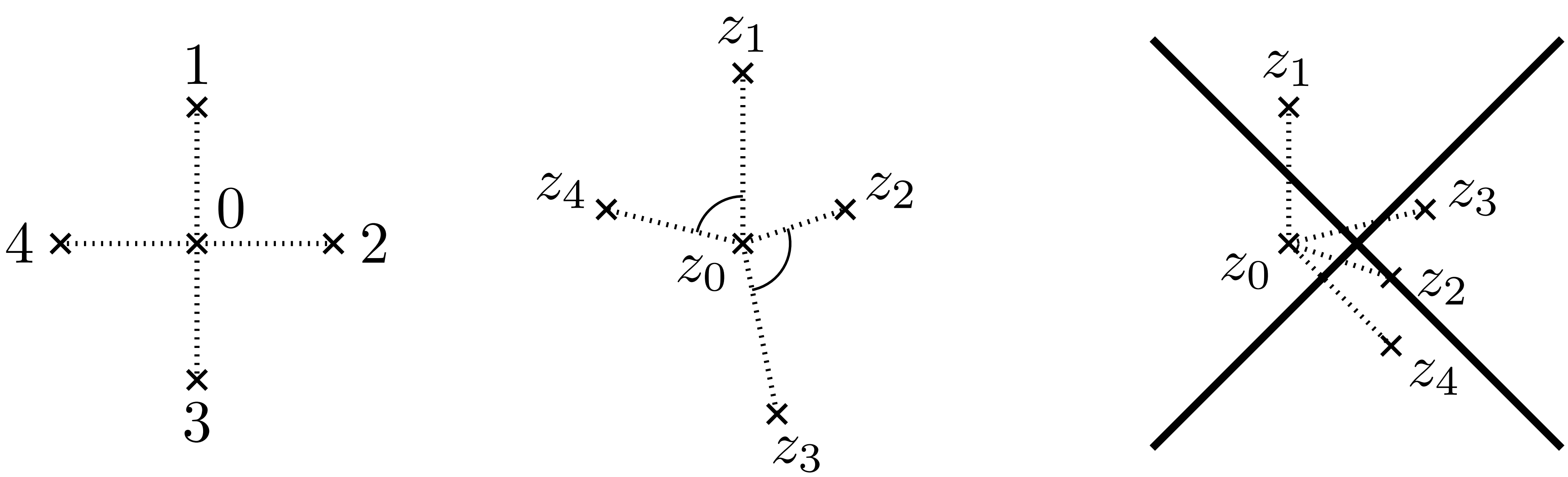
- 2-parameter family of patterns with the same centers.

- A drawing of the dual graph of G corresponds to circle centers for G if and only if around each dual vertex the sum of every other angle is 0 mod π.

- An *embedding* of the dual graph of G corresponds to circle centers for G if and only if around each dual vertex the sum of every other angle is equal to π.



1
0
4
2
3
dual of G
z_1
z_4
z_2
z_0
z_3
embedding
z_1
z_3
z_0
z_2
z_4
not embedding



dual of G embedding not embedding

$$\arg \frac{z_4 - z_0}{z_1 - z_0} + \arg \frac{z_2 - z_0}{z_3 - z_0} = \pi \Leftrightarrow \frac{(z_2 - z_0)(z_4 - z_0)}{(z_1 - z_0)(z_3 - z_0)} \in \mathbb{R}_{<0}$$

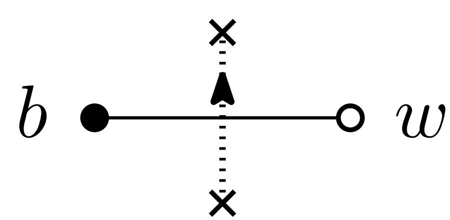
3 Dimer models and circle centers

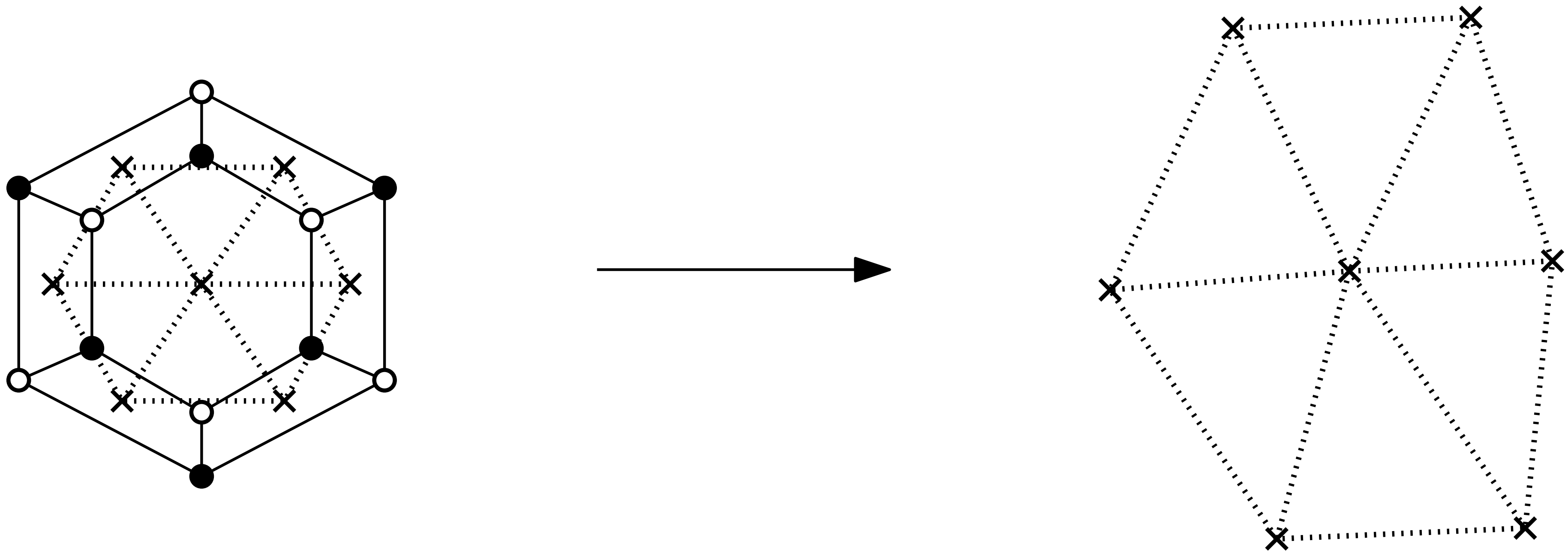
From circle centers to dimer weights

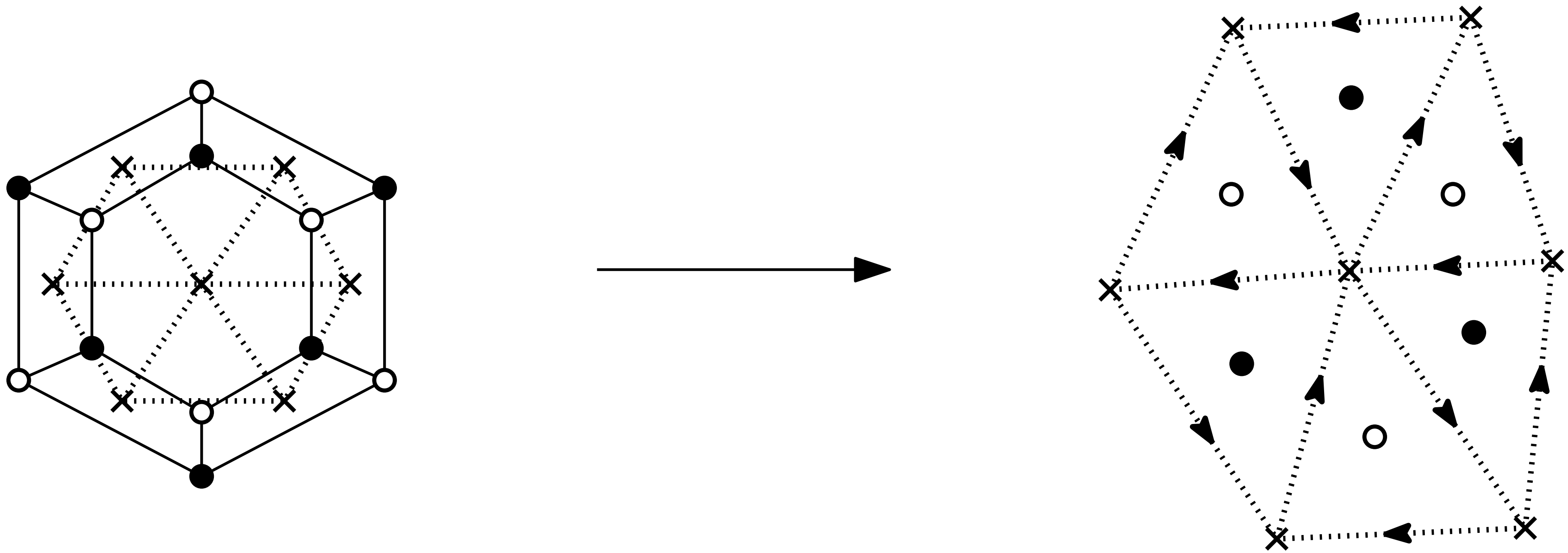
- Fix G a planar unweighted bipartite graph. Start with an embedding of the dual of G as circle centers.

