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Introduction générale

Sommaire

Le présent mémoire est constitué de deux parties indépendantes. La première, constituée du chapitre I, porte sur une démonstration alternative du théorème d'aplatissement par éclatements de Raynaud-Gruson, reposant sur la construction et l'étude de certains espaces valuatifs. Ce thème est présenté dans la section 0.1 de cette introduction. La seconde, qui comprend les chapitres II et III, porte sur l'existence de facteurs ε locaux dans un cadre géométrique et aboutit sur une formule du produit pour le déterminant de la cohomologie d'un faisceau ℓ-adique sur une courbe en caractéristique $p \neq \ell$ positive. Parmi les outils utilisés figure la théorie du corps de classes géométrique, dont on présente dans le chapitre II une démonstration de nature purement géométrique. Les résultats afférents sont énoncés et mis en contexte dans la section 0.2 de la présente introduction.

0.1. Aplatissement par éclatements et Φ-anneaux

0.1.1. Espaces rigides : de Tate à Raynaud. Soit R un anneau de valuation discrète complet, de corps des fractions K, d'idéal maximal $\mathfrak{m}$ et de corps résiduel k. Notons v la valuation $\mathfrak{m}$-adique sur K. Classiquement, la catégorie des K-*espaces rigides* au sens de Tate, introduite par ce dernier en 1962 [**Ta71**], est construite par recollement à partir de la catégorie des K-espaces rigides affines, définie comme étant la catégorie opposée à celle des K-algèbres qui sont quotients de

$$K\langle \underline{T} \rangle = \{f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha T_1^{\alpha_1} \cdots T_n^{\alpha_n} \mid f_\alpha \in K, f_\alpha \xrightarrow[\alpha \to \infty]{} 0\},$$

pour un certain entier n, avec $\underline{T} = (T_1, \ldots, T_n)$. Les K-algèbres $K\langle \underline{T} \rangle$, dites de Tate, sont naturellement munies de structures de K-algèbres de Banach, en les munissant des normes de Gauss

$$\mid \sum_{\alpha \in \mathbb{N}^n} f_\alpha T_1^{\alpha_1} \cdots T_n^{\alpha_n} \mid = \sup_\alpha 2^{-v(f_\alpha)}.$$

Cela confère aux espaces rigides de Tate une teinte analytique, permettant une analogie avec la théorie des fonctions holomorphes. Par exemple, la définition de la propreté en géométrie rigide, due à Kiehl, est choisie de manière à permettre l'imitation de la démonstration analytique par Cartan et Serre de la finitude des dimensions des groupes de cohomologie d'un module cohérent sur un espace analytique complexe propre. Cependant, certains énoncés d'apparence bénine, telle la stabilité par composition de la notion de propreté susmentionnée, semblent être à l'épreuve de toute démonstration par voie analytique.

Dans une lettre à Serre d'octobre 1961, Grothendieck suggère que les espaces rigides de Tate devraient s'inscrire dans un cadre les unissant aux *schémas formels*, dont l'étude remonte à [**EGA1**]. Remarquons par exemple que la R-algèbre $R\langle \underline{T} \rangle$, formées des éléments de $K\langle \underline{T} \rangle$ dont les coefficients appartiennent à R, est la complétion $\mathfrak{m}$-adique de l'algèbre de polynômes $R[\underline{T}]$. Il s'agit alors de se figurer l'espace rigide associé à $K\langle \underline{T} \rangle$ comme étant la *fibre générique* du R-schéma formel se déduisant de l'espace affine $\mathbb{A}^n_R$ par complétion le long de sa fibre spéciale.

C'est à Raynaud qu'échoit la tâche de formaliser l'idée de Grothendieck. Ses résultats sont exposés dans [**Ray74**] : Raynaud y définit en particulier le foncteur « fibre générique », de la catégorie des R-schémas formels de type fini vers celle des K-espaces rigides quasi-compacts. Ce foncteur est essentiellement surjectif, mais n'est pas pleinement fidèle, quand bien même le restreindrait-on à la sous-catégorie des R-schémas formels plats de type fini. Raynaud fait alors l'élégante observation qui suit : le foncteur « fibre générique » devient une équivalence de catégories si l'on substitue à sa source la catégorie qui s'en déduit par inversion formelle des *éclatements admissibles*, ces derniers étant les éclatements formels dont le centre est défini par un idéal ouvert. Nous renvoyons à [**Abb10**] pour un développement systématique de la théorie des espaces rigides du point de vue de Raynaud.

Cette approche par Raynaud de la théorie des espaces rigides permet de ramener l'étude de ces derniers à celle des R-schémas formels, dès lors que l'on sait comparer leurs propriétés respectives. Se pose également le problème comparer les propriétés, telles la propreté ou la platitude, d'un morphisme de R-schéma formels de type fini et de sa fibre générique. C'est sur cette question, dans le cas particulier de la platitude, que se porte notre attention dans la suite.

0.1.2. Aplatissement par éclatement admissible. Tout K-espace rigide quasi-compact X (cf. 0.1.1) est la fibre générique d'un R-schéma formel *plat* et de type fini ; ce dernier est alors réputé être un *R-modèle* de X. De même, tout morphisme $f_K : X \to Y$ de K-espaces rigides quasi-compacts est la fibre générique d'un morphisme $f : \mathcal{X} \to \mathcal{Y}$ entre des R-modèles $\mathcal{X}$ et $\mathcal{Y}$ de X et Y, respectivement. La platitude de f entraîne celle de f_K, mais la réciproque ne saurait être vraie. Se pose alors l'interrogation suivante : si f_K est plat, est-il possible de choisir f également plat ? La réponse à cette question s'avère positive, et des démonstrations peuvent en être trouvées dans ([**BL93**], 5.2) ou dans ([**Abb10**], 5.8.1). Raynaud et Gruson avaient auparavant démontré ([**RG71**] I.5.2.2) une version schématique de ce résultat, s'énonçant comme suit.

THÉORÈME 0.1.3 (cf. I.1.1). *Soit X un schéma de présentation finie sur une base S quasi-compacte et quasi-séparée, soit U un ouvert quasi-compact de S et soit $\mathcal{F}$ un $\mathcal{O}_X$-module quasi-cohérent de type fini. Supposons que la restriction de $\mathcal{F}$ à $X \times_S U$ est à la fois un $\mathcal{O}_{X \times_S U}$-module de présentation finie et un $\mathcal{O}_U$-module plat. Il existe alors un éclatement $f : S' \to S$ vérifiant les propriétés suivantes :*

(1) *l'éclatement f est centré sur un sous-schéma fermé de présentation finie de S disjoint de U ;*

(2) *si $X' \to S'$ est le transformé strict de X selon f, alors le transformé strict de $\mathcal{F}'$ selon f est à la fois un $\mathcal{O}_{X'}$-module de présentation finie et un $\mathcal{O}_{S'}$-module plat.*

Un éclatement $f : S' \to S$ tel qu'autorisé dans le théorème 0.1.3 est réputé *U-admissible*. Rappelons brièvement la notion de transformé strict selon un éclatement : si Z est le diviseur de Cartier exceptionel de l'éclatement f, alors le *transformé strict X' de X selon f* est le sous-schéma fermé de $S' \times_S X$ défini par l'annulation de l'idéal quasi-cohérent des sections dont le support est contenu dans $Z \times_S X$. Le *transformé strict $\mathcal{F}'$ de $\mathcal{F}$ selon f* est quant à lui la restriction à X' du quotient de $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{S' \times_S X}$ par le sous-module de ses sections dont le support est contenu dans $Z \times_S X$.

Une démonstration du théorème 0.1.3, différente de celle de Raynaud-Gruson, est présentée dans le Chapitre I. Celle-ci consiste à considérer la situation du théorème 0.1.3 non pas sur S, ni même sur un éclatement admissible de S, mais sur la limite inverse

$$\widetilde{S} = \lim_{S' \to S} S',$$

où la limite, cofiltrante, porte sur l'ensemble des éclatements U-admissibles de S. La limite est ici prise dans la catégorie des espaces topologiques localement annelés, de sorte que $\widetilde{S}$ est naturellement muni d'un faisceau d'anneaux $\mathcal{O}_{\widetilde{S}}$, dont la tige en chaque point est un anneau local. On procède alors comme suit : on commence par munir $\mathcal{O}_{\widetilde{S}}$ d'une structure plus riche, en l'occurrence de Φ-anneau Φ-local (cf. 0.1.6, 0.1.8), puis on démontre directement le théorème 0.1.3 lorsque la base S est le spectre d'un Φ-anneau Φ-local (cf. 0.1.8). On note alors que la propriété d'aplatissement, établie sur les tiges Φ-locales de $\mathcal{O}_{\widetilde{S}}$, se propage localement sur $\widetilde{S}$, et on utilise enfin un argument de limite pour descendre la propriété de platitude de $\widetilde{S}$ vers un éclatement U-admissible de S adéquatement choisi.

0.1.4. Φ-anneaux. La catégorie des Φ-anneaux est un substitut algébrique à la catégorie des anneaux topologiques adiques, ces derniers formant les briques élémentaires de la théorie des espaces rigides à la Raynaud (cf. 0.1.1), ou encore de la théorie de Huber.

Définissons la notion de Φ-anneau. Il s'agit d'un anneau A, muni d'une notion abstraite d'admissiblité portant sur ses idéaux de type fini, sujette aux axiomes suivants : un produit fini d'idéaux admissibles l'est aussi, et tout idéal de type fini contenant un idéal admissible de A est lui-même admissible. Étant donné un anneau A, une structure de Φ-anneau sur A correspond naturellement à une famille de supports constructibles sur l'espace topologique $\mathrm{Spec}(A)$ (cf. I.2.9). Cette interprétation explique le choix de la lettre Φ, qui fait écho à la notation de Cartan pour les familles de supports dans [**Car51**].

Parmi les Φ-anneaux se distinguent les deux classes qui suivent.

(1) Les Φ-anneaux de type *adique* : étant donné un idéal I d'un anneau A, on peut munir A d'une structure de Φ-anneau, réputé *I-adique*, en déclarant admissibles les idéaux de type fini de A qui sont ouverts pour la topologie I-adique.

(2) Les Φ-anneaux de type *birationnel* : un anneau noethérien intègre étant donné, on peut munir celui-ci d'une structure de Φ-anneau en déclarant admissible tout idéal non nul.

Si A est un anneau et Φ_0 est un ensemble d'idéaux de type fini de A, alors il existe une plus petite structure de Φ-anneau sur A pour laquelle tous les éléments de Φ_0 sont admissibles. On dit alors que la structure de Φ-anneau de A est *engendrée* par Φ_0. Lorsque Φ_0 est un singleton, on obtient ainsi un Φ-anneau de type adique.

Les Φ-anneaux de type adique sont précisément ceux dont on use dans la démonstration du théorème 0.1.3 : dans le contexte de ce dernier, supposant la base S affine d'anneau A, l'ouvert quasi-compact U est le complémentaire dans S du lieu d'annulation d'un idéal de type fini I de A, et on se contente alors de munir A de sa structure de Φ-anneau I-adique.

Quant aux Φ-anneaux de type birationnel, on peut considérer l'étude de ceux-ci, ou plutôt de leur Φ-localisation (cf. 0.1.5) comme étant l'objet de la géométrie birationnelle.

0.1.5. Profondeur, purification, clôture. Soit A un Φ-anneau (cf. 0.1.5). On définit dans I.2.10 des groupes de cohomologie $H^q_\Phi(M)$ pour tout A-module M et tout entier q : il s'agit de la cohomologie à supports du faisceau quasi-cohérent sur $\mathrm{Spec}(A)$ associé à M, où on choisit pour famille de supports les fermés de $\mathrm{Spec}(A)$ qui sont contenus dans le lieu d'annulation d'un idéal admissible de A. On a par exemple

$$H^0_\Phi(M) = \{m \in M \mid \text{il existe un idéal admissible } I \text{ de } A \text{ tel que } Im = 0\}.$$

On définit alors (cf. I.2.14) la notion de profondeur en s'inspirant de ([**SGA2**] III.2.3) : pour chaque entier d, un A-module M est *d-profond* si $H^q_\Phi(M)$ s'annule pour tout entier $q < d$. On démontre alors (cf. I.2.18) qu'un A-module M est d-profond si et seulement si les groupes de cohomologie locale $H^q_I(M)$ s'annulent pour tout entier $q < d$ et pour tout idéal admissible I de A.

Un module 1-profond est aussi réputé *pur*, tandis que l'on qualifiera de *clos* un module 2-profond : il s'agit de la terminologie issue de ([**EGA4**], 5.9, 5.10).

Pour tout A-module M, il existe un homomorphisme A-linéaire $M \to M^{\mathrm{pur}}$ vers un A-module pur M^{pur}, et qui est universel pour cette propriété : il suffit de considérer le quotient de M par $H^0_\Phi(M)$. Le module M^{pur} est réputé être la *purification* de M.

De même, on démontre (cf. I.2.32) qu'il existe un homomorphisme A-linéaire $M \to M^{\triangleleft}$ vers un A-module clos $M^{\triangleleft}$, et qui est universel pour cette propriété. Le module $M^{\triangleleft}$ est réputé être la *clôture* de M. Si M est pur, on a l'expression suivante :

$$M^{\triangleleft} = \operatorname{colim} \mathrm{Hom}_A(I, M),$$

où la colimite (filtrée) porte sur l'ensemble des idéaux admissibles de A (cf. I.2.27). En particulier, un A-module M est clos si et seulement si l'homomorphisme naturel

$$M \to \mathrm{Hom}_A(I, M), \tag{1}$$

est un isomorphisme pour tout idéal admissible I (cf. I.2.24)

Concluons ce paragraphe en indiquant comment la caractérisation (1) permet de retrouver le critère de normalité de Serre. Considérons un anneau intègre noethérien A vérifiant les deux conditions suivantes, dites de Serre :

(R1) l'anneau A est régulier en codimension 1, i.e. pour tout idéal premier $\mathfrak{p}$ de A de codimension 1 (respectivement 0), l'anneau localisé $A_\mathfrak{p}$ est un anneau de valuation discrète (respectivement un corps) ;

(S2) tout idéal de A de codimension au moins 2 est de profondeur au moins 2.

Montrons que A est normal, i.e. intégralement clos dans son corps des fractions K. Soit a un élément de K qui est entier sur A, et considérons le *conducteur* de a, c'est-à-dire l'idéal I de A constitué de ses éléments b tels que ab appartient à A. Si $\mathfrak{p}$ est un idéal premier de A de codimension au plus 1, alors $A_\mathfrak{p}$ est intégralement clos par (R1), de sorte que a appartient à $A_\mathfrak{p}$, et donc que I n'est pas contenu dans $\mathfrak{p}$. Ainsi la codimension de I est au moins 2, et celui-ci est par conséquent de profondeur au moins 2 par (S2). Munissons l'anneau A de sa structure de Φ-anneau I-adique. Le Φ-anneau A est 2-profond, de sorte que $A^{\triangleleft} = A$. Puisque $Ia \subseteq A$, le critère (1) montre que a appartient à A.

0.1.6. Spectres valuatifs. Nous construisons dans I.4 un espace valuatif associé à un espace topologique muni d'un faisceau de Φ-anneaux. Bornons nous à décrire ici le cas particulier d'un schéma affine.

À un Φ-anneau A on peut fonctoriellement associer un espace topologique $X = \Phi\mathrm{Spec}(A)$, dont un point x est une classe d'équivalence de valuation $|\cdot(x)| : A \to \Gamma_+$, où Γ est un groupe abélien totalement ordonné et Γ_+ est le semigroupe $\Gamma \sqcup \{0\}$, qui satisfait aux propriétés suivantes :

(1) pour tout idéal admissible I dans A, l'élément $|I(x)| = \max_{f\in I}|f(x)|$ de Γ_+ est strictement positif ;

(2) pour tout élément γ de Γ, il existe un idéal admissible I dans A tel que $|I(x)| \leq \gamma$.

La topologie de X est alors engendrée par les *domaines rationnels*, de la forme

$$U\left(g^{-1}I\right) = \{x \in X \mid |I(x)| \leq |g(x)|\},$$

où I est un idéal admissible de A et g en est un élément. On munit alors X d'un faisceau d'anneaux $\mathcal{O}_X^{\triangleleft}$ dont les sections sur $U\left(g^{-1}I\right)$ forment la A-algèbre localisée $A^{\triangleleft}[g^{-1}]$, puis du sous-faisceau d'anneaux $\mathcal{O}_X$ formé des sections de $\mathcal{O}_X^{\triangleleft}$ de valuation au plus 1 en chaque point.

Remarque 0.1.7. Supposant que A est I-adique, on retrouve le spectre valuatif de Huber [**Hu93**] de la paire $(A^{\triangleleft}, A)$ comme étant le fermé de $\Phi\mathrm{Spec}(A)$ défini par l'inéquation $|I| < 1$. Son complémentaire dans $\Phi Spec(A)$ est alors un ouvert isomorphe à $U = \mathrm{Spec}(A) \setminus V(I)$. Ainsi, l'espace $\Phi\mathrm{Spec}(A)$ est un recollement de U et du spectre valuatif de Huber de $(A^{\triangleleft}, A)$, cp. [**Fu95**].

Le faisceau d'anneaux $\mathcal{O}_X$ est naturellement muni d'une structure de faisceau de Φ-anneaux, en déclarant, pour tout ouvert U, admissible tout idéal de type fini I de $\mathcal{O}_X(U)$ tel que $|I(x)| > 0$ pour tout point x de U. On obtient ainsi un espace topologique Φ-annelé $(X, \mathcal{O}_X)$, encore noté $\Phi\mathrm{Spec}(A)$. La tige $\mathcal{O}_{X,x}$ de $\mathcal{O}_X$ en un point x de X est par ailleurs un Φ-anneau Φ-*local* : c'est un anneau local, et tous ses idéaux admissibles sont inversibles (cf. I.2.36).

Ainsi $(X, \mathcal{O}_X)$ est un espace topologique Φ-*localement* Φ-annelé. On démontre (cf. I.4.11) que $(X, \mathcal{O}_X)$ est terminal dans la catégorie des espaces Φ-localement Φ-annelés munis d'un morphisme de Φ-anneaux de A vers $\Gamma(X, \mathcal{O}_X)$.

On démontre (cf. I.2.39) un résultat de structure sur les Φ-anneaux Φ-locaux, dans l'esprit de ([**Abb10**] 1.9.4), qui implique que la classe d'équivalence de valuation sur A correspondant à un point x de X est encodée dans la structure de Φ-anneau de $\mathcal{O}_{X,x}$. Autrement dit, à un Φ-anneau Φ-local est canoniquement associé une valuation, et celle attribuée à $\mathcal{O}_{X,x}$ permet de retrouver la valuation x par composition avec l'homomorphisme naturel $A \to \mathcal{O}_{X,x}$.

0.1.8. Aplatissement local. Conservons les notations de 0.1.6. Supposons que A est un Φ-anneau I-adique, pour un certain idéal I de type fini dans A. L'espace localement annelé $(X, \mathcal{O}_X)$ est alors isomorphe à la limite inverse (cf. I.4.13)

$$\lim_{S' \to \mathrm{Spec}(A)} S',$$

la limite portant sur les éclatements U-admissibles de $\mathrm{Spec}(A)$, où U est le complémentaire de $V(I)$ dans $\mathrm{Spec}(A)$. Ainsi, la démonstration du théorème 0.1.3 dans le cas d'une base affine $S = \mathrm{Spec}(A)$ se ramène par un argument de limite à un énoncé d'aplatissement sur $\Phi\mathrm{Spec}(A)$. Puisque celui-ci est Φ-localement Φ-annelé (cf. 0.1.6), on se ramène à démontrer l'énoncé suivant, qui constitue la version locale du théorème d'aplatissement 0.1.3.

Proposition 0.1.9 (cf. I.5.2, I.5.3). *Soit A un Φ-anneau qui est Φ-local, i.e. qui est local et dont les idéaux admissibles sont tous inversibles, soit B une A-algèbre de type fini et soit M un B-module de type fini. Supposons que les assertions suivantes sont vérifiées :*

(1) le $A^{\triangleleft}$-module $M \otimes_A A^{\triangleleft}$ est plat ;

(2) le $B \otimes_A A^{\triangleleft}$-module $M \otimes_A A^{\triangleleft}$ est de présentation finie.

Le B-module M^{pur} est alors de présentation finie, et est un A-module plat.

Nous consacrons les parties I.5.7 et I.6 du chapitre I à la démonstration du fait que l'énoncé local 0.1.9 se globalise pour donner le théorème de Raynaud-Gruson 0.1.3.

0.1.10. Φ-anneaux normaux et faisceau structural. Un Φ-anneau est réputé *normal* si l'homomorphisme de clôture (cf. 0.1.5)

$$A \to A^{\triangleleft},$$

est injectif et d'image intégralement close dans $A^{\triangleleft}$.

Pour tout Φ-anneau A, il existe un homomorphisme $A \to A^{+}$ vers un Φ-anneau normal A^{+}, et qui est universel pour cette propriété : il suffit en effet de considérer la clôture intégrale de A dans $A^{\triangleleft}$. Le Φ-anneau A^{+} est réputé être la *normalisation* de A.

Il se trouve que tout Φ-anneau Φ-local est normal (cf. I.2.39). En particulier, notant $X = \Phi\mathrm{Spec}(A)$ pour un Φ-anneau A, cela implique que $\Gamma(X, \mathcal{O}_X)$ est également un Φ-anneau normal, et donc que l'homomorphisme naturel de A vers $\Gamma(X, \mathcal{O}_X)$ se factorise uniquement par un homomorphisme

$$A^{+} \to \Gamma(X, \mathcal{O}_X).$$

Ce dernier est toujours un isomorphisme. Plus généralement, si I est un idéal admissible de A et si g en est un élément, alors la A-algèbre formée des sections de $\mathcal{O}_X$ sur l'ouvert rationnel $U\left(g^{-1}I\right)$ est la normalisation du sous-Φ-anneau $A[g^{-1}I]$ de $A[g^{-1}]$: on obtient donc une description explicite du faisceau $\mathcal{O}_X$.

0.1.11. Φ-schémas. À tout Φ-anneau est associé un spectre valuatif (cf. 0.1.6), qui est un espace topologique Φ-localement Φ-annelé, et il est naturel d'introduire les recollements de tels espaces. Définissons simplement un *Φ-schéma* comme étant un espace topologique Φ-localement Φ-annelé, dont chaque point admet un voisinage ouvert isomorphe à $\Phi\mathrm{Spec}(A)$, pour un certain Φ-anneau A, ce dernier pouvant être supposé Φ-normal. En particulier, à chaque point x d'un Φ-schéma X est canoniquement associé une valuation $f \mapsto |f(x)|$ sur $\mathcal{O}_{X,x}$.

Informellement, la catégorie des Φ-schémas s'insère dans le diagramme commutatif suivant.

$$\begin{tikzcd}
\text{Schémas} \arrow[r, "\text{“fibre générique”}"] \arrow[d, "\text{complétion}"'] & \Phi\text{-schémas} \arrow[d, "\text{complétion}"] \\
\text{Schémas formels} \arrow[r, "\text{“fibre générique”}"] & \text{Espaces rigides ou adiques}
\end{tikzcd}$$

Si la pertinence de la notion de Φ-schéma nous est apparue de prime abord dans le contexte du théorème d'aplatissement de Raynaud-Gruson 0.1.3, celle-ci nous semble intéressante en soi. Nous n'étudierons cependant pas les Φ-schémas dans le présent texte.

0.2. Facteurs locaux géométriques

Dans toute cette section sont fixés des nombres premiers distincts ℓ et p, ainsi qu'une clôture algébrique $\overline{\mathbb{Q}}_\ell$ du corps $\mathbb{Q}_\ell$ des nombres ℓ-adiques. Par *faisceau ℓ-adique* sera ici entendu un $\overline{\mathbb{Q}}_\ell$-faisceau constructible (cf. III.3.5).

0.2.1. Les équations fonctionnelles et leurs constantes. Soit X un schéma propre sur un corps fini k, ce dernier étant muni d'un choix de clôture algébrique $\overline{k}$. Considérons un complexe borné $\mathcal{F}$ de faisceaux ℓ-adiques sur X. La fonction L de Weil-Grothendieck associée au couple $(X, \mathcal{F})$ est la série formelle

$$L(X, \mathcal{F}; T) = \prod_{x \in |X|} \det(1 - \mathrm{Frob}_x T^{[k(x):k]} \mid \mathcal{F}_{\bar{x}})^{-1},$$

où $|X|$ est l'ensemble des points fermés de X et pour chaque x dans $|X|$, on note $\bar{x}$ un choix de clôture algébrique de x et $\mathcal{F}_{\bar{x}}$ la tige de $\mathcal{F}$ en ce point géométrique. Enfin, on a noté Frob_x l'élément de Frobenius géométrique du groupe de Galois $\mathrm{Gal}(k(\bar{x})/k(x))$: si $k(x)$ est de cardinal q_x, il s'agit de l'automorphisme $\alpha \mapsto \alpha^{q_x^{-1}}$ de $k(\bar{x})$.

Ainsi, $L(X, \mathcal{F}; T)$ est un élément de $\overline{\mathbb{Q}}_\ell[[T]]$ de terme constant 1. Un théorème de Grothendieck [**Gr66**] affirme que la série formelle $L(X, \mathcal{F}; T)$ est une fraction rationnelle en la variable T, donnée par

$$L(X, \mathcal{F}; T) = \det(1 - \mathrm{Frob}_k T \mid R\Gamma(X_{\overline{k}}, \mathcal{F}))^{-1},$$

où la cohomologie $R\Gamma(X_{\overline{k}}, \mathcal{F})$ est un complexe borné de représentations ℓ-adiques du groupe de Galois $\mathrm{Gal}(\overline{k}/k)$. Notant $H^\nu(X_{\overline{k}}, \mathcal{F})$ la cohomologie de degré ν de ce complexe, on a encore :

$$L(X, \mathcal{F}; T) = \prod_{\nu \in \mathbb{Z}} \det(1 - \mathrm{Frob}_k T \mid H^\nu(X_{\overline{k}}, \mathcal{F}))^{(-1)^{\nu-1}},$$

où chaque $H^\nu(X_{\overline{k}}, \mathcal{F})$ est une représentation ℓ-adique de dimension finie de $\mathrm{Gal}(\overline{k}/k)$. Lorsque X est lisse, une application de la dualité de Poincaré, ou du moins de l'analogue de cette dernière en cohomologie étale ([**SGA4**], XVIII 3.2.5), procure alors une équation fonctionnelle

$$L(X, \mathcal{F}; T) = \varepsilon(X, \mathcal{F}) T^{-\chi(X, \mathcal{F})} L(X, D(\mathcal{F}); T^{-1}),$$

où le *dual* $D(\mathcal{F})$ de $\mathcal{F}$ est encore un complexe borné $\mathcal{F}$ de faisceaux ℓ-adiques sur X, où l'entier $\chi(X, \mathcal{F})$ est la *caractéristique d'Euler* de $(X, \mathcal{F})$, définie par

$$\chi(X, \mathcal{F}) = \dim R\Gamma_c(X_{\overline{k}}, \mathcal{F}) = \sum_\nu (-1)^\nu \dim H^\nu(X_{\overline{k}}, \mathcal{F}),$$

et où le *facteur ε global* $\varepsilon(X, \mathcal{F})$ est donné par

$$\begin{aligned}\varepsilon(X, \mathcal{F}) &= \det\left(-\mathrm{Frob}_k \mid R\Gamma(X_{\overline{k}}, \mathcal{F})\right)^{-1} \\ &= (-1)^{\chi(X, \mathcal{F})} \det\left(\mathrm{Frob}_k \mid R\Gamma(X_{\overline{k}}, \mathcal{F})\right)^{-1}.\end{aligned}$$

Cette dernière quantité est complètement déterminée par la parité de $\chi(X, \mathcal{F})$, et par le déterminant

$$\det\left(R\Gamma(X_{\overline{k}}, \mathcal{F})\right)^{-1},$$

qui est un $\mathrm{Gal}(\overline{k}/k)$-module de rank 1. Ce dernier objet est un invariant du couple $(X, \mathcal{F})$, bien plus grossier que la fonction L elle-même, et qu'il est par conséquent envisageable de calculer. Plus généralement se pose la question suivante.

