Cube moves for s-embeddings and alpha-embeddings

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Joint work with:

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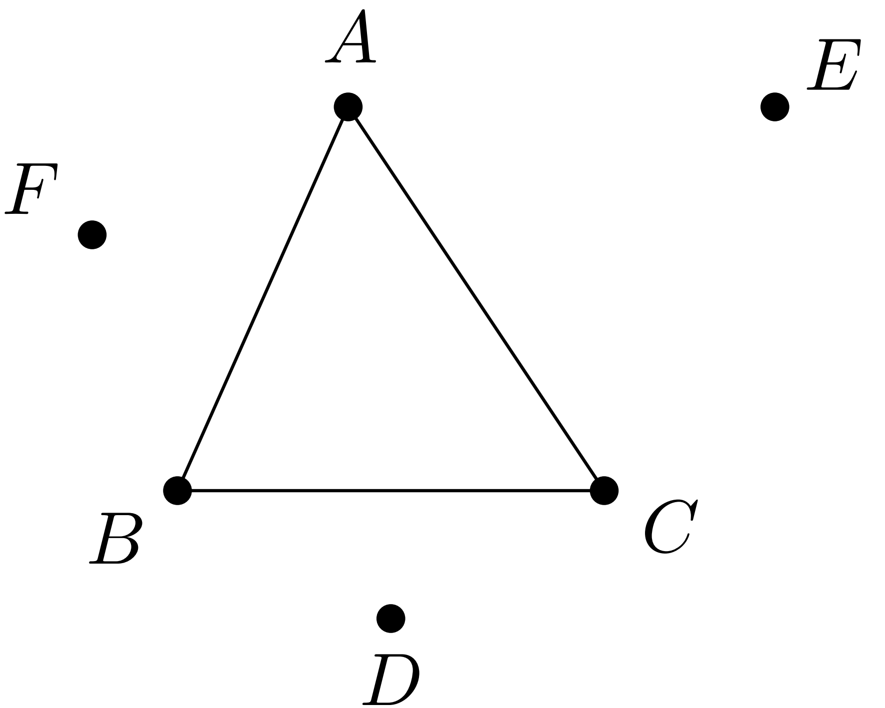
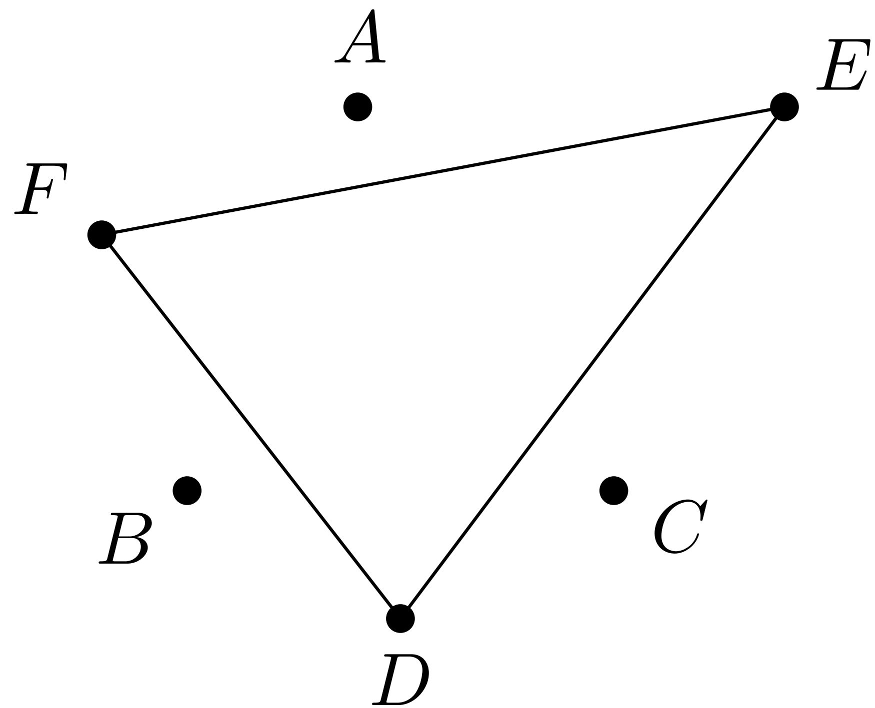
Session on Probability and Combinatorics, March 21 2021

1 Steiner, Kennelly and Tutte

Theorem (Steiner). *The perpendiculars to* (AB), (BC) *and* (AC) *going respectively through* F, D *and* E *have a common point of intersection iff the perpendiculars to* (DE), (EF) *and* (ED) *going respectively through* C, A *and* B *have a common point of intersection.*





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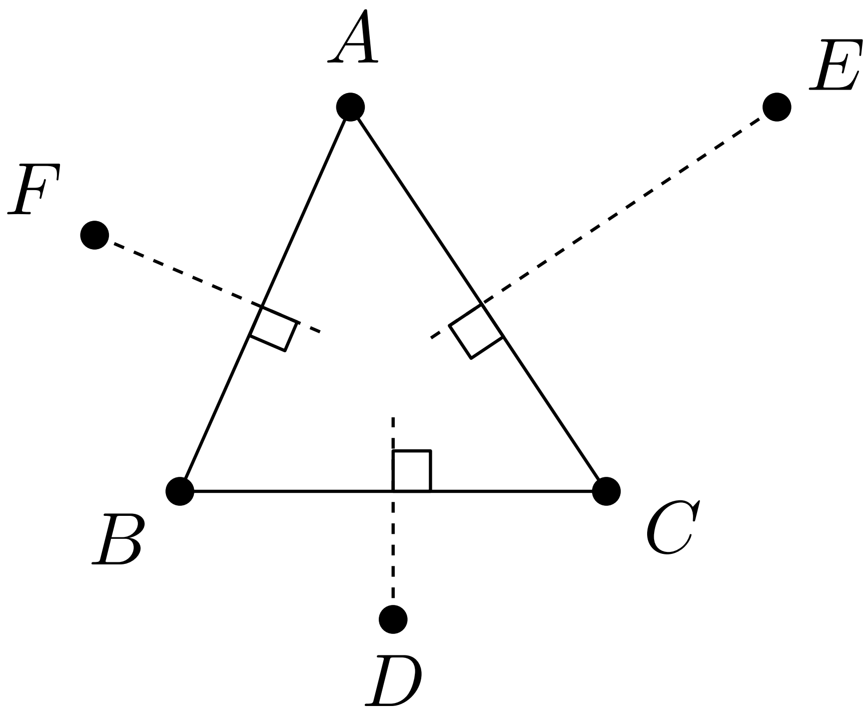

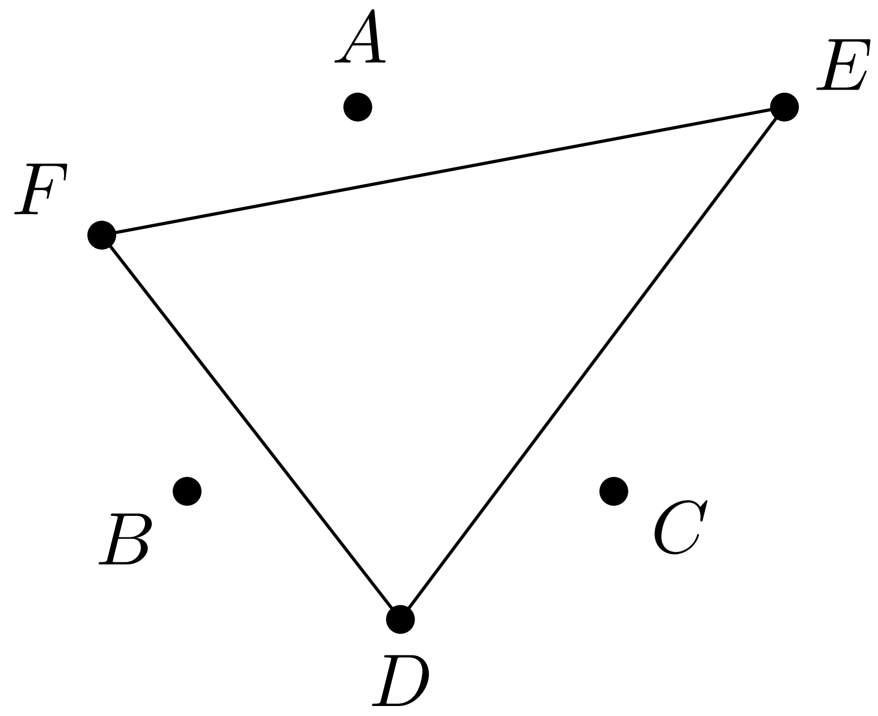

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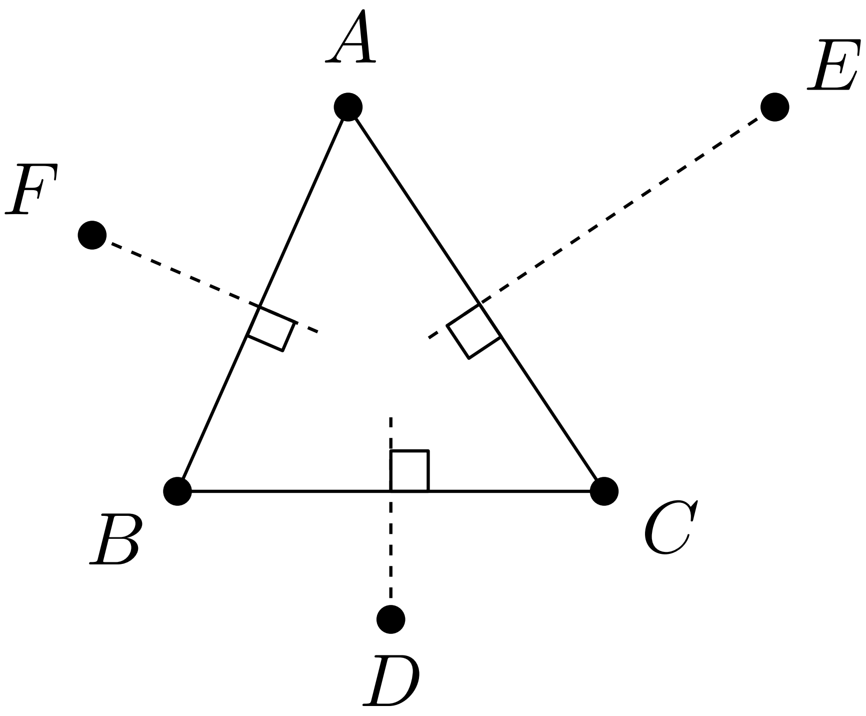