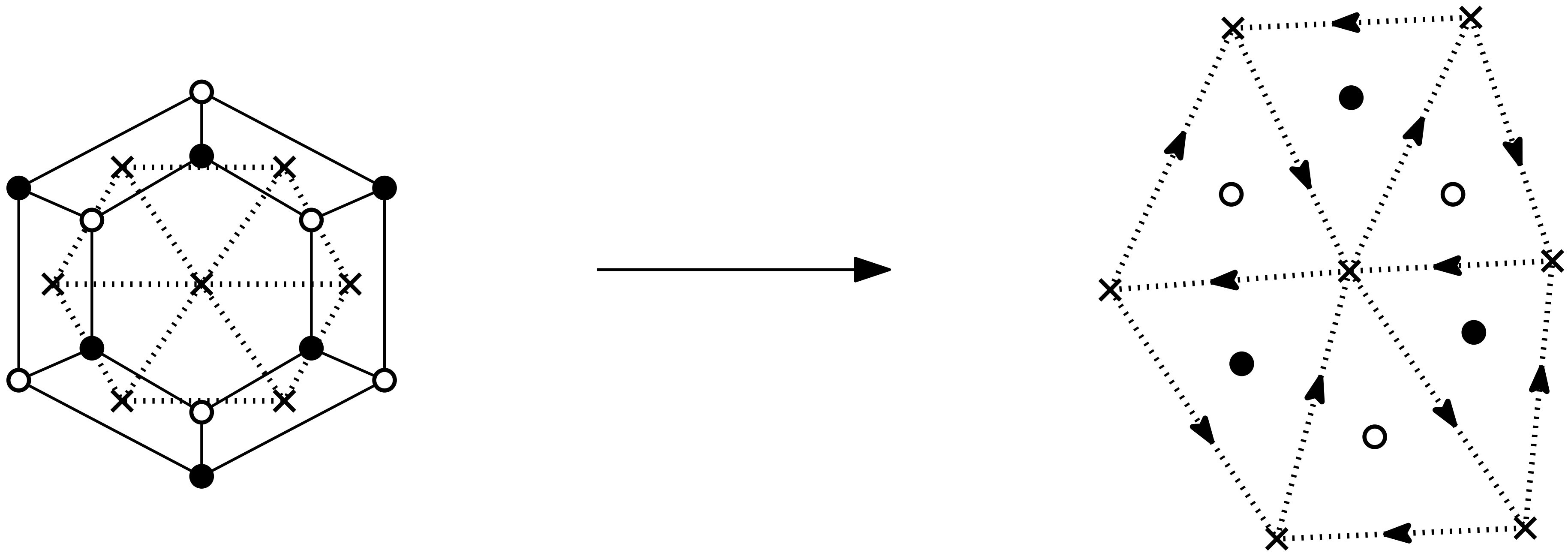
- Construct complex edge weights for G associated to that embedding which satisfy the Kasteleyn condition.

- For an edge in G between b and w, the weight is the vector (complex number) of its corresponding dual edge, oriented so that b lies to its left.

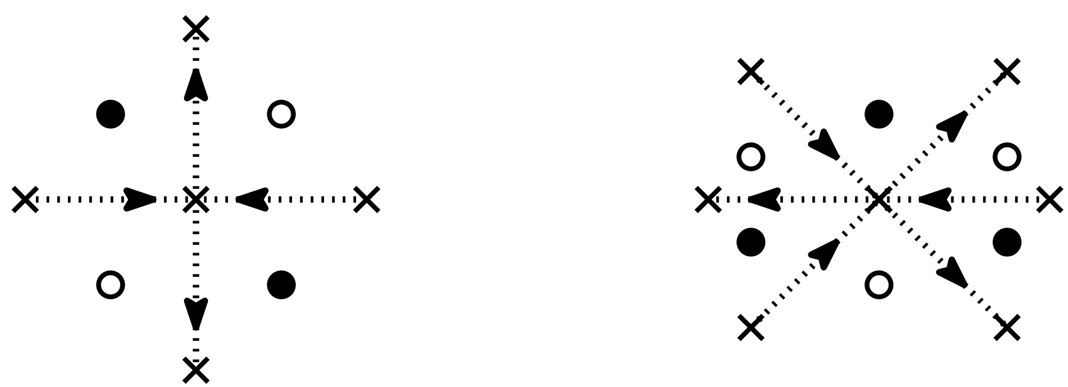


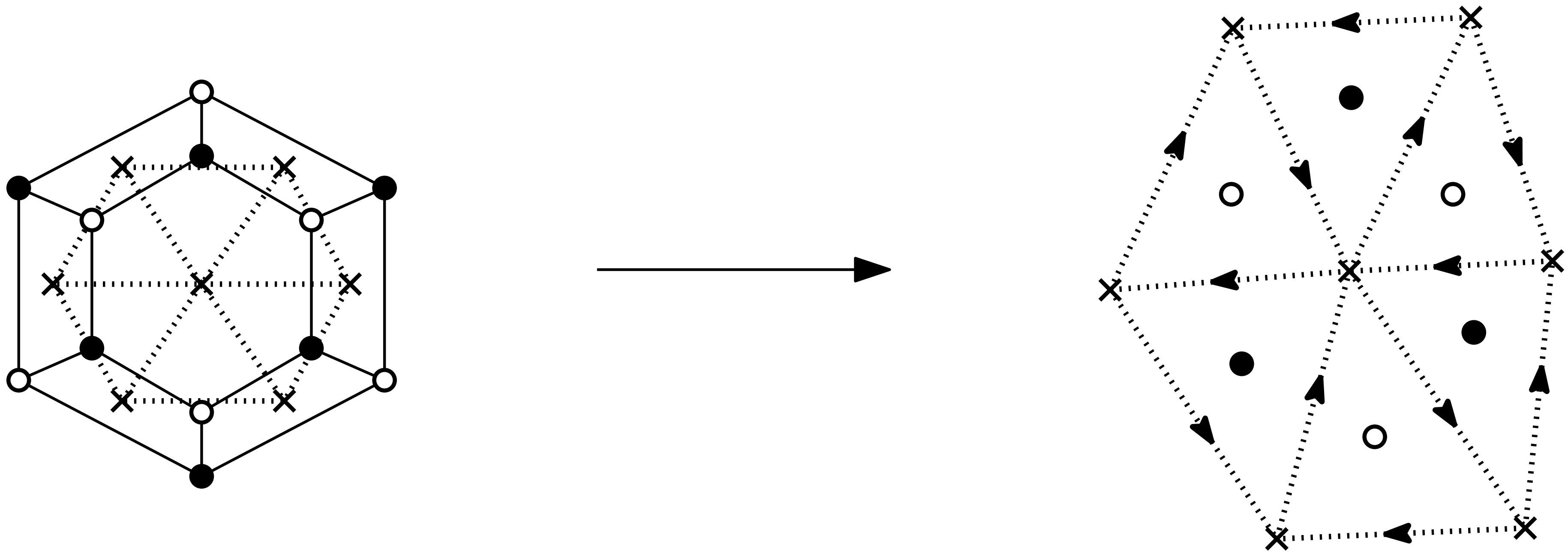






- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree 2 mod 4 (resp. 0 mod 4) is positive (resp. negative).





- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree 2 mod 4 (resp. 0 mod 4) is positive (resp. negative).

- Around every vertex, the sum of the complex edge weights is zero, i.e. the edge weights have zero divergence.

For a bipartite graph, the geometric local condition
"being centers of a circle pattern with embedded dual"
implies the local condition
"being Kasteleyn edge weights with zero divergence"
(Kenyon-Lam-R.-Russkikh, 2018)

- The fact that circle center embeddings satisfy the Kasteleyn condition was also observed by Affolter (2018).

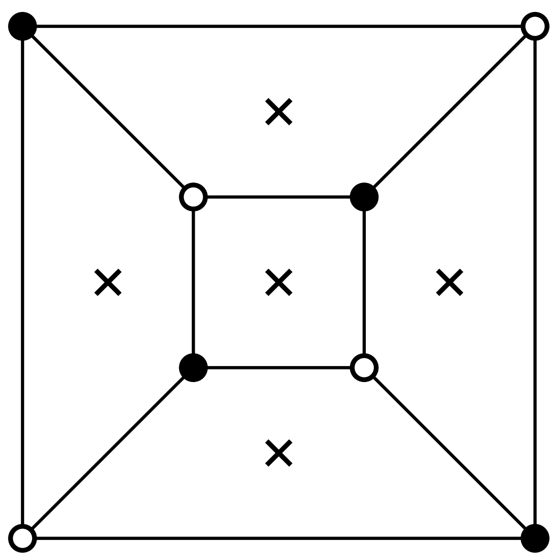
- Positive edge weights are obtained from circle centers as distances between adjacent centers.

- Going from circle centers to dimer edge weights is a local construction.

From dimer weights to circle centers

- Given a bipartite graph with positive edge weights, find gauge equivalent weights coming from circle centers.