Problème 0.2.2. Si X est un schéma propre sur un corps k de caractéristique différente de ℓ, et si $\mathcal{F}$ est faisceau ℓ-adique sur X, comment calculer la classe du déterminant

$$\det\left(R\Gamma(X_{\overline{k}}, \mathcal{F})\right)^{-1}$$

dans le groupe $H^1(\mathrm{Gal}(\overline{k}/k), \overline{\mathbb{Q}}_\ell^\times)$ des caractères ℓ-adiques de $\mathrm{Gal}(\overline{k}/k)$?

Lorsque k est un corps fini et lorsque X est une courbe lisse, les travaux de Tate [**Ta50**], Dwork [**Dw56**], Deligne [**De73**], Langlands [**Lan**] et enfin Laumon [**La87**], assurent que le déterminant de la cohomologie d'un faisceau ℓ-adique se factorise en un produit fini de contributions de nature purement locale. Nous y reviendrons dans le paragraphe 0.2.8 ci-bas.

Dans la suite, nous considérons le problème 0.2.2 exclusivement dans le cas où X est une courbe lisse sur un corps k parfait de caractéristique p.

0.2.3. Le cas des systèmes locaux de rang 1. Dans le cas d'un faisceau ℓ-adique de rang générique 1 sur une courbe projective lisse, une solution au problème 0.2.2 peut être trouvée dans la thèse de Tate [**Ta50**] dans le cas d'un corps de base fini, ou dans une lettre de Deligne à Serre de 1974 dans le cas d'un corps de base arbitraire : on obtient alors une décomposition du déterminant de la cohomologie en un produit de contributions locales, cf. III.8.3. Cette lettre est publiée en tant qu'annexe à l'article [**BE01**]. Dans celle-ci, Deligne suppose le corps de base de caractéristique p positive ; sa méthode donne également un résultat de localisation du déterminant en caractéristique nulle, mais ce n'est qu'en caractéristique p que l'existence du système local d'Artin-Schreier permet de factoriser l'expression obtenue (cf. III.8, et plus spécifiquement le lemme III.8.10). Nous avons choisi de donner une rédaction détaillée de l'argument de Deligne dans la partie III.8. Bornons-nous ici à en décrire l'essence.

Soit X un courbe projective lisse sur un corps k de caractéristique p, et $j : U \to X$ un ouvert non vide de X. Considérons un système local ℓ-adique $\mathcal{F}$ de rang 1 sur U. Il s'agit de calculer le caractère ℓ-adique

$$\det\left(R\Gamma(X_{\overline{k}}, j_!\mathcal{F})\right)^{-1} \cong \det\left(R\Gamma_c(U_{\overline{k}}, \mathcal{F})\right)^{-1}.$$

Supposons que la caractéristique d'Euler de $(X, \mathcal{F})$ est un entier négatif $-d$. La formule de Künneth symétrique (cf. 49) permet alors d'écrire :

$$\det R\Gamma_c(U_{\overline{k}}, \mathcal{F})^{-1} \cong \det R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}}, \mathcal{F}^{[d]})^{(-1)^d}, \tag{2}$$

où $\mathrm{Sym}^d_k(U)$ est le *d-ième produit symétrique* de U, quotient de U^d par le groupe des bijections de $[|1, d|]$ dans lui-même agissant par permutations des coordonnées.

L'étape suivante consiste à invoquer la théorie du corps de classes globale géométrique. On se donne pour cela un diviseur effectif $i : D \to X$ sur X, supporté sur le complémentaire de U dans X, et bornant la ramification de $\mathcal{F}$ au bord de U (cf. III.5.12). À la paire (X, D) est associée le groupe de Picard généralisé $\mathrm{Pic}_k(X, D)$ (cf. III.5.13) qui paramètre les couples $(\mathcal{L}, \alpha)$ constitués d'un fibré en droites $\mathcal{L}$ sur X muni d'une *rigidification* α le long de D, c'est-à-dire d'un isomorphisme $\alpha : \mathcal{O}_D \to i^*\mathcal{L}$: il s'agit d'un k-schéma en groupes séparé lisse commutatif, extension de $\mathbb{Z}$ par un k-schéma en groupes géométriquement connexe. On dispose alors du morphisme d'Abel-Jacobi

$$\Phi : U \to \mathrm{Pic}_k(X, D), \tag{3}$$

qui envoie une section x de U sur la paire $(\mathcal{O}(x), 1)$, où $1 : \mathcal{O}_D \to \mathcal{O}(x) \otimes_{\mathcal{O}_X} \mathcal{O}_D$ est la rigidification de $\mathcal{O}(x)$ sur D induite par la section unité $1 : \mathcal{O}_X \hookrightarrow \mathcal{O}(x)$. Munis de ces notations, nous pouvons énoncer le théorème principal de la théorie du corps de classes globale géométrique comme suit.

Théorème 0.2.4 (Théorie du corps de classes globale géométrique, cf. III.5.15). *Soit $\mathcal{F}$ un système local ℓ-adique de rang 1 sur U, de ramification bornée par D. Il existe alors un unique (à isomorphisme près) couple $(\chi_{\mathcal{F}}, \beta)$ constitué d'un système local ℓ-adique multiplicatif $\chi_{\mathcal{F}}$ sur $\mathrm{Pic}_k(X, D)$ et d'un isomorphisme $\beta : \Phi^{-1}\chi_{\mathcal{F}} \to \mathcal{F}$.*

Un système local ℓ-adique multiplicatif sur un k-schéma en groupes G est un système local ℓ-adique χ de rang 1 sur G muni, pour toutes sections g, h de G d'un isomorphisme

$$g^{-1}\chi \otimes h^{-1}\chi \cong (gh)^{-1}\chi,$$

vérifiant certaines compatibilités naturelles, cf. III.5.1 et III.5.6.

Nous renvoyons au paragraphe 0.2.5 ci-bas pour une discussion plus complète du théorème 0.2.4. Contentons-nous pour l'heure de revenir à la situation ci-dessus. Soit $\chi_{\mathcal{F}}$ le système local ℓ-adique multiplicatif sur $\mathrm{Pic}_k(X,D)$ procuré par l'application à $(X,D,\mathcal{F})$ du théorème 0.2.4, de sorte que le tiré en arrière $\Phi^{-1}\chi_{\mathcal{F}}$ de $\chi_{\mathcal{F}}$ par le morphisme d'Abel-Jacobi Φ, cf. (3), est isomorphe à $\mathcal{F}$.

Le morphisme Φ induit par multiplication un morphisme, cf. 50

$$\Phi_d : \mathrm{Sym}^d_k(U) \to \mathrm{Pic}_k(X,D),$$

et la suite spectrale de Leray donne alors un isomorphisme

$$\begin{aligned}\det R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}}, \mathcal{F}^{[d]}) &\cong \otimes_{q\in\mathbb{Z}} \det R\Gamma_c(\mathrm{Pic}_k(X,D)_{\overline{k}}, R^q\Phi_{d!}\Phi_d^{-1}\chi_{\mathcal{F}})^{(-1)^q}\\ &\cong \otimes_{q\in\mathbb{Z}} \det R\Gamma_c(\mathrm{Pic}_k(X,D)_{\overline{k}}, \chi_{\mathcal{F}}\otimes R^q\Phi_{d!}\mathbb{Q}_\ell)^{(-1)^q}.\end{aligned}$$

Il se trouve que les complexes $R^q\Phi_{d!}\mathbb{Q}_\ell$ se concentrent sur une seule fibre du morphisme de projection $\pi : \mathrm{Pic}_k(X,D) \to \mathrm{Pic}_k(X)$, à savoir celle du fibré en droites $\Omega^1_X(D)$ (cf. III.8.6). Les fibres de π sont par ailleurs des torseurs sous l'action du noyau de π, ce dernier se décomposant comme un produit de groupes de Picard locaux (cf. III.8.11). On use alors de la compatibilité locale-globale en théorie du corps de classes géométrique, ainsi que de la formule de Künneth, ce qui résulte en l'obtention d'une décomposition du déterminant de la cohomologie en un produit de facteurs locaux ; nous renvoyons à III.8.11 pour plus de détails.

0.2.5. Théorie du corps de classes géométrique. Revenons sur l'ingrédient essentiel de la méthode de Deligne en rang 1, à savoir la théorie du corps de classes géométrique, sous la forme du théorème 0.2.4. Ce théorème est initialement obtenu par Serre et Lang, cf. ([**La56**], 6) et [**Se59**], chacun d'entre eux en attribuant à l'autre l'idée essentielle de la démonstration. Cette dernière repose sur la propriété dite "Albanaise" des Jacobiennes généralisées $\mathrm{Pic}_k(X,D)$, due à Rosenlicht [**Ro54**]. En sus de la factorisation du déterminant de la cohomologie d'un système local de rang 1 évoquée dans la section 0.2.3, Deligne en esquisse par ailleurs une démonstration similaire à celle de Serre et Lang dans sa lettre à Serre de 1974.

Deligne a également donné une démonstration de nature géométrique du théorème 0.2.4, dans le cas d'un système local de rang 1 non ramifié : on peut en trouver une exposition, dans le cas d'un corps de base fini, dans ([**La90**], Sect. 2). La méthode de Deligne s'étend aisément au cas d'un système local modérément ramifié. Dans le chapitre II, nous étendons la méthode de Deligne au cas d'un faisceau arbitrairement ramifié, en nous inspirant de notes non publiées de Genestier sur la théorie du corps de classes globale arithmétique. Une telle extension a également été obtenue peu après par Takeuchi dans [**Ta18**] par une méthode différente. Nous étendons par la même occasion le résultat au cas des courbes relatives, comme suit.

Théorème 0.2.6 (cf. II.1.1). *Soit $X \to S$ un morphisme de schémas, propre et lisse à fibres géométriquement connexes de dimension 1. Soit D un diviseur de Cartier effectif relatif sur X, tel que les fibres du morphisme $D \to S$ sont non vides. Supposons que ℓ soit inversible sur S. Soit $\mathcal{F}$ un système local ℓ-adique sur le complémentaire U de D dans X, à ramification bornée par D. Il existe alors un unique (à isomorphisme près) couple $(\chi_{\mathcal{F}}, \beta)$ constitué d'un système local ℓ-adique multiplicatif $\chi_{\mathcal{F}}$ sur $\mathrm{Pic}_S(X,D)$ et d'un isomorphisme $\beta : \Phi^{-1}\chi_{\mathcal{F}} \to \mathcal{F}$.*

Nous renvoyons à la section II.4 pour la définition et l'étude du schéma de Picard généralisé $\mathrm{Pic}_S(X,D)$. Contentons-nous ici d'indiquer que sa formation commute au changement de base, et qu'il coïncide avec le schéma en groupes introduit dans 0.2.3 lorsque S est le spectre d'un corps.

Notre démonstration du théorème 0.2.6 utilise la théorie du corps de classes géométrique locale (cf. II.3), due à Serre [**Se61**] pour un corps de base algébriquement clos et à Contou-Carrere et Suzuki [**Su13**] en général, pour pouvoir établir la proposition clé II.3.14. Nous aurions pu également démontrer cette dernière en usant de la théorie d'Artin-Schreier-Witt, comme dans [**Ta18**], obtenant ainsi le théorème 0.2.6 sans utiliser la théorie du corps de classes géométrique locale.

Nous expliquons dans III.5.34 comment déduire du théorème 0.2.6 et de la théorie des extensions de Gabber-Katz (cf. 0.2.14 ci-après) la version suivante du théorème principal de la théorie du corps de classes locale géométrique.

THÉORÈME 0.2.7 (cf. III.5.26). *Soit T un trait hensélien d'équicaractéristique distincte de ℓ, de point générique η et de point fermé s, soit D un diviseur de Cartier effectif sur T et soit π une uniformisante de T. Soit $\mathrm{Pic}(T,D)_s$ le s-schéma de Picard local de (T,D) (cf. III.5.20) et soit $\Phi_{\eta,\pi}:\eta\to\mathrm{Pic}(T,D)_s$ le morphisme d'Abel-Jacobi local (cf. III.5.23). Soit $\mathcal{F}$ un système local ℓ-adique sur η, à ramification bornée par D. Il existe alors un unique (à isomorphisme près) couple $(\chi_{\mathcal{F}},\beta)$ constitué d'un système local ℓ-adique multiplicatif $\chi_{\mathcal{F}}$ sur $\mathrm{Pic}(T,D)_s$ et d'un isomorphisme $\beta:\Phi_{\eta,\pi}^{-1}\chi_{\mathcal{F}}\to\mathcal{F}$.*

En passant à la limite sur le diviseur D, on retrouve le théorème de Suzuki ([**Su13**], Th. A (1)), cf. III.5.30. Nous donnons également une version du théorème 0.2.7 où $\mathrm{Pic}(T,D)_s$ est remplacé par un T-schéma en groupes $\mathrm{Pic}(T,D)$ de fibre spéciale $\mathrm{Pic}(T,D)_s$, et où le morphisme d'Abel-Jacobi local devient indépendant de π (cf. III.5.25). Cette dernière formulation, essentiellement due à Gaitsgory (cf. III.5.28), se révèle être, de par son caractère canonique, particulièrement adaptée à la démonstration d'énoncés de compatibilité, tels la compatibilité local-global (cf. III.5.36) ou la fonctorialité par rapport à la norme (cf. III.5.37). Il est notable que nous démontrons ces deux dernières propriétés sans jamais nous référer aux constructions proprement dites de la théorie du corps de classes géométrique.

0.2.8. Facteurs locaux ℓ-adiques. Revenons au problème 0.2.2, restreint au cas d'une courbe projective lisse sur un corps parfait k de caractéristique p. Nous avons évoqué dans 0.2.3 la solution apportée par Deligne dans le cas d'un faisceau ℓ-adique de rang générique 1, en faisant usage de la théorie du corps de classes géométrique explicitée dans 0.2.5. Intéressons-nous maintenant au cas des faisceaux ℓ-adiques de rang quelconque, ce qui nous permettra par la même occasion d'expliciter la décomposition obtenue par Deligne en rang 1.

Il s'agit donc de décomposer le déterminant de la cohomologie d'un faisceau ℓ-adique sur une courbe projective lisse définie sur un corps parfait k de caractéristique p, en un produit de contributions locales. Précisons à présent la nature de ces facteurs locaux. Ceux-ci dépendent du choix d'un caractère injectif $\psi:\mathbb{F}_p\to\overline{\mathbb{Q}}_\ell^\times$, fixé dans cette exposition. On se donne également une clôture algébrique $\overline{k}$ de k. Nous faisons ici, pour simplifier l'exposition, l'hypothèse suivante, en renvoyant à III.1.6 pour une présentation qui l'omet :

$(\star)$ aucune extension finie de k contenue dans $\overline{k}$ ne contient toutes les racines de l'unité ℓ-primaires de $\overline{k}$.

L'hypothèse $(\star)$ est par exemple vérifiée si k est fini, ou encore si celui-ci est la perfection d'un corps de type fini sur $\mathbb{F}_p$. Considérons à présent les triplets $(T,\mathcal{F},\omega)$ où T est un trait hensélien sur k, de point fermé s fini sur k, où ω est une 1-forme différentielle méromorphe non nulle sur T (cf. III.7.1) et où $\mathcal{F}$ est un complexe borné de faisceaux ℓ-adiques sur T. On définit le *conducteur* d'un tel triplet comme étant l'entier

$$a(T,\mathcal{F},\omega)=a(T,\mathcal{F})+\mathrm{rg}(\mathcal{F}_\eta)v(\omega),$$

où on a posé

$$a(T,\mathcal{F})=\mathrm{rg}(\mathcal{F}_\eta)+\mathrm{sw}(\mathcal{F}_\eta)-\mathrm{rg}(\mathcal{F}_{\overline{s}}),$$

où η est le point générique de T, où sw est le conducteur de Swan, et où $v(\omega)$ est la valuation de la 1-forme ω.

Une *théorie des facteurs locaux ℓ-adiques sur k, de caractère ψ,* est une application ε qui à un tel triplet $(T, \mathcal{F}, \omega)$ associe un système local ℓ-adique $\varepsilon(T, \mathcal{F}, \omega)$ de rang 1 sur le point fermé s de T, satisfaisant aux propriétés suivantes :

(i) la classe d'isomorphisme de $\varepsilon(T, \mathcal{F}, \omega)$ ne dépend que de la classe d'isomorphisme du triplet $(T, \mathcal{F}, \omega)$;

(ii) pour tout triangle distingué

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \xrightarrow{[1]}$$

de complexes bornés de faisceaux ℓ-adiques sur T, on a

$$\varepsilon(T, \mathcal{F}, \omega) \cong \varepsilon(T, \mathcal{F}', \omega) \otimes \varepsilon(T, \mathcal{F}'', \omega);$$

(iii) si $\mathcal{F}$ est concentré sur le point fermé s de T, alors $\varepsilon(T, \mathcal{F}, \omega)$ est isomorphe à $\det(\mathcal{F}_s)^{-1}$;

(iv) pour toute extension finie génériquement étale $f : T' \to T$ de traits henséliens sur k, de points fermés respectifs s et s', on a

$$\varepsilon(T, f_*\mathcal{F}, \omega) \cong \delta_{s'/s}^{\otimes a(T', \mathcal{F})} \otimes \mathrm{Ver}_{s'/s}\left(\varepsilon(T', \mathcal{F}, f^*\omega)\right),$$

pour tout complexe borné de faisceaux ℓ-adiques $\mathcal{F}$ sur T', *de rang générique nul*, où le caractère $\delta_{s'/s}$ est le morphisme signature pour l'action de $\mathrm{Gal}(\overline{s}/s)$ sur $\mathrm{Hom}_s(\overline{s}, s')$ et $\mathrm{Ver}_{s'/s}$ est la composition avec l'homomorphisme de transfert $\mathrm{Gal}(\overline{s}/s)^{\mathrm{ab}} \to \mathrm{Gal}(\overline{s'}/s')^{\mathrm{ab}}$ (cf. III.3.22) ;

(v) si $j : \eta \to T$ est le point générique de T et si $\mathcal{F}$ est un système local ℓ-adique de rang 1 sur η, de conducteur de Swan $\nu - 1$, alors

$$\varepsilon(T, j_!\mathcal{F}[0], \omega) \cong R^\nu f_!\left(\chi_{\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}\right)(-v(\omega)),$$

où $f : \mathrm{Pic}^{\nu+v(\omega)}(T, \nu s)_s \to s$ est la composante de degré $\nu+v(\omega)$ du s-schéma en groupes $\mathrm{Pic}^{\nu+v(\omega)}(T, \nu s)_s$ (cf. 0.2.7), où $\chi_{\mathcal{F}}$ est le système local ℓ-adique multiplicatif associé à $\mathcal{F}$ par la théorie du corps de classes locale géométrique (cf. 0.2.7), et où $\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ est le faisceau d'Artin-Schreier associé au morphisme résidu Res_ω, cf III.7.5 pour plus de détails ;

(vi) si $f : X \to s$ est une courbe projective lisse géométriquement connexe de genre g sur une extension finie s de k, si ω est une 1-forme non nulle sur X et si $\mathcal{F}$ est un complexe borné de faisceaux ℓ-adiques sur X, alors on a la formule du produit

$$\det(Rf_*\mathcal{F})^{-1}(1-g) \cong \bigotimes_{x \in |X|} \delta_{x/s}^{\otimes a(X_{(x)}, \mathcal{F}_{|X_{(x)}})} \otimes \mathrm{Ver}_{x/s}\left(\varepsilon(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})\right),$$

où $|X|$ est l'ensemble des points fermés de X et $X_{(x)}$ est l'hensélisé de X en un point fermé x, le produit tensoriel au membre de droite ne comptant qu'un nombre fini de termes non triviaux.

La méthode de Deligne, esquissée ci-haut dans 0.2.3, et décrite en détail dans III.8, donne alors le résultat qui suit.

THÉORÈME 0.2.9 (cf. III.7.7, III.7.10, III.8.3). *Soit k un corps parfait de caractéristique p et soit $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ un caractère injectif. Les axiomes $(i), (ii), (iii)$ et (v) caractérisent (à isomorphisme près) les facteurs locaux ℓ-adiques des triplets $(T, \mathcal{F}, \omega)$ où $\mathcal{F}$ est de rang générique* 1*, et la formule du produit (vi) est alors vérifiée pour les faisceaux ℓ-adiques de rang générique* 1.

Il s'agit donc d'étendre ce résultat de Deligne en rang supérieur. La première difficulté, avant même de démontrer la formule du produit (vi) en rang quelconque, est de construire des facteurs locaux vérifiant les axiomes $(i) - (v)$. Nous reviendrons sur ce point dans les paragraphes 0.2.12

et 0.2.13 ci-après. Contentons-nous pour l'heure d'énoncer l'instance suivante des résultats du chapitre III.

Théorème 0.2.10 (cf. III.1.7, III.1.8, III.1.9). *Soit k un corps parfait de caractéristique p vérifiant $(\star)$, et soit $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ un caractère injectif. On a alors :*

(1) il existe une unique théorie des facteurs locaux ℓ-adiques sur k, de caractère ψ (à isomorphisme près) ;

(2) si k est fini, l'application qui à un triplet $(T, \mathcal{F}, \omega)$ associe
$$(-1)^{a(T,\mathcal{F})}\mathrm{Tr}\left(\mathrm{Frob}_s \mid \varepsilon(T, \mathcal{F}, \omega)\right),$$
où s est le point fermé de T et Frob_s est la substitution de Frobenius géométrique de s, coïncide avec le facteur local classique de $(T, \mathcal{F}, \omega)$, normalisé comme dans ([La87], Th. 3.1.5.4).

Les énoncés III.1.7, III.1.8 et III.1.9 n'imposent pas à k de vérifier $(\star)$; il faut alors se restreindre aux triplets $(T, \mathcal{F}, \omega)$ tels que $\mathcal{F}$ est *potentiellement unipotent* (cf. III.9.8). Le théorème de monodromie locale de Grothendieck assure que cette dernière hypothèse est nécéssairement vérifiée lorsque k vérifie $(\star)$ (cf. III.9.9).

Pour un corps de base k *fini*, l'existence et l'unicité de facteurs locaux vérifiant les axiomes $(i) - (v)$ est un théorème démontré indépendamment par Langlands [**Lan**] et Deligne [**De73**], améliorant un résultat antérieur de Dwork [**Dw56**], qui construisait les facteurs locaux au signe près. La formule du produit (vi) est alors démontrée, toujours dans le cas d'un corps de base fini, par Deligne [**De73**] pour les faisceaux ℓ-adiques à monodromie géométrique finie, puis par Laumon ([**La87**], Th. 3.2.1.1) en général. L'ingrédient clé de la démonstration de Laumon est sa variante ℓ-adique de la méthode de la phase stationnaire (cf. III.11.5), utilisée en particulier pour démontrer une formule cohomologique pour les facteurs locaux ℓ-adiques ([**La87**], 3.5.1.1). Nous généralisons cette dernière au cas d'un corps de base non nécéssairement fini, comme suit.

Théorème 0.2.11 (cf. III.1.10). *Soit k un corps parfait de caractéristique p vérifiant $(\star)$, et soit $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ un caractère injectif. Soit $(T, \mathcal{F}, \omega)$ un triplet sur k, tel que $\omega = d\pi$ pour une certaine uniformisante π sur T ; on note s le point fermé de T. Soit $\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F})$ la transformée de Fourier locale de Laumon du faisceau ℓ-adique $\mathcal{F}$, cf. ([**La87**], 2.4.1) : il s'agit d'un système local ℓ-adique sur le point générique de l'hensélisation $(\mathbb{P}^1_s)_{(\infty)}$ de $\mathbb{P}^1_s$ à l'infini. On a alors :*

(1) le caractère $\det(\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F}))$ est modérément ramifié, et la théorie du corps de classes locale géométrique (cf. 0.2.7) lui associe donc un système local multiplicatif $\chi_{\det(\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F}))}$ sur $\mathrm{Pic}((\mathbb{P}^1_s)_{(\infty)}, \infty)_s$;

(2) considérant la variable t de $\mathbb{P}^1_s$ comme un s-point de $\mathrm{Pic}^1((\mathbb{P}^1_s)_{(\infty)}, \infty)_s$, on a un isomorphisme
$$\varepsilon(T, \mathcal{F}, d\pi) \cong t^{-1}\chi_{\det(\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F}))}.$$

Nous aurions pu démontrer le théorème 0.2.11 en adaptant fidèlement l'argument de Laumon à ce cadre plus général. Nous choisissons d'en présenter dans III.11.7 une démonstration plus directe, exploitant le caractère géométrique de nos facteurs locaux.

0.2.12. Théorie de Brauer. Revenons à présent sur le théorème 0.2.10, et considérons le problème consistant à définir des facteurs locaux ℓ-adiques vérifiant les axiomes 0.2.8$(i)-(v)$. La stratégie utilisée à cet effet par Deligne et Langlands dans le cas d'un corps de base fini consiste à utiliser le *théorème d'induction de Brauer* pour se ramener par la formule d'induction 0.2.8(iv) au cas d'un système local de rang générique 1 dont le facteur local est alors prescrit par 0.2.8(v).

Le théorème de Brauer auquel nous venons de faire allusion est le suivant. Si G est un groupe fini, alors le groupe des classes de $\overline{\mathbb{Q}}_\ell$-représentations virtuelles de G *de rang virtuel nul* est engendré par les classes de la forme $\mathrm{Ind}_H^G\chi - \mathrm{Ind}_H^G\overline{\mathbb{Q}}_\ell$, où H est un sous-groupe de G et χ est une $\overline{\mathbb{Q}}_\ell$-représentation de dimension 1 de H (cf. III.2.35). L'ensemble des classes de cette forme, auquel on adjoint la classe de la représentation triviale de G, engendre alors le groupe des classes de $\overline{\mathbb{Q}}_\ell$-représentations virtuelles de G.

Partant d'un triplet $(T, \mathcal{F}, \omega)$ avec $\mathcal{F}$ génériquement de monodromie finie, le théorème d'induction de Brauer montre que les axiomes 0.2.8$(i)-(v)$ caractérisent (à isomorphisme près) le facteur local $\varepsilon(T, \mathcal{F}, \omega)$, et donne par ailleurs un candidat pour ce dernier : toute la difficulté consiste alors à démontrer que le facteur local ainsi construit est indépendant des choix effectués. Cette dernière question est résolue par Langlands dans [**Lan**] par une méthode purement locale, tandis que Deligne en donne dans [**De73**] une démonstration par un argument de comparaison local-global.