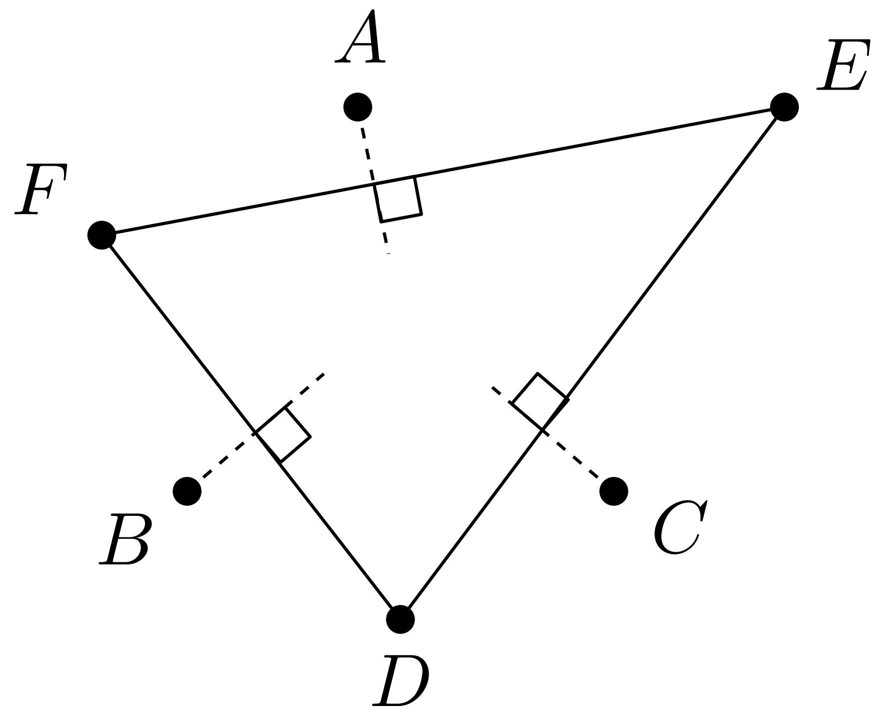

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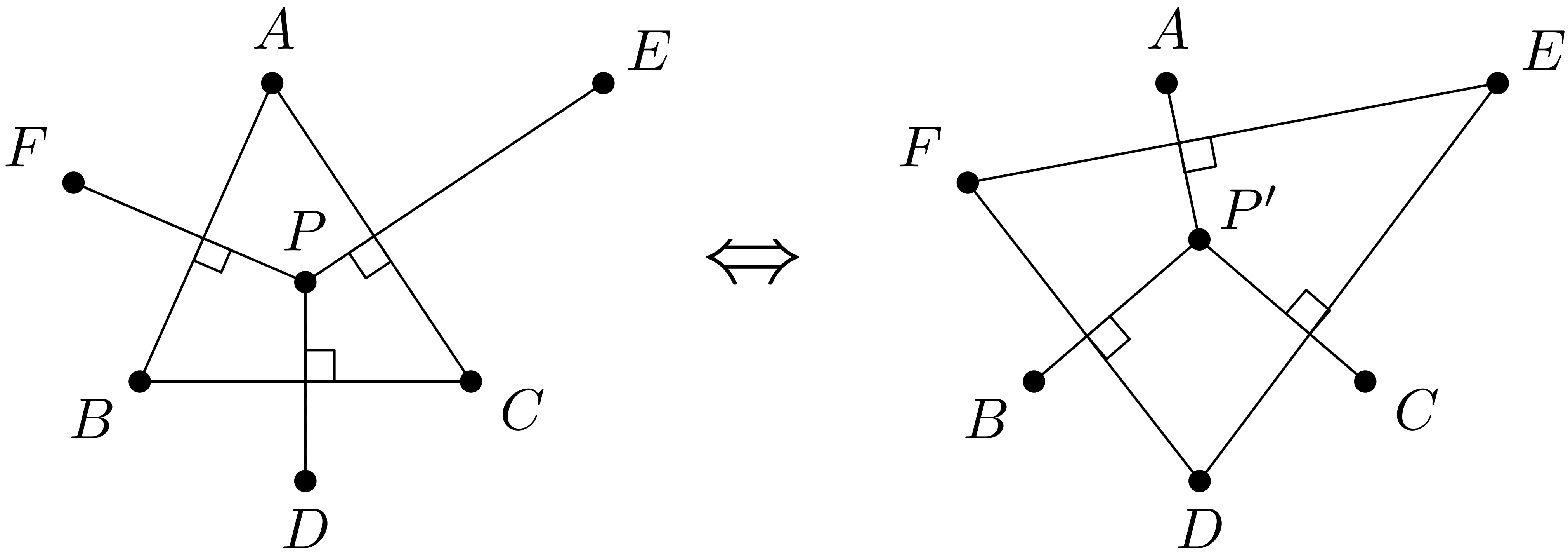
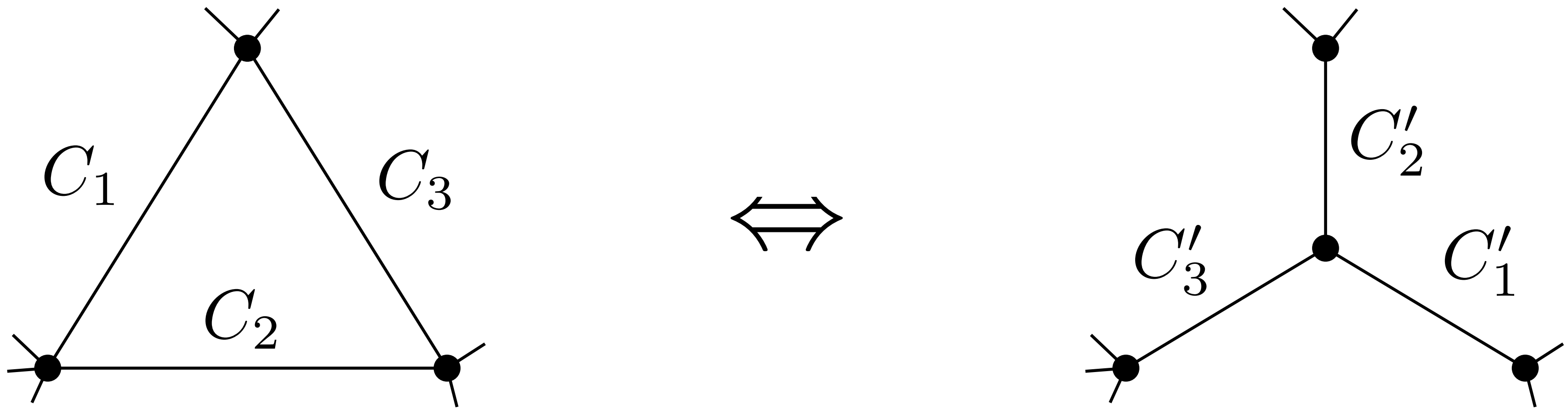


Theorem (Kennelly, 1899). *Resistor networks are equivalent under the Δ-Y transformation.*



$$C_1 = \frac{C'_2 C'_3}{C'_1 + C'_2 + C'_3} \qquad C'_1 = \frac{C_1C_2 + C_2C_3 + C_1C_3}{C_1}$$

$$C_2 = \frac{C'_1 C'_3}{C'_1 + C'_2 + C'_3} \qquad C'_2 = \frac{C_1C_2 + C_2C_3 + C_1C_3}{C_2}$$

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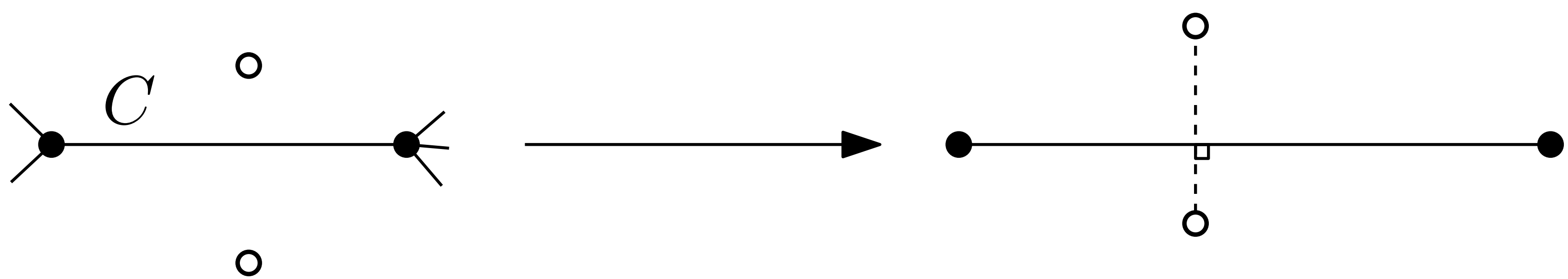
- Random walks on weighted graphs are invariant under Kennelly's Δ-Y transformation.

- The harmonic embedding (Tutte 1963) of a planar graph G makes the random walk on G a martingale.

- Embedding of G and its dual such that pairs of a primal and its dual edge are orthogonal and satisfy:



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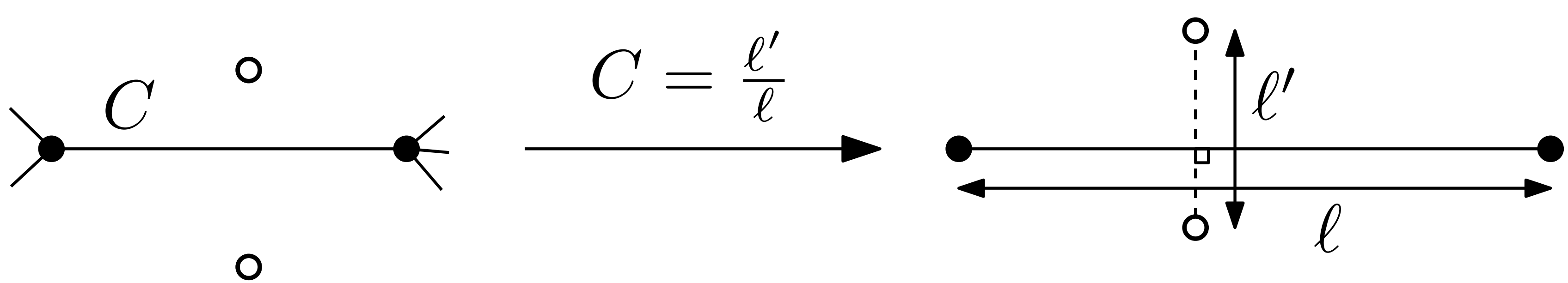
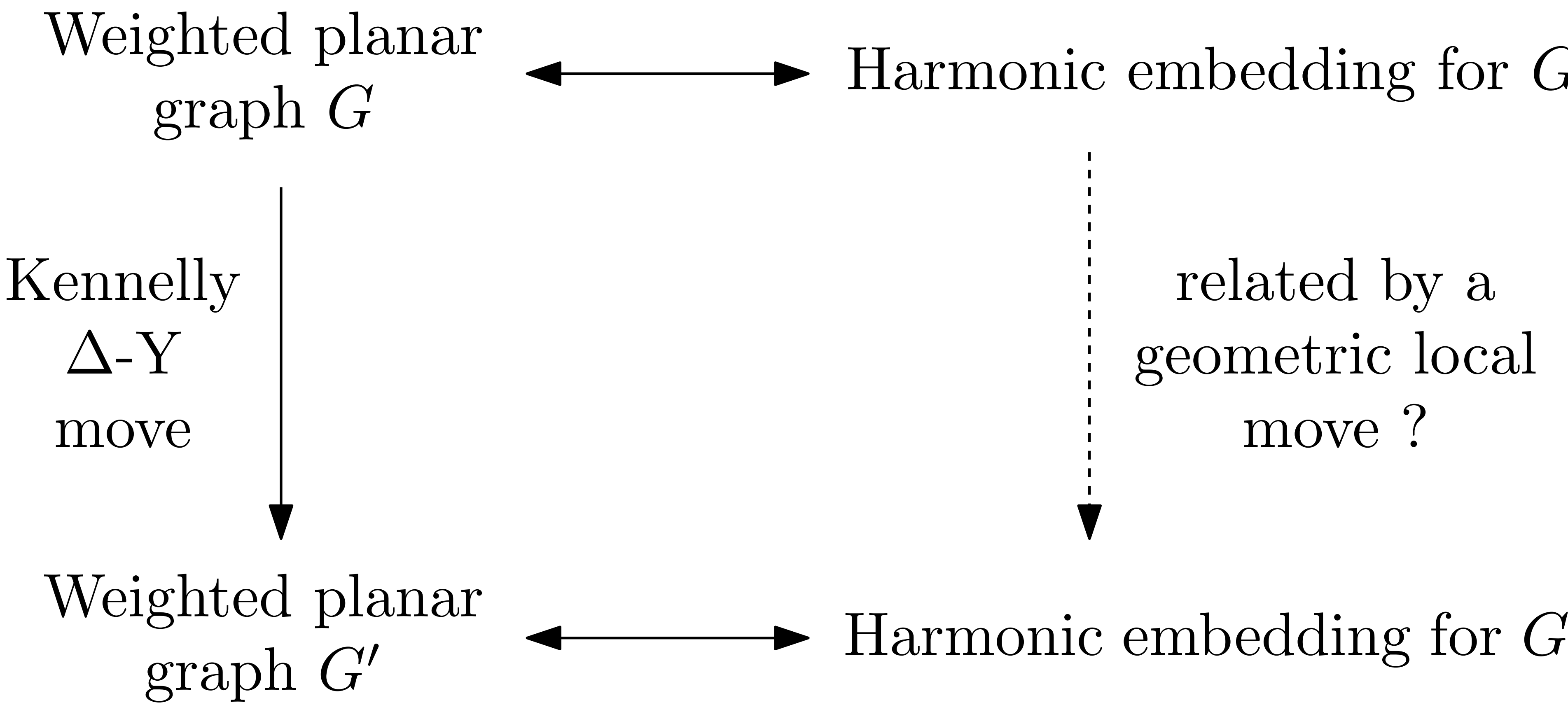
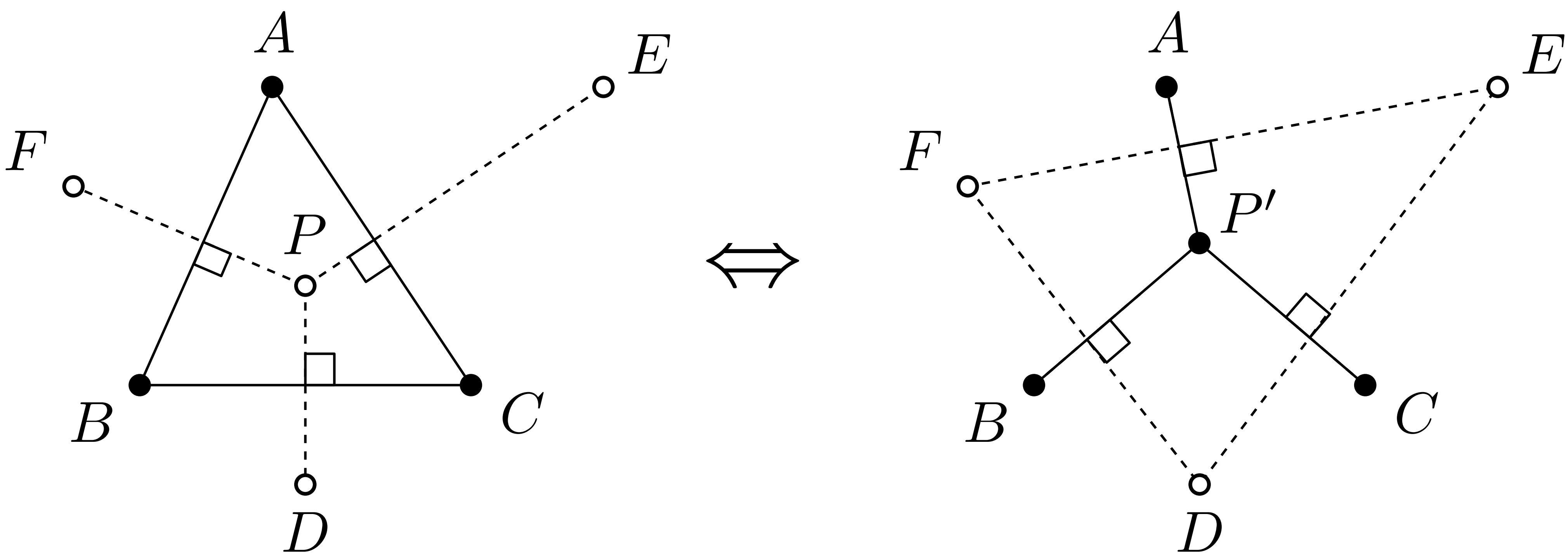