Coulomb gauge



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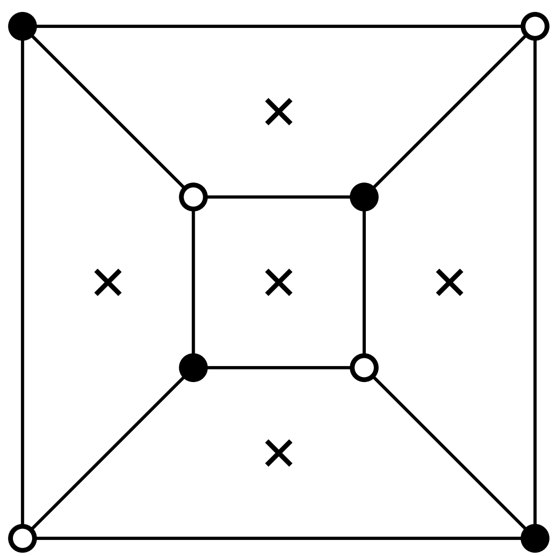
× → *Coulomb gauge*



From dimer weights to circle centers

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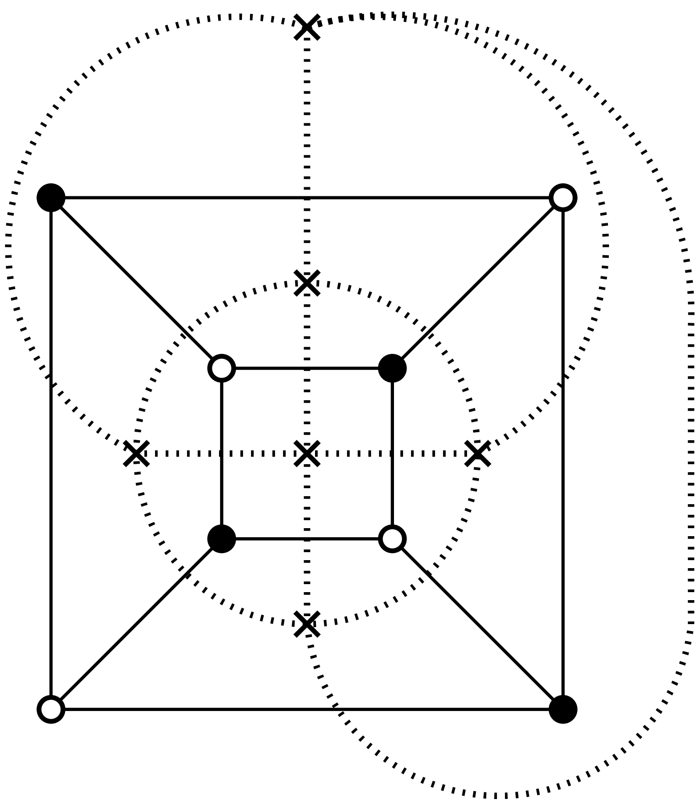

dual

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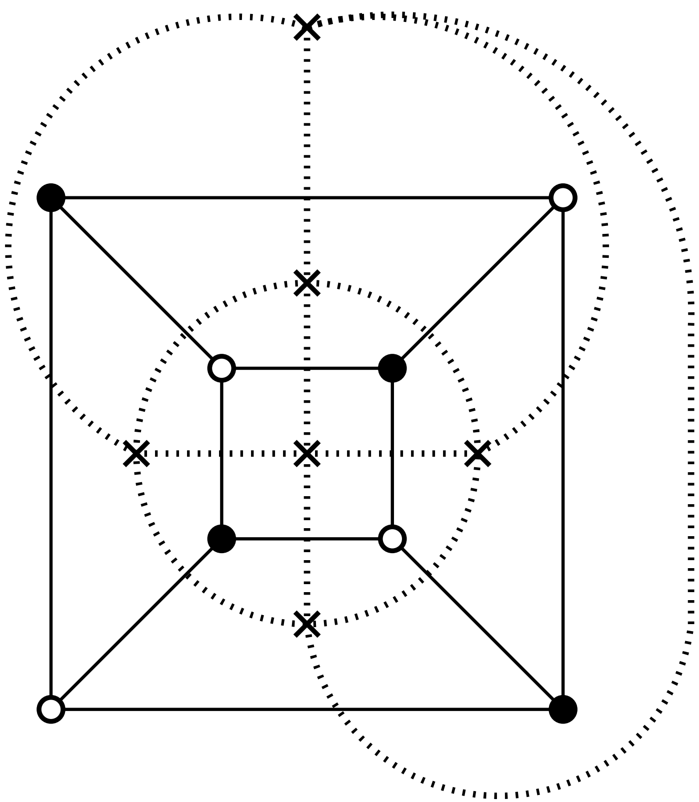

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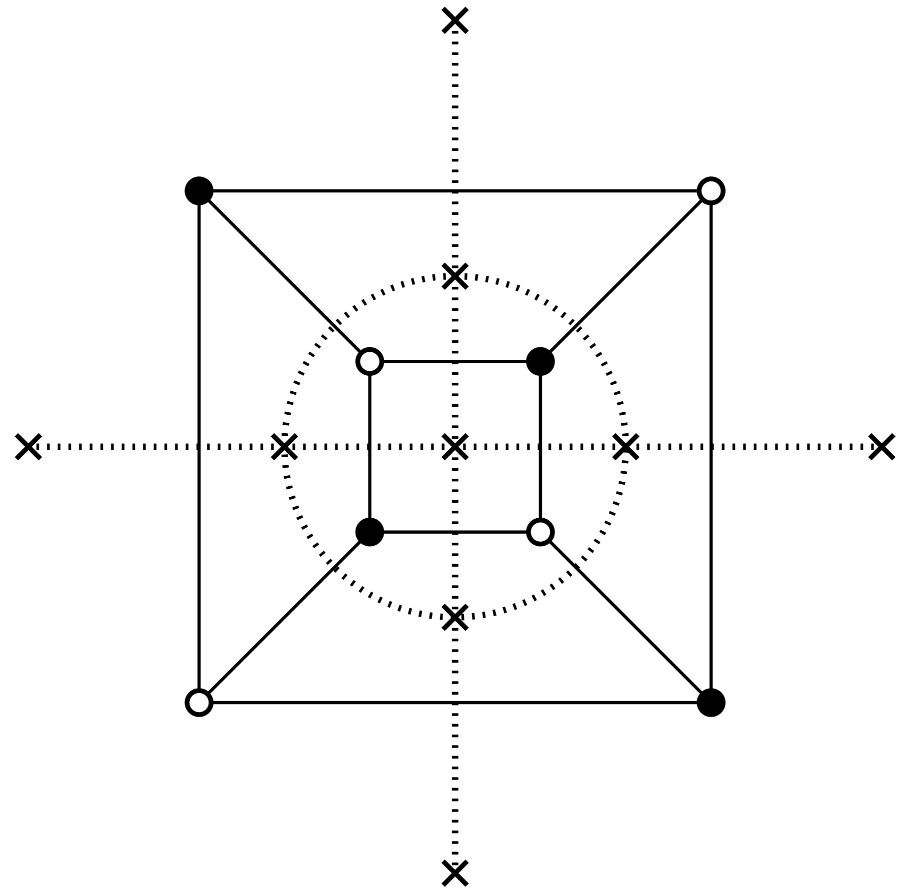

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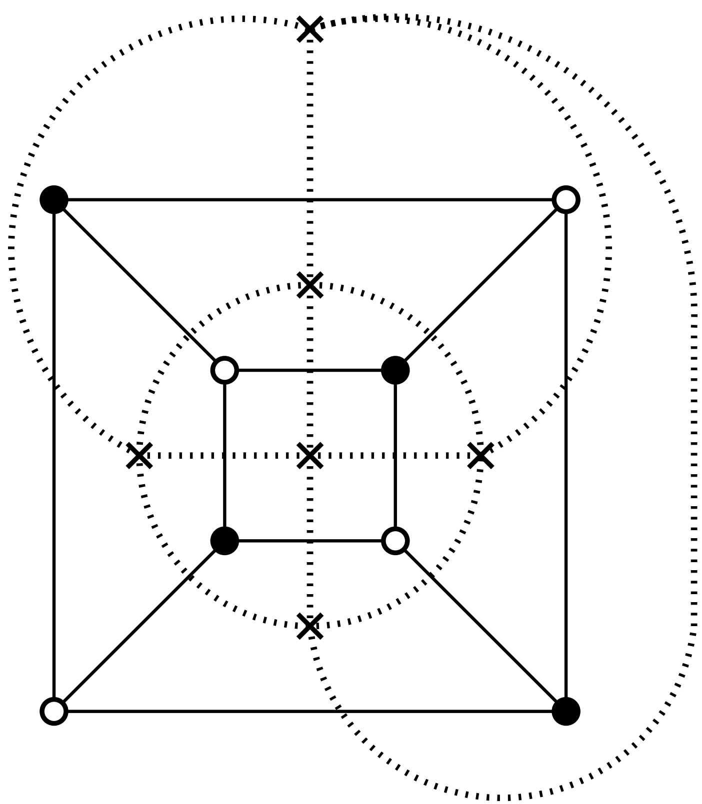
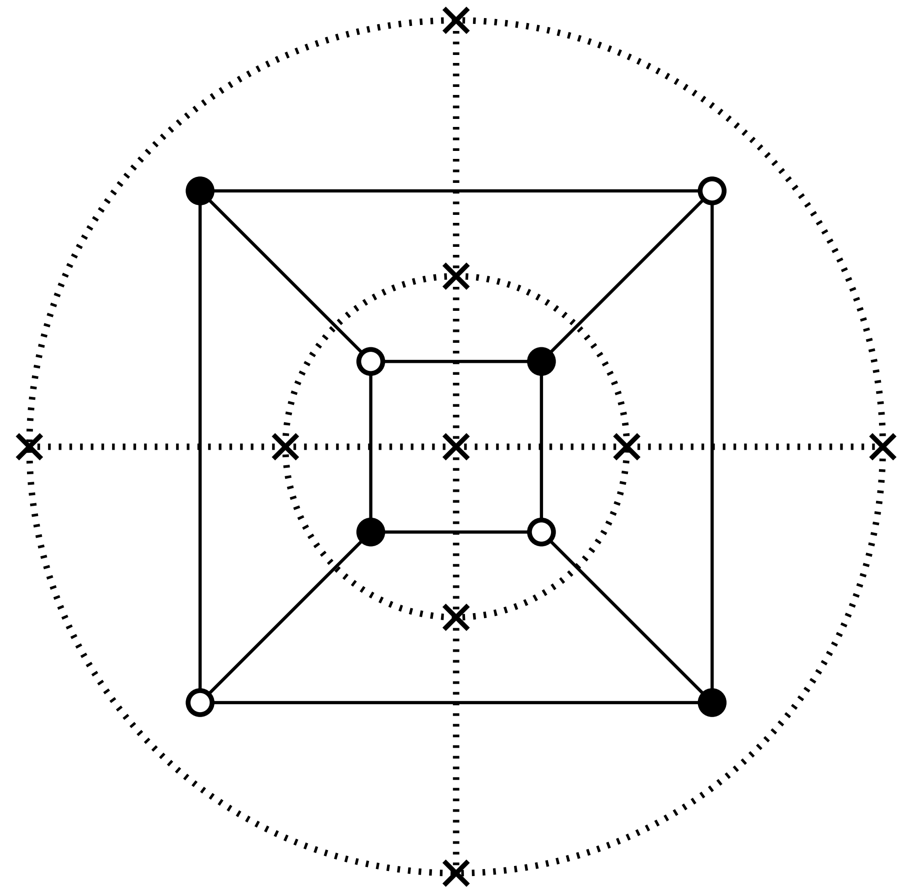
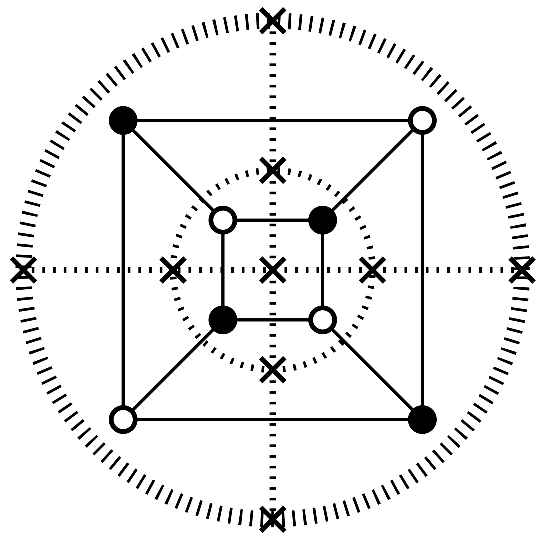