0.2.13. Monodromie locale et représentations tordues par un 2-cocycle. Soit k un corps parfait de caractéristique p vérifiant $(\star)$, et soit $(T, \mathcal{F}, \omega)$ un triplet sur k. Supposons $\mathcal{F}$ supporté sur le point générique η de T, et irréductible. Par le théorème de monodromie locale de Grothendieck (cf. III.9.9), le faisceau $\mathcal{F}$ est alors à monodromie géométrique finie. Si l'on souhaite définir le facteur local $\varepsilon(T, \mathcal{F}, \omega)$ en suivant la stratégie de 0.2.12, c'est-à-dire en appliquant le théorème d'induction de Brauer pour se ramener par induction et multiplicativité au rang 1, il faut tout d'abord se ramener au cas où $\mathcal{F}$ est à monodromie finie, et pas seulement à monodromie géométrique finie.

Soit s le point fermé de T. Si k est fini, alors le corps résiduel $k(s)$ l'est également, de sorte que le groupe de Galois $G = \mathrm{Gal}(\overline{s}/s)$ est pro-cyclique. On vérifie alors, cf. ([**De73**], 4.10), que le faisceau ℓ-adique $\mathcal{F}$, à monodromie géometrique finie, est de la forme $\mathcal{E} \otimes \mathcal{W}$, où $\mathcal{E}$ est à monodromie finie et $\mathcal{W}$ est non ramifié, de rang 1. Pour un corps de base k général, le groupe de Galois absolu G n'est pas procyclique, ni même abélien, et la réduction au cas d'un faisceau à monodromie finie se révèle plus délicate. Convenablement interprétée, elle est néanmoins possible, comme nous allons à présent l'expliquer.

Notant I le groupe de monodromie géométrique de $\mathcal{F}$, la fibre $V = \mathcal{F}_{\overline{\eta}}$ du faisceau $\mathcal{F}$ est une représentation ℓ-adique d'un quotient Q de $\mathrm{Gal}(\overline{\eta}/\eta)$, s'insérant dans une suite exacte

$$1 \to I \xrightarrow{\iota} Q \xrightarrow{\pi} G \to 1.$$

Le groupe G agit alors par conjugaison sur l'ensemble des caractères irréductibles du groupe fini I, et cette action permute les facteurs isotypiques de la représentation V de I. Comme V est supposée irréductible, une unique G-orbite de caractères de I apparaît dans V, de sorte que l'on peut écrire

$$V \cong \mathrm{Ind}_{Q_\chi}^{Q} V[\chi],$$

où Q_χ est l'image réciproque dans Q du stabilisateur G_χ dans G d'un certain caractère irréductible χ du groupe fini I, et $V[\chi]$ est la composante χ-isotypique de V, considérée comme représentation de I.

Soit E une représentation de I de caractère χ, et soit W l'espace vectoriel des homomorphismes I-équivariants de E dans $V[\chi]$. On dispose alors d'un isomorphisme

$$\begin{aligned} \theta : E \otimes W &\to V[\chi] \\ e \otimes w &\mapsto w(e). \end{aligned}$$

Celui-ci devient I-équivariant si l'on munit W de l'action triviale de I. Se pose alors la question suivante : peut-on munir E et W d'actions de Q_χ et G_χ respectivement qui rendent Q_χ-équivariant l'isomorphisme θ ?

La réponse à cette question est négative en général, mais on a néanmoins le résultat suivant (cf. III.2.39) : il existe des applications $\rho_E : Q_\chi \to \mathrm{GL}(E)$ et $\rho_W : G_\chi \to \mathrm{GL}(W)$ telles que le

produit tensoriel
$$\rho_E \otimes \rho_W : Q_\chi \to GL(E \otimes W),$$
correspond par θ à l'action de Q_χ sur $V[\chi]$, et telles que l'on a
$$\rho_E(q_1)\rho_E(q_2) = \mu(\pi(q_1), \pi(q_2))\rho_E(q_1q_2)$$
$$\rho_W(g_1)\rho_W(g_2) = \mu(g_1, g_2)^{-1}\rho_W(g_1g_2),$$
pour tout couple q_1, q_2 (respectivement g_1, g_2) d'éléments de Q_χ (respectivement G_χ), pour une certaine application continue $\mu : G_\chi \times G_\chi \to \overline{\mathbb{Q}}_\ell^\times$ à valeurs dans le groupe des racines de l'unité de $\overline{\mathbb{Q}}_\ell$. En particulier, l'application ρ_E induit un homomorphisme de Q_χ vers $\mathrm{PGL}(E)$: on pourra donc qualifier ρ_E de *représentation projective* de Q_χ, ou encore de *représentation tordue par* μ.

On notera que l'application μ est nécessairement un 2-cocycle sur G_χ (cf. III.2.6). Remplacer ρ_E et ρ_W par $(\lambda \circ \pi)\rho_E$ et $\lambda^{-1}\rho_W$ respectivement, pour une certaine application $\lambda : G_\chi \to \overline{\mathbb{Q}}_\ell^\times$, à valeurs dans le groupe des racines de l'unité de $\overline{\mathbb{Q}}_\ell$, a pour effet de multiplier μ par le 2-cobord de λ, c'est-à-dire par l'application
$$(g_1, g_2) \to \lambda(g_1)\lambda(g_2)\lambda(g_1g_2)^{-1}.$$
Le 2-cocycle μ apparaissant ci-dessus n'est donc pas uniquement déterminé ; en revanche, sa classe de cohomologie l'est. Lorsque le groupe de cohomologie continue $H^2_{\mathrm{cont}}(G_\chi, \mathbb{Q}/\mathbb{Z})$ est nul, ce qui est par exemple le cas lorsque k est fini, alors tout 2-cocycle sur G_χ est un 2-cobord, et on peut donc supposer $\mu = 1$ ci-dessus ; c'est la réduction utilisée par Deligne et Langlands.

On vérifie que dans le contexte ci-dessus, la représentation tordue ρ_E est nécéssairement d'image projective finie (cf. III.2.38). Nous démontrons par ailleurs dans III.2.36 que le théorème d'induction de Brauer s'étend au cadre des représentations tordues par un 2-cocycle, d'image projective finie. Ceci nous permet dans le reste du chapitre III de nous ramener à des situations de rang 1 par induction et additivité.

La définition et l'étude des représentations tordues est l'objet de la section III.2. La section III.3 est quant à elle consacrée à la contrepartie géométrique des représentations tordues : les faisceaux ℓ-adiques tordus. Le reste du chapitre III est entièrement rédigé en termes de faisceaux tordus. Nous donnons par exemple dans III.1.6 une caractérisation purement locale des facteurs locaux ℓ-adiques, formulée dans ce langage, ce qui permet d'obtenir un énoncé plus précis que le théorème 0.2.10.

0.2.14. Extensions de Gabber-Katz et définition des facteurs locaux. Plutôt que de définir des facteurs locaux à l'aide du procédé de réduction au cas d'un faisceau (tordu) à monodromie (projective) finie (cf. 0.2.13) et du théorème d'induction de Brauer, tels Deligne et Langlands, nous choisissons dans III.9 d'adopter une approche quelque peu différente. Nous commençons par donner une définition cohomologique simple de ces facteurs locaux (cf. III.9.2), et ce n'est qu'ensuite que nous utilisons la théorie de Brauer pour vérifier que ceux-ci vérifient les propriétés 0.2.8$(i) - (v)$.

La définition cohomologique susmentionnée repose sur la théorie des extensions de Gabber-Katz, que nous décrivons dans la section III.4. Contentons-nous ici d'en donner un aperçu. Considérons un triplet $(T, \mathcal{F}, \omega)$ comme dans 0.2.8, avec $\omega = d\pi$ pour une certaine uniformisante π sur T. Notant s le point fermé de T, l'élément π induit un morphisme
$$\pi : T \to \mathbb{A}^1_s.$$
Le théorème de Gabber-Katz (cf. III.4.19) assure alors l'existence (et l'unicité à isomorphisme près) d'un faisceau ℓ-adique $\pi_\diamond\mathcal{F}$ sur $\mathbb{A}^1_s$ satisfaisant aux propriétés suivantes :

(1) la restriction $\pi^{-1}\pi_\diamond\mathcal{F}$ est isomorphe à $\mathcal{F}$;

(2) le faisceau ℓ-adique $\pi_\diamond\mathcal{F}$ est modérément ramifié à l'infini ;

(3) la restriction de $\pi_\diamond \mathcal{F}$ à $\mathbb{G}_{m,s}$ est un système local dont le groupe de monodromie géométrique admet un unique sous-p-groupe de Sylow.

Notant $f : \mathbb{A}^1_s \to s$ la projection naturelle, on se contente alors de poser

$$\varepsilon_\pi(T, \mathcal{F}, d\pi) = \det\left(Rf_!(\pi_\diamond \mathcal{F} \otimes \mathcal{L}_\psi^{-1})\right)^{-1},$$

où $\mathcal{L}_\psi$ est le système local d'Artin-Schreier associé à ψ. On utilise alors la théorie du corps de classes locale géométrique pour définir $\varepsilon(T, \mathcal{F}, \omega)$ lorsque ω est une 1-forme méromorphe non nulle arbitraire sur T, cf. III.9.2. Nous démontrons ensuite, par réduction au rang 1 via la théorie de Brauer, que ε_π est indépendant de π à isomorphisme près (cf. III.9.16), et satisfait aux propriétés 0.2.8$(i) - (v)$.

0.2.15. Formule du produit. Comme l'avait déjà observé Deligne dans [**De73**], disposer d'un formalisme des facteurs locaux ℓ-adiques satisfaisant aux propriétés 0.2.8$(i) - (v)$ implique la validité de la formule 0.2.8(vi) pour les faisceaux ℓ-adiques *à monodromie géométrique finie* : après réduction au cas d'un faisceau à monodromie finie (cf. 0.2.13), appliquer le théorème de Brauer permet de supposer que le faisceau ℓ-adique considéré est de rang générique 1, auquel cas la formule du produit résulte de la thèse de Tate [**Ta50**] si k est fini. Pour un corps parfait non nécessairement fini, on applique plutôt le théorème 0.2.9, qui donne la formule du produit en rang 1. Nous menons dans la section III.10 les vérifications nécéssaires, menant ainsi à la formule du produit (cf. III.10.3) pour les faisceaux ℓ-adiques à monodromie géométrique finie.

Pour démontrer la formule du produit dans le cas général, tel qu'énoncé dans 0.2.8(vi), nous adaptons dans III.11 la démonstration par Laumon [**La87**] de l'énoncé analogue pour les corps finis. Celle-ci repose sur la méthode de la phase stationnaire ℓ-adique (cf. III.11.5), ainsi que sur la formule donnée dans le théorème 0.2.11 :

$$\varepsilon(T, \mathcal{F}, d\pi) \cong t^{-1} \chi_{\det(\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F}))}.$$

Cette formule est elle-même une conséquence de la méthode de la phase stationnaire ℓ-adique. Pour un corps de base fini, la démonstration par Laumon de cette formule débute par une réduction au cas modérément ramifié ([**La87**], 3.5.3.1), et poursuit avec un calcul explicite dans ce dernier cas ([**La87**], 2.5.3.1). Nous choissons de passer outre ces étapes en donnant une démonstration directe à partir de la méthode de la phase stationnaire ℓ-adique (cf. III.11.5) ; le point clé est l'indépendance en le choix de l'uniformisante des facteurs locaux tels qu'introduits dans 0.2.14, que nous appliquons à un trait hensélien dont le corps résiduel n'est pas une extension finie de k, mais plutôt la perfection d'un corps discrètement valué hensélien d'équicaractéristique p, de corps résiduel k.

Chapitre I

Aplatissement par éclatement admissible

Sommaire

I.1. Introduction

In Raynaud's approach to rigid geometry, the category of quasi-compact quasi-separated rigid spaces over a completely valued field K of rank 1 is defined as the localization with respect to "admissible blow-ups" of the category of finitely presented formal schemes over the ring of integers $R \subseteq K$ (cf. [**Ray74**]). It has been known since then that a flat morphism of quasi-compact quasi-separated rigid spaces over K can be represented by a flat morphism between appropriate R-models. A precise statement and a proof can be found in ([**BL93**], 5.2) or ([**Abb10**], 5.8.1). The schematic version of this flattening theorem had been previously proved by Raynaud and Gruson in 1971:

Theorem I.1.1 (([**SP**] 0815), cp. ([**RG71**] *I*.5.2.2)). *Let Y be a scheme of finite presentation over a quasi-compact and quasi-separated scheme X, let U be a quasi-compact open subset of X and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type. Assume that the restriction of $\mathcal{F}$ to $Y \times_X U$ is a finitely presented $\mathcal{O}_{Y\times_X U}$-module which is flat over U. Then there exists a blow-up $f : X' \to X$ such that:*

(1) *The center of the blow-up f is a finitely presented closed subscheme of X, disjoint from U.*
(2) *If Y' is the strict transform of the X-scheme Y along f, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ along f is finitely presented over $\mathcal{O}_{Y'}$ and flat over X'.*

Here, by "strict transform" we mean the following: if Z is the exceptional Cartier divisor of the blow-up f, then Y' is the closed subscheme of $X' \times_X Y$ defined by the vanishing of the quasi-coherent ideal of sections supported on $Z \times_X Y$. Similarly, the strict transform $\mathcal{F}'$ of $\mathcal{F}$ along f is the pullback to Y' of the quotient of $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{X'\times_X Y}$ by the submodule of its sections supported on $Z \times_X Y$.

Our proof of Raynaud-Gruson's theorem proceeds in two steps.We first consider the (filtered) limit

$$\widetilde{X} = \lim_{X' \to X} X'$$

of all blow-ups as allowed in the statement of the theorem. This is not a scheme, but a ringed space which we identify to a valuative space I.4.13. These valuative spaces are constructed and studied in greater generality in Section I.4, using the langage of "rings with constructible supports", whose set-up is the purpose of Sections I.2 and I.3. We obtain in this broader context a more general version of Raynaud-Gruson's theorem, namely Theorem I.5.9, which implies that a variant of the conclusion of Theorem I.1.1 holds over $\widetilde{X}$ (cf. Section I.5). The second and last step consists in descending the properties shown to hold on the limit $\widetilde{X}$ to some blow-up (cf. Section I.6). We now give a more detailed description of the content of this text.

In Section I.2, we define the notion of "rings with constructible supports" or "Φ-rings" (cf. I.2.3) and we consider their basic properties. We include a few sorites on the notion of depth (cf. I.2.13 and I.2.26), and give an interpretation of the closure and purification functors from ([**EGA4**], 5.9, 5.10) as adjoint functors, cf. I.2.31 and I.2.32. We conclude Section I.2 with the definition and the study of the notion of "Φ-local" Φ-rings.

Section I.3 is devoted to the definition of sheaves of Φ-rings and to their basic properties. It is noticed there that the notion of Φ-ring is well-defined in any topos, cf. I.3.5, and that the notions and properties from Section I.2 extend to this broader context.

In Section I.4, to any locally Φ-ringed topological space X we associate a valuative space $\widetilde{X}$, which is itself a locally Φ-ringed topological space, endowed with a morphism $\widetilde{X} \to X$, cf. I.4.1, I.4.3 and I.4.5. We call this valuative space the "Φ-localization" of X. It is actually a Φ-locally Φ-ringed topological space, and the morphism $\widetilde{X} \to X$ is universal for this property, cf. I.4.10. We then consider the context of Theorem I.1.1, in which we endow the base scheme X with a structure of locally Φ-ringed topological space; we show that the valuative space associated to X coincides with the limit of all blow-ups as allowed in Theorem I.1.1, cf. I.4.13.

In Section I.5, Φ-localizations are shown to have a flattening property, cf. I.5.8 and I.5.9. The main argument is purely local, cf. I.5.2, and rests upon the equational criterion for flatness. Theorem I.1.1 is then deduced from I.5.9 in Section I.6.

Ahmed Abbes has brought to our attention during the preparation of this text that a proof of Raynaud-Gruson's theorem similar to ours has been announced by Kazuhiro Fujiwara and Fumiharu Kato in [**FK06**]. Besides the sketch given in ([**FK06**] 5.6), their proof does not seem to have appeared in print. We notice that our treatment of the local case, namely I.5.2 and I.5.3, seems to differ from theirs.

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I.2. Rings with constructible supports

I.2.1. Let A be a ring. A **family of constructible supports** on A is a set Φ of finitely generated ideals of A which satisfies the following two properties:

(1) Φ is stable under finite products of ideals. In particular the unit ideal belongs to Φ.
(2) If a finitely generated ideal I of A contains an element of Φ, then I is itself in Φ.

Examples I.2.2. (*i*) Given an arbitrary set Φ_0 of ideals of A, the **family of constructible supports generated by** Φ_0 is the set of finitely generated ideals of A which contain a finite product of elements of Φ_0. If Φ_0 consists only of finitely generated ideals, then this family of constructible supports is the smallest one containing Φ_0.

(*ii*) If A is a preadmissible topological ring in the sense of ([**EGA1**] 0.7.1.2), i.e. a linearly topologized ring with an ideal of definition (an open ideal I of A such that any neighbourhood of 0 in A contains I^N for some integer N), then one can consider the family of constructible supports generated by all the ideals of definition. This family of constructible supports consists precisely of the finitely generated open ideals of A whenever A is preadic in the sense of ([**EGA1**] 0.7.1.9), i.e. when A has an ideal of definition I such that I^N is open in A for any integer N.

I.2.3. The category ΦRings of **rings with constructible supports** or **Φ-rings** is defined as follows:

- ▷ Its objects are the pairs (A, Φ_A) consisting of a ring A and a family of constructible supports Φ_A on A (cf. I.2.1). An ideal of A is said to be **admissible** if it belongs to Φ_A.
- ▷ Its morphisms are the ring homomorphisms $f : A \to B$ such that for any admissible ideal I of A, the ideal $f(I)B$ of B is admissible.

From now on, we commit the abuse of notation where we use the same name for a ring with constructible supports and for its underlying ring, just as it is customary to use the same symbol for a ring and for its underlying set.

Proposition I.2.4. *The category* ΦRings *is complete and cocomplete, and the forgetful functor from* ΦRings *to the category of rings preserve all small limits and colimits.*

Indeed let us consider a functor F from a small category J to ΦRings. Let us endow the ring $\lim F$ with the family of constructible supports which consists of all the finitely generated ideals I such that $IF(j)$ is admissible in $F(j)$ for all j. The resulting ring with supports is a limit of F in ΦRings. Moreover if colim F is endowed with the family of constructible supports generated in the sense of I.2.2 (i) by all the ideals generated by an admissible ideal of $F(j)$ for some j, then the resulting ring with supports is a colimit of F in ΦRings.

Proposition I.2.5. *The forgetful functor from* ΦRings *to the category of rings has both a left adjoint and a right adjoint.*

Indeed the functor which endows a ring A with the family of constructible supports consisting only of the unit ideal is a left adjoint to the forgetful functor, while the functor which endows a ring A with the family of constructible supports consisting of all of its finitely generated ideals is a right adjoint to the forgetful functor.

Remark I.2.6. Let $f : A \to B$ be a ring homomorphism between preadic topological rings in the sense of ([**EGA1**] 0.7.1.9). Let us endow A and B with the family of constructible supports consisting of all their finitely generated open ideals as in Example I.2.2 (ii), and let us assume that A has a finitely generated ideal of definition. Then f yields a morphism of ΦRings if and only if the topology on B is finer that the topology induced by that of A. In particular f may be a morphism of ΦRings without being continuous.

I.2.7. A **family of supports** on a topological space X is a set Φ of closed subsets of X which satisfies the following two properties:

(1) Φ is stable under finite unions. In particular the empty set belongs to Φ.

(2) If a closed subset of Z of X is contained in an element of Φ, then Z is itself in Φ.

This definition is taken from ([**God73**] II.2.5), besides the fact that we require a family of supports to be nonempty.[1] The notion of family of supports was introduced in [**Car51**], but additional conditions on Φ were required, e.g. paracompactness of its elements.

1. It is indeed mistakenly asserted in ([**God73**] II.2.5) that for a family of supports Φ in the sense defined there and for a sheaf of abelian groups on X, the set of global sections of this sheaf with supports in Φ is an abelian group. This requires Φ to contain the support of the zero section, hence to be nonempty.

Example I.2.8. Given an arbitrary set Φ_0 of closed subsets of X, the set of closed subsets of X which are contained in a finite union of elements of Φ_0 is a family of supports on X. It is the **family of supports generated by** Φ_0, i.e. the smallest family of supports on X containing Φ_0

I.2.9. If Φ is a family of constructible supports on a ring A in the sense of I.2.3, then the set of closed subsets $Z \subseteq X$ contained in $V(I)$ for some $I \in \Phi$ is a family of supports on $\mathrm{Spec}(A)$ which is generated (cf. I.2.8) by its globally constructible elements ([**EGA1**], 0.2.3.2). Conversely, if Φ is a family of supports on $\mathrm{Spec}(A)$ generated by its globally constructible elements then the set of finitely generated ideals I such that $V(I)$ belongs to Φ is a family of constructible supports on A.

These two constructions are inverse to each other. Indeed if J is a finitely generated ideal such that $V(J) \subseteq V(I)$ for some admissible ideal $I = (f_1, \dots, f_r)$, then $f_i^{N_i}$ belongs to J for some integer N_i. For $N = \sum_i N_i$ this yields $I^N \subseteq J$, so that J is admissible. On the other hand, if we start from a family of supports Φ on $\mathrm{Spec}(A)$ generated by its globally constructible elements, then for any globally constructible element Z of Φ, the complement of Z is a globally constructible open subset of $\mathrm{Spec}(A)$, hence is quasi-compact, so that it is a finite union of standard open subsets $(D(f_i))_{1 \leq i \leq r}$. Thus $Z = V(I)$ where I is the ideal generated by $(f_i)_{1 \leq i \leq r}$.

I.2.10. Let X be a topological space and let Φ be a family of supports (cf. I.2.7). Let us consider as in ([**God73**] *II*.2.5) the functor $\Gamma_\Phi(X, -)$ which to a sheaf of abelian groups $\mathcal{F}$ on X associates the abelian group

$$\Gamma_\Phi(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) \mid \mathrm{supp}(s) \in \Phi\}.$$

It is a left exact additive functor between abelian categories, and its source has enough injectives ([**God73**] 7.1.1). Consequently, it has right derived functors ([**God73**] 7.2), which are denoted by $H^q_\Phi(X, \mathcal{F})$. When Φ is the set of closed subsets of a given closed set Z, these groups are also denoted by $H^q_Z(X, \mathcal{F})$.

Proposition I.2.11. *For any sheaf of abelian groups $\mathcal{F}$ on X, one has*

$$H^q_\Phi(X, \mathcal{F}) \cong \mathrm{colim}\ H^q_Z(X, \mathcal{F}),$$

where the (filtered) colimit runs over elements Z of Φ.

Indeed, if $\mathcal{F} \to \mathcal{J}^\bullet$ is an injective resolution of $\mathcal{F}$, then $\Gamma_\Phi(X, \mathcal{J}^\bullet)$ is the (cofiltered) colimit of the complexes of abelian groups $\Gamma_Z(X, \mathcal{J}^\bullet)$ when Z runs over elements of Φ. Since filtered colimits are exact, taking cohomology groups of these complexes yields I.2.11.

Proposition I.2.12. *Let Z and Z' be closed subsets of X, and let $\mathcal{F}$ be a sheaf of abelian groups on X.*

(a) *There exists a long exact sequence*

$$\cdots \to H^q_{Z \cap Z'}(X, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \oplus H^q_{Z'}(X, \mathcal{F}) \to H^q_{Z \cup Z'}(X, \mathcal{F}) \to H^{q+1}_{Z \cap Z'}(X, \mathcal{F}) \to \cdots$$

(b) *Let $U = X \setminus Z$. There exists a long exact sequence*

$$\cdots \to H^q_{Z \cap Z'}(X, \mathcal{F}) \to H^q_{Z'}(X, \mathcal{F}) \to H^q_{Z' \cap U}\left(U, \mathcal{F}_{|U}\right) \to H^{q+1}_{Z \cap Z'}(X, \mathcal{F}) \to \cdots$$

Let $\mathcal{F} \to \mathcal{J}^\bullet$ be a flasque resolution ([**KS90**] II.2.4.6vi). Part (a) follows from the exactness of the sequence of complexes

$$0 \to \Gamma_{Z \cap Z'}(X, \mathcal{J}^\bullet) \to \Gamma_Z(X, \mathcal{J}^\bullet) \oplus \Gamma_{Z'}(X, \mathcal{J}^\bullet) \to \Gamma_{Z \cup Z'}(X, \mathcal{J}^\bullet) \to 0,$$

while part (b) follows from the exactness of the sequence of complexes

$$0 \to \Gamma_{Z \cap Z'}(X, \mathcal{J}^\bullet) \to \Gamma_{Z'}(X, \mathcal{J}^\bullet) \to \Gamma_{Z' \cap U}(U, \mathcal{J}^\bullet_{|U}) \to 0.$$

Alternatively, part (b) is an instance of ([**SGA2**] 2.2.8, 2.2.2).

I.2.13. Let A be a Φ-ring (cf. I.2.3), and let Φ_A be the corresponding family of supports on $X = \mathrm{Spec}(A)$ as in I.2.7. If M is an A-module, corresponding to the quasi-coherent sheaf $\widetilde{M}$ on X, then we set

$$H_I^q(M) = H_{V(I)}^q(X, \widetilde{M}) \quad \text{and} \quad H_{\Phi_A}^q(M) = H_{\Phi_A}^q(X, \widetilde{M}).$$

Definition I.2.14. Let $d \geq 0$ be an integer. A module M over the Φ-ring A is said to be d-deep if $H_{\Phi_A}^q(M)$ vanishes for each integer $q < d$.

Remark I.2.15. Assume that A is noetherian, that its family of constructible supports is generated by a single ideal I, and that M is finitely generated. The I-depth of M in the sense of ([**SGA2**] III.2.3) is at least d if and only if M is d-deep in the sense of the definition I.2.14. This follows from ([**SGA2**] III.3.1, III.3.3).

Lemma I.2.16. *Let $d \geq 0$ be an integer. Let I be a finitely generated ideal of a ring A, and let M be an A-module such that $H_I^q(M) = 0$ for any $q < d$. Then for any finitely generated ideal J containing I, we have $H_J^q(M) = 0$ for any $q < d$, and the canonical morphism $H_J^d(M) \to H_I^d(M)$ is injective.*

By induction on the number of generators of J, one can assume that $J = (I, g)$ for some element g. One then applies I.2.12 (b) to $Z = V(g)$ and $Z' = V(I)$, so that $U = D(g)$. The resulting long exact sequence takes the form

$$\cdots \to H_J^q(M) \to H_I^q(M) \to H_I^q\left(M[g^{-1}]\right) \to H_J^{q+1}(M) \to \cdots.$$

By ([**SGA2**] II.2) and by the hypothesis, one has

$$H_I^q\left(M[g^{-1}]\right) \cong H_I^q(M)\,[g^{-1}] \cong 0$$

for $q < d$. This proves the injectivity of $H_J^q(M) \to H_I^q(M)$ for $q \leq d$ and concludes the proof.