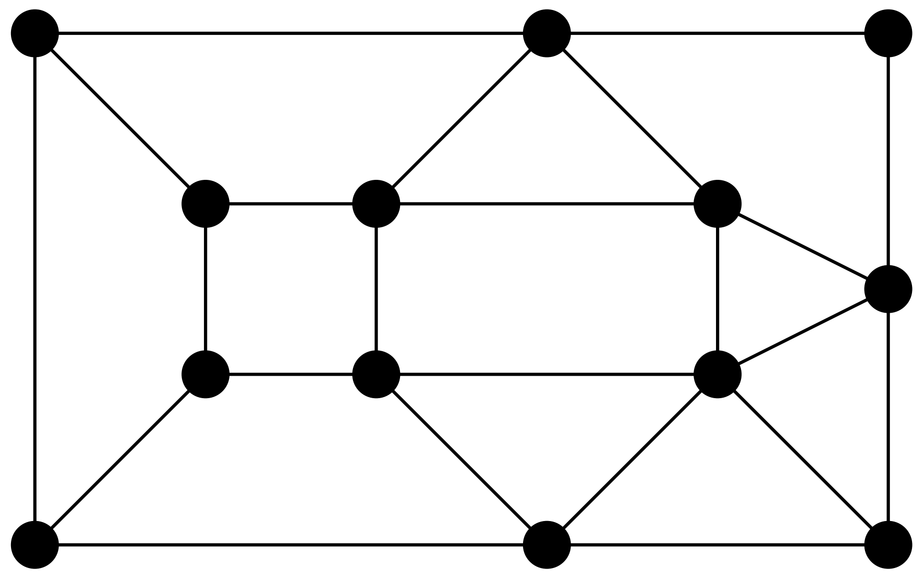


Weighted planar graph G
Harmonic embedding for G
Kennelly Δ-Y move
related by a geometric local move ?
Weighted planar graph G'
Harmonic embedding for G'

Theorem (Konopelchenko-Schief, 2002). *Kennelly's* Δ*-*Y *transformation applied to Tutte's harmonic embeddings is possible thanks to Steiner's theorem.*



- Diamond graph $G^\diamond$ of a planar graph G:



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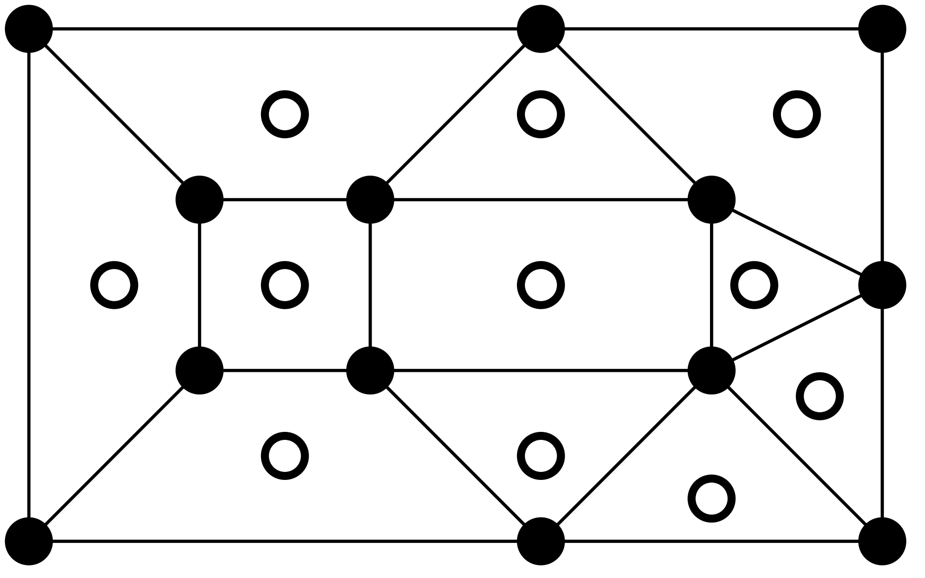

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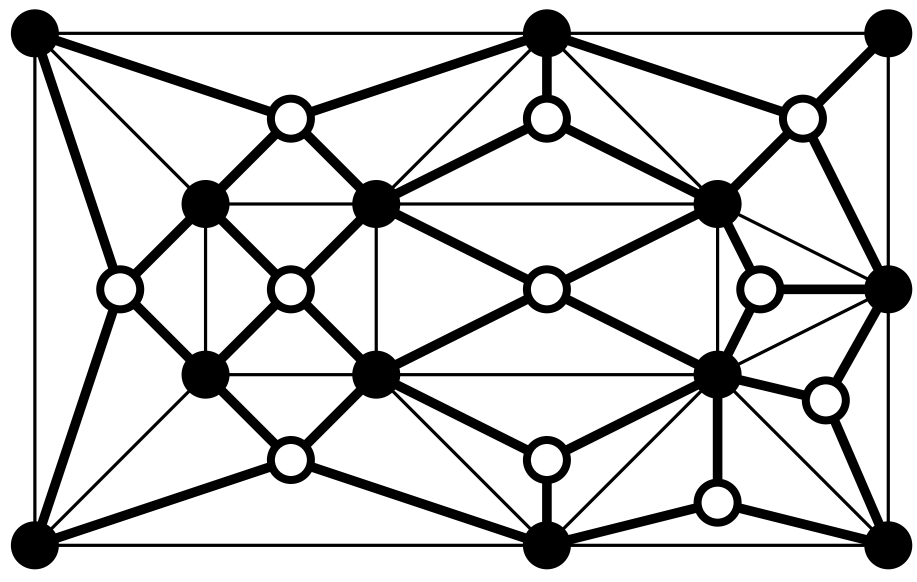

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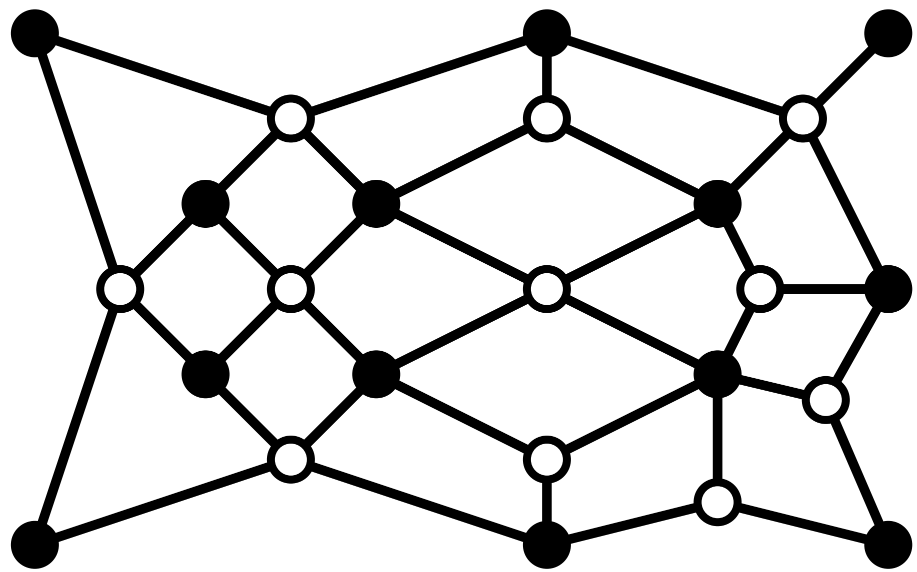
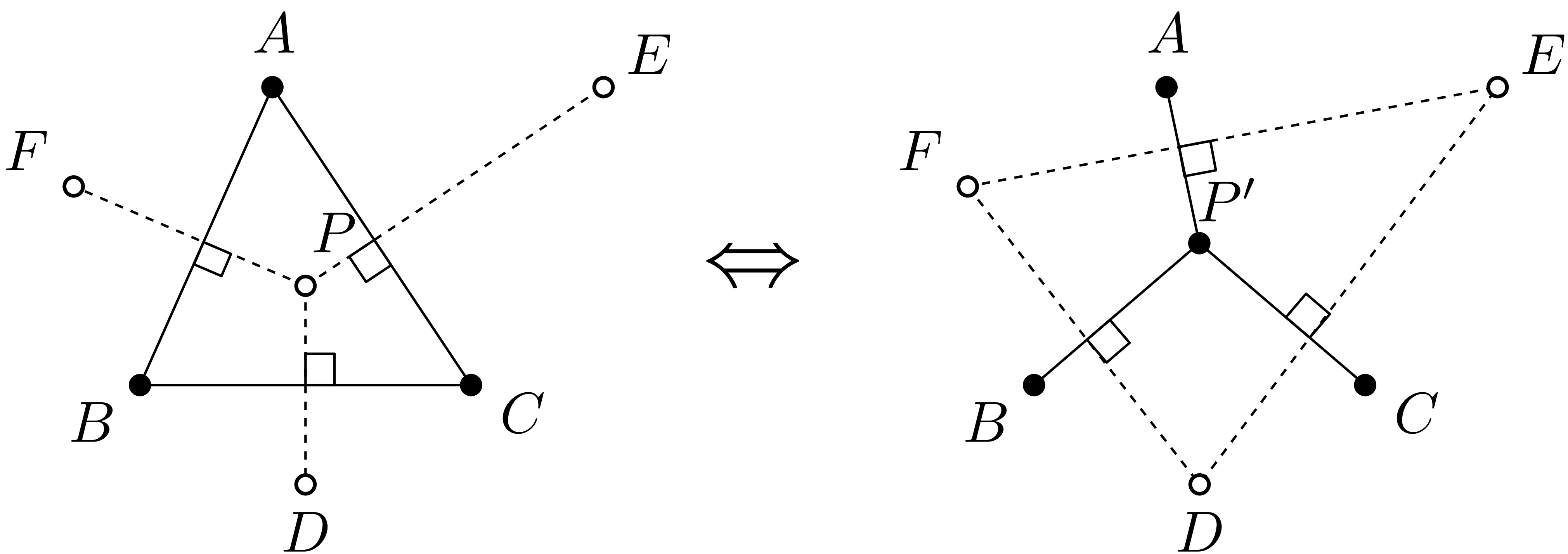


- A harmonic embedding of G is an embedding of $G^\diamond$ where all the faces are orthodiagonal quads.