dual



augmented dual



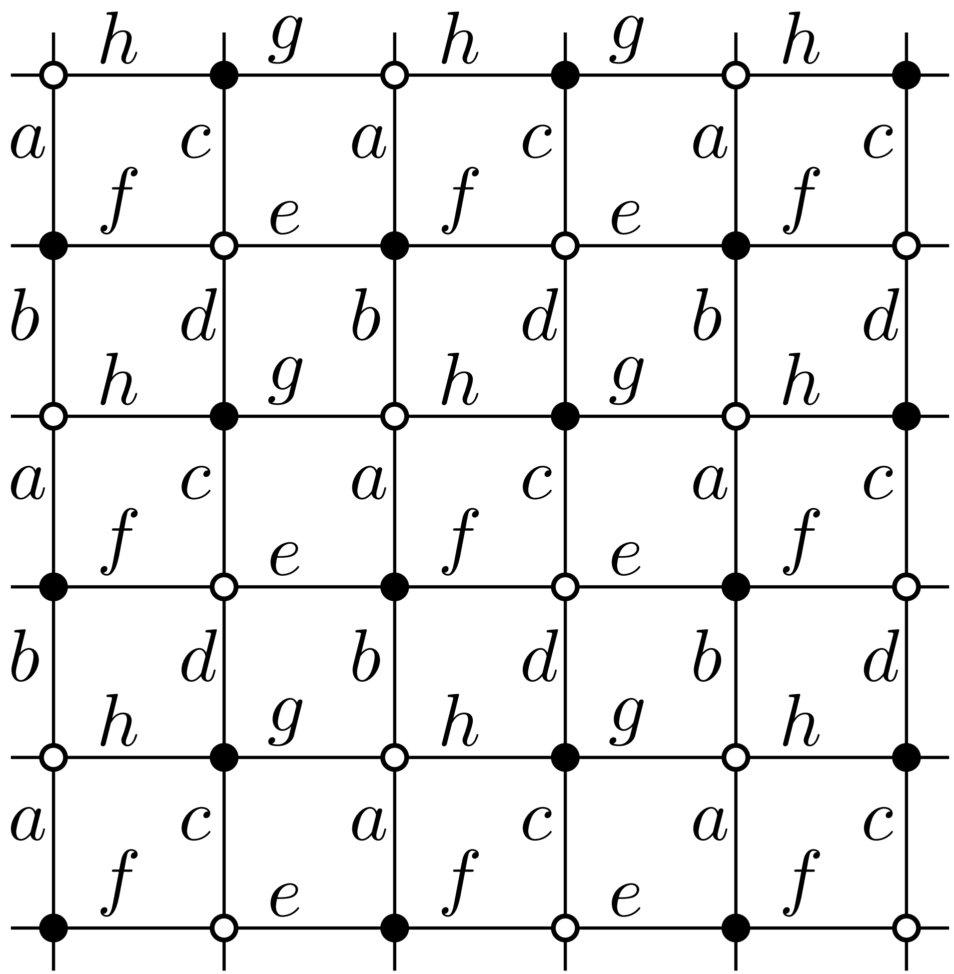
Theorem (Kenyon-Lam-R.-Russkikh 2018). *Let G be a planar bipartite weighted graph with outer face of degree 4. Fix a convex quadrilateral P.*
There are two circle center embeddings of the augmented dual of G which produce weights that are gauge equivalent to the original weights and such that the four outer dual vertices are mapped to the vertices of P.

- Given
  - an unweighted bipartite planar graph G with boundary of length 4
  - a convex quadrilateral (boundary condition)

  there is a 2-to-1 correspondence between embeddings of the augmented dual of G as circle centers and dimer Boltzmann measures on G.

- Expected to hold in some form for other boundary lengths.

- Other setting: infinite planar bipartite graphs, periodic in two directions with edge weights also periodic.



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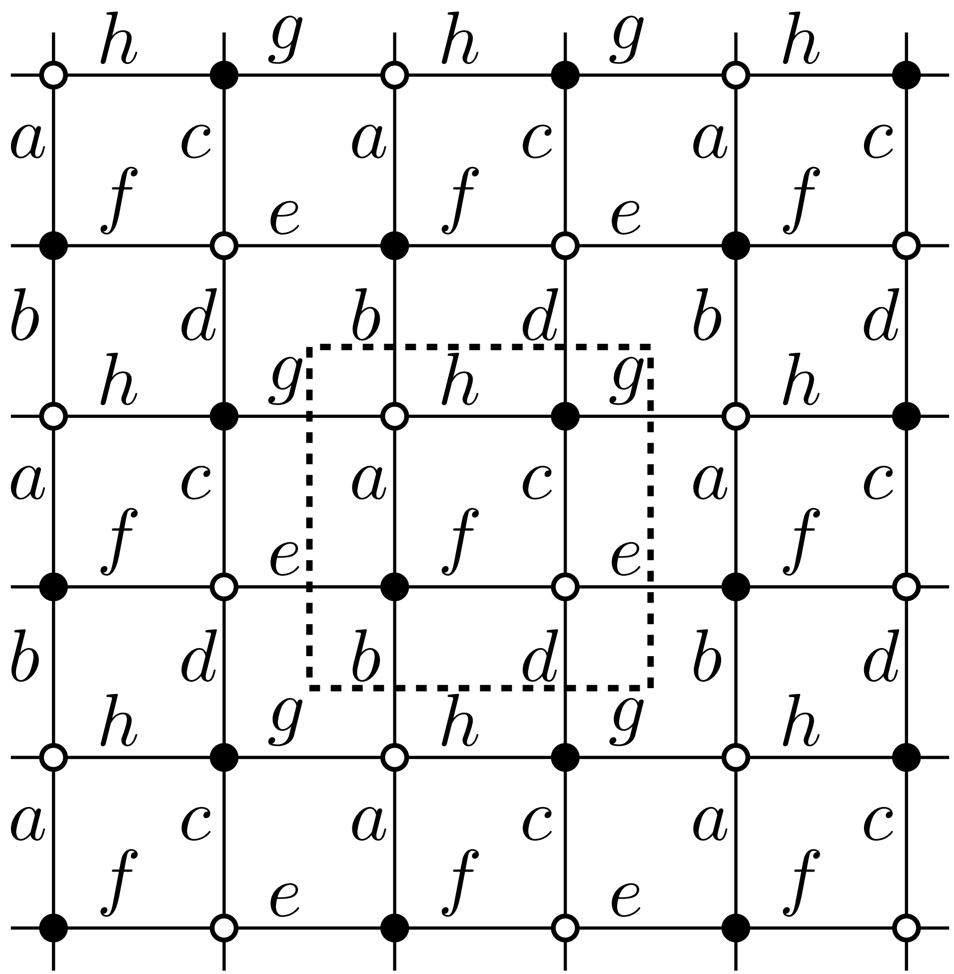