Proposition I.2.17. *Let A be a Φ-ring and let M be a d-deep A-module. Then $H_I^q(M) = 0$ for any admissible ideal I and any $q < d$.*

We prove Proposition I.2.17 by induction on d, the case $d = 0$ being tautological. We are thus led to assume that $d \geq 1$ and that the result has been proved for $(d-1)$-deep modules. By I.2.11 one has

$$H_{\Phi_A}^q(M) \cong \operatorname{colim} H_I^q(M),$$

where I runs over the admissible ideals of A. It is therefore sufficient to prove that for any pair $I \subseteq J$ of admissible ideals and for any $q < d$ the homomorphism $H_J^q(M) \to H_I^q(M)$ is injective. The latter fact follows from Lemma I.2.16, since $H_I^q(M) = 0$ for $q < d-1$ and for any admissible ideal I of A.

Corollary I.2.18. *Let A be a Φ-ring and let Φ_0 be a set of finitely generated ideals of A generating the family of constructible supports of A as in I.2.2(i). An A-module M is d-deep if and only if $H_I^q(M) = 0$ for any I in Φ_0 and any $q < d$.*

The forward direction follows from I.2.17. For the converse we proceed by induction on d, the case $d = 0$ being tautological. We first note that $H_I^q(M) = 0$ for any $q < d$ whenever I is a finite product of elements of Φ_0. Indeed, if I and I' satisfy $H_I^q(M) = H_{I'}^q(M) = 0$ for any $q < d$, then by I.2.12(a) we have an exact sequence

$$0 \to H_{II'}^q(M) \to H_{I+I'}^{q+1}(M) \to H_I^{q+1}(M) \oplus H_{I'}^{q+1}(M),$$

for any $q < d$. Since the canonical homomorphism $H_{I+I'}^{q+1}(M) \to H_I^{q+1}(M)$ is injective by Lemma I.2.16, we obtain the vanishing of $H_{II'}^q(M)$ for any $q < d$.

Any admissible ideal J must contain an ideal I which is a finite product of elements of Φ_0. By Lemma I.2.16, we obtain $H_J^q(M) = 0$ for any admissible ideal J and any $q < d$. By I.2.11 the A-module M must be d-deep.

Proposition I.2.19. *Let M be a 1-deep A-module, and let $I = (f_1, \ldots, f_r)$ be an admissible ideal of A. Then the homomorphism*

$$\mathrm{Hom}_A(I, M) \to M^{\oplus r}$$

sending an element ψ to $(\psi(f_i))_{1 \leq i \leq r}$ is injective, and its image consists of the r-uples $(m_i)_{1 \leq i \leq r}$ such that $f_i m_j = f_j m_i$ for all i, j.

As I is generated by $f_1, \ldots, f_r$, the injectivity is clear. In order to characterize the image, let us consider a free A-module F of rank r with basis $(e_i)_{1 \leq i \leq r}$ and let G be its quotient by the relations $f_i e_j - f_j e_i$. Consider the homomorphism $G \to I$ sending e_i to f_i, and let H be its kernel. The exact sequence

$$0 \to H \to G \to I$$

yields the exact sequence

$$\mathrm{Hom}_A(I, M) \to \mathrm{Hom}_A(G, M) \to \mathrm{Hom}_A(H, M) \to 0.$$

We thus have to show that $\mathrm{Hom}_A(H, M)$ vanishes. Since M has no nonzero I-torsion, it is sufficient to show that $IH = 0$. But if $x = \sum_i a_i e_i$ belongs to H then

$$f_j x = f_j \left(\sum_i a_i e_i \right) = \left(\sum_i a_i f_i \right) e_j = 0$$

holds in G for any j, so that $Ix = 0$.

Proposition I.2.20. *Let M be an A-module. If I is an ideal of A generated by $f_1, \ldots, f_r$, then $H^q_I(M)$ is isomorphic to the q-th cohomology group of the Čech complex*

$$C^\bullet(M, f_\bullet) \ : \ 0 \to M \to \prod_i M\left[\frac{1}{f_i}\right] \to \prod_{i<j} M\left[\frac{1}{f_i f_j}\right] \to \cdots \to M\left[\frac{1}{f_1 \ldots f_r}\right] \to 0$$

where M is in degree 0. This isomorphism is functorial in M.

This is ([**SGA2**] II.5).

Remark I.2.21. It follows from Proposition I.2.20 that for any finitely generated ideal I of A, any A-module M, and any flat ring homomorphism $A \to A'$, the base change homomorphism

$$H^q_I(M) \otimes_A A' \to H^q_{IA'}(M \otimes_A A')$$

is bijective for each integer q.

Corollary I.2.22. *Let $f : A \to B$ be a homomorphism of Φ-rings, an let M be a B-module. If M is 1-deep (respectively, 2-deep) as a B-module, then it is so as an A-module. The converse holds if the family of constructible supports of B is generated by that of A.*

Indeed, it follows from Proposition I.2.20 that for any admissible ideal I of A and any integer q, we have

$$H^q_I(M) \cong H^q_{IB}(M),$$

where M is considered as an A-module on the left hand side, and as a B-module on the right hand side.

Corollary I.2.23. *An A-module M is 1-deep if and only if for any admissible ideal I of A, M has no nonzero I-torsion.*

Indeed by Proposition I.2.20 (or by definition) the module $H^0_I(M)$ consists of all m in M such that $f_i^{N_i} m = 0$ for some integers $(N_i)_{1 \leq i \leq r}$. For $N = \sum_i N_i$ this gives $I^N m = 0$, so that $H^0_{\Phi_A}(M)$ is the submodule of elements m in M such that $Im = 0$ for some admissible ideal I.

COROLLARY I.2.24. *An A-module M is 2-deep if and only if for any admissible ideal I of A, the homomorphism*

$$M \to \mathrm{Hom}_A(I, M)$$

which sends an element m to $(x \mapsto xm)$ is bijective.

Assume first that M is 2-deep. The injectivity follows from I.2.23, since M is also 1-deep. For the surjectivity, let ψ be an element of $\mathrm{Hom}_A(I, M)$, for some admissible ideal $I = (f_1, \ldots, f_r)$. Then $(\psi(f_i)f_i^{-1})_{1\leq i\leq r}$ is a 1-cycle in the Čech complex $C^\bullet(M, f_\bullet)$. Since M is 2-deep, the module $H^1_I(M)$ vanishes by I.2.17. By I.2.20, the 1-cycle $(\psi(f_i)f_i^{-1})_{1\leq i\leq r}$ must be a 1-boundary, so that $\psi(f_i) = f_i m$ in the localization of M by f_i, for some m in M. Thus $f_i^{N_i}\psi(f_i) = f_i^{N_i+1}m$ in M for some integers N_i. For $N = 1 + \sum_i N_i$ and x in I^N one has $\psi(x) = xm$, so that $x\psi(y) = y\psi(x) = xym$ for any y in I. Since M has no nonzero I^N-torsion this implies $\psi(y) = ym$, so that ψ is the image of m.

Conversely, assume that the homomorphim

$$M \to \mathrm{Hom}_A(I, M)$$

is bijective for any admissible ideal I. The injectivity yields that M is 1-deep by I.2.23. Let $I = (f_1, \ldots, f_r)$ be an admissible ideal. By I.2.20 we have to show that any 1-cycle $(m_i f_i^{-N_i})_{1\leq i\leq r}$ in the Čech complex $C^\bullet(M, f_\bullet)$ is a 1-boundary. By increasing the integers N_i if necessary, one can assume that $f_i^{N_i}m_j = f_j^{N_j}m_i$ in M. For $N = \sum_i N_i$ and for any r-uple $\alpha = (\alpha_1, \ldots, \alpha_r)$ of nonnegative integers summing to N, the element $m_\alpha = f^{\alpha - N_i e_i}m_i$ does not depend on the choice of an index i such that $\alpha_i \geq N_i$. Here $f^\alpha = f_1^{\alpha_1} \ldots f_r^{\alpha_r}$ and e_i is the i-th basis vector. Since $f^\alpha m_\beta = f^\beta m_\alpha$, one must have $m_\alpha = f^\alpha m$ for some m by I.2.19 and by the hypothesis applied to I^N. Thus our initial 1-cycle $(m_i f_i^{-N_i})_{1\leq i\leq r}$ is the boundary of m.

COROLLARY I.2.25. *Let Φ_0 be a set of finitely generated ideals of A generating the family of constructible supports of A as in* I.2.2(*i*). *An A-module M is 2-deep if and only if for any I in Φ_0 the homomorphism*

$$M \to \mathrm{Hom}_A(I, M)$$

which sends an element m to $(x \mapsto xm)$ is bijective.

Assume indeed that the homomorphism

$$M \to \mathrm{Hom}_A(I, M)$$

is bijective for any I in Φ_0. It follows that for any I in Φ_0, M has no nonzero I-torsion. However, if I and J are admissible ideal and if M has no nonzero I-torsion then

$$\mathrm{Hom}_A(I, \mathrm{Hom}_A(J, M)) \cong \mathrm{Hom}_A(IJ, M),$$

since the product homomorphism $I \otimes_A J \to IJ$ has I-torsion kernel. From this one deduces that the homomorphism

$$M \to \mathrm{Hom}_A(I, M)$$

is bijective whenever I is a product of element of Φ_0. The proof of Corollary I.2.24 then shows that $H^1_I(M)$ vanishes whenever I is a product of elements of Φ_0. One concludes with Corollary I.2.11.

I.2.26. In ([**EGA4**], 5.9, 5.10) the notions of "*modules purs*" and "*modules clos*" are introduced in a noetherian setting. We define here the corresponding purification and closure functors in a more general setting.

Definition I.2.27. Let A be a Φ-ring and let M be an A-module. The **purification** of M is defined as the quotient

$$M^{\mathrm{pur}} = M/H^0_{\Phi_A}(M).$$

If M is 1-deep (cf. I.2.14), the **closure** of M is defined as the colimit

$$M^{\lhd} = \operatorname{colim} \operatorname{Hom}_A(I, M),$$

where the colimit runs over all the admissible ideals of A. In general, we set $M^{\lhd} = (M^{\mathrm{pur}})^{\lhd}$ (cf. I.2.29 below).

Example I.2.28. Let A be a valuation ring, and let us endow A with the family of supports consisting of all the nonzero principal ideals of A. Then, for any A-module M, the purification M^{pur} is the largest torsion-free quotient of M, while we have

$$M^{\lhd} = S^{-1}M,$$

where $S = A \setminus \{0\}$.

Proposition I.2.29. *Let A be a Φ-ring and let M be an A-module. Then M^{pur} is 1-deep and $M^{\lhd}$ is 2-deep.*

The A-module M^{pur} is 1-deep by the characterization I.2.23. In order to prove that $M^{\lhd}$ is 2-deep, we can assume that M is 1-deep. Let J be an admissible ideal of A. By I.2.19 the functor $\operatorname{Hom}_A(J, -)$ coincides on 1-deep A-modules with $\operatorname{Hom}_A(G, -)$ for some finitely presented A-module G. In particular it commutes with filtered colimits of 1-deep A-modules. Thus

$$\begin{aligned}
\operatorname{Hom}_A(J, M^{\lhd}) &\cong \operatorname{colim} \operatorname{Hom}_A(J, \operatorname{Hom}_A(I, M)) \\
&\cong \operatorname{colim} \operatorname{Hom}_A(I \otimes_A J, M) \\
&\cong \operatorname{colim} \operatorname{Hom}_A(IJ, M) \\
&\cong M^{\lhd}.
\end{aligned}$$

The third identification above follows from the fact that the product map $I \otimes_A J \to IJ$ has I-torsion kernel (and J-torsion kernel as well). We conclude by Proposition I.2.24 that $M^{\lhd}$ is 2-deep.

Definition I.2.30. The category $\Phi\mathrm{Rings}^{\geq d}$ is the full subcategory of ΦRings whose objects are the Φ-rings which are d-deep as modules over themselves (cf. I.2.14).

There is a fully faithful functor from the category of rings to the category of Φ-rings, which sends a ring A to the Φ-ring whose underlying ring is A and whose only admissible ideal is the unit ideal. The image of this fully faithful functor is the intersection over all integers d of the subcategories $\Phi\mathrm{Rings}^{\geq d}$ of ΦRings. Indeed if for some ideal $I = (f_1, \ldots, f_r)$ the Čech complex $C^\bullet(A, f_\bullet)$ of Proposition I.2.20 is exact then it is an exact complex of flat A-modules so that $C^\bullet(A, f_\bullet) \otimes_A A/I = A/I[0]$ is exact as well, and thus $I = A$.

Proposition I.2.31. *The* ***purification functor***

$$A \mapsto A^{\mathrm{pur}} = A/H^0_{\Phi_A}(A)$$

from ΦRings to $\Phi\mathrm{Rings}^{\geq 1}$ yields a left adjoint to the canonical inclusion functor. Here $A/H^0_{\Phi_A}(A)$ is endowed with the family of constructible supports generated by that of A.

This follows from the characterization of 1-deep modules in I.2.23.

PROPOSITION I.2.32. *The* ***closure functor***

$$A \mapsto A^{\triangleleft} = \text{colim } \mathrm{Hom}_A(I, A),$$

from $\Phi\mathrm{Rings}^{\geq 1}$ *to* $\Phi\mathrm{Rings}^{\geq 2}$ *yields a left adjoint to the canonical inclusion functor. Here* $A^{\triangleleft}$ *is endowed with the family of constructible supports generated by that of* A*.*

Let A be an object of $\Phi\mathrm{Rings}^{\geq 1}$. We notice as above that for any pair I, J of admissible ideals, the product map $I \otimes_A J \to IJ$ has I-torsion kernel. This enables us to define an A-bilinear morphism

$$\mathrm{Hom}_A(I, A) \times \mathrm{Hom}_A(J, A) \to \mathrm{Hom}_A(I \otimes_A J, A) \cong \mathrm{Hom}_A(IJ, A),$$

which yields the A-algebra structure on $A^{\triangleleft}$. By Proposition I.2.29, the A-module $A^{\triangleleft}$ is 2-deep. By Corollary I.2.22, we deduce that $A^{\triangleleft}$, when endowed with the family of constructible supports generated by that of A, is also 2-deep as a module over itself. Using I.2.24 and I.2.19 again, one checks that any Φ-ring morphism from A to a 2-deep Φ-ring B factors as an A-module homomorphism through $\mathrm{Hom}_A(I, A)$ for any admissible ideal I, hence through $A^{\triangleleft}$. The resulting homomorphism $A^{\triangleleft} \to B$ of A-modules must be a Φ-ring morphism, and this shows that the morphism $A \to A^{\triangleleft}$ is universal among Φ-ring morphisms from A to a 2-deep Φ-ring.

The composition of the functors of propositions I.2.31 and I.2.32 will still be denoted by $A \to A^{\triangleleft}$.

DEFINITION I.2.33. The category $\Phi\mathrm{Rings}^{+}$ of **normal Φ-rings** is the full subcategory of $\Phi\mathrm{Rings}$ whose objects are the Φ-rings A such that the adjunction morphism

$$A \to A^{\triangleleft}$$

is injective, so that A can be considered as a subring of $A^{\triangleleft}$, and such that A is integrally closed in $A^{\triangleleft}$: any element of $A^{\triangleleft}$ which is integral over A is required to lie in A.

In particular, any object of $\Phi\mathrm{Rings}^{\geq 2}$ is an object of $\Phi\mathrm{Rings}^{+}$, and any object of $\Phi\mathrm{Rings}^{+}$ is an object in $\Phi\mathrm{Rings}^{\geq 1}$.

PROPOSITION I.2.34. *The* ***normalization functor*** *which sends a* Φ*-ring* A *to the integral closure* A^{+} *of* A *in its closure* $A^{\triangleleft}$ *yields a left adjoint to the canonical inclusion functor from* $\Phi\mathrm{Rings}^{+}$ *to* $\Phi\mathrm{Rings}$*. Here* A^{+} *is endowed with the family of constructible supports generated by that of* A*.*

Let $f : A \to B$ be a morphism in $\Phi\mathrm{Rings}$ such that B belongs to $\Phi\mathrm{Rings}^{+}$. This induces a morphism $A^{\triangleleft} \to B^{\triangleleft}$ sending the image of A into B. Since $B \subseteq B^{\triangleleft}$ is an integrally closed subring, the morphism $A \to B$ factors uniquely through $A \to A^{+}$, the integral closure of the image of A in $A^{\triangleleft}$. Thus the morphism $A \to A^{+}$ is universal among Φ-ring morphisms from A to a normal Φ-ring.

I.2.35. We now turn to the definition and study of the local objects in our category of Φ-rings.

DEFINITION I.2.36. A Φ-ring A is **Φ-local** if it is a local ring and if all of its admissible ideals are invertible.

In ([**Abb10**] 1.9.1), these objects are called “prevaluative” rings, in the particular case where the admissible ideals are the open finitely generated ideals for some adic topology.

EXAMPLE I.2.37. Let A be a valuation ring, and let us endow A with the family of supports consisting of all the nonzero principal ideals of A. Then A is a Φ-local Φ-ring.

Remark I.2.38. Let A be a Φ-local Φ-ring. The condition that any admissible ideal of A is invertible implies that whenever an admissible ideal I of A is generated by a family $(x_\lambda)_{\lambda\in\Lambda}$ of elements of A, there exists λ in Λ such that x_λ is a nonzero divisor generating I. Indeed, the ideal I is an invertible ideal in a local ring, hence is generated by some nonzero divisor x of A. For each λ in Λ, we then have $x_\lambda = a_\lambda x$ for a unique element a_λ of A. Moreover, we can write x as $\sum_{\lambda\in\Lambda} b_\lambda x_\lambda$ for some family $(b_\lambda)_{\lambda\in\Lambda}$ of elements of A, only finitely many of which being non zero. Since x is a non zero divisor, we must have $\sum_{\lambda\in\Lambda} a_\lambda b_\lambda = 1$. Since A is local, there exists λ in Λ such that a_λ is a unit, in which case x_λ is a nonzero divisor generating I.

A structure theorem similar to ([**Abb10**] 1.9.4) holds in the general case:

Proposition I.2.39. *Let A be a Φ-local Φ-ring with maximal ideal $\mathfrak{n}$.*
(i) *The canonical homomorphism $A \to A^{\triangleleft}$ is injective.*
(ii) *The ring $A^{\triangleleft}$ is a local ring, and its maximal ideal $\mathfrak{m}$ is contained in A. More precisely, $A^{\triangleleft}$ is the localization of A at $\mathfrak{m}$.*
(iii) *An ideal of A is admissible if and only if it is generated by an element of $A \setminus \mathfrak{m}$.*
(iv) *The ring $A/\mathfrak{m}$ is a valuation subring of the field $A^{\triangleleft}/\mathfrak{m}$.*
(v) *We have*

$$\mathfrak{m} = \bigcap_{s\in A\setminus\mathfrak{m}} s\mathfrak{n}.$$

(vi) *A is a normal Φ-ring.*
(vii) *Let I be a finitely generated ideal of A and let g be an element of A such that (I, g) is admissible. Then $I(A/\mathfrak{m}) \subseteq g(A/\mathfrak{m})$ if and only if $I \subseteq gA$.*

Let S be the subset of elements of A which generate an admissible ideal. Then S is a multiplicative subset of the set of nonzero divisors of A and any admissible ideal of A is equal to sA for some element s of S. In particular we have

$$A^{\triangleleft} \cong \operatorname*{colim}_{s\in S} \operatorname{Hom}_A(sA, A) \xrightarrow{\sim} \operatorname*{colim}_{s\in S} s^{-1}A \xrightarrow{\sim} S^{-1}A.$$

This proves in particular (i). Let us define

$$\mathfrak{m} = \bigcap_{s\in S} s\mathfrak{n}.$$

Then $\mathfrak{m}$ is a proper ideal of $A^{\triangleleft}$ contained in A. We show that $S = A \setminus \mathfrak{m}$.

First, let a be an element of $A \setminus \mathfrak{m}$. Then there exists an element s of S such that a does not belong to $s\mathfrak{n}$. The finitely generated ideal (s, a) is admissible, as it contains the admissible ideal sA. Thus (s, a) is invertible, since A is Φ-local, and is in particular generated by either a or s, cf. I.2.38. If (s, a) is generated by s, then $a = bs$ for some b in A. But then b does not belong to $\mathfrak{n}$, so it is a unit of A, and hence $s = b^{-1}a$ belongs to aA. If (s, a) is generated by a, it is also the case that s belongs to aA. Thus aA contains the admissible ideal sA and is therefore admissible itself, so that a belongs to S. Conversely, if a is an element of S, then a does not belong to $a\mathfrak{n}$, and thus a is in $A \setminus \mathfrak{m}$.

This shows (iii) and that $A^{\triangleleft}$ is the localization of A at $\mathfrak{m}$, hence (ii) and (v). Moreover, for any pair a, s of elements of $A \setminus \mathfrak{m}$, the ideal (s, a) of A is admissible, hence it is generated by either a or s, cf. I.2.38. Thus, either a belongs to sA or s belongs to aA. This shows that $A/\mathfrak{m}$ is a valuation subring of the field $A^{\triangleleft}/\mathfrak{m}$, hence ($iv$).

In particular, $A/\mathfrak{m}$ is integrally closed in $A^{\triangleleft}/\mathfrak{m}$, so that A is integrally closed in $A^{\triangleleft}$. Thus A is a normal Φ-ring, hence (vi).

Let I be a finitely generated ideal of A and let g be an element of A such that (I, g) is admissible, hence equal to sA for some element s of S. If we have the inclusion $I(A/\mathfrak{m}) \subseteq g(A/\mathfrak{m})$ then $s(A/\mathfrak{m}) \subseteq g(A/\mathfrak{m})$ also holds, so that $s = ag + b$ with $a \in A$ and $b \in \mathfrak{m}$. Since

$\mathfrak{m} \subseteq s\mathfrak{n}$ we have $b = cs$ for some $c \in \mathfrak{n}$. Thus we have $(1-c)s = ag$, and consequently $s = (1-c)^{-1}ag$, so that $(I,g) = sA$ is contained in gA. In particular, we have $I \subseteq gA$, hence (vii).

Remark I.2.40. The datum of a Φ-local Φ-ring is the same as the datum of a local ring together with a valuation subring of its residue field. The inverse construction is simply the following: given a local ring B and a valuation subring R of its residue field, let A be the inverse image of R in B, and declare an ideal of A to be admissible if it is generated by an element which is invertible in B. Then A is a Φ-ring and $A^{\triangleleft} = B$. Moreover, the valuation subring of the residue field of $A^{\triangleleft}$ produced by Proposition I.2.39 is precisely R.

Construction I.2.41. Let A be a local Φ-ring and let $f : A \to S$ be a local homomorphism from A to a valuation ring S such that $f(I)S \neq 0$ for any admissible ideal I of A. Let us endow S with the family of constructible supports generated by that of A. Then S is a Φ-local Φ-ring. Let $\mathfrak{m}$ be the maximal ideal of $S^{\triangleleft}$ (cf. I.2.39) and let $\mathfrak{p}$ be its inverse image in $A^{\triangleleft}$. The homomorphism $A^{\triangleleft} \to S^{\triangleleft}/\mathfrak{m}$ induced by f factors through an injective homomorphism from $\kappa(\mathfrak{p}) = \mathrm{Frac}(A^{\triangleleft}/\mathfrak{p})$ to $S^{\triangleleft}/\mathfrak{m}$. Let $R \subseteq \kappa(\mathfrak{p})$ be the inverse image of the valuation ring $S/\mathfrak{m}$ under this homomorphism. The pair $(\mathfrak{p}, R)$ has the following properties:

(a) The ring R is a valuation ring, and the homomorphism $A \to \kappa(\mathfrak{p})$ factors through a local homomorphism $A \to R$.

(b) For any $a \in \kappa(\mathfrak{p})$ there is an admissible ideal I of A such that $Ia \subseteq R$.

(c) The prime ideal $\mathfrak{p}$ contains no admissible ideal of $A^{\triangleleft}$.

(d) Let I be a finitely generated ideal of A and let g be an element of A such that (I,g) is admissible. Then $IR \subseteq gR$ if and only if $IS \subseteq gS$.

If a is a non zero element of R, then a or a^{-1} belongs to the valuation ring $S/\mathfrak{m}$, hence to R. Thus R is a valuation ring. Since its maximal ideal is the inverse image of that of $S/\mathfrak{m}$, the local homomorphism $A \to S/\mathfrak{m}$ factors through a local homomorphism $A \to R$.

For any $a \in \kappa(\mathfrak{p})$, there is an admissible ideal I of A such that $f(Ia)(S/\mathfrak{m}) \subseteq S/\mathfrak{m}$. For such an ideal I, we have $Ia \subseteq R$.

The maximal ideal $\mathfrak{m}$ of $S^{\triangleleft}$ is the intersection of all $f(I)\mathfrak{n}$ where I runs over the admissible ideals of A, and where $\mathfrak{n}$ is the maximal ideal of S. For any admissible ideal I of A, the invertible ideal $f(I)S$ is not contained in $f(I)\mathfrak{n}$, and is consequently not contained in $\mathfrak{m}$. In particular, the ideal I is not contained in $\mathfrak{p}$.

Let I be a finitely generated ideal of A and let g be an element of A such that (I,g) is admissible. Then $IR \subseteq gR$ if and only if $I(S/\mathfrak{m}) \subseteq g(S/\mathfrak{m})$, if and only if $IS \subseteq gS$ by I.2.39 (vii).

I.3. Sheaves of Φ-rings

Let $\mathcal{U}$ be a universe ([**SGA4**], I.0). For any category C, we denote by $\mathcal{U}$-C the full subcategory of C whose objects are the objects of C which belong to $\mathcal{U}$.

I.3.1. Let C be a $\mathcal{U}$-site ([**SGA4**], II.3.0.2). A **presheaf of Φ-rings** on C (with respect to $\mathcal{U}$) is a contravariant functor from C to $\mathcal{U}$-ΦRings. A presheaf of Φ-rings A (with respect to $\mathcal{U}$) is a **sheaf of Φ-rings** if for any Φ-ring B in $\mathcal{U}$, the functor

$$U \in C \mapsto \mathrm{Hom}_{\Phi\mathrm{Rings}}(B, A(U))$$

is a sheaf of sets on C (with respect to $\mathcal{U}$). We also define a morphism of sheaves of Φ-rings to be a morphism of presheaves of Φ-rings between sheaves of Φ-rings. We henceforth omit references to $\mathcal{U}$ when no confusion arise from the lack thereof.

This definition is a specialization of ([**SGA4**], II.6.1). Since the category ΦRings is complete I.2.4, a presheaf of Φ-rings on C is a sheaf of Φ-rings if and only if for any covering sieve S of an object U of C, the natural morphism

$$A(U) \to \lim_{C_{/S}} A \tag{4}$$

is an isomorphism of Φ-rings ([**SGA4**], II.6.2).

Proposition I.3.2. *Let A be a presheaf of Φ-rings on C. Then A is a sheaf of Φ-rings if and only if its underlying presheaf of sets is a sheaf and for each integer r the contravariant functor*

$$\mathrm{Adm}_r(A) : U \mapsto \{(f_i)_{1\leq i\leq r} \in A(U)^r \mid (f_i)_{1\leq i\leq r} \textit{ generates an admissible ideal of } A(U)\}$$

is a sheaf of sets on C.