- ABCD orthodiagonal quad iff $AB^2 + CD^2 = BC^2 + AD^2$.

In terms of $G^\diamond$, Steiner's theorem becomes a *cube move*:

3 orthodiagonal quads $\longleftrightarrow$ 3 orthodiagonal quads



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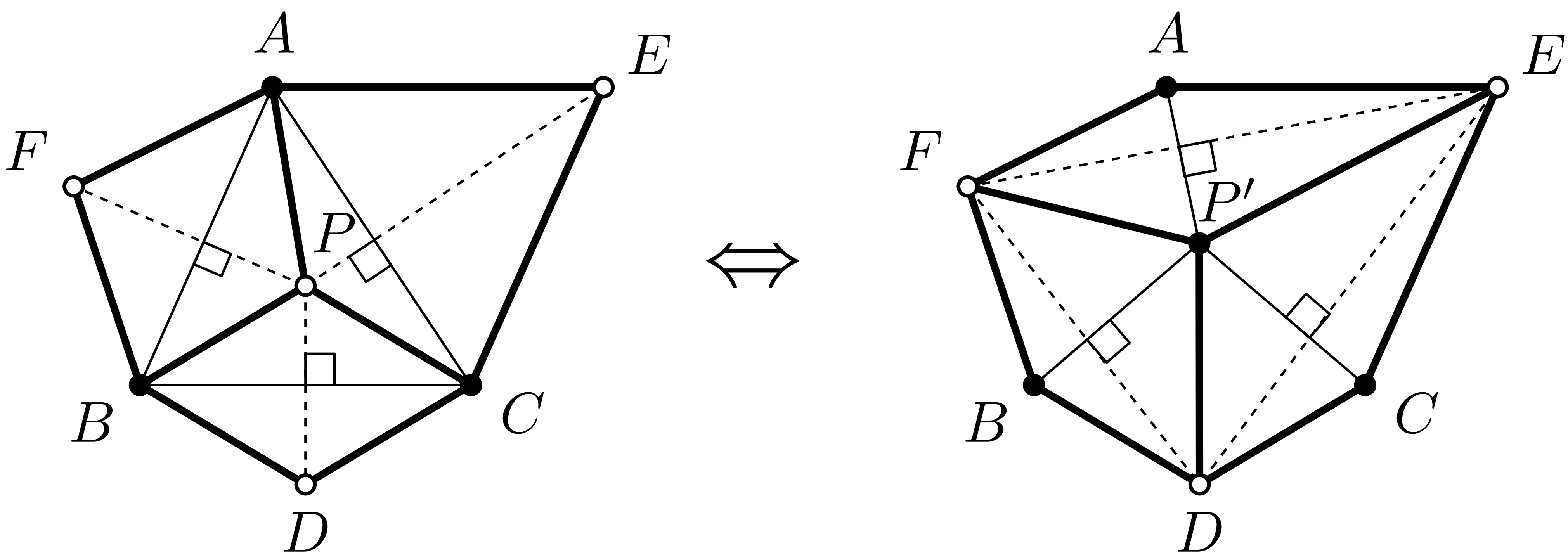

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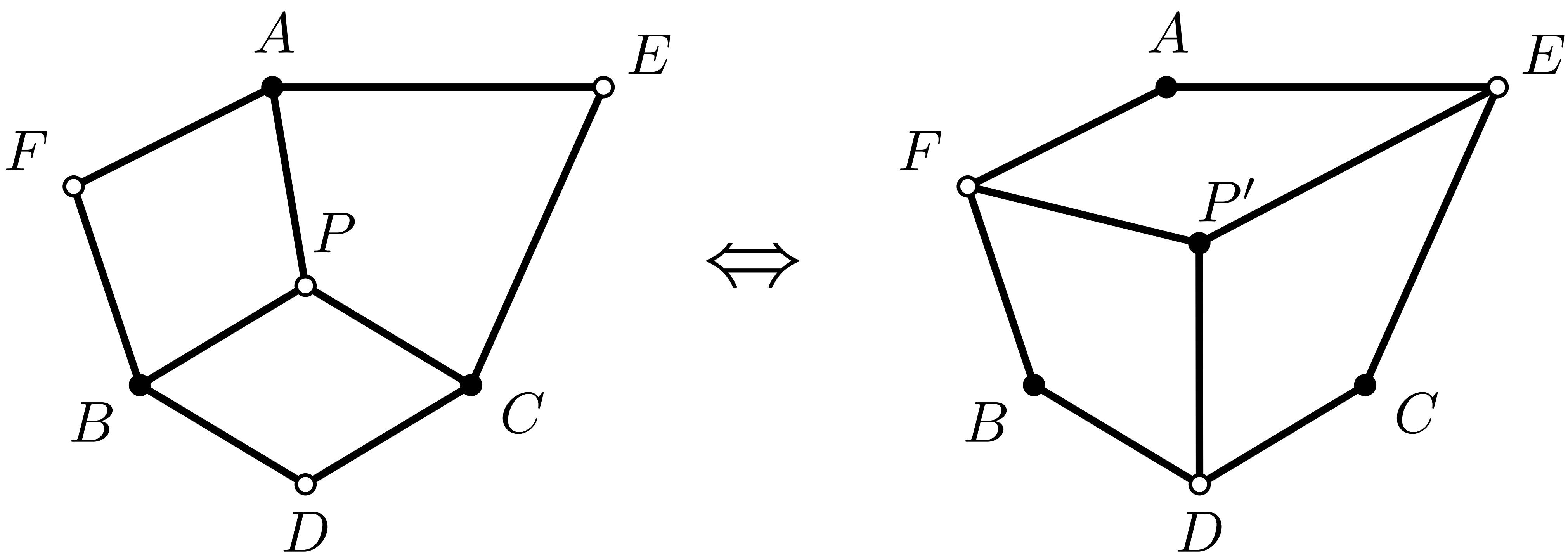
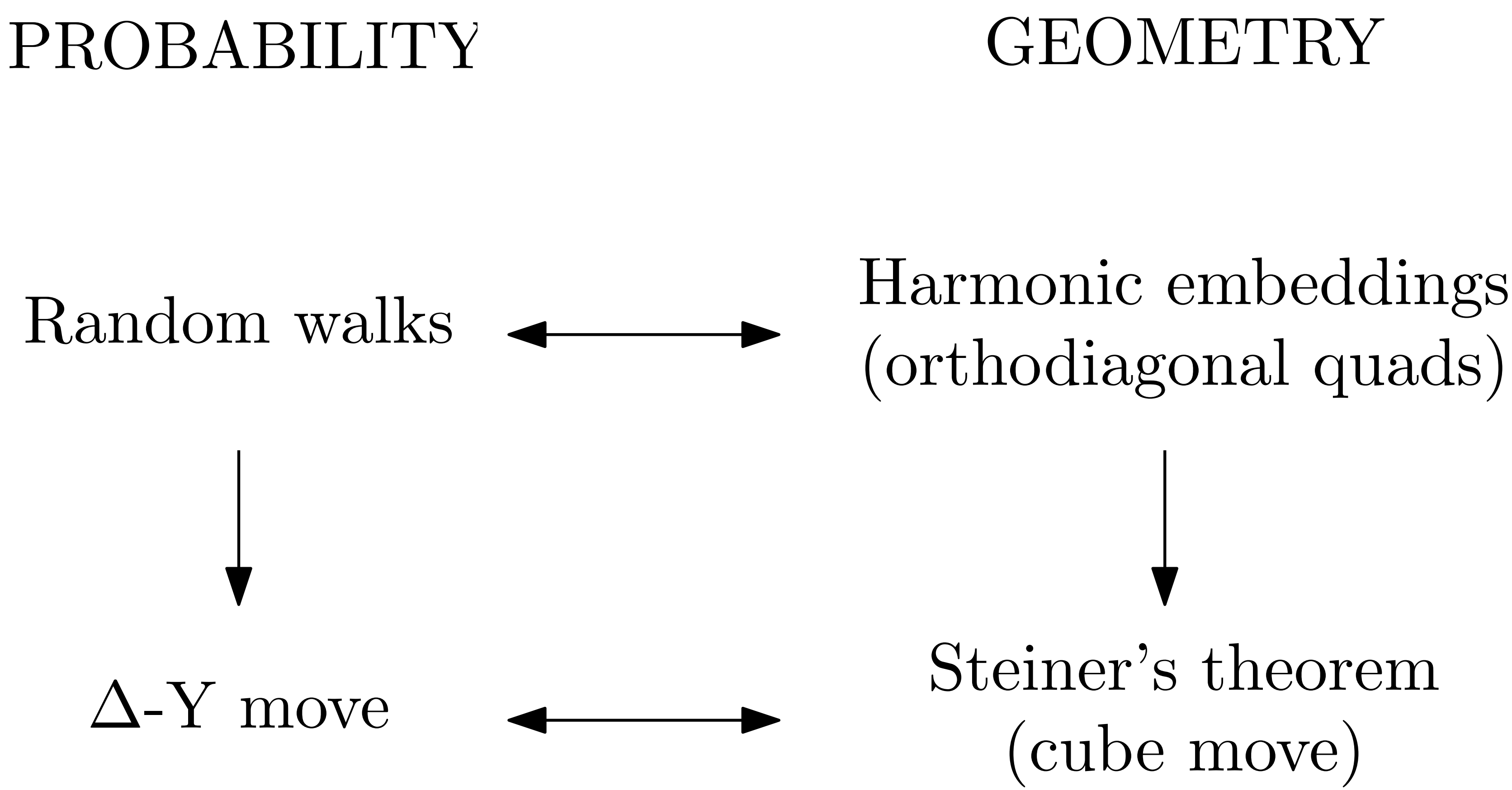



PROBABILITY
GEOMETRY
Random walks
Harmonic embeddings
(orthodiagonal quads)
Δ-Y move
Steiner's theorem
(cube move)

2 Δ-Y for the Ising model

The (ferromagnetic) Ising model

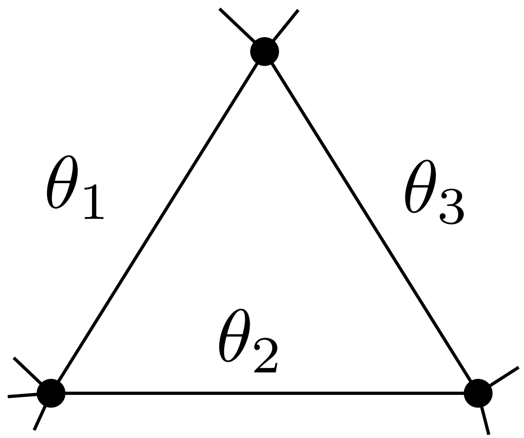
- Setting: graph $G = (V, E)$ equipped with coupling constants $J_{vv'} > 0$ attached to each edge $\{v, v'\} \in E$.