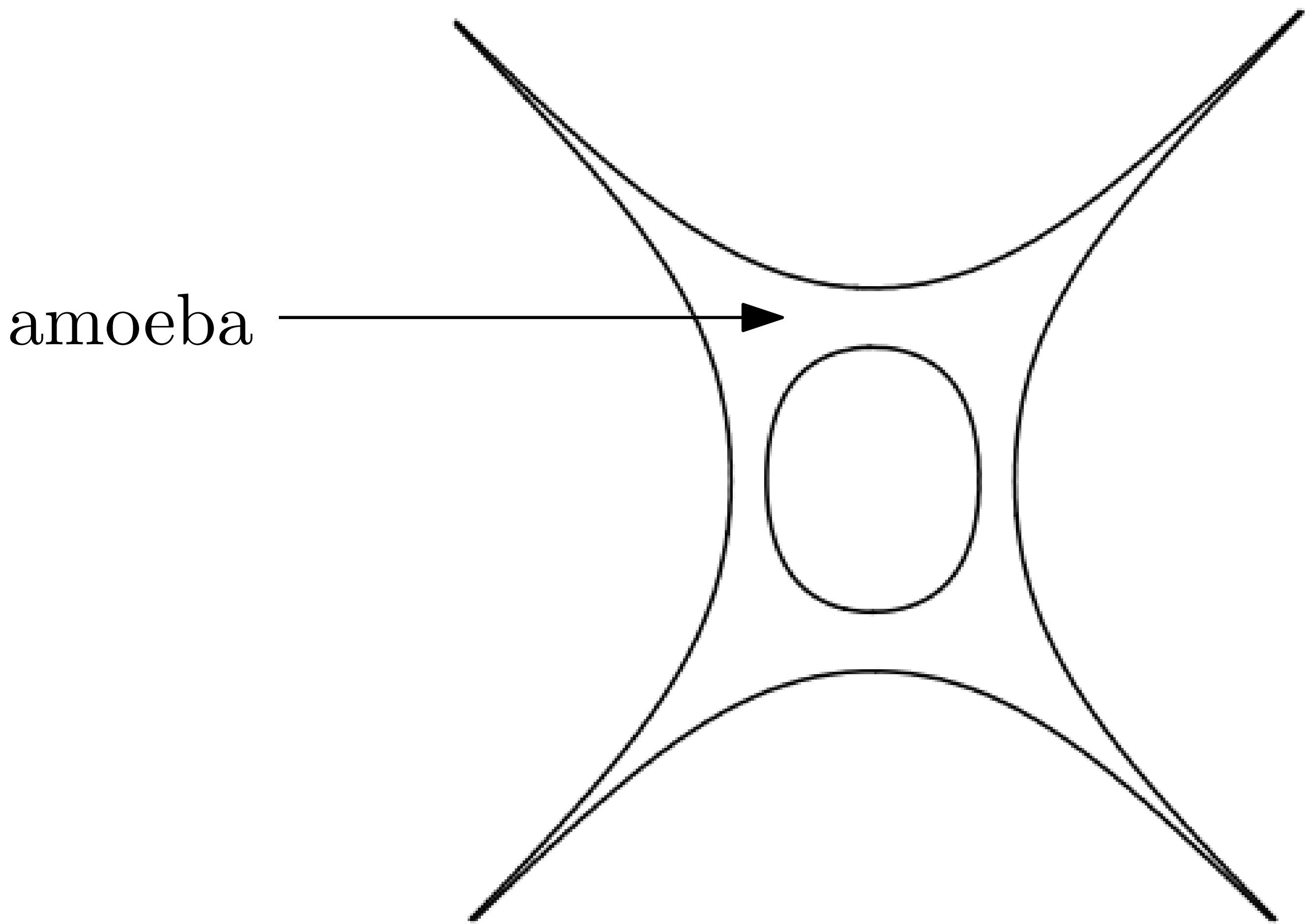
- Let G be an infinite periodic weighted graph.

- *Gibbs measure*: probability measure on the dimer coverings of G, whose restriction to finite subgraphs are Boltzmann measures induced by the edge weights.

- *Ergodic Gibbs measure*: not a convex combination of other Gibbs measures.

- *Liquid*: correlations decay polynomially.

- The interior points of the amoeba (log-log representation of the spectral curve of G) parametrize the liquid ergodic Gibbs measures on G (Kenyon-Okounkov-Sheffield).

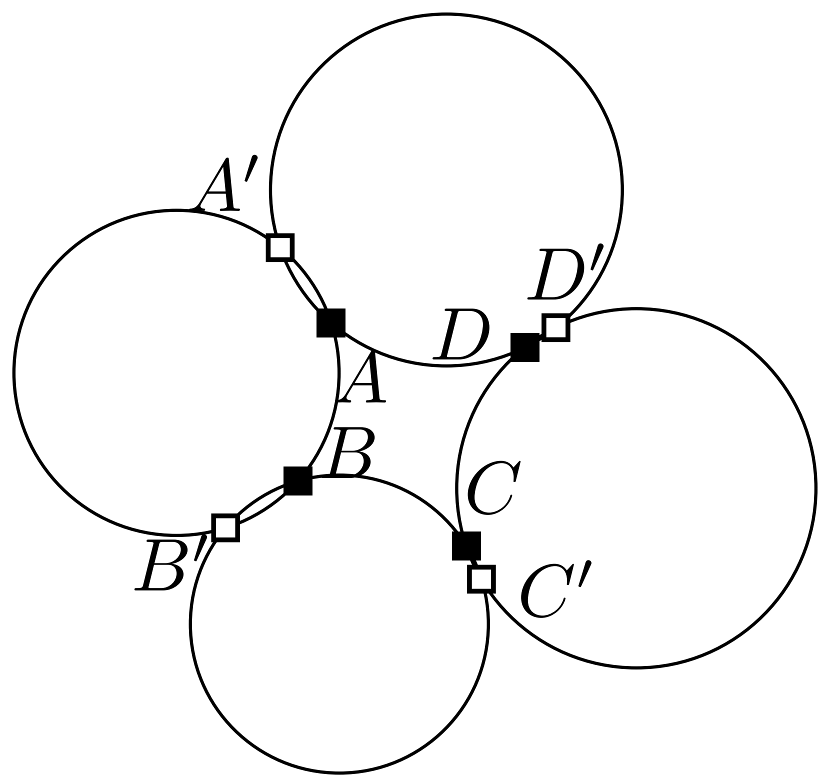

amoeba

Theorem (Kenyon-Lam-R.-Russkikh 2018). *Let G be an infinite weighted bipartite graph, periodic in two directions. Periodic circle center embeddings of the dual of G producing edge weights that are gauge equivalent to the original ones are in bijection with liquid ergodic Gibbs measures on G.*

- In both the finite and the infinite case, the construction of a circle center embedding associated with a weighted planar graph G depends globally (not locally) on G.

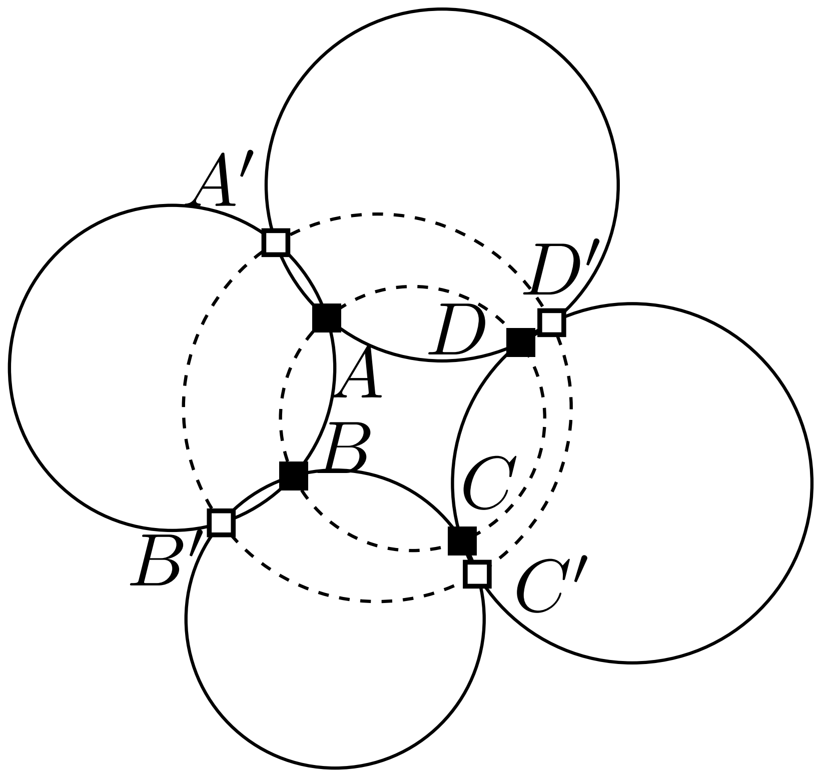
4 Motivation and perspectives

Miquel's theorem



Theorem (Miquel, 1838). *In this setting,*
A, B, C, D *concyclic* $\Leftrightarrow$ A', B', C', D' *concyclic.*