Indeed if A is a sheaf of Φ-rings then its underlying presheaf of rings is a sheaf since the forgetful functor from ΦRings to the category of rings has a left adjoint by I.2.5 and thus commutes with the limit in 4. For each integer r, taking the Φ-ring B in the definition of a sheaf of Φ-rings to be the polynomial ring $\mathbb{Z}[X_1, \dots, X_r]$ endowed with the family of constructible supports generated by the ideal $(X_1, \dots, X_r)$ yields that the presheaf $\mathrm{Adm}_r(A)$ is a sheaf of sets.

Conversely, let us assume that the presheaf of rings underlying A is a sheaf and that $\mathrm{Adm}_r(A)$ is a sheaf for each integer r. For each covering sieve S of an object U of C, the natural homomorphism

$$A(U) \to \lim_{C_{/S}} A \tag{5}$$

is bijective since A is a sheaf of rings. Moreover an element $(f_i)_{1\leq i\leq r}$ of $A(U)^r$ generates an admissible ideal if and only if it generates an admissible ideal of $A(V)^r$ for each $V \to U$ in S, since $\mathrm{Adm}_r(A)$ is a subsheaf of A^r. According to the description of limits in ΦRings given in the proof of I.2.4 the latter condition holds if and only if (f_i) generates an admissible ideal of $\lim_{C_{/S}} A$. Thus 5 is an isomorphims of Φ-rings and A is a sheaf of Φ-rings.

Corollary I.3.3. *Let A be a presheaf of Φ-rings on C and let $\widetilde{A}$ be the sheaf of rings associated to its underlying presheaf of rings. Then there is a unique sheaf of Φ-rings with underlying sheaf of rings $\widetilde{A}$, such that for each r the sheaf $\mathrm{Adm}_r(\widetilde{A})$ (cf. I.3.2) is the sheaf associated to $\mathrm{Adm}_r(A)$. Moreover the morphism $A \to \widetilde{A}$ is initial among the morphisms of presheaves of Φ-rings from A to a sheaf of Φ-rings.*

Let $\mathrm{Adm}_r(\widetilde{A}) \subseteq \widetilde{A}^r$ be the sheaf associated to $\mathrm{Adm}_r(A)$. For any object U of C, let us declare an ideal of $\widetilde{A}(U)$ admissible if it is generated by elements $f_1, \dots, f_r$ such that $(f_1, \dots, f_r)$ belongs to $\mathrm{Adm}_r(\widetilde{A})(U)$.

Let r and s be integers and let us consider the morphism of presheaves of sets given by

$$\alpha_{r,s} : A^{rs} \times A^s \to A^r$$

$$((a_{ij})_{1\leq i\leq r, 1\leq j\leq s}, (g_j)_{1\leq j\leq s}) \mapsto \left(\sum_j a_{ij} g_j\right)_{1\leq i\leq r}.$$

The fiber product $\alpha_{r,s}^{-1}(\mathrm{Adm}_r(A)) = (A^{rs} \times A^s) \times_{A^r} \mathrm{Adm}_r(A)$ is a subobject of $A^{rs} \times A^s$ contained in $A^{rs} \times \mathrm{Adm}_s(A)$ since A is a presheaf of Φ-rings (cf. I.2.1(2)). Since the "associated sheaf" functor ([**SGA4**], II.3.4) commutes with finite limits ([**SGA4**], II.4.1), we obtain that the morphism $\widetilde{\alpha}_{r,s}$ between the associated sheaves of sets satisfies $\widetilde{\alpha}_{r,s}^{-1}(\mathrm{Adm}_r(\widetilde{A})) \subseteq \widetilde{A}^{rs} \times \mathrm{Adm}_s(\widetilde{A})$. If U is an object of C then this shows that the axiom I.2.1(2) is satisfied for $\widetilde{A}(U)$ and that an element $(f_1, \dots, f_r)$ of $\widetilde{A}^r(U)$ generates an admissible ideal if and only if it belongs to $\mathrm{Adm}_r(\widetilde{A})(U)$.

By considering

$$\begin{aligned}\beta_{r,s} : \mathrm{Adm}_r(A) \times \mathrm{Adm}_s(A) &\to \mathrm{Adm}_{rs}(A)\\ ((f_i)_{1\le i\le r}, (g_j)_{1\le j\le s}) &\mapsto (f_i g_j)_{1\le i\le r, 1\le j\le s}\end{aligned}$$

instead of $\alpha_{r,s}$ and by taking the induced morphism between the associated sheaves of sets, one obtains the axiom I.2.1(1) as well. Thus we have endowed the sheaf of rings $\widetilde{A}$ with a structure of presheaf of Φ-rings. Since a local section $(f_1, \dots, f_r)$ of $\widetilde{A}^r$ generates an admissible ideal if and only if it belongs to $\mathrm{Adm}_r(\widetilde{A})$, the notation $\mathrm{Adm}_r(\widetilde{A})$ is consistent with I.3.2. For each r, $\mathrm{Adm}_r(\widetilde{A})$ is a sheaf, hence $\widetilde{A}$ is a sheaf of Φ-rings by Proposition I.3.2.

Let $A \to B$ be a morphism of presheaves of Φ-rings from A to a sheaf of Φ-rings. The underlying morphism of presheaves of rings uniquely factors through the natural morphism $A \to \widetilde{A}$. Since for each integer the morphism $A^r \to B^r$ maps $\mathrm{Adm}_r(A)$ into $\mathrm{Adm}_r(B)$, the induced morphism $\widetilde{A}^r \to B^r$ maps $\mathrm{Adm}_r(\widetilde{A})$ into $\mathrm{Adm}_r(B)$ as well. Thus the morphism $\widetilde{A} \to B$ is a morphism of sheaves of Φ-rings.

Corollary I.3.4. *Let A be a sheaf of Φ-rings on C, and let U be a sheaf of sets on C. Then the ring $\mathrm{Hom}(U, A)$ is endowed with a family of constructible supports by declaring admissible any ideal of $\mathrm{Hom}(U, A)$ which is generated by elements $f_1, \dots, f_r$ such that $(f_1, \dots, f_r)$ belongs to the subset $\mathrm{Hom}(U, \mathrm{Adm}_r(A))$ of $\mathrm{Hom}(U, A^r) = \mathrm{Hom}(U, A)^r$.*

This follows from the proof of I.3.3 by applying the limit preserving functor $\mathrm{Hom}(U, \cdot)$ to $\alpha_{r,s}$ and $\beta_{r,s}$.

I.3.5. Let X be a topos. We define a **Φ-ring of** X to be a ring A in X together with a subobject $\mathrm{Adm}_r(A)$ of A^r for each positive integer r, such that:

(1) For all U in X, the element $1 \in A(U)$ belongs to $\mathrm{Adm}_1(A)(U)$. For all positive integers r, s, for all $(f_i)_{1\le i\le r}$ and $(g_j)_{1\le j\le s}$ in $\mathrm{Adm}_r(A)(U)$ and $\mathrm{Adm}_s(A)(U)$ respectively, the element $(f_i g_j)_{1\le i\le r, 1\le j\le s}$ of $A(U)^{rs}$ belongs to $\mathrm{Adm}_{rs}(A)(U)$.

(2) For all U in X, for all positive integers r, s, and for all $(f_i)_{1\le i\le r}$ in $\mathrm{Adm}_r(A)(U)$, any element $(g_j)_{1\le j\le s}$ of $A(U)^s$ such that $\sum_i f_i A(U) \subseteq \sum_j g_j A(U)$ belongs to $\mathrm{Adm}_s(A)(U)$.

We define a morphism of Φ-rings of X to be a ring homomorphism $f : A \to B$ such that for each positive integer r, the subobject $f(\mathrm{Adm}_r(A))$ of B^r is a subobject of $\mathrm{Adm}_r(B)$. This defines a category $\Phi\mathrm{Rings}_X$.

Proposition I.3.6. *Let C be a site and let $\widetilde{C}$ be the topos of sheaves of sets on C. The functor which associates to a sheaf of Φ-rings A on C the Φ-ring $(A, (\mathrm{Adm}_r(A))_{r\ge 0})$ in $\widetilde{C}$ (cf. I.3.2) is an equivalence of categories.*

This follows from I.3.2 and I.3.4.

I.3.7. Let $f = (f_*, f^{-1}) : X \to Y$ be a morphism of toposes[2]. If A is a Φ-ring of X then setting $\mathrm{Adm}_r(f_*A) = f_*(\mathrm{Adm}_r(A))$ gives f_*A the structure of a Φ-ring of Y. Similarly if B is a Φ-ring of X then setting $\mathrm{Adm}_r(f^{-1}B) = f^{-1}\mathrm{Adm}_r(B)$ gives $f^{-1}B$ the structure of a Φ-ring of Y, since the functor f^{-1} commutes with finite limits.

In particular if A is a Φ-ring of X and x is a point of X, then **the stalk** A_x **of** A **at** x is defined to be $x^{-1}A$. It is a Φ-ring of the topos of sets, or in other words a Φ-ring. We have

$$A_x = \mathrm{colim}_{(U,u)} A(U),$$

in the category of Φ-rings, where the colimit is indexed by pairs (U, u) where U runs over the members of a small generating family of X and u is an element of $U_x = x^{-1}U$.

2. We choose here the notation f^{-1} in order to avoid confusion with the pullback functor f^* between categories of modules, which is associated to a morphism a ringed toposes.

Proposition I.3.8. *The pair of functors* (f_*, f^{-1}) *between the categories* $\Phi\mathrm{Rings}_X$ *and* $\Phi\mathrm{Rings}_Y$ *is a pair of adjoint functors.*

This follows from the corresponding adjunction property of (f_*, f^{-1}) between the categories X and Y.

I.3.9. A Φ-ringed topos is a pair $(X, \mathcal{O}_X)$ where X is a topos and $\mathcal{O}_X$ is a Φ-ring of X (cf. I.3.5). We define a morphism of Φ-ringed topos from a Φ-ringed topos $(X, \mathcal{O}_X)$ to an other one $(Y, \mathcal{O}_Y)$ to be a morphism of toposes $f = (f_*, f^{-1}) : X \to Y$ together with a morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ (cf. I.3.7) of Φ-rings in Y.

I.3.10. Let us recall that a ring A in a topos X is **local** if the natural morphism

$$A^\times \sqcup (1 + A^\times) \to A$$

is an epimorphism and if the limit of the diagram $e \xrightarrow{1} A \xleftarrow{0} e$ is the initial object of X [3].

Proposition I.3.11. *Let* $(X, \mathcal{O}_X)$ *be a ringed topos with enough points. Then* $\mathcal{O}_X$ *is local if and only if for each point* x *of* X *the stalk* $\mathcal{O}_{X,x}$ *of* $\mathcal{O}_X$ *at* x *(cf.* I.3.7*) is a local ring.*

This appears as an exercise in ([**SGA4**], IV.13.9). The proposition follows from the fact that a ring A (in the punctual topos) is local if and only if for any element a of A, either a or $1 + a$ is invertible, and if moreover $0 \neq 1$ in A.

Definition I.3.12. A Φ-ringed topos $(X, \mathcal{O}_X)$ (cf. I.3.9) is locally Φ-ringed if the underlying ring of $\mathcal{O}_X$ is local. A morphism of locally Φ-ringed toposes is a morphism of Φ-ringed toposes (cf. I.3.9) which is also a morphism of locally ringed toposes ([**SGA4**], IV.13.9).

I.3.13. A Φ-ring A in a topos is Φ-local if the following three conditions are satisfied:
(1) The ring A is local (cf. I.3.10).
(2) For all U in X, each element of $\mathrm{Adm}_1(A)(U)$ is not a zero divisor in $A(U)$.
(3) For each integer r the morphism

$$\coprod_{j=1}^{r} \left(\mathrm{Adm}_1(A) \times A^{[1,r]\setminus\{j\}} \right) \to \mathrm{Adm}_r(A)$$

$$(f, (a_i)_{1 \leq i \leq r, i \neq j}) \mapsto (a_i f)_{1 \leq i \leq r} \text{ where } a_j = 1$$

is epimorphic.

Definition I.3.14. A Φ-ringed topos $(X, \mathcal{O}_X)$ is Φ-locally Φ-ringed if $\mathcal{O}_X$ is Φ-local. A morphism of Φ-locally Φ-ringed toposes is a morphism of locally Φ-ringed toposes (cf. I.3.12) between Φ-locally Φ-ringed toposes.

Proposition I.3.15. *Let* $(X, \mathcal{O}_X)$ *be a* Φ*-ringed topos with enough points. Then* $(X, \mathcal{O}_X)$ *is* Φ*-locally* Φ*-ringed if and only if for each point* x *of* X *the stalk* $\mathcal{O}_{X,x}$ *of* $\mathcal{O}_X$ *at* x *(cf.* I.3.7*) is a* Φ*-local* Φ*-ring.*

This follows from Proposition I.3.11 and from the fact that a Φ-ring (in the punctual topos) is Φ-local if and only if its underlying ring is local and if for any admissible ideal $I = (f_1, \ldots, f_r)$ of A, there exists i such that f_i generates I and is not a zero divisor.

3. The last condition, which can be reworded as "$0 \neq 1$", is missing from the definition of a local ring in a topos which can be found in ([**SGA4**], IV.13.9)

I.3.16. Let us recall that we denote by $\Phi\mathrm{Rings}_X$ the category of Φ-rings of a topos X (cf. I.3.5).

Definition I.3.17. Let $(X, \mathcal{O}_X)$ be a Φ-ringed topos. For $d \leq 2$, an $\mathcal{O}_X$-module $\mathcal{M}$ is said to be d-deep if for all U in X, the $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ is d-deep (cf. I.2.14).

Proposition I.3.18. *Let $d \leq 2$ and let $(X, \mathcal{O}_X)$ be a ringed topos with enough points. Then an $\mathcal{O}_X$-module $\mathcal{M}$ is d-deep if and only if all of its stalks are d-deep.*

The case $d = 0$ is tautological. By I.2.23 an $\mathcal{O}_X$-module $\mathcal{M}$ is 1-deep if and only if for each integer r, the inverse image of the neutral subobject $0 \to \mathcal{M}^r$ of $\mathcal{M}^r$ by the morphism

$$\begin{aligned} \mathrm{Adm}_r(\mathcal{O}_X) \times \mathcal{M} &\to \mathcal{M}^r \\ ((f_j)_{1 \leq j \leq r}, m) &\mapsto (f_j m)_{1 \leq j \leq r} \end{aligned}$$

is a subobject of $\mathrm{Adm}_r(\mathcal{O}_X) \times 0$. This yields the proposition for $d = 1$. For $d = 2$, one rather uses I.2.19 and I.2.24, which imply that a $\mathcal{M}$ is 2-deep if and only if for each integer r the natural morphism

$$\begin{aligned} \mathrm{Adm}_r(\mathcal{O}_X) \times \mathcal{M} &\to \lim \left(\mathrm{Adm}_r(\mathcal{O}_X) \times \mathcal{M}^r \rightrightarrows \mathcal{M}^{r^2} \right) \\ ((f_j)_{1 \leq j \leq r}, m) &\mapsto ((f_j)_{1 \leq j \leq r}, (f_j m)_{1 \leq j \leq r}) \end{aligned}$$

is an isomorphism, where the transition morphisms in the limit are given by $((f_j)_j, (m_j)_j) \mapsto (f_i m_j)_{i,j}$ and $((f_j)_j, (m_j)_j) \mapsto (f_j m_i)_{i,j}$.

Definition I.3.19. Let X be a topos. For $d \leq 2$ we define the category $\Phi\mathrm{Rings}_X^{\geq d}$ of d-deep Φ-rings to be the full subcategory of $\Phi\mathrm{Rings}_X$ whose objects are the Φ-rings A which are d-deep as modules over themselves.

Proposition I.3.20. *Let $d \leq 2$ be an integer, and let A be a presheaf of Φ-rings on a site C such that $A(U)$ is d-deep for any object U of C. Then the sheaf of Φ-rings associated to A (cf. I.3.3) is a d-deep Φ-ring in the topos of sheaves of sets on C.*

This follows from the characterizations of d-deep modules in a Φ-ringed topos given in the proof of the proposition I.3.18.

Proposition I.3.21. *Let X be a topos. The canonical inclusion functor from $\Phi\mathrm{Rings}_X^{\geq 1}$ to $\Phi\mathrm{Rings}_X$ admits a left adjoint. We call this left adjoint the **purification** functor and we denote it by $A \mapsto A^{\mathrm{pur}}$.*

Let C be a site such that X is (equivalent to) the topos of sheaves of sets on C (e.g. the canonical site of X). By I.3.6 the category $\Phi\mathrm{Rings}_X$ is equivalent to the category of sheaves of Φ-rings on C. A sheaf of Φ-rings A on C corresponds to a 1-deep Φ-ring object in X if and only if for each object U of C the Φ-ring $A(U)$ is 1-deep. Thus the functor which associated to a sheaf of Φ-rings A on C the sheaf associates to the presheaf of Φ-rings $U \mapsto A(U)^{\mathrm{pur}}$ (cf. I.3.3 and I.3.20) defines a left adjoint to the canonical inclusion functor.

Proposition I.3.22. *Let X be a topos. The canonical inclusion functor from $\Phi\mathrm{Rings}_X^{\geq 2}$ to $\Phi\mathrm{Rings}_X$ admits a left adjoint. We call it the **closure** functor and we denote it by $A \mapsto A^{\triangleleft}$.*

This is proved similarly as I.3.21 using the closure functor on Φ-rings instead of the purification functor.

Proposition I.3.23. *Let X be a topos, let x be a point of X and let A be a Φ-ring of X. Then $(A^{\triangleleft})_x \cong (A_x)^{\triangleleft}$ and $(A^{\mathrm{pur}})_x \cong (A_x)^{\mathrm{pur}}$.*

This follows from the proofs of I.3.21 and I.3.22 since the purification and closure functors on Φ-rings have right adjoints and thus commute with filtered colimits.

I.4. Valuative spaces as Φ-localizations

Let X be a locally Φ-ringed topological space (cf. I.3.12).

I.4.1. We define the relative valuative spectrum $\widetilde{X}$ of X to be the set of triples $\tilde{x} = (x, \mathfrak{p}, R)$, where x is a point of X, $\mathfrak{p}$ is a prime ideal of $\mathcal{O}^{\triangleleft}_{X,x}$ (cf. I.3.22, I.3.23) which does not contain an admissible ideal of $\mathcal{O}^{\triangleleft}_{X,x}$ and R is a valuation subring of $\kappa(\mathfrak{p}) = \mathrm{Frac}(\mathcal{O}^{\triangleleft}_{X,x}/\mathfrak{p})$, such that the following two conditions are satisfied:

(a) The homomorphism $\mathcal{O}_{X,x} \to \kappa(\mathfrak{p})$ factors through a local homomorphism $\mathcal{O}_{X,x} \to R$.

(b) For any $a \in \kappa(\mathfrak{p})$ there is an admissible ideal I of $\mathcal{O}_{X,x}$ satisfying $Ia \subseteq R$.

If $\tilde{x} = (x, \mathfrak{p}, R)$ is a point of $\widetilde{X}$, we write $\Gamma(\tilde{x}) = \kappa(\mathfrak{p})^{\times}/R^{\times}$, and $\Gamma(\tilde{x})_+ = \Gamma(\tilde{x}) \sqcup \{0\}$. The multiplicative valuation on $\mathcal{O}^{\triangleleft}_{X,x}$ associated to R is a multiplicative map $\mathcal{O}^{\triangleleft}_{X,x} \to \Gamma(\tilde{x})_+$ denoted by $f \mapsto |f(\tilde{x})|$.

Remark I.4.2. Here and hereafter we use the term "valuation" as a synonym for "higher-rank norm". One recovers the usual notion of valuation by reversing the order on $\Gamma(\tilde{x})_+$ and by relabelling $0 \in \Gamma(\tilde{x})_+$ as ∞.

I.4.3. Let $\pi : \widetilde{X} \to X$ be the map $(x, \mathfrak{p}, R) \mapsto x$. If U is an open subset of X, I is a finitely generated ideal of $\Gamma(U, \mathcal{O}_X)$ and g is an element of $\Gamma(U, \mathcal{O}_X)$ such that (I, g) is an admissible ideal of $\Gamma(U, \mathcal{O}_X)$ then we set

$$U\left(g^{-1}I\right) = \{\tilde{x} \in \pi^{-1}(U) \mid |I(\tilde{x})| \leq |g(\tilde{x})|\}.$$

Here, by $|I(\tilde{x})|$ we denote the maximum of $|f(\tilde{x})|$ where f runs over all elements of I; this is equal to $\max(|f_i(\tilde{x})|)_{i=1}^{r}$ whenever I is generated by $f_1, \ldots, f_r$. We endow $\widetilde{X}$ with the topology generated by subsets of the form $U\left(g^{-1}I\right)$.

Proposition I.4.4. *The map π is continuous and its image is equal to the support of the sheaf $\mathcal{O}^{\triangleleft}_X$, or equivalently of its subsheaf of rings $\mathcal{O}^{\mathrm{pur}}_X$.*

Since $\mathcal{O}^{\mathrm{pur}}_X$ is a subsheaf of rings of $\mathcal{O}^{\triangleleft}_X$, its support is equal to the support of $\mathcal{O}^{\triangleleft}_X$. For any open subset U of X, the inverse image $\pi^{-1}(U) = U(1^{-1}\Gamma(U, \mathcal{O}_X))$ is open in $\widetilde{X}$, hence the continuity of π. Next we notice that if $\mathcal{O}^{\triangleleft}_{X,x}$ vanishes then this ring has no prime ideals, so that $\pi^{-1}(x)$ is empty. Let x be a point of X such that $\mathcal{O}^{\triangleleft}_{X,x}$ is nonzero, and let $\mathfrak{p}'$ be a minimal prime ideal of $\mathcal{O}^{\triangleleft}_{X,x}$. Were $\mathfrak{p}'$ to contain an admissible ideal I of $\mathcal{O}^{\triangleleft}_{X,x}$, this finitely generated ideal would be nilpotent in the localization $(\mathcal{O}^{\triangleleft}_{X,x})_{\mathfrak{p}'}$ and we would obtain an element f of $\mathcal{O}^{\triangleleft}_{X,x} \setminus \mathfrak{p}'$ and an integer N such that $fI^N = 0$, which would contradict the fact that $\mathcal{O}^{\triangleleft}_{X,x}$ is 1-deep. Thus $\mathfrak{p}'$ does not contain any admissible ideal of $\mathcal{O}^{\triangleleft}_{X,x}$. Let R' be a valuation subring of $\kappa(\mathfrak{p}')$ dominating the image of $\mathcal{O}_{X,x}$. The construction I.2.41 applied to the local homomorphism $\mathcal{O}_{X,x} \to R'$ yields a pair $(\mathfrak{p}, R)$ such that $(x, \mathfrak{p}, R)$ belongs to $\widetilde{X}$, and therefore the fiber $\pi^{-1}(\{x\})$ is not empty.

I.4.5. Let us consider the following sheaf of monoids on $\widetilde{X}$:

$$S : U \mapsto S(U) = \{s \in \Gamma(U, \pi^{-1}\mathcal{O}^{\triangleleft}_X) \mid \forall \tilde{x} \in U,\ |s(\tilde{x})| > 0\}.$$

We define $\mathcal{O}^{\triangleleft}_{\widetilde{X}}$ to be the sheafification of the presheaf of rings $U \mapsto S(U)^{-1}\Gamma(U, \pi^{-1}\mathcal{O}^{\triangleleft}_X)$.

Proposition I.4.6. *Let $\tilde{x} = (x, \mathfrak{p}, R)$ be a point of $\widetilde{X}$. The stalk $\mathcal{O}^{\triangleleft}_{\widetilde{X},\tilde{x}}$ of $\mathcal{O}^{\triangleleft}_{\widetilde{X}}$ at $\tilde{x}$ is the localization at $\mathfrak{p}$ of $\mathcal{O}^{\triangleleft}_{X,x}$.*

The stalk at $\tilde{x}$ of the presheaf given by $U \mapsto S(U)^{-1}\Gamma(U, \pi^{-1}\mathcal{O}^{\triangleleft}_X)$ is canonically isomorphic to $S_{\tilde{x}}^{-1}\mathcal{O}^{\triangleleft}_{X,x}$ where $S_{\tilde{x}}$ is the stalk at $\tilde{x}$ of the sheaf S. If $|s(\tilde{x})| > 0$ for a section s of $\mathcal{O}^{\triangleleft}_{\widetilde{X}}$ on a neighbourhood U of x then up to shrinking U if necessary we have $|s(\tilde{x})| \geq |I(\tilde{x})|$ for some admissible ideal I of $\Gamma(U, \mathcal{O}_X)$ by I.4.1(b), and thus s is a section of S on the open set $\{\tilde{y} \in \pi^{-1}(U) \mid |I(\tilde{y})| \leq |s(\tilde{y})|\}$. We thus have

$$S_{\tilde{x}} = \{s \in \mathcal{O}^{\triangleleft}_{X,x} \mid |s(\tilde{x})| > 0\}.$$

Since $S_{\tilde{x}}$ is the complement of $\mathfrak{p}$ we obtain $\mathcal{O}^{\triangleleft}_{\widetilde{X},\tilde{x}} \cong (\mathcal{O}^{\triangleleft}_{X,x})_{\mathfrak{p}}$.

For each point $\widetilde{x}$ of $\widetilde{X}$, the valuation $|\cdot(\tilde{x})|$ on $\mathcal{O}^{\triangleleft}_{X,x}$ extends uniquely to its localization $\mathcal{O}^{\triangleleft}_{\widetilde{X},\tilde{x}}$. This allows us to consider the subsheaf $\mathcal{O}_{\widetilde{X}}$ of $\mathcal{O}^{\triangleleft}_{\widetilde{X}}$ defined as follows:

$$\mathcal{O}_{\widetilde{X}} : U \mapsto \mathcal{O}_{\widetilde{X}}(U) = \{f \in \mathcal{O}^{\triangleleft}_{\widetilde{X}}(U) \mid \forall \tilde{x} \in U,\ |f(\tilde{x})| \leq 1\}.$$

We endow $\mathcal{O}_{\widetilde{X}}$ with a structure of a presheaf of Φ-rings by declaring a finitely generated I of $\mathcal{O}_{\widetilde{X}}(U)$ to be admissible whenever $|I(\tilde{x})| > 0$ at each point $\tilde{x}$ of U. Then $\mathcal{O}_{\widetilde{X}}$ is a sheaf of Φ-rings on $\widetilde{X}$ by I.3.2.