- Probability measure on functions $\sigma \in \{-1, +1\}^V$, where the probability of σ is proportional to

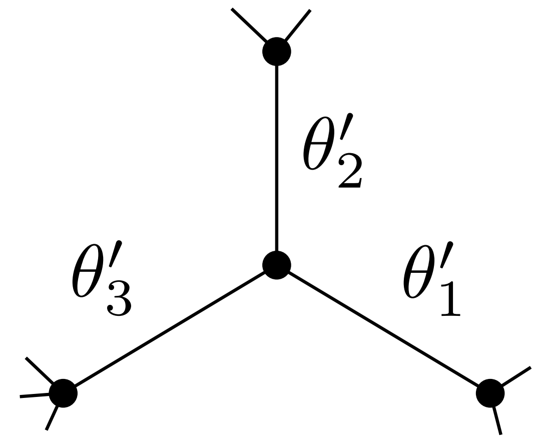
$$\exp\left(\sum_{e=\{v,v'\}\in E} J_e \sigma(v)\sigma(v') \right).$$

- Change of variables $J_e = \frac{1}{2} \log \frac{1+\sin\theta_e}{\cos\theta_e}$, with $\theta_e \in (0, \frac{\pi}{2})$.

Theorem (Wannier 1945, Melotti-R.-Thévenin 2020). *Ising models are probabilistically equivalent under the Δ-Y transformation.*



$\Leftrightarrow$



$$\sin\theta_1 = \frac{\cos\theta'_1 \sin\theta'_2 \sin\theta'_3}{\cos\theta'_1 + \cos\theta'_2 \cos\theta'_3}$$

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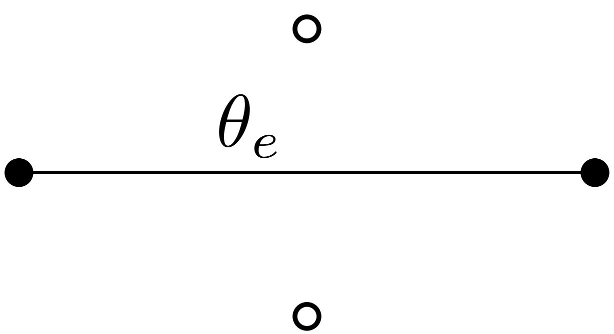
s-embeddings (Chelkak 2017)

- An s-embedding of a planar graph G is an embedding of $G^\diamond$ such that each quad is tangential and:



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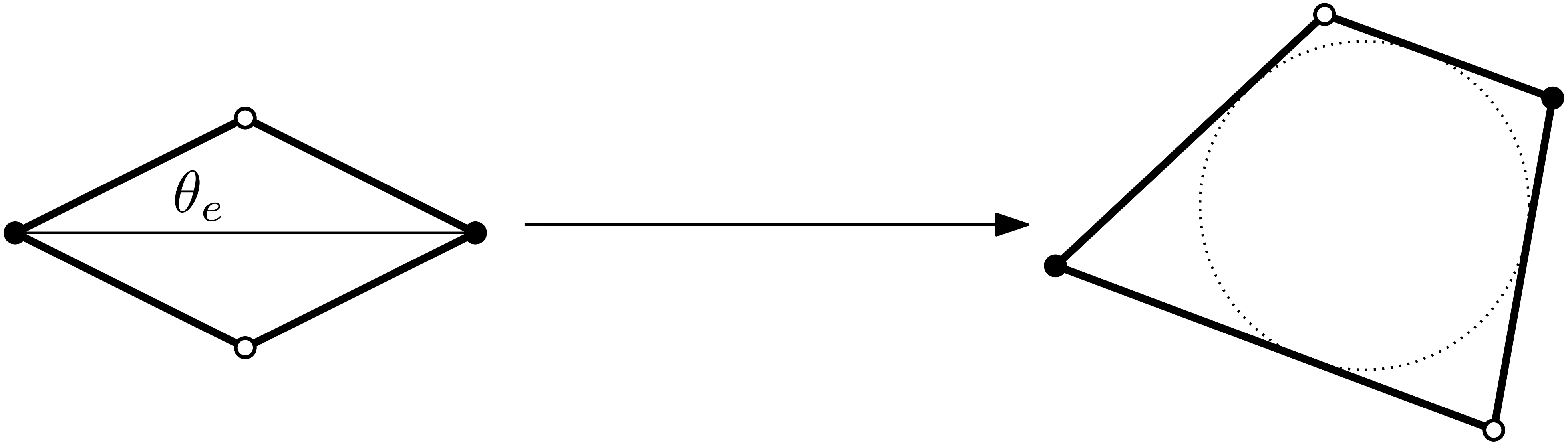

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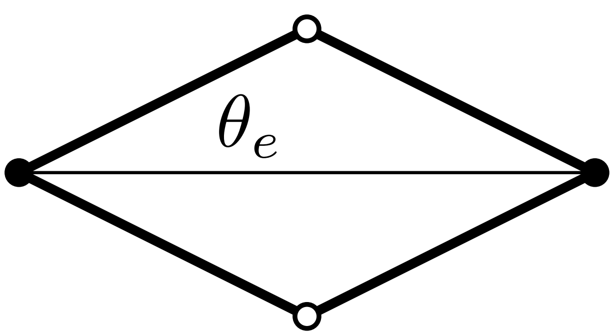
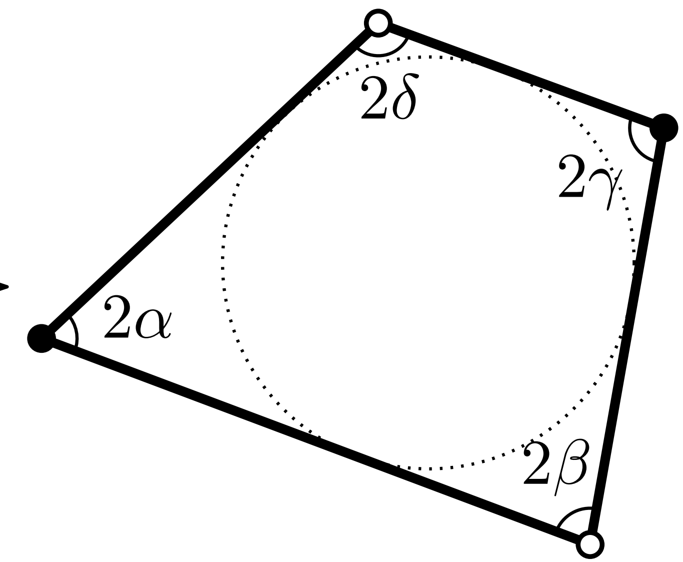
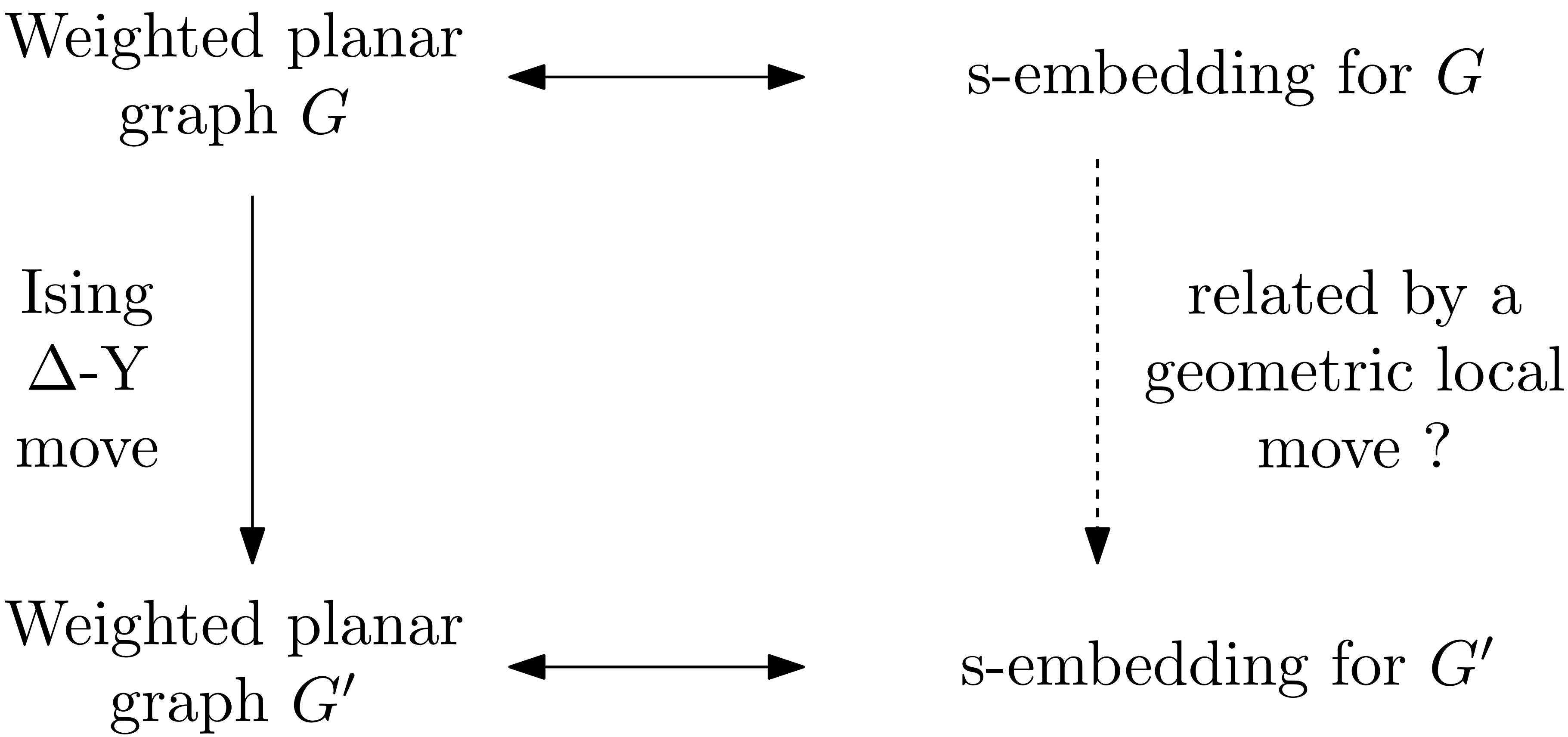