Miquel's theorem



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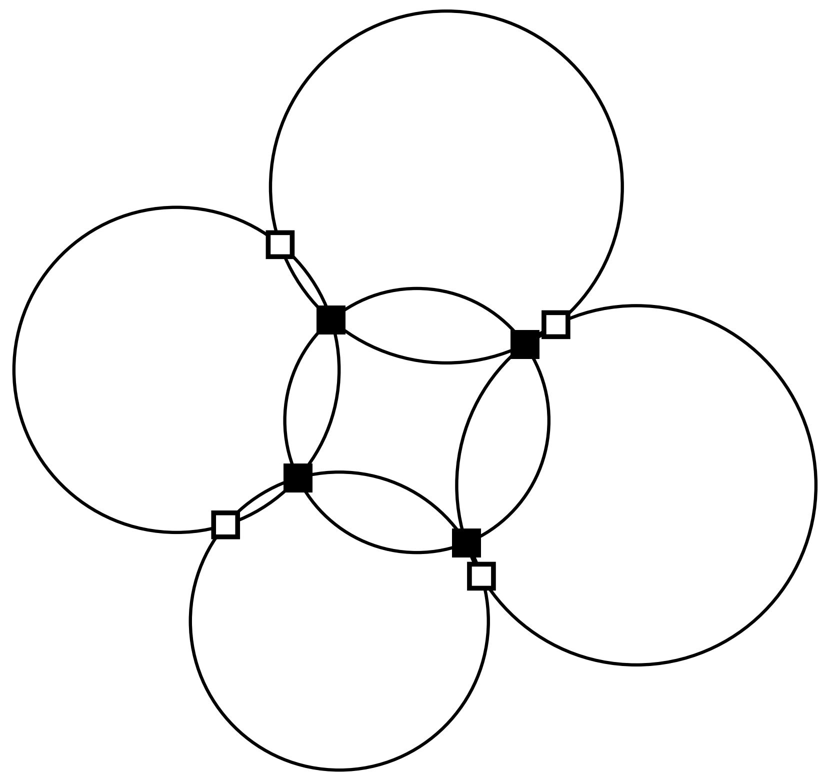

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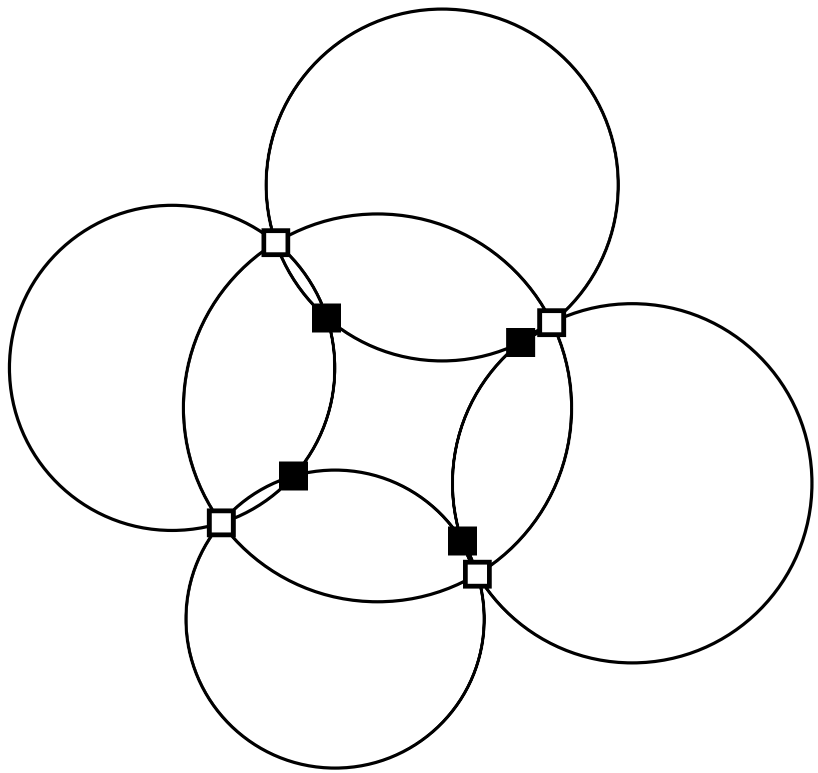
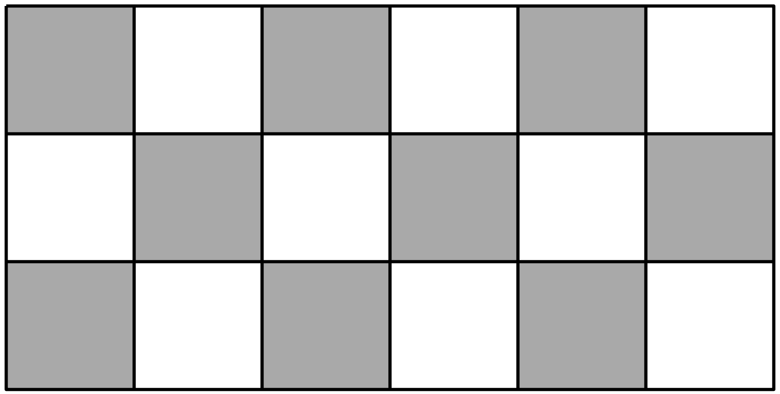


Theorem (Miquel, 1838)**.** *In this setting,* A, B, C, D *concyclic* $\Leftrightarrow$ A', B', C', D' *concyclic.*

Miquel dynamics

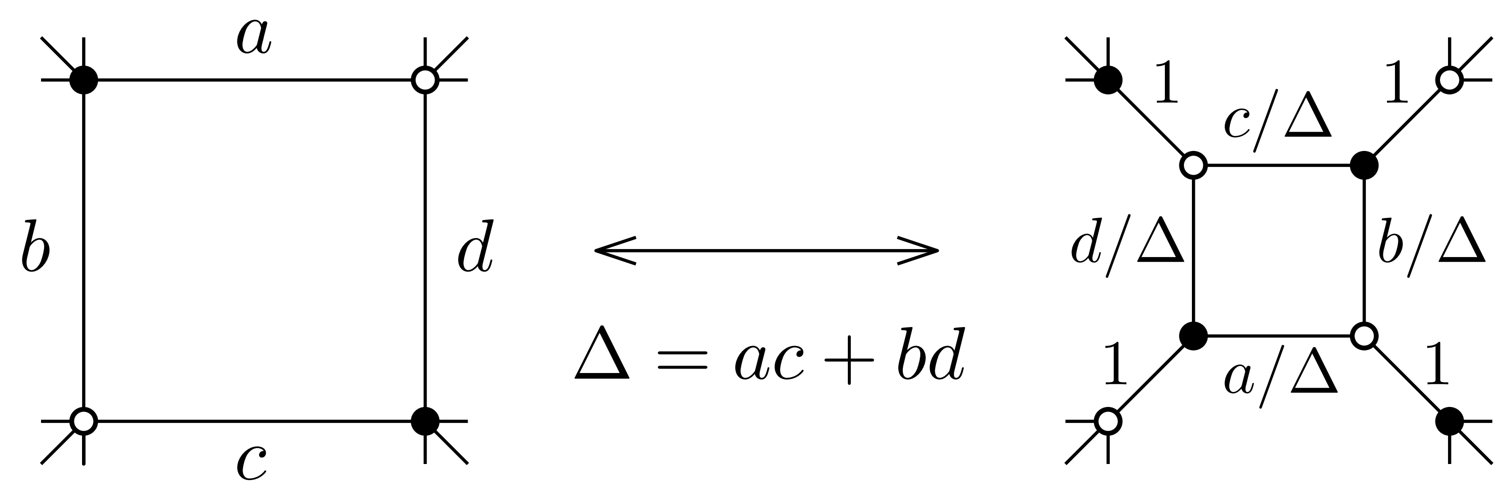
- Discrete-time dynamics on the space of square-grid circle patterns: alternate Miquel moves on all the white faces then on all the black faces.
- Invented by Kenyon, first properties studied in [R. 2018] and [Glutsyuk-R. 2018].



[Mathematica]

Dimer local move

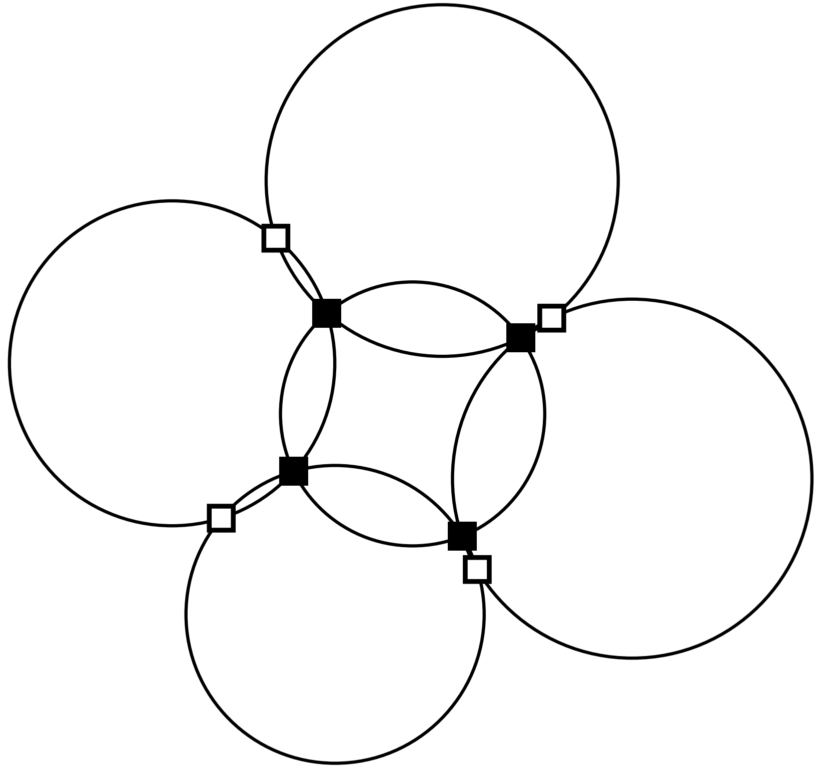
- Urban renewal:



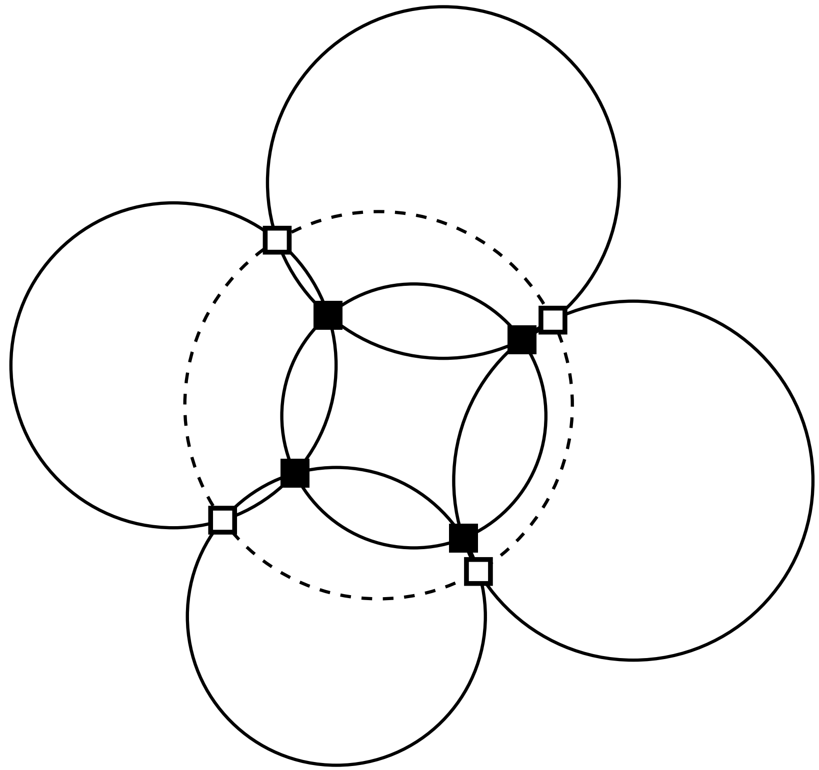
Theorem (Affolter 2018, Kenyon-Lam-R.-Russkikh, 2018). *The Miquel move for circle patterns commutes with the urban renewal for dimer models.*

- The identification with the Goncharov-Kenyon dimer dynamics yields the integrability of Miquel dynamics.

Miquel's theorem revisited



Miquel's theorem revisited



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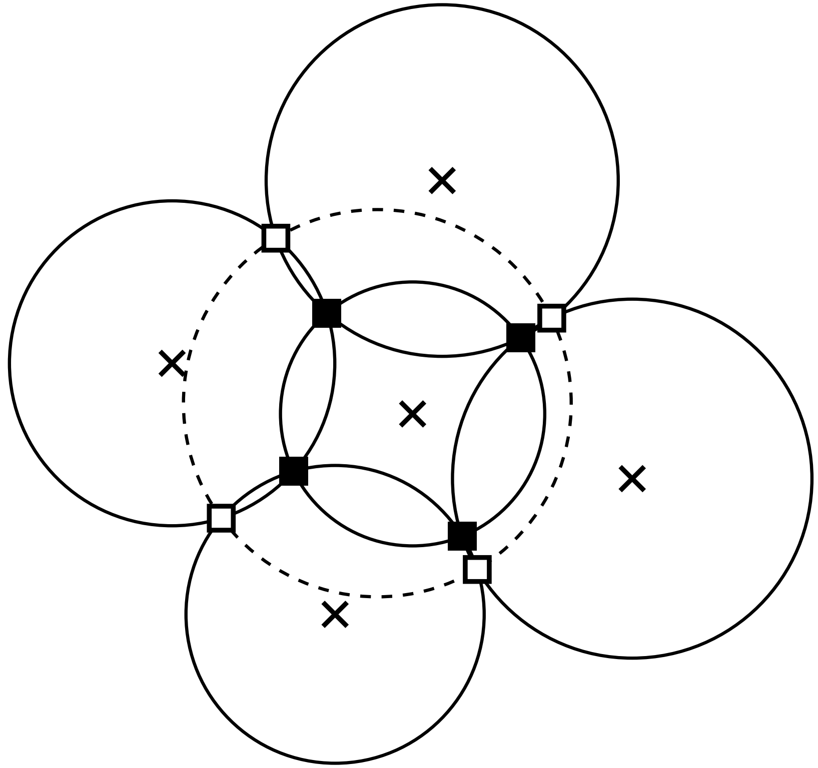

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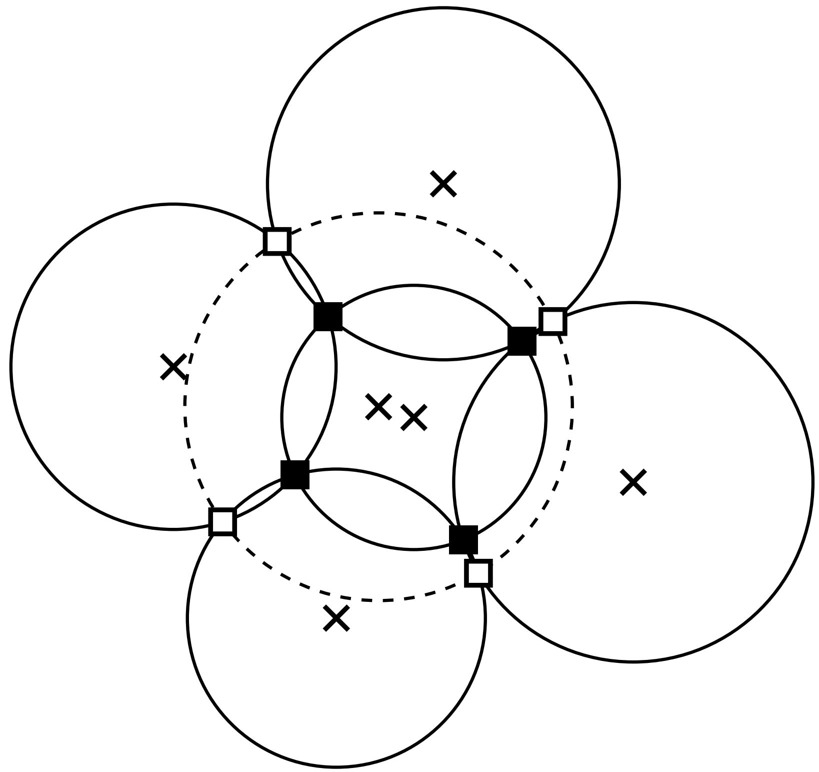
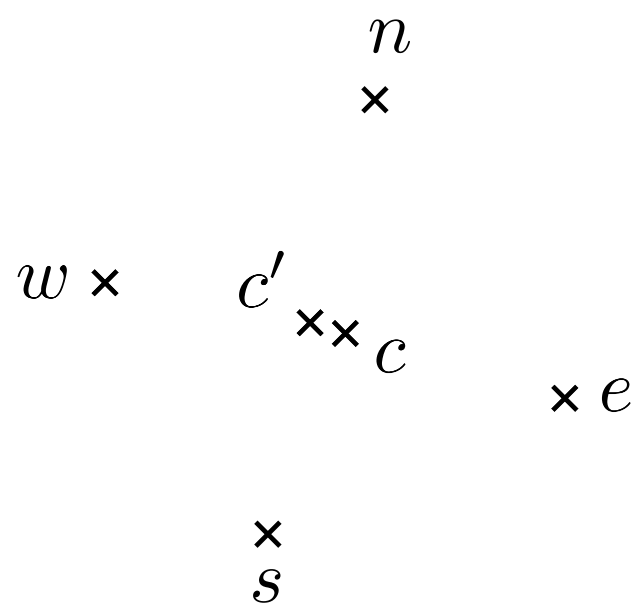


Miquel's theorem revisited



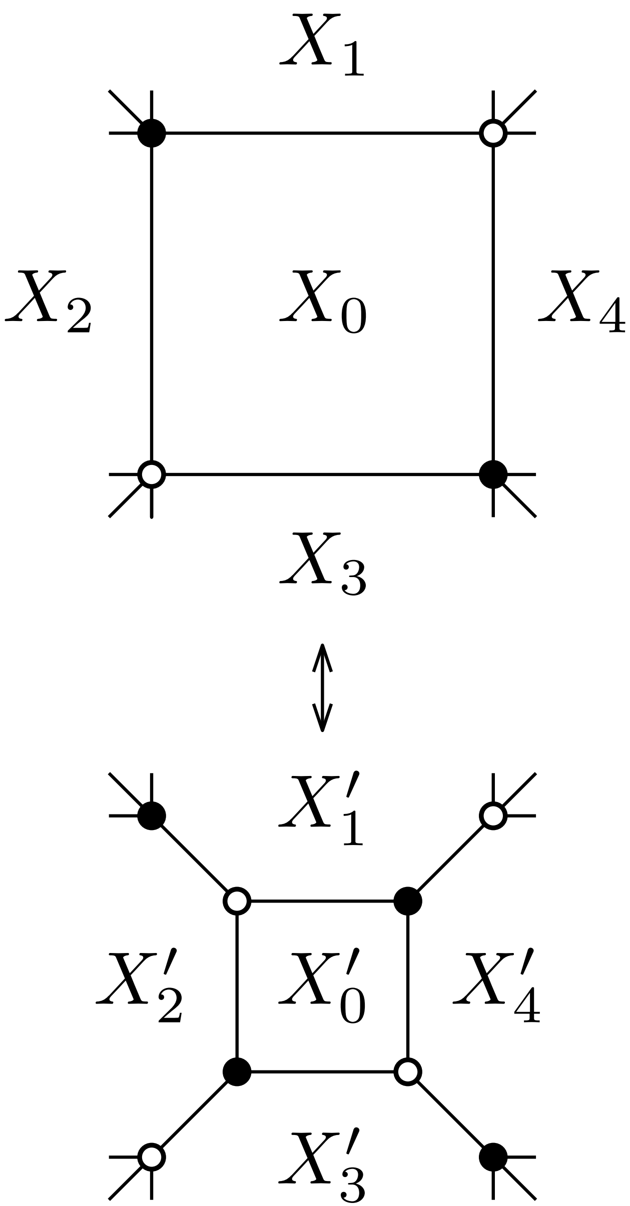
$n, w, s, e, c, c' \in \mathbb{C}$

Theorem (Affolter 2018, Kenyon-Lam-R.-Russkikh 2018)**.**

$$\frac{(c-w)(s-c')(e-n)}{(w-s)(c'-e)(n-c)} = -1$$

Discrete Schwarzian KP equation

Miquel move and cluster algebras



X variables evolution for a Miquel move:

$$X'_0 = X_0^{-1}$$
$$X'_1 = X_1(1 + X_0)$$
$$X'_2 = X_2/(1 + X_0^{-1})$$
$$X'_3 = X_3(1 + X_0)$$
$$X'_4 = X_4/(1 + X_0^{-1})$$

Embeddings in statistical mechanics

- Consider an infinite planar graph periodic in two directions on which we study a statistical mechanical model (random walk, dimers, Ising,...).