Proposition I.4.7. *Let $\tilde{x} = (x, \mathfrak{p}, R)$ be a point of $\widetilde{X}$.*
(i) *The stalk $\mathcal{O}_{\widetilde{X},\tilde{x}}$ of $\mathcal{O}_{\widetilde{X}}$ at $\tilde{x}$ is the inverse image of $R \subseteq \kappa(\mathfrak{p})$ in $\mathcal{O}^{\triangleleft}_{\widetilde{X},\tilde{x}}$.*
(ii) *The stalk $\mathcal{O}_{\widetilde{X},\tilde{x}}$ is a Φ-local Φ-ring with residual valuation ring R (cf. I.2.39).*

The stalk $\mathcal{O}_{\widetilde{X},\tilde{x}}$ is the preimage of R under the canonical surjective homomorphism $(\mathcal{O}^{\triangleleft}_{X,x})_{\mathfrak{p}} \to \kappa(\mathfrak{p})$, and a finitely generated ideal I of $\mathcal{O}_{\widetilde{X},\tilde{x}}$ is admissible if and only if $|I(\tilde{x})| > 0$, i.e. if and only if I is generated by an element which is invertible in $(\mathcal{O}^{\triangleleft}_{X,x})_{\mathfrak{p}}$. By I.2.40 one concludes that $\mathcal{O}_{\widetilde{X},\tilde{x}}$ is a Φ-local Φ-ring.

Corollary I.4.8. *The canonical morphism $(\mathcal{O}_{\widetilde{X}})^{\triangleleft} \to \mathcal{O}^{\triangleleft}_{\widetilde{X}}$ of Φ-rings is an isomorphism.*

This follows from I.3.23 and I.4.7.

Definition I.4.9. The **Φ-localization** of a locally Φ-ringed topological space X is the Φ-locally Φ-ringed topological space $\widetilde{X}$ endowed with the sheaf of Φ-rings $\mathcal{O}_{\widetilde{X}}$.

I.4.10. The canonical morphism $\pi^{-1}\mathcal{O}_X \to \mathcal{O}^{\triangleleft}_{\widetilde{X}}$ uniquely factors through a morphism of sheaves of Φ-rings from $\pi^{-1}\mathcal{O}_X$ to $\mathcal{O}_{\widetilde{X}}$. This provides π with the structure of a morphism of locally Φ-ringed topological spaces (cf. I.4.7). Moreover the source $\widetilde{X}$ of π is Φ-locally Φ-ringed by I.4.7.

Proposition I.4.11. *The morphism $\pi : \widetilde{X} \to X$ is terminal among morphisms of locally Φ-ringed topological spaces from a Φ-locally Φ-ringed topological space to X, and hence is an isomorphism if the stalks of $\mathcal{O}_X$ are Φ-local. In particular, the assignment $X \mapsto \widetilde{X}$ is functorial and provides a right adjoint to the canonical inclusion functor from Φ-locally Φ-ringed topological spaces to locally Φ-ringed topological spaces.*

Let us consider a morphism $\varphi : Y \to X$ of locally Φ-ringed topological spaces such that the stalks of $\mathcal{O}_Y$ are Φ-local Φ-rings. For each point y of Y we denote by $f \mapsto |f(y)|$ the valuation on $\mathcal{O}^{\triangleleft}_{Y,y}$ associated to its residual valuation ring (cf. I.2.39).

We first construct a map $\widetilde{\varphi} : Y \to \widetilde{X}$ such that $\pi\widetilde{\varphi} = \varphi$. Let y be a point of Y and consider the morphism of Φ-rings $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$. Let $\mathfrak{m}_y$ be the maximal ideal of $\mathcal{O}^{\triangleleft}_{Y,y}$, and let $(\mathfrak{p}, R)$ be the pair obtained from the local homomorphism $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}/\mathfrak{m}_y$ by the construction I.2.41. We set $\widetilde{\varphi}(y) = (\varphi(y), \mathfrak{p}, R)$.

The map $\widetilde{\varphi}$ is continuous. Indeed if $I = (f_1, \ldots, f_r)$ is an admissible ideal of $\mathcal{O}_X(U)$ and if g is an element of $\mathcal{O}_X(U)$ then for each $y \in \pi^{-1}(U)$ we have $|\varphi^{-1}I(y)| \leq |\varphi^{-1}g(y)|$ if and only if $|I(\widetilde{\varphi}(y))| \leq |g(\widetilde{\varphi}(y))|$ (cf. I.2.41(d)). We thus have

$$\widetilde{\varphi}^{-1}\left(U(g^{-1}I)\right) = \{y \in \varphi^{-1}(U) \mid |\varphi^{-1}I(y)| \leq |\varphi^{-1}g(y)|\}.$$

This is an open subset of Y since if y is one of its points then $f_i = a_i g$ for some elements $a_1, \ldots, a_r$ of $\mathcal{O}_{Y,y}$ (cf. I.2.39), so that $I\mathcal{O}_V \subseteq g\mathcal{O}_V$ for some open neighbourhood V of y in $\varphi^{-1}(U)$.

Let us consider the homomorphism

$$\widetilde{\varphi}^{-1}\pi^{-1}\mathcal{O}_X^{\triangleleft} \xrightarrow{\sim} \varphi^{-1}\mathcal{O}_X^{\triangleleft} \to \mathcal{O}_Y^{\triangleleft}.$$

This homomorphism sends the subsheaf of monoids $\widetilde{\varphi}^{-1}S$ (cf. I.4.5) into the sheaf of invertible elements of $\mathcal{O}_{Y,y}^{\triangleleft}$ (cf. I.2.39), so that it uniquely factors though a local homomorphism $\widetilde{\varphi}^{-1}\mathcal{O}_{\widetilde{X}}^{\triangleleft} \to \mathcal{O}_Y^{\triangleleft}$. Moreovever the image of $\widetilde{\varphi}^{-1}\mathcal{O}_{\widetilde{X}}$ is contained in the subsheaf $\mathcal{O}_{Y,y}$, so that we obtain a morphism $\widetilde{\varphi}^{-1}\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_Y$ which endows $\widetilde{\varphi}$ with a structure of morphism of locally Φ-ringed topological spaces.

I.4.12. We assume for the remainder of this section that the locally ringed topological space underlying X is a scheme. We make the following two definitions:

- ▹ A quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ is **admissible** if for any affine open subscheme $V \subseteq X$, the ideal $\mathcal{I}(V)$ is admissible in the Φ-ring $\mathcal{O}_X(V)$.
- ▹ An **admissible blow-up** $X' \to X$ is the blow-up of X along a closed subscheme defined by an admissible quasi-coherent ideal sheaf.

Let $\pi : \widetilde{X} \to X$ be the Φ-localization of X (cf. I.4.9). Let $\mathcal{I}$ be an admissible quasi-coherent ideal sheaf on X, and let V be an affine open subset of X. Then $\mathcal{I}(V) = (f_1, \ldots, f_r)$ is an admissible ideal in $\mathcal{O}_X(V)$. Hence $\mathcal{I}(V)$ becomes invertible on $\pi^{-1}(V)$. More precisely, we have the decomposition by I.2.39

$$\pi^{-1}(V) = \bigcup_{i=1}^{r} V(f_i^{-1}\mathcal{I}(V)),$$

and the restriction of $\mathcal{I}\mathcal{O}_{\widetilde{X}}$ to $V(f_i^{-1}\mathcal{I}(V))$ is freely generated by f_i. Thus $\mathcal{I}\mathcal{O}_{\widetilde{X}}$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module of rank 1. In particular if we denote by $X_{\mathcal{I}}$ the blow-up in X of the closed subscheme defined by $\mathcal{I}$, then from the universal property of the blow-up we obtain a unique factorisation $\widetilde{X} \to X_{\mathcal{I}} \to X$ of π.[4] This yields a morphism

$$\widetilde{X} \to \lim_{X' \to X} X' \tag{6}$$

of locally Φ-ringed topological spaces, where the limit runs over the category of admissible blow-ups of X. The latter category is cofiltered; indeed it is non empty since the identity from X to itself is an admissible blow-up, any pair $X_{\mathcal{I}}, X_{\mathcal{J}}$ of admissible blow-ups of X is dominated by the admissible blow-up $X_{\mathcal{I}\mathcal{J}}$ and there are no pair of distinct parallel arrows since for any admissible quasi-coherent ideals $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{O}_X$, the existence of an X-morphism $X_{\mathcal{I}} \to X_{\mathcal{J}}$ implies that $X_{\mathcal{I}}$ is $\mathcal{J}$-torsion free, since $\mathcal{J}$ is invertible on its schematically dense open subspace $X \setminus V(\mathcal{I})$, so that the universal property of the blow-up ensures that there exists a unique such morphism.

The limit (6) of locally Φ-ringed topological spaces is given by taking the corresponding limit in the category of locally ringed topological spaces, and by declaring the canonical isomorphism of its structure sheaf with the colimit

$$\operatorname{colim}_{f : X' \to X} \; f^{-1}\mathcal{O}_{X'}$$

to be an isomorphism of sheaves of Φ-rings.

Theorem I.4.13. *Assume that X is quasi-compact and quasi-separated and that there exists a quasi-compact open subscheme $U \subseteq X$ such that for any open subset $V \subseteq X$, a finitely*

4. The universal property of the blow-up, while usually stated in the category of schemes, also holds in the larger category of locally ringed topological spaces.

generated ideal I of $\mathcal{O}_X(V)$ is admissible if and only if $I\mathcal{O}_{U\cap V} = \mathcal{O}_{U\cap V}$. Then the canonical morphism

$$\widetilde{X} \to \lim_{X'\to X} X'$$

from $\widetilde{X}$ to the inverse limit of the admissible blow-ups of X is an isomorphism of locally Φ-ringed topological spaces. In particular $\widetilde{X}$ is quasi-compact and is even a spectral space, cf. ([**SP**] 0A2V*,* 0A2Z)

For any admissible blow-up $f : X' \to X$ we endow X' with the unique structure of locally Φ-ringed topological space such that for any open subset $V \subseteq X'$, a finitely generated ideal I of $\mathcal{O}_{X'}(V)$ is admissible if and only if $I\mathcal{O}_{f^{-1}(U)\cap V} = \mathcal{O}_{f^{-1}(U)\cap V}$.

In order to prove I.4.13, it is sufficient to show that the limit in (6), which is a locally Φ-ringed topological space, is actually Φ-locally Φ-ringed. Indeed, a morphism from a Φ-locally Φ-ringed topological space to X uniquely factors through the limit in (6) by the universal property of the blow-up, hence if the latter is Φ-locally Φ-ringed then it is terminal among morphisms of locally Φ-ringed topological spaces from a Φ-locally Φ-ringed topological space to X, so that the canonical morphism (6) is an isomorphism by virtue of I.4.11.

Let us therefore prove that the limit in (6) is Φ-locally Φ-ringed. Let $x = (x_{\mathcal{I}})_{\mathcal{I}}$ be a point on this limit. The stalk of the structure sheaf at x is the colimit of $\mathcal{I} \mapsto \mathcal{O}_{X_{\mathcal{I}},x_{\mathcal{I}}}$ where the index $\mathcal{I}$ runs over all admissible quasi-coherent ideal sheaves on X. Let J be an admissible ideal of $\mathcal{O}_{X_{\mathcal{I}},x_{\mathcal{I}}}$. We must show that $J\mathcal{O}_{X_{\mathcal{I}\mathcal{K}},x_{\mathcal{I}\mathcal{K}}}$ is an invertible ideal for some admissible quasi-coherent ideal sheaf $\mathcal{K}$.

We first prove that there exists an admissible quasi-coherent ideal sheaf $\mathcal{J}$ on $X_{\mathcal{I}}$ such that $\mathcal{J}_{x_{\mathcal{I}}} = J$. The ideal J is the stalk at $x_{\mathcal{I}}$ of an admissible quasi-coherent ideal sheaf $\mathcal{J}_0$ on an open neighbourhood V of $x_{\mathcal{I}}$ in $X_{\mathcal{I}}$. Since $\mathcal{J}_0\mathcal{O}_{U\cap V} = \mathcal{O}_{U\cap V}$ where U denotes the (isomorphic) inverse image of U in $X_{\mathcal{I}}$, there exists a unique quasi-coherent ideal sheaf $\mathcal{J}_1$ on $U \cup V$ such that $\mathcal{J}_{1|U} = \mathcal{O}_U$ and $\mathcal{J}_{1|V} = \mathcal{J}_0$. Since $X_{\mathcal{I}}$ and U are quasi-compact, there exists by ([**EGA1**], 9.4.7) a quasi-coherent ideal sheaf $\mathcal{J}$ of finite type on $X_{\mathcal{I}}$ such that $\mathcal{J}_{|U\cup V} = \mathcal{J}_1$. The condition $\mathcal{J}\mathcal{O}_U = \mathcal{O}_U$ implies that $\mathcal{J}$ is admissible.

By ([**RG71**], *I*.5.1.4) the composition of blow-ups $(X_{\mathcal{I}})_{\mathcal{J}} \to X$ is isomorphic to the blow-up of an admissible quasi-coherent ideal sheaf $\mathcal{K}$ on X. Thus $J\mathcal{O}_{X_{\mathcal{I}\mathcal{K}},x_{\mathcal{I}\mathcal{K}}}$ is an invertible ideal and we have shown that the stalks of the structure sheaf of the limit appearing in (6) are Φ-local Φ-rings. As explained above, this together with I.4.11 and the universal property of the blow-up concludes the proof of Theorem I.4.13.

I.5. The flattening property of Φ-localizations

I.5.1. If M is a module over a valuation ring R then M is R-flat if and only if for any nonzero element r of R the module M has no nonzero r-torsion, see ([**Bou98**], VI §3.6 Lemme 1) or ([**SP**] 0539). We generalize this fact to modules over Φ-local Φ-rings:

PROPOSITION I.5.2. *Let A be a Φ-local Φ-ring, and let M be a A-module. The following are equivalent:*

(i) *The A-module M is flat.*

(ii) *The $A^{\triangleleft}$-module $M^{\triangleleft} = M \otimes_A A^{\triangleleft}$ is flat, and the map $M \to M^{\triangleleft}$ is injective.*

We have $(i) \implies (ii)$, since A is 1-deep by I.2.39. We thus focus on the converse implication. We use the equational criterion of flatness, see ([**Bou98**], I §2.11 Corollaire 1) or ([**SP**] 00HK). Namely, given a relation $\sum_{d\in D} a_d x_d = 0$ with a_d in A and x_d in M, for some finite set D, we would like to show that this relation is trivial, in the sense that we can find relations of the form

$$x_d = \sum_{e\in E} b_{de} y_e,$$

such that $\sum_{d\in D} a_d b_{de} = 0$ for any e. We proceed by induction on the cardinality of D, the case $D = \emptyset$ being empty.

First, assume that we have a relation of the form $\sum_{d\in D} a_d c_d = 0$, where $(c_d)_{d\in D} \subseteq A$ and $c_{d_0} = 1$ for some d_0 in D. We then have

$$\sum_{d\in D\setminus\{d_0\}} a_d(x_d - c_d x_{d_0}) = \sum_{d\in D} a_d(x_d - c_d x_{d_0}) = 0.$$

Since this relation has fewer terms than the original one, the induction hypothesis ensures that we can find relations

$$x_d - c_d x_{d_0} = \sum_{e\in E} b_{de} y_e$$

with $\sum_{d\in D\setminus\{d_0\}} a_d b_{de} = 0$ for any e. We set $E' = E \sqcup \{\star\}$, $y_\star = x_{d_0}$, $b_{d_0 e} = 0$ for all $e \in E$, and $b_{d\star} = c_d$ for all $d \in D$, so that $b_{d_0\star} = 1$. We then have

$$x_d = \sum_{e\in E'} b_{de} y_e,$$

for any d in D, and $\sum_{d\in D} a_d b_{de} = 0$ for any e in E'. Thus our relation is trivial.

Let us now assume that the previous case does not happen. Since $M^{\triangleleft}$ is flat over $A^{\triangleleft}$, our relation is trivial in $M^{\triangleleft}$, hence we can find relations

$$x_d = \sum_{e\in E} b_{de} y_e,$$

such that $\sum_{d\in D} a_d b_{de} = 0$ for any e, with y_e in M. Here the coefficients b_{de} are elements of $A^{\triangleleft}$, hence it is sufficient for our purpose to show that they belong to A.

Actually, we have the stronger result that for all $(d, e) \in D \times E$, the element b_{de} belongs to the maximal ideal of $A^{\triangleleft}$. Indeed, if it were not the case, then, for some $e_0 \in E$, the ideal generated by $(b_{de_0})_{d\in D}$ would not be contained in the maximal ideal of $A^{\triangleleft}$. We would then have an element s of A, invertible in $A^{\triangleleft}$, such that all the sb_{de_0} belong to A. The ideal of A generated by the family $(sb_{de_0})_{d\in D}$ would then be admissible, hence generated by $sb_{d_0 e_0}$ for some d_0 in D. This would imply that $sb_{de_0} = c_d s b_{d_0 e_0}$ for some c_d in A, with $c_{d_0} = 1$, and the relation $\sum_{d\in D} a_d b_{de_0} = 0$ would yield $\sum_{d\in D} a_d c_d = 0$. This is a contradiction.

The following lemma is both a consequence and a generalization of ([**SP**] 053E):

Proposition I.5.3. *Let A be a Φ-local Φ-ring, let $A \to B$ be a ring homomorphism of finite type, and let M be a B-module of finite type. If M is flat over A, and if $M \otimes_A A^{\triangleleft}$ is a finitely presented $B \otimes_A A^{\triangleleft}$-module, then M is a finitely presented B-module.*

Since M is a B-module of finite type, we can find an exact sequence

$$0 \to K \to B^{\oplus r} \to M \to 0.$$

Let $\mathfrak{m}$ be the maximal ideal of $A^{\triangleleft}$, which is contained in A by I.2.39 (ii), and let $S = A \setminus \mathfrak{m}$. Since M is flat over A, the following exact sequence is still exact:

$$0 \to K/\mathfrak{m}K \to (B/\mathfrak{m}B)^{\oplus r} \to M/\mathfrak{m}M \to 0.$$

By ([**SP**] 053E), the kernel $K/\mathfrak{m}K$ is a finitely generated $B/\mathfrak{m}B$-module. Thus there is a finite subset $\mathfrak{S}$ of K such that $K = B\mathfrak{S} + \mathfrak{m}K$. But we also have an exact sequence

$$0 \to S^{-1}K \to \left(S^{-1}B\right)^{\oplus r} \to S^{-1}M \to 0,$$

and since by hypothesis $S^{-1}M = M \otimes_A A^{\triangleleft}$ is a finitely presented module over $S^{-1}B = B \otimes_A A^{\triangleleft}$, the kernel $S^{-1}K$ is a finitely generated $S^{-1}B$-module. Thus, by enlarging $\mathfrak{S}$ if necessary, we can also assume that $S^{-1}K = S^{-1}B\mathfrak{S}$. We claim that $K = B\mathfrak{S}$. Indeed, the relations $S^{-1}K = S^{-1}B\mathfrak{S}$ and $\mathfrak{m} = S^{-1}\mathfrak{m}$ imply

$$\mathfrak{m}K = \mathfrak{m}S^{-1}K = \mathfrak{m}S^{-1}B\mathfrak{S} = \mathfrak{m}B\mathfrak{S},$$

so that

$$K = B\mathfrak{S} + \mathfrak{m}K = B\mathfrak{S} + \mathfrak{m}B\mathfrak{S} = B\mathfrak{S}.$$

In particular, K is a finitely generated B-module, and consequently M is a finitely presented B-module.

Corollary I.5.4. *Let A be a Φ-local Φ-ring, let $A \to B$ be a ring homomorphism of finite type. If B is flat over A, and if $B \otimes_A A^{\triangleleft}$ is a finitely presented $A^{\triangleleft}$-algebra, then B is a finitely presented A-algebra.*

Indeed, if we choose a surjection $B' = A[X_1, \dots, X_n] \to B$, then B is flat over A, and $B \otimes_A A^{\triangleleft}$ is a finitely presented $B' \otimes_A A^{\triangleleft}$-module since $B \otimes_A A^{\triangleleft}$ is a finitely presented $A^{\triangleleft}$-algebra. Thus, by I.5.3, we conclude that B is a finitely presented B'-module, and thus that B is a finitely presented A-algebra.

I.5.5. Let $(X, \mathcal{O}_X)$ be a ringed topological space. A commutative $\mathcal{O}_X$-algebra $\mathcal{A}$ is said to be *finitely presented* if for any point x of X there exists an open neighbourhood U of x in X and an isomorphism

$$A_U \otimes_{\Gamma(U,\mathcal{O}_X)} \mathcal{O}_U \to \mathcal{A}_{|U},$$

of $\mathcal{O}_U$-algebras, for some finitely presented $\Gamma(U, \mathcal{O}_X)$-algebra A_U.

Proposition I.5.6. *Let $(X, \mathcal{O}_X)$ be a ringed topological space, let $\mathcal{A}$ be finitely presented commutative $\mathcal{O}_X$-algebra, and let $\mathcal{G}$ be a finitely presented $\mathcal{A}$-module. Let x be a point of X such that the stalk $\mathcal{M}_x$ is a flat $\mathcal{O}_{X,x}$-module. Then there exists an open neighbourhood U of x such that the restriction $\mathcal{M}_{|U}$ is a flat $\mathcal{O}_U$-module.*

Up to replacing X by some open neighbourhood of x in X, we can assume (and we do) that $\mathcal{A}$ is of the form $A \otimes_{\Gamma(X,\mathcal{O}_X)} \mathcal{O}_X$, for some finitely presented $\Gamma(X, \mathcal{O}_X)$-algebra A, and that we have a global presentation

$$\mathcal{A}^{\oplus m} \xrightarrow{\varphi} \mathcal{A}^{\oplus n} \to \mathcal{M} \to 0.$$

For any integer j between 1 and m, let $(a_{i,j})_{i=1}^n$ be the image by φ of the j-th basis vector. Each $a_{i,j}$ belongs to $\Gamma(X, \mathcal{A})$. Let V be an open neighbourhood of x in X such that each restriction $a_{i,j|V}$ is the image in $\Gamma(V, \mathcal{A})$ of an element $b_{i,j}$ of the finitely presented $\Gamma(V, \mathcal{O}_X)$-algebra $A_V = A \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(V, \mathcal{O}_X)$. Let M_V be the cokernel of the matrix $(b_{i,j})_{i,j}$. It is a finitely presented A_V-module, and we have an isomorphism

$$M_V \otimes_{A_V} \mathcal{A}_{|V} \to \mathcal{M}_{|V},$$

of $\mathcal{A}_{|V}$-modules. For each open neighbourhood U of x in V, the $\Gamma(U, \mathcal{O}_X)$-algebra $A_U = A_V \otimes_{\Gamma(V,\mathcal{O}_X)} \Gamma(U, \mathcal{O}_X)$ is finitely presented, and the A_U-module $M_U = M_V \otimes_{A_V} A_U$ is finitely presented. Moreover, by hypothesis the filtered colimit

$$\mathrm{colim}_U M_U \cong \mathcal{M}_x,$$

where U runs over the cofiltered set of open neighbourhood of x in V, is a flat module over the colimit

$$\mathrm{colim}_U \Gamma(U, \mathcal{O}_X) = \mathcal{O}_{X,x}.$$

By ([**SP**] 02JO(3)), this implies that there exists an open neighbourhood U of x in V such that M_U is a flat $\Gamma(U, \mathcal{O}_X)$-module. In particular, the $\mathcal{O}_U$-module

$$\mathcal{M}_{|U} \cong M_U \otimes_{A_U} \mathcal{A}_{|U} \cong M_U \otimes_{\Gamma(U,\mathcal{O}_X)} \mathcal{O}_U,$$

is flat, hence the conclusion of Proposition I.5.6.

I.5.7. Let X be a Φ-ringed topological space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the **closure** $\mathcal{F}^{\triangleleft}$ of $\mathcal{F}$ to be the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\triangleleft}$ and the **purification** $\mathcal{F}^{\mathrm{pur}}$ of $\mathcal{F}$ to be the subsheaf of $\mathcal{F}^{\triangleleft}$ generated by the image of $\mathcal{F}$.

THEOREM I.5.8. *Let X be a locally Φ-ringed topological space with the Φ-localization $\pi : \widetilde{X} \to X$ (cf. I.4.9), and let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that $\mathcal{F}^{\triangleleft}$ is a flat $\mathcal{O}_X^{\triangleleft}$-module (cf. I.5.7). Then $(\pi^*\mathcal{F})^{\mathrm{pur}}$ is a flat $\mathcal{O}_{\widetilde{X}}$-module.*

The stalk of $(\pi^*\mathcal{F})^{\mathrm{pur}}$ at a point $\tilde{x} = (x, \mathfrak{p}, R)$ of $\widetilde{X}$ is the purification of $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{\widetilde{X},\tilde{x}}$. Since $(\pi^*\mathcal{F})^{\triangleleft}_{\tilde{x}} \cong (\mathcal{F}_x^{\triangleleft})_{\mathfrak{p}}$ is a flat module over $\mathcal{O}^{\triangleleft}_{\widetilde{X},\tilde{x}} \cong (\mathcal{O}^{\triangleleft}_{X,x})_{\mathfrak{p}}$ (cf. I.4.7), one concludes by I.5.2 that $(\pi^*\mathcal{F})^{\mathrm{pur}}_{\tilde{x}}$ is flat over the Φ-local Φ-ring $\mathcal{O}_{\widetilde{X},\tilde{x}}$.

THEOREM I.5.9. *Let X be a locally Φ-ringed topological space with the Φ-localization $\pi : \widetilde{X} \to X$ (cf. I.4.9). Let $\mathcal{A}$ be a finitely presented commutative $\mathcal{O}_X$-algebra (cf. I.5.5) and let $\mathcal{F}$ be an $\mathcal{A}$-module of finite type such that $\mathcal{F}^{\triangleleft}$ is finitely presented as an $\mathcal{A}^{\triangleleft}$-module and flat as an $\mathcal{O}_X^{\triangleleft}$-module (cf. I.5.7). Then $(\pi^*\mathcal{F})^{\mathrm{pur}}$ is finitely presented as a $\pi^*\mathcal{A}$-module and flat as an $\mathcal{O}_{\widetilde{X}}$-module.*

By using I.5.8 and replacing X with $\widetilde{X}$ it is sufficient for the purpose of proving I.5.9 to show the following:

PROPOSITION I.5.10. *Let X be a Φ-locally Φ-ringed topological space. Let $\mathcal{A}$ be a finitely presented commutative $\mathcal{O}_X$-algebra (cf. I.5.5) and let $\mathcal{F}$ be an $\mathcal{A}$-module of finite type such that $\mathcal{F}^{\triangleleft}$ is finitely presented as an $\mathcal{A}^{\triangleleft}$-module and $\mathcal{F}$ is flat as an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is finitely presented as an $\mathcal{A}$-module.*

Let x be a point of X. Let U be an open neighbourhood of x such that there exists a surjective homomorphism $\psi : \mathcal{A}_{|U}^{\oplus n} \to \mathcal{F}_{|U}$. The $\mathcal{A}^{\triangleleft}_{|U}$-module $\mathcal{F}^{\triangleleft}_{|U}$ is finitely presented, and by ([**SP**] 01BP) this implies that the $\mathcal{A}^{\triangleleft}_{|U}$-module $(\ker\psi)^{\triangleleft}$ is of finite type. By I.5.3, the $\mathcal{A}_x$-module $\mathcal{F}_x$ is finitely presented, and hence the stalk of $\ker\psi$ at x is a finitely generated $\mathcal{A}_x$-module. Thus up to shrinking U if necessary there exists a finitely generated sub-module $\mathcal{H} \subseteq \ker\psi$ such that $\mathcal{H}_x = (\ker\psi)_x$ and $\mathcal{H}^{\triangleleft} = (\ker\psi)^{\triangleleft}$. Let $\mathcal{G}$ be the quotient of $\mathcal{A}_{|U}^{\oplus n}$ by $\mathcal{H}$. Then $\mathcal{G}$ is a finitely presented $\mathcal{A}_{|U}$-module and we have a surjective homomorphism $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}_x \cong \mathcal{F}_x$ and $\mathcal{G}^{\triangleleft} \cong \mathcal{F}^{\triangleleft}$. Moreover, $\mathcal{G}$ and $\mathcal{F}$ have the same image in $\mathcal{F}^{\triangleleft}$, so that $\mathcal{F}$ is the purification of $\mathcal{G}$. Since $\mathcal{A}_{|U}$ is a finitely presented $\mathcal{O}_U$-algebra, since $\mathcal{G}_x$ is a flat $\mathcal{O}_{X,x}$-module and since $\mathcal{G}$ is a finitely presented $\mathcal{A}_{|U}$-module, we conclude by Proposition I.5.6 that there exists an open neighbourhood $V \subseteq U$ of x such that $\mathcal{G}_{|V}$ is a flat $\mathcal{O}_V$-module. In particular $\mathcal{G}_{|V}$ is isomorphic to its purification $\mathcal{F}_{|V}$, so that $\mathcal{F}_{|V}$ is finitely presented as an $\mathcal{A}_{|V}$-module.