$$\tan^2 \theta_e = \frac{\cot \beta + \cot \delta}{\cot \alpha + \cot \gamma}$$

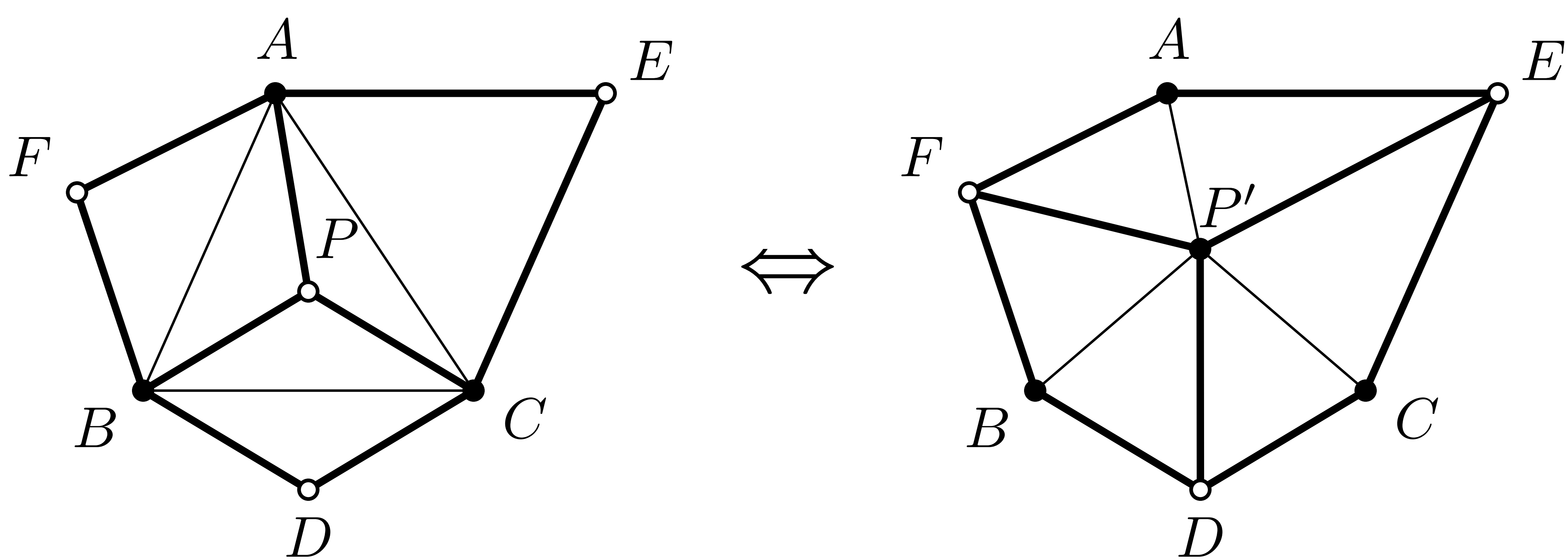


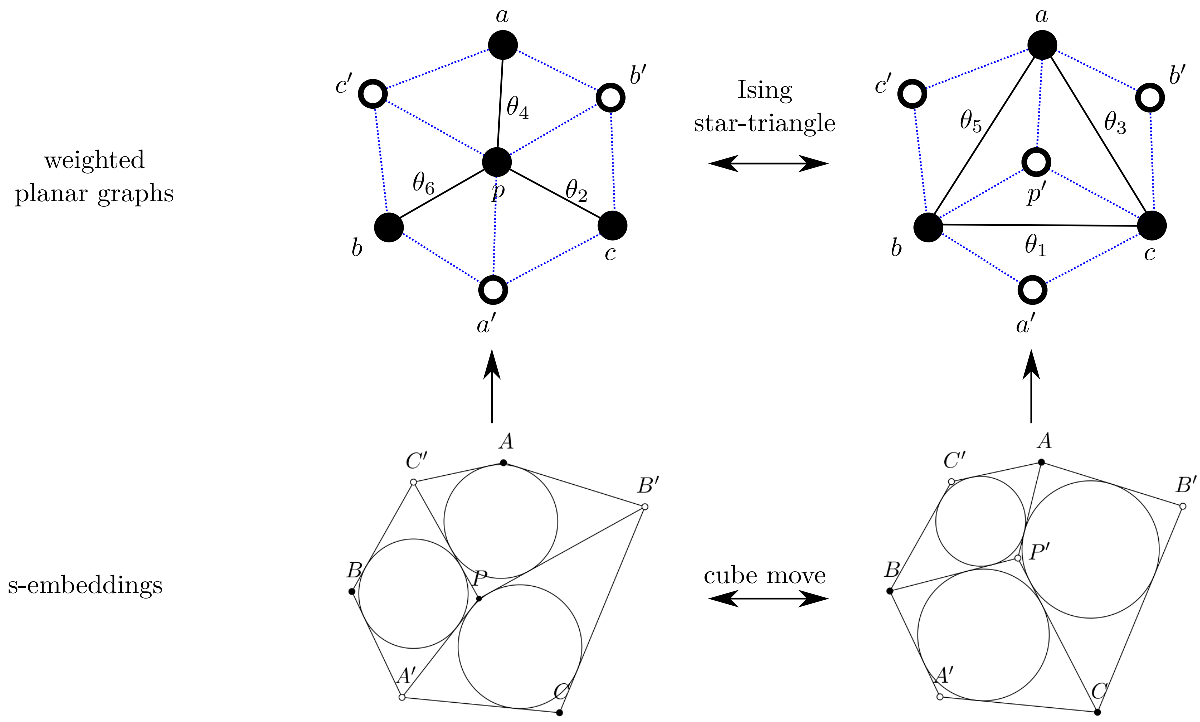
- $ABCD$ tangential quad iff $AB + CD = BC + AD$.

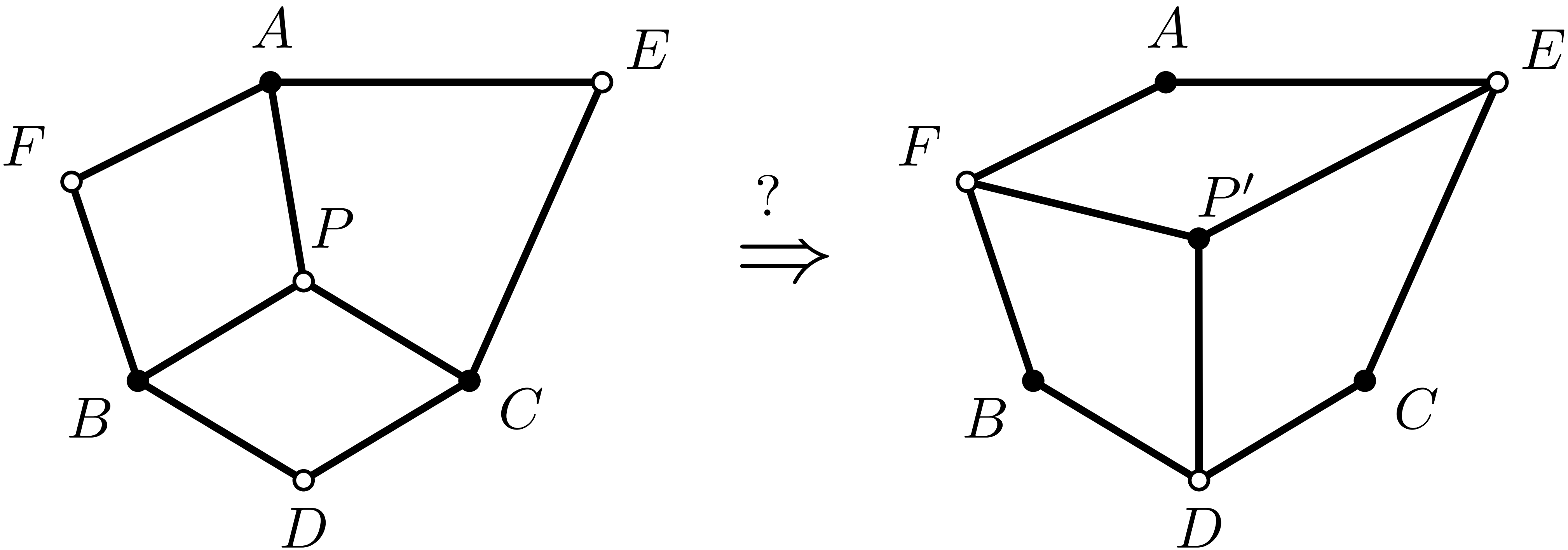

Weighted planar graph G
s-embedding for G
Ising Δ-Y move
related by a geometric local move ?
Weighted planar graph G'
s-embedding for G'

Theorem (Melotti-R.-Thévenin 2020)**.** *The cube move for s-embeddings is uniquely defined and is conjugated to the Ising* Δ*-*Y*.*

3 tangential quads ⟷ 3 tangential quads

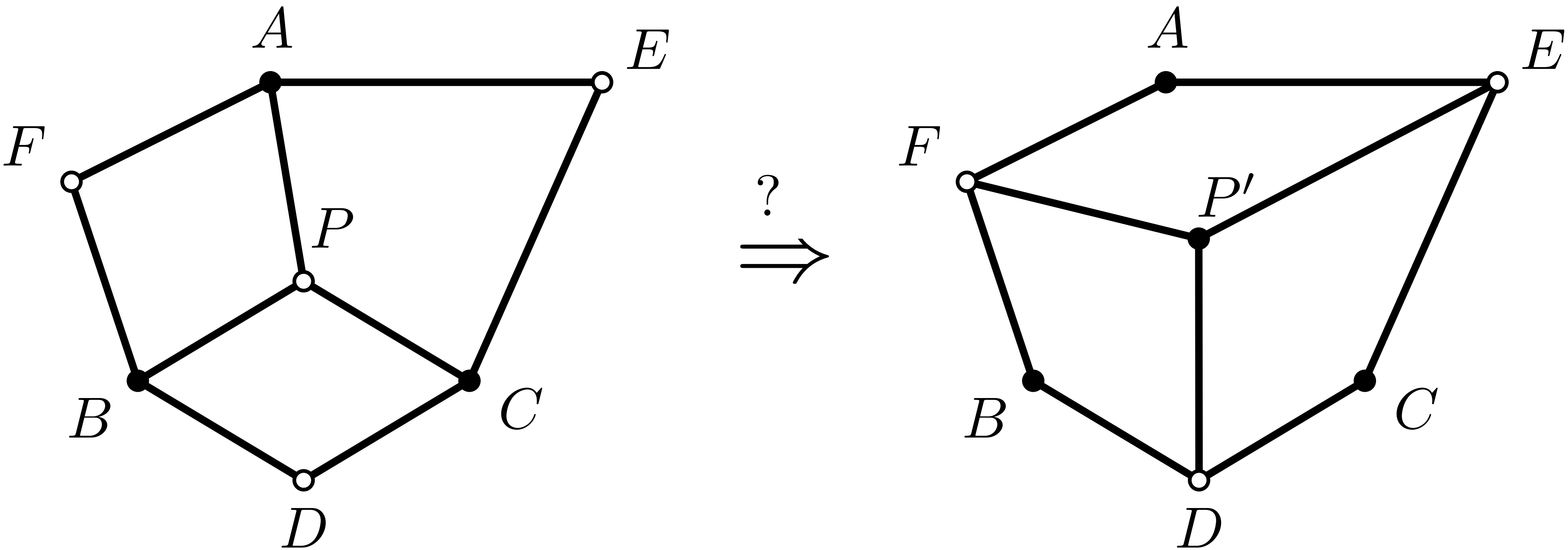



weighted
planar graphs
s-embeddings
Ising
star-triangle
cube move
a
b
c
p
a′
b′
c′
p′
θ1
θ2
θ3
θ4
θ5
θ6
A
B
C
P
A′
B′
C′
P′



$$PA - PB = AF - FB$$
$$PB - PC = BD - DC$$
$$PC - PA = CE - EA$$

$$AF - FB + BD - DC + CE - EA = 0$$



$$PA - PB = AF - FB \qquad P'F - P'D = FB - BD$$
$$PB - PC = BD - DC \qquad P'D - P'E = DC - CE$$
$$PC - PA = CE - EA \qquad P'E - P'F = EA - AF$$