- Find an embedding of it such that universal conformally invariant objects appear in the scaling limit.

- Same issue for formulating the convergence to Liouville quantum gravity of random planar maps decorated with some statistical mechanical model.

Embeddings in statistical mechanics

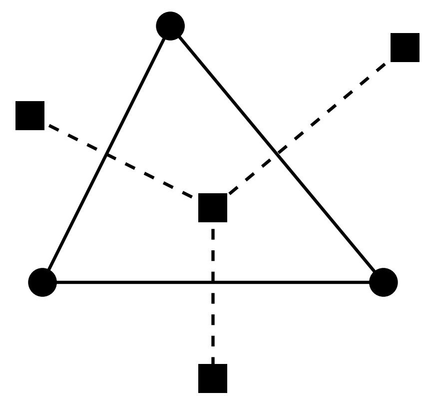
Theorem (Kenyon-Lam-R.-Russkikh, 2018)**.** *Circle center embeddings for dimers generalize the Tutte embedding adapted to spanning trees and the s-embeddings adapted to the Ising model.*

Tutte embeddings and circle centers

embedding of
$V(G) \cup F(G) \cup E(G)$

planar weighted graph G → Tutte embedding of G

Temperley bijection ↓ ↓



planar bipartite weighted graph G' → circle centers for G'

$V(G') \leftrightarrow$
$V(G) \cup F(G) \cup E(G)$

Tutte embeddings and circle centers

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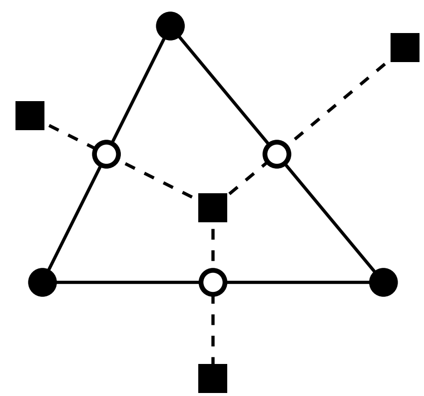

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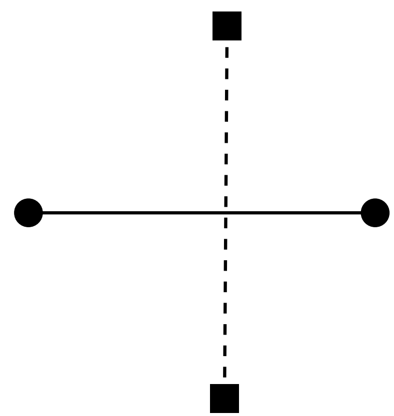
s-embeddings and circle centers

(Chelkak 2017)

embedding of
$V(G) \cup F(G) \cup E(G)$

planar weighted graph G → s-embedding of G

Wu-Lin
Dubédat



planar bipartite weighted graph $\widetilde{G}$ → circle centers for $\widetilde{G}$

$F(\widetilde{G}) \leftrightarrow$
$V(G) \cup F(G) \cup E(G)$

s-embeddings and circle centers

(Chelkak 2017)

embedding of
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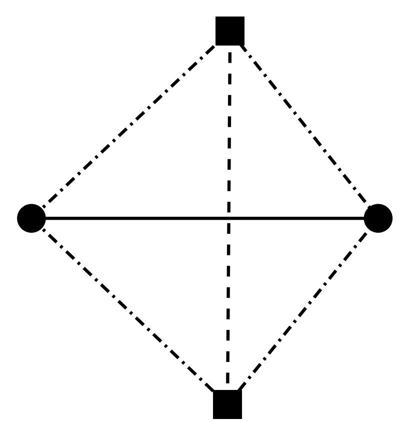
planar weighted graph G → s-embedding of G

Wu-Lin
Dubédat

↓ ↓



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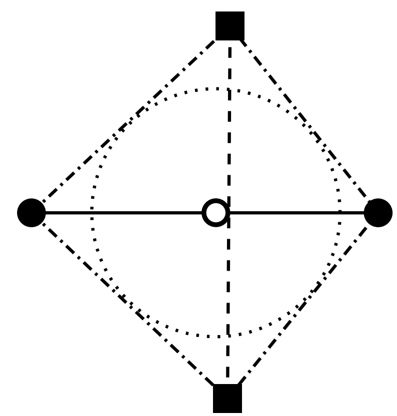

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$F(\widetilde{G}) \leftrightarrow$
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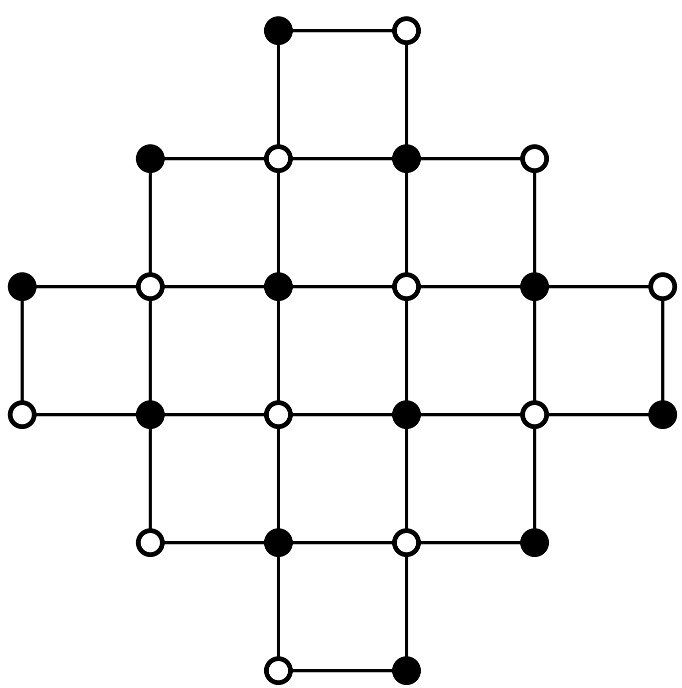
What to expect ?

- Local formulas for the correlations and the free energy (Kenyon, Boutillier, de Tilière, Raschel).

- Limit shape results.



Aztec diamond of size 3

What to expect ?

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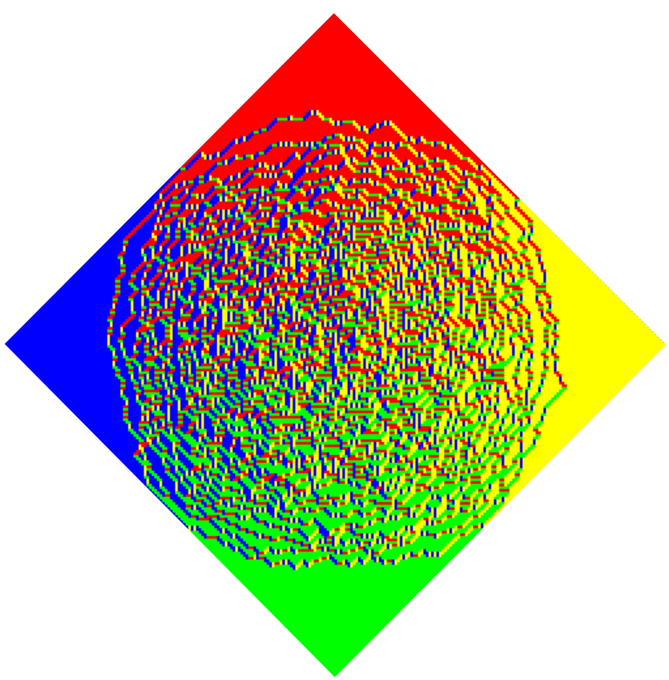

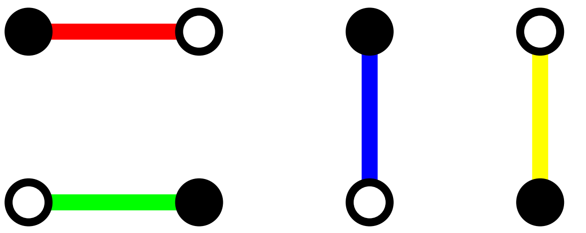

picture by Cris Moore

tuvalu.santafe.edu/~moore/aztec256.gif

What to expect ?

- Local formulas for the correlations and the free energy (Kenyon, Boutillier, de Tilière, Raschel).

- Limit shape results.

- Notion of a discrete holomorphic function on such embeddings and conditions for converging to a continuous holomorphic function (Chelkak-Laslier-Russkikh, 2019).

THANK YOU !