I.6. Proof of Raynaud-Gruson's theorem

I.6.1. We first prove the following variant of Raynaud-Gruson's theorem I.1.1:

THEOREM I.6.2. *Let X be quasi-compact and quasi-separated scheme and let U be a quasi-compact open subset of X. Let $\mathcal{A}$ be a (quasi-coherent) finitely presented commutative $\mathcal{O}_X$-algebra and let $\mathcal{F}$ be a quasi-coherent $\mathcal{A}$-module of finite type. Assume that $\mathcal{F}_{|U}$ is finitely presented over $\mathcal{A}_{|U}$ and flat over $\mathcal{O}_U$. Then there exists a blow-up $f : X' \to X$ such that:*
1. *The center of f is a finitely presented closed subscheme of X, disjoint from U.*
2. *The strict transform $\mathcal{F}'$ of $\mathcal{F}$ along f is finitely presented over $f^*\mathcal{A}$ and flat over $\mathcal{O}_{X'}$.*

Since X is quasi-compact and quasi-separated and U is quasi-compact, the complement of U is the closed subscheme defined by a finitely generated quasi-coherent ideal sheaf $\mathcal{J}$ on X: indeed, the quasi-coherent ideal defining this complement with its reduced structure is a filtered union of the family $(\mathcal{J}_\lambda)_{\lambda\in\Lambda}$ of its finitely generated quasi-coherent subsheaves by

([**EGA1**], 6.9.9), hence the quasi-compact open subset U is the filtered union of the open subsets $(X \setminus V(\mathcal{J}_\lambda))_{\lambda\in\Lambda}$, and thus U is equal to $X \setminus V(\mathcal{J}_\lambda)$ for some λ. Let $f : X' \to X$ be the blowing-up of X along $\mathcal{J}$. For any blow-up $g : X'' \to X'$ whose center is a finitely presented closed subscheme of X' disjoint from $f^{-1}(U)$, the composition fg is a blow-up whose center is a finitely presented closed subscheme of X disjoint from U, cf. ([**RG71**], *I*.5.1.4). Hence by replacing X with X', we can assume (and we do) that $\mathcal{J}$ is invertible.

We endow X with the structure of a locally Φ-ringed topological space by declaring for any open subset $V \subseteq X$ a finitely generated ideal I of $\mathcal{O}_X(V)$ to be admissible whenever $I\mathcal{O}_{U\cap V} = \mathcal{O}_{U\cap V}$. Since $\mathcal{J}$ is invertible we have $\mathcal{O}_X^{\triangleleft} = \operatorname{colim}_{N>0}\ \mathcal{J}^{-N} = j_*\mathcal{O}_U$ where $j : U \to X$ is the canonical inclusion. In particular the assumptions of Theorem I.6.2 imply that $\mathcal{F}^{\triangleleft} = \operatorname{colim}_{N>0}\ \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{J}^{-N}$ is flat over $\mathcal{O}_X^{\triangleleft} = \operatorname{colim}_{N>0}\mathcal{J}^{-N}$ and finitely presented over $\mathcal{A}^{\triangleleft} = \operatorname{colim}_{N>0}\ \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{J}^{-N}$.

Let $\pi : \widetilde{X} \to X$ be the Φ-localization of X. By ([**EGA1**], 6.9.10) applied to the relative spectrum of $\mathcal{A}$, there exists a finitely presented $\mathcal{A}$-module $\mathcal{G}$ together with a surjective homomorphism $\mathcal{G} \to \mathcal{F}$. By Theorem I.5.9 the $\pi^*\mathcal{A}$-module $(\pi^*\mathcal{F})^{\mathrm{pur}}$ is finitely presented, hence the kernel $\mathcal{K}$ of the surjective homomorphism $\pi^*\mathcal{G} \to (\pi^*\mathcal{F})^{\mathrm{pur}}$ is of finite type.

By I.4.13 we have an isomorphism of locally Φ-ringed topological spaces

$$\widetilde{X} \to \lim_{X'\to X} X'$$

from $\widetilde{X}$ to the inverse limit of the admissible blow-ups of X (cf. I.4.12). We endow each blow-up $g : X' \to X$ with the structure of a locally Φ-ringed topological space by declaring for any open subset $V \subseteq X'$ a finitely generated ideal I of $\mathcal{O}_{X'}(V)$ to be admissible whenever $I\mathcal{O}_{g^{-1}(U)\cap V} = \mathcal{O}_{g^{-1}(U)\cap V}$. Since $\mathcal{K}$ is a $\pi^*\mathcal{A}$-module of finite type, there exists a factorization $\widetilde{X} \xrightarrow{\pi'} X' \xrightarrow{f} X$ of π through an admissible blow-up and a finitely generated sub-$f^*\mathcal{A}$-module $\mathcal{K}'$ of $\operatorname{Ker}(f^*\mathcal{G} \to (f^*\mathcal{F})^{\mathrm{pur}})$ such that the image of $\pi'^*\mathcal{K}'$ in $\pi^*\mathcal{G}$ is equal to $\mathcal{K}$.

Let us consider the finitely presented $f^*\mathcal{A}$-module $\mathcal{G}' = f^*\mathcal{G}/\mathcal{K}'$. We have $\pi'^*\mathcal{G}' \cong (\pi^*\mathcal{F})^{\mathrm{pur}}$, hence an isomorphism $(\pi'^*\mathcal{G}')^{\triangleleft} \cong (\pi^*\mathcal{F})^{\triangleleft}$. By I.4.4 and I.4.6, the morphism of ringed space

$$(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^{\triangleleft}) \to (X', \mathcal{O}_{X'}^{\triangleleft})$$

is flat and surjective, hence we deduce that the canonical homomorphism $\mathcal{G}'^{\triangleleft} \to (f^*\mathcal{F})^{\triangleleft}$ is an isomorphism. Since the homomorphism $\mathcal{G}' \to f^*\mathcal{F}$ is surjective, we obtain that the purification of $\mathcal{G}'$ is isomorphic to $(f^*\mathcal{F})^{\mathrm{pur}}$. More generally, for any admissible blow-up $X'' \xrightarrow{f'} X'$, the purification of $f'^*\mathcal{G}'$ is isomorphic to $(f'^*f^*\mathcal{F})^{\mathrm{pur}}$.

For each admissible blow-up $X'' \xrightarrow{f'} X' \xrightarrow{f} X$, the set of points x'' of X'' such that $f'^*\mathcal{G}'_{x''}$ is flat over $\mathcal{O}_{X'',x''}$ is an open subset $U_{f'}$ of X'' by ([**SP**], 0399(2)). The $\mathcal{O}_{\widetilde{X}}$-module $\pi'^*\mathcal{G}' \cong (\pi^*\mathcal{F})^{\mathrm{pur}}$ is flat by Theorem I.5.9, hence by ([**SP**] 02JO(3)), any point of $\widetilde{X}$ is in the inverse image of $U_{f'}$ for some f'. Since $\widetilde{X}$ is quasi-compact by Theorem I.4.13, we can find f' as above such that the inverse image of $U_{f'}$ in $\widetilde{X}$ is $\widetilde{X}$. By ([**SP**], 0A2W(1)) we can assume up to taking a further refinement of f' that we have $U_{f'} = X''$. Since $f'^*\mathcal{G}'$ is flat over $\mathcal{O}_{X''}$, it must coincide with its purification $(f'^*f^*\mathcal{F})^{\mathrm{pur}}$. In particular the strict transform $(f'^*f^*\mathcal{F})^{\mathrm{pur}}$ of $\mathcal{F}$ on X'' is flat over $\mathcal{O}_{X''}$ and finitely presented over $f'^*f^*\mathcal{A}$. This concludes the proof of I.6.2.

I.6.3. We now prove Raynaud-Gruson's theorem I.1.1. Let $g : Y \to X, U$ and $\mathcal{F}$ be as in the statement of the theorem. Let $(X_\lambda)_{\lambda\in\Lambda}$ and $(Y_\lambda)_{\lambda\in\Lambda}$ be covers of X and Y by finitely many affine open subsets such that for each $\lambda \in \Lambda$, we have $g(Y_\lambda) \subseteq X_\lambda$. Then for each $\lambda \in \Lambda$, Theorem I.6.2 yields a finitely generated quasi-coherent ideal sheaf $\mathcal{J}_\lambda \subseteq \mathcal{O}_{X_\lambda}$ such that $V(\mathcal{J}_\lambda)$ is disjoint from $U \cap X_\lambda$ and such that the blow-up $\mathcal{J}_\lambda$ has the desired flattening property with respect to the affine morphism $Y_\lambda \to X_\lambda$ and $\mathcal{F}_{|Y_\lambda}$.

Since $\mathcal{J}_\lambda \mathcal{O}_{U\cap X_\lambda} = \mathcal{O}_{U\cap X_\lambda}$, there exists a unique quasi-coherent ideal sheaf $\mathcal{J}'_\lambda$ on $U \cup X_\lambda$ such that $\mathcal{J}'_{\lambda|U} = \mathcal{O}_U$ and $\mathcal{J}'_{\lambda|X_\lambda} = \mathcal{J}_\lambda$. Since X and $U \cup X_\lambda$ are quasi-compact, there exists by ([**EGA1**], 9.4.7) a quasi-coherent ideal sheaf $\mathcal{I}_\lambda$ of finite type on X such that $\mathcal{I}_{\lambda|U\cup X_\lambda} = \mathcal{J}'_\lambda$. The blow-up of $\prod_{\lambda\in\Lambda} \mathcal{I}_\lambda$ then satisfies the conclusion of Theorem I.1.1.

Chapitre II

Théorie du corps des classes géométrique

Sommaire

II.1. Introduction

Let $X \to S$ be a relative curve, i.e. a smooth morphism of schemes of relative dimension 1, with connected geometric fibers, which is Zariski-locally projective over S. Let $Y \hookrightarrow X$ be a relative effective Cartier divisor over S (cf. II.4.10), and let U be the complement of Y in X.

The pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module and α is a rigidification of $\mathcal{L}$ along Y, are parametrized by an S-group scheme $\mathrm{Pic}_S(X, Y)$, the relative rigidified Picard scheme (cf. II.4.8). The Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_S(X, Y)$$

is the morphism which sends a section x of U to the pair $(\mathcal{O}(x), 1)$, cf. II.4.14. We prove the following relative version of the main theorem of geometric global class field theory:

Theorem II.1.1. *(Th. II.5.3) Let Λ be a finite ring of cardinality invertible on S, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by Y (cf. II.5.2). Then, there exists a unique (up to isomorphism) multiplicative étale sheaf of Λ-modules $\mathcal{G}$ on $\mathrm{Pic}_S(X, Y)$, locally free of rank 1, such that the pullback of $\mathcal{G}$ by Φ is isomorphic to $\mathcal{F}$.*

The notion of multiplicative locally free Λ-module of rank 1 is defined in II.2.5, and it corresponds to isogenies $G \to \mathrm{Pic}_S(X, Y)$ with constant kernel $\Lambda^\times$. We restrict ourself in this article to $\Lambda^\times$-torsors, with Λ as in Theorem II.1.1, in order to simplify the exposition, since we are able to apply directly our main descent tool in this context, namely Lemma II.5.9. However, the latter lemma, and hence Theorem II.1.1 can be extended to G-torsors, where G is an arbitrary locally constant finite abelian group on $S_{\text{ét}}$.

The case where S is the spectrum of a perfect field is originally due Serre and Lang, cf. ([**La56**], 6) and [**Se59**]. Their proof relies on the Albanese property of Rosenlicht's generalized Jacobians [**Ro54**]. A similar proof was sketched by Deligne in his 1974 letter to Serre, published as an appendix in [**BE01**]. However, a more geometric proof was given by Deligne in the tamely ramified case; an account of Deligne's proof in the unramified case over a finite field can be found in ([**La90**], Sect. 2). We generalize the latter approach by Deligne to allow arbitrary ramification and an arbitrary base S. This generalization is inspired by notes by Alain Genestier (unpublished) on arithmetic global class field theory.

Deligne's approach has the advantage over Serre and Lang's to yield an explicit geometric construction of the isogeny over $\mathrm{Pic}_S(X, Y)$ corresponding to a local system of rank 1 over U.

This feature of Deligne's approach carries over to ours, and is in fact crucial in order to handle the case of an arbitrary base S.

The author was informed during the preparation of this manuscript that Daichi Takeuchi has independently obtained a different proof of II.1.1 in the case where S is the spectrum of a perfect field, also by generalizing Deligne's approach to handle arbitrary ramification.

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Notation and conventions. We fix a universe $\mathcal{U}$ ([**SGA4**], I.0). Thoughout this paper, all sets are assumed to belong to $\mathcal{U}$ and we will use the term "topos" as a shorthand for "$\mathcal{U}$-topos" ([**SGA4**], IV.1.1). The category of sets belonging to $\mathcal{U}$ is simply denoted by Sets.

For any integers e, d we denote by $[\![e, d]\!]$ the set of integers n such that $e \leq n \leq d$ and by $\mathfrak{S}_d$ the group of bijections of $[\![1, d]\!]$ onto itself.

In this paper, all rings are unital and commutative. For any ring A, we denote by Alg_A the category of A-algebras. For any scheme S, we denote by $\mathrm{Sch}_{/S}$ the category of S-schemes. We denote by $S_{\text{ét}}$ (resp. $S_{\text{Ét}}$) the small étale topos (resp. big étale topos) of a scheme S, i.e. the topos of sheaves of sets for the étale topology ([**SGA4**], VII.1.2) on the category of étale S-schemes (resp. on $\mathrm{Sch}_{/S}$), and by S_{Fppf} the big fppf topos of S, i.e. the topos of sheaves of sets for the fppf topology on $\mathrm{Sch}_{/S}$ ([**SGA4**], VII.4.2). If $f : X \to S$ is a morphism of schemes, then we denote by (f^{-1}, f_*) the induced morphism of topos from $X_{\text{Ét}}$ to $S_{\text{Ét}}$. The symbol f^* will exclusively denote the pullback functor from $\mathcal{O}_S$-modules to $\mathcal{O}_X$-modules.

II.2. Preliminaries

II.2.1. Let E be a topos and let G be an abelian group in E. We denote by GE the category of objects of E endowed with a left action of G. If X is an object of E, we denote by $E_{/X}$ the topos of X-objects in E. If X is considered as an object of GE by endowing it with the trivial left G-action, then we have $(GE)_{/X} = G(E_{/X})$ and this category will be simply denoted by $GE_{/X}$.

DEFINITION II.2.2. A G**-torsor** over an object X of E is an object P of $GE_{/X}$ such that $P \to X$ is an epimorphism and the morphism

$$G \times_X P \to P \times_X P$$
$$(g, p) \mapsto (g \cdot p, p)$$

is an isomorphism. We denote by $\mathrm{Tors}(X, G)$ the full subcategory of $GE_{/X}$ whose objects are the G-torsors over X. If $f : Y \to X$ is a morphism in E, we denote by $f^{-1} : \mathrm{Tors}(X, G) \to \mathrm{Tors}(Y, G)$ the functor which associates $f^{-1}P = P \times_{X,f} Y$ to a G-torsor P over X.

The category $\mathrm{Tors}(X, G)$ is monoidal, with product

$$P_1 \otimes P_2 = G_2 \setminus P_1 \times_X P_2,$$

where G_2 is the kernel of the multiplication morphism $G \times G \to G$, and where $G_2 \hookrightarrow G \times G$ acts diagonally on $P_1 \times_X P_2$. The neutral element for this product is the trivial G-torsor over X, namely $G \times X$, and each G-torsor P over X is invertible with respect to $\otimes$, with inverse given by

$$P^{-1} = \underline{\mathrm{Hom}}_{GE_{/X}}(P, G \times X),$$

where $\underline{\mathrm{Hom}}_{GE_{/X}}$ denotes the internal Hom functor in $GE_{/X}$.

EXAMPLE II.2.3. If $G = \Lambda^\times$ for some ring Λ in E, then the monoidal category $\mathrm{Tors}(X, G)$ is equivalent to the groupoid of locally free Λ-modules of rank 1 in $E_{/X}$. The equivalence is given by the functor which sends an object P of $\mathrm{Tors}(X, G)$ to the Λ-module $G \setminus (\Lambda \times P)$, where the action of $G = \Lambda^\times$ on $\Lambda \times P$ is given by the formula $g \cdot (\lambda, p) = (g\lambda, g \cdot p)$. The functor which sends a locally free Λ-module M of rank 1 of $E_{/X}$ to the G-torsor of isomorphisms of Λ-modules from M to Λ defines a quasi-inverse to the latter functor.

II.2.4. Let E be a topos, and let us denote by 1 its terminal object. Let us consider an exact sequence

$$1 \to G \xrightarrow{i} P \xrightarrow{r} Q \to 1$$

of abelian groups in E. The morphism

$$\begin{aligned} G \times_Q P &\to P \times_Q P \\ (g, p) &\mapsto (i(g) + p, p) \end{aligned}$$

is an isomorphism, so that P is a G-torsor over Q. Moreover, the multiplication morphism

$$P \times P \to P$$

factors though $G_2 \setminus P \times P$, where $G_2 \hookrightarrow G \times G$ is the kernel of the multiplication morphism of G, acting diagonally on $P \times P$. We thus obtain a morphism

$$p_1^{-1}P \otimes p_2^{-1}P \to m^{-1}P$$

of G-torsors over $Q \times Q$, where p_1 and p_2 are the canonical projections and m is the multiplication morphism of Q.

The following definition is inspired by ([**MB85**], I.2.3):

DEFINITION II.2.5. Let G be an abelian group of E and let Q be a commutative semigroup of E (with or without identity). Let $m : Q \times Q \to Q$ be the multiplication morphism of Q. A **multiplicative G-torsor** over Q is a G-torsor $P \to Q$, together with an isomorphism $\theta : p_1^{-1}P \otimes p_2^{-1}P \to m^{-1}P$ of G-torsors over $Q \times Q$ where p_1 and p_2 are the canonical projections, which satisfy the following two properties.

- ▷ **Symmetry:** if σ is the involution of $Q \times Q$ which switches the two factors, then the isomorphism

$$p_2^{-1}P \otimes p_1^{-1}P \to \sigma^{-1}(p_1^{-1}P \otimes p_2^{-1}P) \xrightarrow{\sigma^{-1}\theta} \sigma^{-1}m^{-1}P \to m^{-1}P$$

  is the composition of θ with the canonical isomorphism $p_2^{-1}P \otimes p_1^{-1}P \to p_1^{-1}P \otimes p_2^{-1}P$.
- ▷ **Associativity:** if $q_i : Q \times Q \times Q \to Q$ (resp. $q_{ij} : Q \times Q \times Q \to Q \times Q$) is the projection on the i-th factor for $i \in [\![1, 3]\!]$ (resp. on the i-th and j-th factors for $(i, j) \in [\![1, 3]\!]^2$ such that $i < j$) and if $m_3 : Q \times Q \times Q \to Q$ is the multiplication morphism, then the diagram of G-torsors over $Q \times Q \times Q$

$$\begin{tikzcd}
 & q_1^{-1}P \otimes (mq_{23})^{-1}P \arrow[dr, "(q_1 \times mq_{23})^{-1}\theta"] & \\
q_1^{-1}P \otimes q_2^{-1}P \otimes q_3^{-1}P \arrow[ur, "id \otimes q_{23}^{-1}\theta"] \arrow[dr, "q_{12}^{-1}\theta \otimes id"'] & & m_3^{-1}P \\
 & (mq_{12})^{-1}P \otimes q_3^{-1}P \arrow[ur, "(mq_{12} \times q_3)^{-1}\theta"'] &
\end{tikzcd}$$

  is commutative.

The category of multiplicative G-torsors is fibered in groupoids over the category of commutative semigroups of E. We denote by $\mathrm{Tors}^{\otimes}(Q, G)$ the groupoid of multiplicative G-torsors over a commutative semigroup Q of E.

Remark II.2.6. If $G = \Lambda^{\times}$ for some ring Λ in E, we use the term "**multiplicative locally free Λ-module of rank** 1" as a synonym for "multiplicative G-torsor", when we want to emphasize the locally free Λ-module of rank 1 corresponding to a given G-torsor, rather than the G-torsor itself (cf. II.2.3).

Proposition II.2.7. *Let G be an abelian group in E, let Q be a commutative semigroup in E and let I be an ideal of Q. If the projection morphisms $Q \times I \to Q$ and $I \times I \to I$ onto the first factors are morphisms of descent for the fibered category of multiplicative G-torsors (cf. II.2.5), then the restriction functor*

$$\mathrm{Tors}^{\otimes}(Q, G) \to \mathrm{Tors}^{\otimes}(I, G)$$

is fully faithful.

Let $i : I \to Q$ be the canonical injection morphism. Let p_1 and p_2 be the projection morphisms of $Q \times I$ onto its first and second factors respectively, and let $m : Q \times I \to I$ be the multiplication morphism. Let (P, θ) and (P', θ') be multiplicative G-torsors over Q. We have an isomorphism

$$\beta_P : p_1^{-1}P \xrightarrow{(\mathrm{id}\times i)^{-1}\theta} m^{-1}i^{-1}P \otimes p_2^{-1}i^{-1}P^{-1},$$

and similarly for P'. If $\alpha : i^{-1}P \to i^{-1}P'$ is a morphism of multiplicative G-torsors over I, then $\beta_{P'}^{-1}(m^{-1}\alpha \otimes p_2^{-1}\alpha)\beta_P$ is an isomorphism from $p_1^{-1}P$ to $p_1^{-1}P'$, which is compatible with the canonical descent datum for p_1 associated to $p_1^{-1}P$ and $p_1^{-1}P'$: indeed, if $q_1 : Q \times I \times I \to Q$ and $q_2, q_3 : Q \times I \times I \to I$ (resp. q_{ij}) are the projections on the first, second and third factors of $Q \times I \times I$ respectively (resp. on the product of its i-th and j-th factors for $(i, j) \in [\![1, 3]\!]^2$ such that $i < j$) and if $m_3 : Q \times I \times I \to I$ is the multiplication morphism, then the diagram of G-torsors over $Q \times I \times I$

$$\begin{tikzcd}
& (im_3)^{-1}P \otimes (imq_{23})^{-1}P^{-1} \arrow[dr, "(mq_{12} \times q_3)^{-1}\theta^{-1} \otimes q_{23}^{-1}\theta^{-1}"] & \\
q_{12}^{-1}p_1^{-1}P \arrow[ur, "(q_1 \times mq_{23})^{-1}\theta"] \arrow[dr, "q_{12}^{-1}\beta_P"'] & & (imq_{12})^{-1}P \otimes (iq_3)^{-1}P \otimes (ip_2q_{12})^{-1}P^{-1} \otimes (iq_3)^{-1}P^{-1} \\
& (imq_{12})^{-1}P \otimes (ip_2q_{12})^{-1}P^{-1} \arrow[ur] &
\end{tikzcd}$$

is commutative, and similarly for P', so that the pullback of $\beta_{P'}^{-1}(m^{-1}\alpha \otimes p_2^{-1}\alpha)\beta_P$ by q_{12} is given by the composition

$$q_1^{-1}P \to (im_3)^{-1}P \otimes (imq_{23})^{-1}P^{-1} \xrightarrow{m_3^{-1}\alpha\otimes(mq_{23})^{-1}\alpha} (im_3)^{-1}P \otimes (imq_{23})^{-1}P''^{-1} \to q_1^{-1}P',$$

and therefore coincides with its pullback by q_{13}.

Since p_1 is a morphism of descent for the fibered category of multiplicative G-torsors, there is a unique morphism $\gamma : P \to P'$ of multiplicative G-torsors over Q such that $p_1^{-1}\gamma = \beta_{P'}^{-1}(m^{-1}\alpha \otimes p_2^{-1}\alpha)\beta_P$. The restriction of $p_1^{-1}\gamma$ to $I \times I$ is the pullback of α by the first projection, which is a morphism of descent for the fibered category of multiplicative G-torsors, so that the restriction of γ to I is α.

Proposition II.2.8. *Let G be an abelian group in E, and let $\rho : M \to Q$ be a morphism of commutative semigroups in E. If ρ (resp. $\rho \times \rho$ and $\rho \times \rho \times \rho$) is a morphism of effective descent (resp. of descent) for the fibered category of G-torsors, then ρ is a morphism of effective descent for the fibered category of multiplicative G-torsors.*

A descent datum of multiplicative G-torsors for ρ yields a descent datum of G-torsors for ρ, hence a G-torsor over Q by hypothesis. Since $\rho \times \rho$ and $\rho \times \rho \times \rho$ are morphisms of descent for the fibered category of G-torsors, the structure of multiplicative G-torsor descends as well. Details are omitted.

Proposition II.2.9. *Let G and Q be abelian groups in E. The groupoid $\mathrm{Tors}^{\otimes}(Q, G)$ of multiplicative G-torsors over Q is equivalent as a monoidal category to the groupoid of extensions of Q by G in E, with the Baer sum as a monoidal structure.*

We have already seen how to associate a multiplicative G-torsor to an extension of Q by G. This construction is functorial, and the corresponding functor is an equivalence by ([**MB85**], I.2.3.10).