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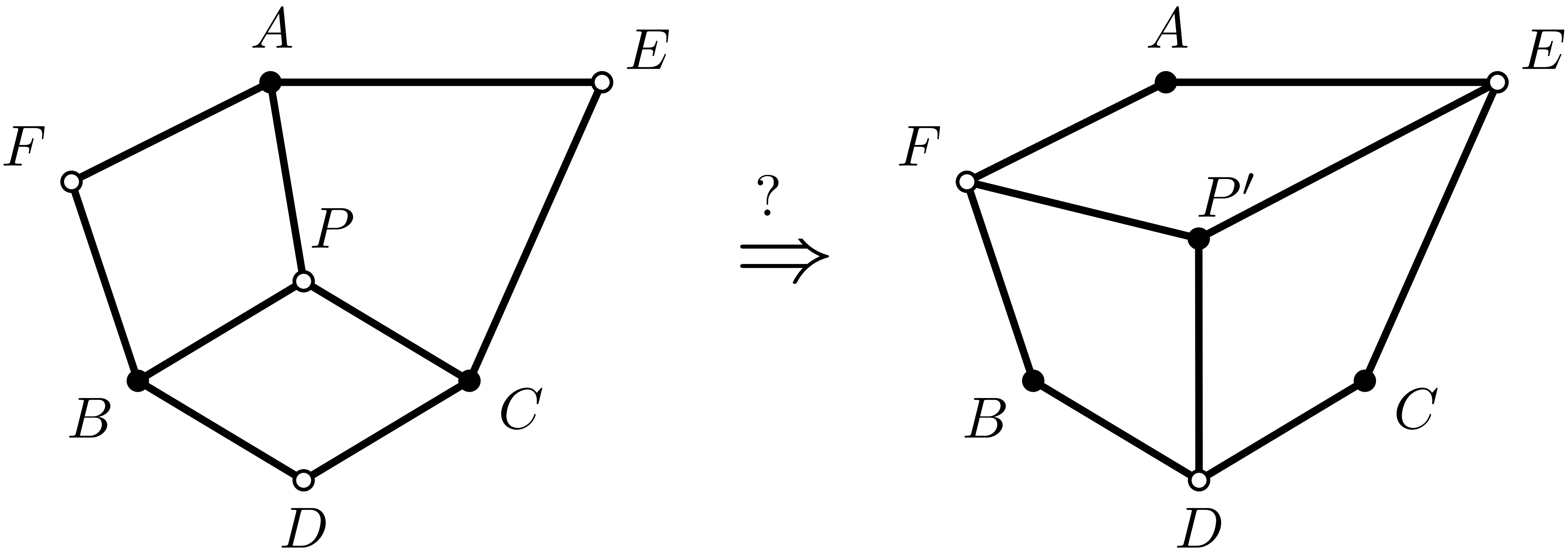

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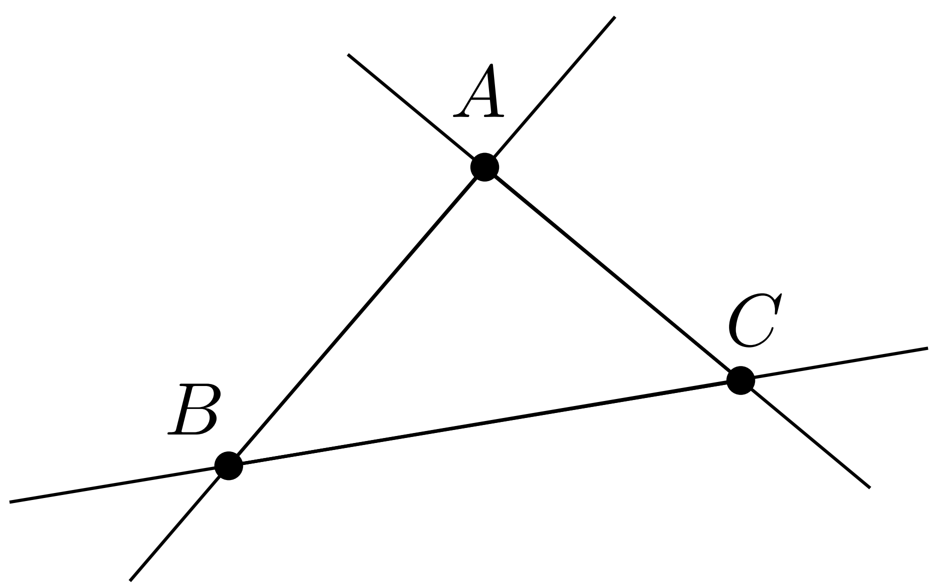
$$\boxed{\begin{aligned} P'F - P'D &= FB - BD \\ P'D - P'E &= DC - CE \end{aligned}}$$
$$P'E - P'F = EA - AF$$

$$AF - FB + BD - DC + CE - EA = 0$$

- Existence via Chelkak's propagation equations.

- Uniqueness via a geometric result:

Proposition (Melotti-R.-Thévenin 2020)**.** *If S is a branch of hyperbola with foci $\{A, B\}$ and S' is a branch of hyperbola with foci $\{A, C\}$, then $S \cap S'$ has at most one point in the shaded area.*



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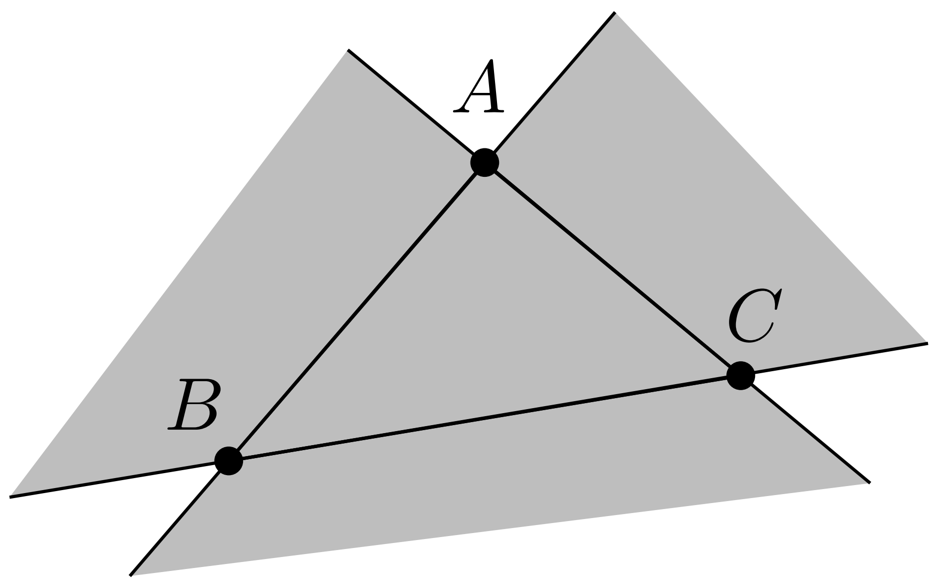

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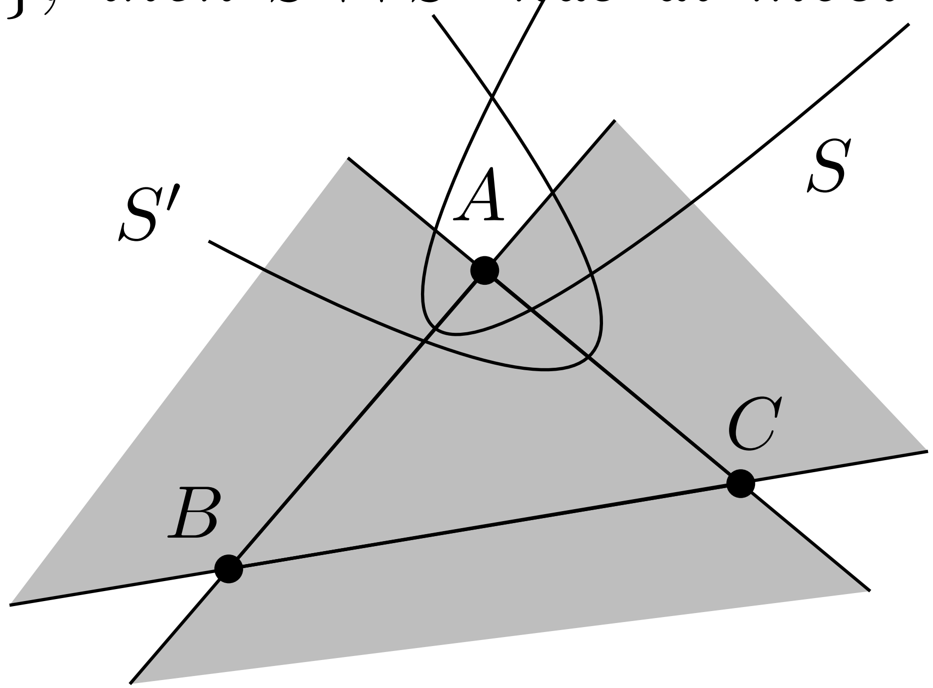
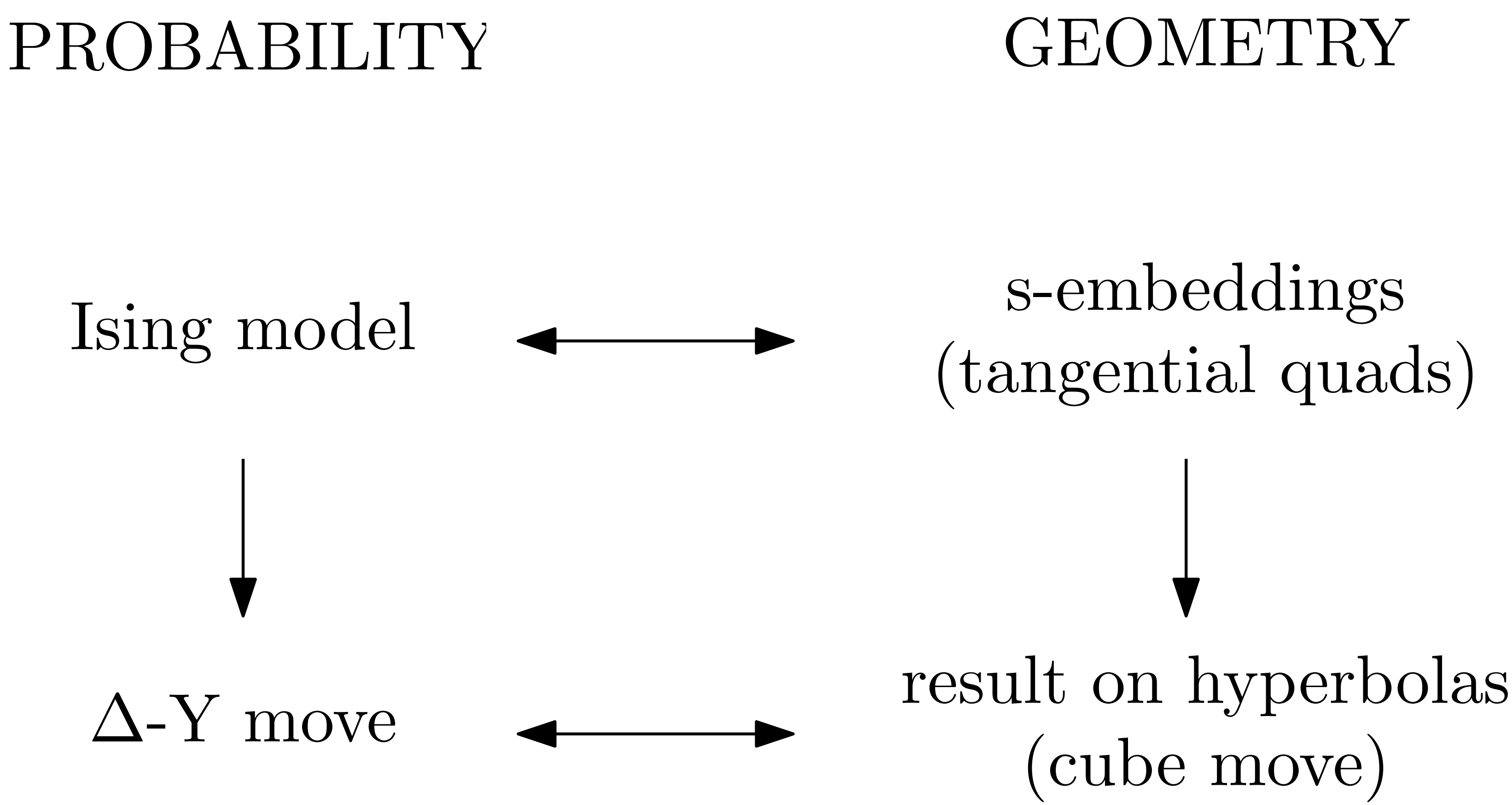



PROBABILITY
GEOMETRY
Ising model
s-embeddings
(tangential quads)
Δ-Y move
result on hyperbolas
(cube move)

3 Common framework

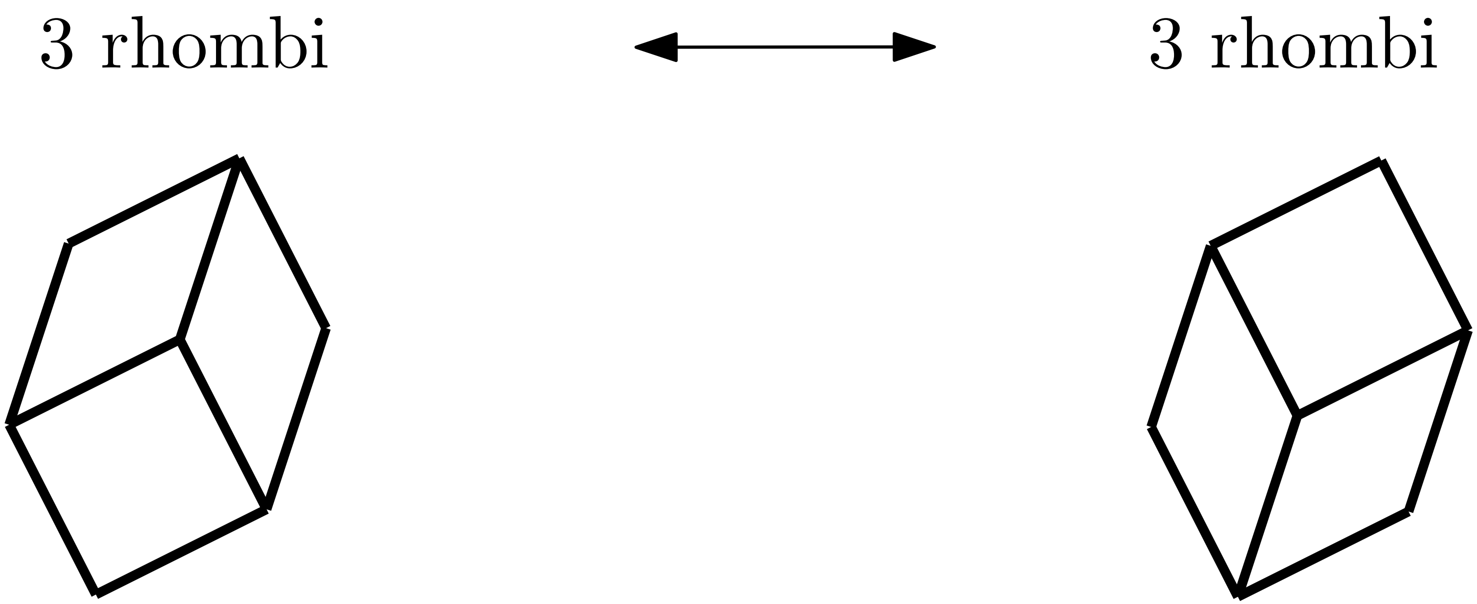
Spanning trees (same Δ-Y as Kennelly) and Ising model can be mapped to some bipartite dimer models.

PROBABILITY		GEOMETRY
Bipartite dimer model	⟷	t-embeddings Affolter, Kenyon-Lam-R.-Russkikh, Chelkak-Laslier-Russkikh
↓		↓
Spider move	⟷	Miquel's theorem

- Quads that are both orthodiagonal and tangential are rhombi.

- G is called isoradial is $G^\diamond$ has rhombic faces (Duffin, Mercat, Kenyon,...).

- Cube move for isoradial graphs uniquely defined.



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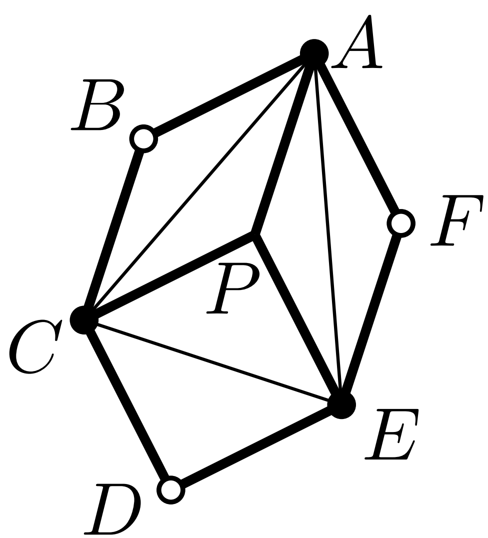
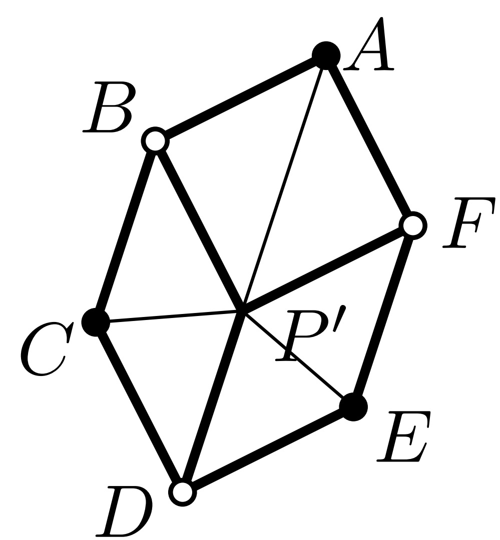
3 rhombi ⟷ 3 rhombi





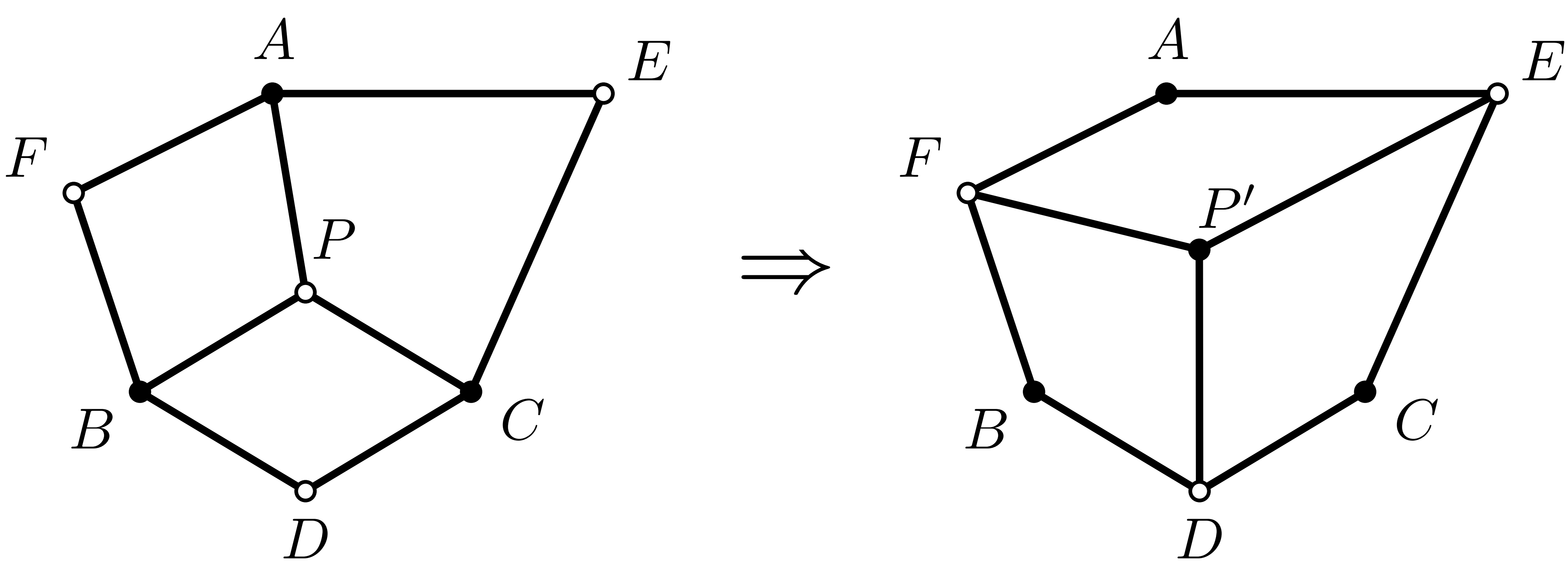
α-quads and α-realizations

- Let α be non-zero real. The quad $ABCD$ is called an α-quad if $AB^\alpha + CD^\alpha = BC^\alpha + AD^\alpha$.

- $\alpha = 1$ gives tangential quads. $\alpha = 2$ gives orthodiagonal quads. Rhombi are α-quads for all α.

- Given a planar graph G, an α-realization of G is a map from the vertex set of $G^\diamond$ to $\mathbb{R}^2$ such that each quad of $G^\diamond$ gets mapped to an α-quad.

Theorem (Melotti-R.-Thévenin 2020)**.** *Let* $\alpha > 1$*. There exists an* α*-realization* A, B, C, D, E, F, P *of the left picture iff there exists an* α*-realization* A, B, C, D, E, F, P' *of the right picture.*



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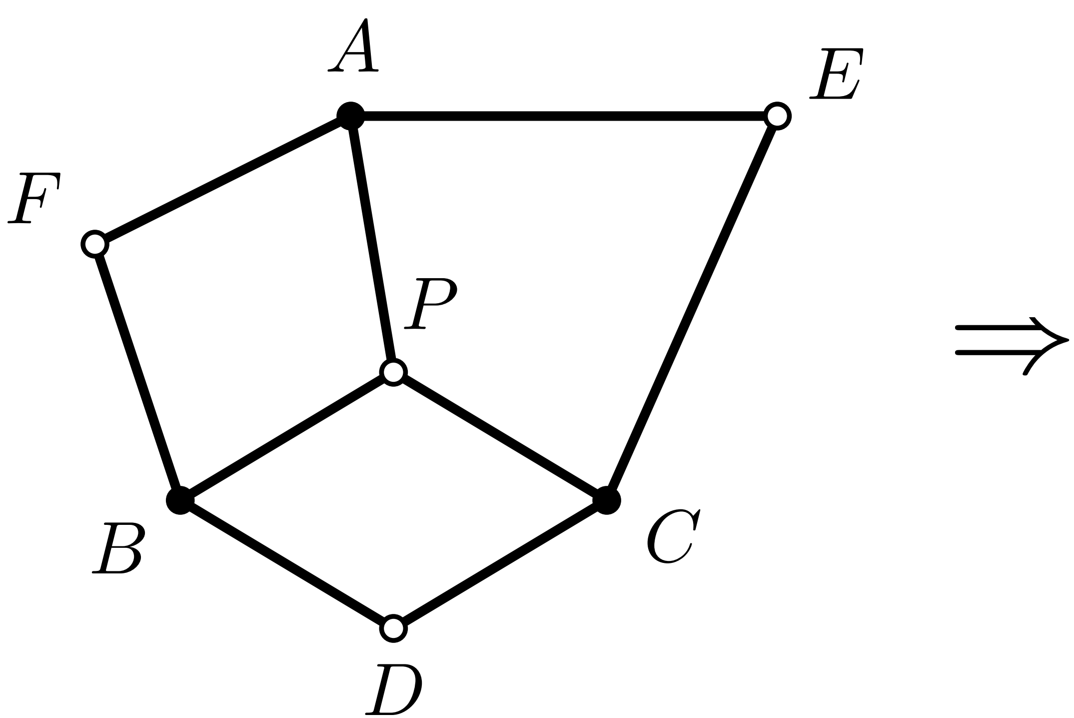
- Existence of the cube move within the class of α-realizations.



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- Existence of the cube move within the class of α-realizations.
- No uniqueness in general.
- Not expected to be true for $\alpha \leq 1$.

Can we extend the parallel ?

- Is there some probability model having a local move associated to general α-realizations ?

- Spanning trees and the Ising model arise as special cases of FK-percolation.

- FK-percolation possesses a Δ-Y transformation for certain choices of weights, some of which are related to isoradial graphs.

- Are there nice embeddings generalizing isoradial embeddings associated to FK-percolation ?

P. Melotti, S. Ramassamy, P. Thévenin, Cube moves for s-embeddings and alpha-realizations, arXiv:2003.08941 (2020).

THANK YOU !