Corollary II.2.10. *Let G and Q be abelian groups in E. The group of isomorphism classes of multiplicative G-torsors over Q is isomorphic to the group $\mathrm{Ext}^1(Q, G)$ of isomorphism classes of extensions of Q by G in E.*

II.2.11. Let S be a scheme, let X be an S-scheme, and let G be a finite abelian group. Let P be a G-torsor over X in $S_{\text{Ét}}$. Since $P \to X$ is an epimorphism in $S_{\text{Ét}}$, there is an étale cover $(X_i \to X)_{i \in I}$ such that for each $i \in I$, the morphism $X_i \to X$ factors through $P \to X$. In particular, for each $i \in I$ the G-torsor $P \times_X X_i \to X_i$ is isomorphic to the trivial G-torsor $G \times X_i \to X_i$, so that $P \times_X X_i$ is representable by a finite étale X_i-scheme. By étale descent of affine morphisms, we obtain:

Proposition II.2.12. *Let G be a finite abelian group, let S be a scheme, and let P be a G-torsor over an S-scheme X in $S_{\text{Ét}}$. Then the étale sheaf $P : \mathrm{Sch}_{/S} \to \mathrm{Sets}$ is representable by a finite étale X-scheme.*

The topos $(S_{\text{Ét}})_{/X}$ coincides with $X_{\text{Ét}}$. The category of G-torsors over X in $S_{\text{Ét}}$ is therefore equivalent to the category of G-torsors over the terminal object in $X_{\text{Ét}}$, and Proposition II.2.12 yields:

Corollary II.2.13. *Let G be a finite abelian group, let S be a scheme, and let X be an S-scheme. Then the category of G-torsors over X in $S_{\text{Ét}}$ is equivalent to the category of G-torsors over the terminal object in $X_{ét}$.*

II.2.14. Let S be a scheme, and let G be a finite abelian group. Let Q be a commutative S-group scheme, and let M be a sub-S-semigroup scheme of Q.

Proposition II.2.15. *Assume that the morphism*

$$\begin{aligned}\rho : M \times_S M &\to Q \\ (x, y) &\mapsto xy^{-1}\end{aligned}$$

is faithfully flat and quasi-compact, and that M is flat over S. Then the restriction functor

$$\mathrm{Tors}^{\otimes}(Q, G) \to \mathrm{Tors}^{\otimes}(M, G),$$

is an equivalence of categories.

Let (P, θ) be a multiplicative G-torsor over M. For $i \in [\![1, 4]\!]$, let r_i be the projection of $R = (M \times_S M) \times_{\rho, Q, \rho} (M \times_S M)$ onto its i-th factor. Similarly, for $i, j \in [\![1, 4]\!]$ such that $i < j$, we denote by $r_{ij} : R \to M \times_S M$ the projection on the i-th and j-th factors. We then have a sequence of isomorphisms

$$\begin{aligned}\left(r_1^{-1}P \otimes r_2^{-1}P^{-1}\right) \otimes \left(r_3^{-1}P \otimes r_4^{-1}P^{-1}\right)^{-1} &\to r_{14}^{-1}(p_1^{-1}P \otimes p_2^{-1}P) \otimes r_{23}^{-1}(p_1^{-1}P \otimes p_2^{-1}P)^{-1} \\ &\xrightarrow{r_{14}^{-1}\theta \otimes (r_{23}^{-1}\theta)^{-1}} (mr_{14})^{-1}P \otimes ((mr_{23})^{-1}P)^{-1},\end{aligned}$$

of G-torsors over R, where $m : M \times_S M \to M$ is the multiplication of M. Since $mr_{14} = mr_{23}$, the latter G-torsor is canonically trivial. We thus obtain an isomorphism

$$\psi : r_1^{-1}P \otimes r_2^{-1}P^{-1} \to r_3^{-1}P \otimes r_4^{-1}P^{-1},$$

of G-torsors over R. The associativity of θ (cf. II.2.5) implies that ψ is a cocycle, i.e. $(p_1^{-1}P \otimes p_2^{-1}P^{-1}, \psi)$ is a descent datum for ρ. By Proposition II.2.12 and since faithfully flat and quasi-compact morphisms of schemes are of effective descent for the fibered category of affine morphisms, the conditions of Proposition II.2.8 are satisfied, and thus there exists a multiplicative G-torsor P' over Q and an isomorphism $\alpha : \rho^{-1}P' \to p_1^{-1}P \otimes p_2^{-1}P^{-1}$ such that ψ is given by the composition

$$r_1^{-1}P \otimes r_2^{-1}P^{-1} \xrightarrow{r_{12}^{-1}\alpha^{-1}} (\rho r_{12})^{-1}P' = (\rho r_{34})^{-1}P' \xrightarrow{r_{34}^{-1}\alpha} r_3^{-1}P \otimes r_4^{-1}P^{-1}.$$

The association $P \mapsto P'$ then defines a functor from $\mathrm{Tors}^{\otimes}(M, G)$ to $\mathrm{Tors}^{\otimes}(Q, G)$. For any multiplicative G-torsor U over Q, we have an isomorphism $U \to (U \times_Q M)'$ by multiplicativity, which is functorial in U.

We now construct, for any multiplicative G-torsor (P, θ) over M, an isomorphism $P \to P' \times_Q M$ of multiplicative G-torsors which is functorial in P. Let $\nu : M \times_S M \to M \times_S M$ be the morphism which sends a section (x, y) to (xy, y). We have an isomorphism

$$(\rho\nu)^{-1}P' \xrightarrow{\nu^{-1}\alpha} \nu^{-1}(p_1^{-1}P \otimes p_2^{-1}P^{-1}) \to m^{-1}P \otimes p_2^{-1}P^{-1} \xrightarrow{\theta^{-1}} p_1^{-1}P.$$

The diagram

$$\begin{tikzcd} & M \times_S M \arrow[dr, "\rho"] & \\ M \times_S M \arrow[ur, "\nu"] \arrow[dr, "p_1"'] & & Q \\ & M \arrow[ur] & \end{tikzcd}$$

is commutative, hence $(\rho\nu)^{-1}P'$ is isomorphic to $p_1^{-1}(P' \times_Q M)$. We thus obtain an isomorphism

$$\beta : p_1^{-1}P \to p_1^{-1}(P' \times_Q M),$$

of multiplicative G-torsors. The morphism β is compatible with the canonical descent data for p_1 associated to $p_1^{-1}P$ and $p_1^{-1}(P' \times_Q M)$. Since p_1 is a covering for the fpqc topology, Proposition II.2.8 applies, hence there is a unique isomorphism $\gamma : P \to P' \times_Q M$ of multiplicative G-torsors such that $\beta = p_1^{-1}\gamma$. The construction of this isomorphism of multiplicative G-torsors is functorial in P, hence the result.

II.2.16. Let A be a ring. If M is an A-module, we denote by $\underline{M}$ the functor $B \mapsto M \otimes_A B$ from Alg_A to Sets.

DEFINITION II.2.17. ([**SGA4**], XVII 5.5.2.2) Let M and N be A-modules. A **polynomial map** from M to N is a morphism of functors $\underline{M} \to \underline{N}$. A polynomial map $f : \underline{M} \to \underline{N}$ is **homogeneous of degree** d if for any A-algebra B, any element λ of B and any element m of $\underline{M}(B)$, we have $f(\lambda m) = \lambda^d f(m)$.

For each integer d and any A-module M, let $\mathrm{TS}^d_A(M) = (M^{\otimes_A d})^{\mathfrak{S}_d}$ be the A-module of symmetric tensors of degree d with coefficients in M. If M is a free A-module with basis $(e_i)_{i\in I}$, then we have a decomposition

$$TS^d_A(M) = \left(\bigoplus_{\beta:[\![1,d]\!]\to I} Ae_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)}\right)^{\mathfrak{S}_d} = \bigoplus_{\substack{\alpha:I\to\mathbb{N}\\ \sum_{i\in I}\alpha(i)=d}} Ae_\alpha, \tag{7}$$

where we have set

$$e_\alpha = \sum_{\substack{\beta:[\![1,d]\!]\to I\\ \forall i, |\beta^{-1}(\{i\})|=\alpha(i)}} e_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)}.$$

In particular $TS^d_A(M)$ is a free A-module, and its formation commutes with base change by any ring morphism $A \to B$.

Proposition II.2.18. *Let M be a flat A-module and let $d \geq 0$ be an integer. Then $\mathrm{TS}^d_A(M)$ is a flat module, and for any A-algebra B the canonical homomorphism*

$$\mathrm{TS}^d_A(M) \otimes_A B \to \mathrm{TS}^d_B(M \otimes_A B)$$

is bijective.

Any flat A-module is a filtered colimit of finite free modules. We have already seen that the conclusion of Proposition II.2.18 holds whenever M is free, hence the conclusion in general since the functor TS^d_A commutes with filtered colimits.

Proposition II.2.19. *Let M be a flat A-module and let $d \geq 0$ be an integer. Let $\gamma_d : \underline{M} \to \underline{\mathrm{TS}^d_A(M)}$ be the functor which sends, for any A-algebra B, an element m of $\underline{M}(B)$ to the element $m^{\otimes d}$ of $\mathrm{TS}^d_B(M \otimes_A B) = \mathrm{TS}^d_A(M) \otimes_A B$ (cf. II.2.18). Then, for any homogeneous polynomial map $f : \underline{M} \to \underline{N}$ of degree d, there is a unique A-linear homomorphism $\tilde{f} : \mathrm{TS}^d_A(M) \to N$ such that $f = \tilde{f}\gamma_d$.*

As in Proposition II.2.18, we can assume that M is free of finite rank over A. Let $(e_i)_{i\in I}$ be a basis of M. Let us write

$$f\left(\sum_{i\in I} X_i e_i\right) = \sum_{\alpha:I\to\mathbb{N}} X^\alpha f_\alpha$$

in $\underline{N}(A[(X_i)_{i\in I}])$ for some elements $(f_\alpha)_\alpha$ of N, where $X^\alpha = \prod_{i\in I} X_i^{\alpha_i}$. Accordingly, we have for any A-algebra B and any element $m = \sum_{i\in I} b_i e_i$ of $\underline{M}(B)$, the formula

$$f(m) = \sum_{\alpha:I\to\mathbb{N}} b^\alpha f_\alpha,$$

where $b^\alpha = \prod_{i\in I} b_i^{\alpha_i}$, by using the naturality of f with the unique morphism of A-algebras $A[(X_i)_{i\in I}] \to B$ which sends X_i to b_i for each i. By applying this to the element $m = \sum_{i\in I} TX_i e_i$ of $\underline{M}(A[T,(X_i)_{i\in I}])$, we obtain

$$f\left(\sum_{i\in I} TX_i e_i\right) = \sum_{\alpha:I\to\mathbb{N}} T^{|\alpha|} X^\alpha f_\alpha,$$

where we have set $|\alpha| = \sum_{i\in I}\alpha(i)$. Since f is homogeneous of degree d, the left side of this equation is also equal to

$$T^d f\left(\sum_{i\in I} X_i e_i\right) = \sum_{\alpha:I\to\mathbb{N}} T^d X^\alpha f_\alpha.$$

We conclude that $T^d f_\alpha = T^{|\alpha|} f_\alpha$ in $N \otimes_A A[T]$ for any $\alpha : I \to \mathbb{N}$, and thus that $f_\alpha = 0$ whenever $|\alpha|$ differs from d. We therefore have

$$f(m) = \sum_{\substack{\alpha: I \to \mathbb{N} \\ |\alpha| = d}} b^\alpha f_\alpha,$$

for any A-algebra B and any element $m = \sum_{i \in I} b_i e_i$ of $\underline{M}(B)$. Using the decomposition (7), we also have

$$\gamma_d(m) = \sum_{\beta : [\![1,d]\!] \to I} \otimes_{j=1}^d b_{\beta(j)} e_{\beta(j)} = \sum_{\substack{\alpha: I \to \mathbb{N} \\ |\alpha| = d}} b^\alpha e_\alpha.$$

The conclusion of Proposition II.2.19 is achieved by taking $\tilde{f}$ to be the unique morphism of A-modules from $\mathrm{TS}^d_A(M)$ to N which sends e_α to f_α.

II.2.20. Let $A \to C$ be a ring morphism such that C is a finitely generated projective A-module of rank d. For any A-algebra B and any element c of $\underline{C}(B)$, we set

$$N_{C/A}(c) = \det_{\underline{A}(B)}(m_c),$$

where m_c is the $\underline{A}(B)$-linear endomorphism of $\underline{C}(B)$ induced by the multiplication by c. This defines a homogeneous polynomial map $N_{C/A} : \underline{C} \to \underline{A}$ of degree d (cf. II.2.17). By II.2.19, there is a unique morphism of A-modules $\varphi : \mathrm{TS}^d_A(C) \to A$ such that $N_{C/A} = \varphi\gamma_d$.

Proposition II.2.21 ([**SGA4**], XVII 6.3.1.6). *The morphism of A-modules $\varphi : \mathrm{TS}^d_A(C) \to A$ is a morphism of A-algebras.*

Let x be an element of C, and let us consider the morphism of A-modules $f : y \to \varphi(\gamma_d(x)y)$ from $\mathrm{TS}^d_A(C)$ to A. For any A-algebra B and any element c of $\underline{C}(B)$, we have

$$f(\gamma_d(c)) = \varphi(\gamma_d(x)\gamma_d(c)) = \varphi(\gamma_d(xc)) = N_{C/A}(xc) = N_{C/A}(x)N_{C/A}(c)$$

by the multiplicativity of determinants, so that $f(\gamma_d(c)) = N_{C/A}(x)\varphi(\gamma_d(c))$. By the uniqueness statement in II.2.19, we obtain $f = N_{C/A}(x)\varphi$, i.e. for all y in $\mathrm{TS}^d_A(C)$ we have

$$\varphi(\gamma_d(x)y) = N_{C/A}(x)\varphi(y). \tag{8}$$

For any A-algebra B, one can apply this argument to the morphism $B \to \underline{C}(B)$ instead of $A \to C$. Thus (8) also holds for any element x of $\underline{C}(B)$ and any element y of $\underline{\mathrm{TS}^d_A(C)}(B) = \mathrm{TS}^d_{\underline{A}(B)}(\underline{C}(B))$ (cf. II.2.18). Now, let y be an element of $\mathrm{TS}^d_A(C)$ and let us consider the morphism of A-modules $g : z \to \varphi(zy)$ from $\mathrm{TS}^d_A(C)$ to A. We have proved that $g\gamma_d = \varphi(y)N_{C/A}$, hence $g = \varphi(y)\varphi$ by II.2.19. Thus φ is a morphism of rings. Since φ is also A-linear, it is a morphism of A-algebras.

II.2.22. Let S be a scheme.

Definition II.2.23 ([**SGA1**], V.1.7).

- ▷ Let T be an object of a category C endowed with a right action of a group Γ. We say that **the quotient T/Γ exists** in C if the covariant functor

$$C \to \mathrm{Sets}$$
$$U \mapsto \mathrm{Hom}_C(T, U)^\Gamma$$

  is representable by an object of C.
- ▷ Let T be an S-scheme. An action of a finite group Γ on T is **admissible** if there exists an affine Γ-invariant morphism $f : T \to T'$ such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^\Gamma$.

Proposition II.2.24 ([**SGA1**], V.1.3). *Let T be an S-scheme endowed with an admissible right action of a finite group Γ. If $f : T \to T'$ is an affine Γ-invariant morphism such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^\Gamma$, then the quotient T/Γ exists and is isomorphic to T'.*

Proposition II.2.25 ([**SGA1**], V.1.8). *Let T be an S-scheme endowed with a right action of a finite group Γ. Then, the action of Γ on T is admissible if and only if T is covered by Γ-invariant affine open subsets.*

Proposition II.2.26 ([**SGA1**], V.1.9). *Let T be an S-scheme endowed with an admissible right action of a finite group Γ, and let S' be a flat S-scheme. Then, the action of Γ on the S'-scheme $T \times_S S'$ is admissible, and the canonical morphism*

$$(T \times_S S')/\Gamma \to (T/\Gamma) \times_S S'$$

is an isomorphism.

Let X be an S-scheme and let $d \geq 0$ be an integer. The group $\mathfrak{S}_d$ of permutations of $[\![1, d]\!]$ acts on the right on the S-scheme $X^{\times_S d} = X \times_S \cdots \times_S X$ by the formula

$$(x_i)_{i \in [\![1,d]\!]} \cdot \sigma = (x_{\sigma(i)})_{i \in [\![1,d]\!]}.$$

Proposition II.2.27. *If X is Zariski-locally quasi-projective over S, then the right action of $\mathfrak{S}_d$ on $X^{\times_S d}$ is admissible. In particular, the quotient $\mathrm{Sym}^d_S(X) = X^{\times_S d}/\mathfrak{S}_d$ exists in the category of S-schemes.*

Since X is Zariski-locally quasi-projective over S, any finite set of points in X with the same image in S is contained in an affine open subset of X. Thus $X^{\times_S d}$ is covered by open subsets of the form $U^{\times_S d}$ where U is an affine open subset of X whose image in S is contained in an affine open subset of S. These particular open subsets are affine and $\mathfrak{S}_d$-invariant, so that the action of $\mathfrak{S}_d$ on $X^{\times_S d}$ is admissible by Proposition II.2.25.

Remark II.2.28. If $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine, then for any S-scheme T we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sch}_{/S}}(X^{\times_S d}, T)^{\mathfrak{S}_d} &= \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma(T, \mathcal{O}_T), B^{\otimes_A d})^{\mathfrak{S}_d}\\ &= \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma(T, \mathcal{O}_T), \mathrm{TS}^d_A(B)),\end{aligned}$$

cf. II.2.16. Thus $\mathrm{Sym}^d_S(X)$ is representable by the S-scheme $\mathrm{Spec}(\mathrm{TS}^d_A(B))$.

Proposition II.2.29. *If X is flat and Zariski-locally quasi-projective over S, then $\mathrm{Sym}^d_S(X)$ is flat over S. Moreover, for any S-scheme S', the canonical morphism*

$$\mathrm{Sym}^d_{S'}(X \times_S S') \to \mathrm{Sym}^d_S(X) \times_S S'$$

is an isomorphism.

This follows from Remark II.2.28 and from Proposition II.2.18.

Proposition II.2.30 ([**SGA1**], IX.5.8). *Let G be a finite abelian group, let P be a G-torsor over an S-scheme X in $S_{\text{Ét}}$. Assume that P and X are endowed with right actions from a finite group Γ such that the morphism $P \to X$ is Γ-equivariant, and that the following properties hold:*

(a) *The right Γ-action on P commutes with the left G-action.*
(b) *The right Γ-action on X is admissible (cf. II.2.23), and the quotient morphism $X \to X/\Gamma$ is finite.*
(c) *For any geometric point $\bar{x}$ of X, the action of the stabilizer $\Gamma_{\bar{x}}$ of $\bar{x}$ in Γ on the fiber $P_{\bar{x}}$ of P at $\bar{x}$ is trivial.*

Then the action of Γ on P is admissible, and P/Γ is a G-torsor over X/Γ in $S_{\text{Ét}}$.

II.2.31. Let S be a scheme, let X be an S-scheme and let $d \geq 1$ be an integer. Let G be a finite abelian group, and let $P \to X$ be a G-torsor over X in $S_{\text{Ét}}$. By II.2.12, the sheaf P is representable by a finite étale X-scheme.

For each $i \in [\![1,d]\!]$ let $p_i : X^{\times_S d} \to X$ be the projection on i-th factor, and let us consider the G-torsor

$$p_1^{-1}P \otimes \cdots \otimes p_d^{-1}P = G_d \setminus P^{\times_S d}$$

over $X^{\times_S d}$, where $G_d \subseteq G^d$ is the kernel of the multiplication morphism $G^d \to G$. By II.2.12, the object $G_d \setminus P^{\times_S d}$ of $S_{\text{Ét}}$ is representable by an S-scheme which is finite étale over $X^{\times_S d}$. The group $\mathfrak{S}_d$ acts on the right on $G_d \setminus P^{\times_S d}$ by the formula

$$(p_i)_{i\in[\![1,d]\!]} \cdot \sigma = (p_{\sigma(i)})_{i\in[\![1,d]\!]}.$$

This action of $\mathfrak{S}_d$ commutes with the left action of G on $G_d \setminus P^{\times_S d}$.

Proposition II.2.32. *If X is Zariski-locally quasi-projective on S, then the right action of $\mathfrak{S}_d$ on $G_d \setminus P^{\times_S d}$ is admissible (cf.* II.2.23*), so that the quotient $P^{[d]}$ of $G_d \setminus P^{\times_S d}$ by $\mathfrak{S}_d$ exists in $\mathrm{Sch}_{/S}$. Moreover, the canonical morphism $P^{[d]} \to \mathrm{Sym}^d_S(X)$ is a G-torsor, and the morphism*

$$p_1^{-1}P \otimes \cdots \otimes p_d^{-1}P \to r^{-1}P^{[d]}$$

where $r : X^{\times_S d} \to \mathrm{Sym}^d_S(X)$ is the canonical projection, is an isomorphism of G-torsors over $X^{\times_S d}$.

By II.2.27 and II.2.30, it is sufficient to show that if $\bar{x} = (\bar{x}_i)_{i=1}^d$ is a geometric point of $X^{\times_S d}$, then the stabilizer of $\bar{x}$ in $\mathfrak{S}_d$ acts trivially on $(G_d \setminus P^{\times_S d})_{\bar{x}}$. Assume that the finite set $\{\bar{x}_i \mid i \in [\![1,d]\!]\}$ has exactly r distinct elements $\bar{y}_1, \ldots, \bar{y}_r$, where $\bar{y}_j$ appears with multiplicity d_j. Then the stabilizer of $\bar{x}$ in $\mathfrak{S}_d$ is isomorphic to the subgroup $\prod_{j=1}^r \mathfrak{S}_{d_j}$ of $\mathfrak{S}_d$. For each $j \in [\![1,r]\!]$, the G-torsor $P_{\bar{y}_j}$ is trivial, and if e is a section of this torsor then $(e)_{i=1}^{d_j}$ is a section of $G_{d_j} \setminus P_{\bar{y}_j}^{d_j}$ which is $\mathfrak{S}_{d_j}$-invariant. The action of $\mathfrak{S}_{d_j}$ on $G_{d_j} \setminus P_{\bar{y}_j}^{d_j}$ is therefore trivial, so that the action of $\prod_{j=1}^r \mathfrak{S}_{d_j}$ on the G-torsor

$$(G_d \setminus P^{\times_S d})_{\bar{x}} = G_r \setminus \left(\prod_{j=1}^r G_{d_j} \setminus P_{\bar{y}_j}^{d_j} \right)$$

is trivial as well.

Proposition II.2.33. *If X is flat and Zariski-locally quasi-projective on S, then for any S-scheme S', the canonical morphism*

$$(P \times_S S')^{[d]} \to P^{[d]} \times_S S'$$

is an isomorphism.

By Proposition II.2.29, the canonical morphism

$$\mathrm{Sym}^d_{S'}(X \times_S S') \to \mathrm{Sym}^d_S(X) \times_S S'$$

is an isomorphism. Thus the second morphism in the composition

$$(P \times_S S')^{[d]} \to (P^{[d]} \times_S S') \times_{\mathrm{Sym}^d_S(X)\times_S S'} \mathrm{Sym}^d_{S'}(X \times_S S') \to P^{[d]} \times_S S'$$

is an isomorphism, while the first morphism is a morphism of G-torsors, hence an isomorphism.

II.3. Geometric local class field theory

Let k be a perfect field, and let L be a complete discretely valued extension of k with residue field k. We denote by $\mathcal{O}_L$ its ring of integers, and by $\mathfrak{m}_L$ the maximal ideal of $\mathcal{O}_L$.

II.3.1. Let us consider the functor

$$\begin{aligned}\mathbb{O}_L : \mathrm{Alg}_k &\to \mathrm{Alg}_{\mathcal{O}_L}\\ A &\mapsto \lim_n A \otimes_k \mathcal{O}_L/\mathfrak{m}_L^n,\end{aligned}$$

with values in the category of $\mathcal{O}_L$-algebras.

PROPOSITION II.3.2. *The functor $\mathbb{O}_L$ is representable by a k-scheme.*

Indeed, if π is a uniformizer of L, then we have an isomorphism $k((t)) \to L$ which sends t to π, so that the functor $\mathbb{O}_L$ is isomorphic to the functor $A \mapsto A[[t]]$, which is representable by an affine space over k of countable dimension.

COROLLARY II.3.3. *The functor $\mathbb{L} = \mathbb{O}_L \otimes_{\mathcal{O}_L} L$ is representable by an ind-k-scheme.*

We can assume that L is the field of Laurent series $k((t))$. In this case, we have

$$\mathbb{L}(A) = A((t)) = \mathrm{colim}_n\ t^{-n}A[[t]]$$

for any k-algebra A, and for each integer n the functor $A \mapsto t^{-n}A[[t]]$ is representable by a k-scheme, cf. II.3.2.

PROPOSITION II.3.4. *Let G (resp. H) be the functor from Alg_k to the category of groups which associates to a k-algebra A the subgroup $G(A)$ of $A((t))^\times$ consisting of Laurent series of the form $1 + \sum_{r>0} a_r t^{-r}$ where a_r is a nilpotent element of A for each $r > 0$ and vanishes for r large enough (resp. of Laurent series of the form $1 + \sum_{r>0} a_r t^r$ where a_r belongs to A for each $r > 0$). Let $\underline{\mathbb{Z}}$ be the functor which sends a k-algebra A to the group of locally constant functions $\mathrm{Spec}(A) \to \mathbb{Z}$. Then for any uniformizer π of L, the morphism*

$$\begin{aligned}\mathbb{G}_{m,k} \times \underline{\mathbb{Z}} \times G \times H &\to \mathbb{L}^\times,\\ (a, n, g, h) &\mapsto a\pi^n g(\pi)h(\pi),\end{aligned}$$

is an isomorphism of group-valued functors.

Let A be a k-algebra. By ([**CC13**], 0.8), every invertible element u of $A((t))$ uniquely factors as $u = t^n f(t)h(t)$ where $f(t)$ and $h(t)$ are elements of $A[[t]]^\times$ and $G(A)$ respectively, and $n : \mathrm{Spec}(A) \to \mathbb{Z}$ is a locally constant function. Moreover, there is a unique factorisation $f(t) = ag(t)$ where a and $g(t)$ belong to $A^\times$ and $H(A)$ respectively, hence the result.

COROLLARY II.3.5. *The functor $\mathbb{L}^\times$ is representable by an ind-k-scheme. Moreover, its restriction to the category of reduced k-algebras is representable by a reduced k-scheme.*

The groups $\underline{\mathbb{Z}}$ and H from Proposition II.3.4 are representable by reduced k-schemes, and so is $\mathbb{G}_{m,k}$. Moreover, the group G from II.3.4 is the filtered colimit of the functor $n \mapsto G_n$, where G_n is the functor which associates to a k-algebra A the subset $G_n(A)$ of $A((t))^\times$ consisting of Laurent series of the form $1 + \sum_{r=1}^n a_r t^{-r}$ where $a_r^n = 0$ for each $r \in [\![1, n]\!]$. For each n, the functor G_n is representable by an affine k-scheme. Thus G is representable by an ind-k-scheme, and so is $\mathbb{L}^\times$ by II.3.4. The last assertion of Corollary II.3.5 follows from the fact that $G(A)$ is the trivial group for any reduced k-algebra A.

COROLLARY II.3.6. *Let $d \geq 0$ be an integer, and let $\mathbb{U}_L^{(d)}$ be the subfunctor $1 + \mathfrak{m}_L^d\mathbb{O}_L$ (resp. $\mathbb{O}_L^\times$) of $\mathbb{L}^\times$ for $d \geq 1$ (resp. for $d = 0$). Then the functor*

$$\begin{aligned}\mathbb{L}^\times/\mathbb{U}_L^{(d)} : \mathrm{Alg}_k &\to \mathrm{Sets}\\ A &\mapsto \mathbb{L}^\times(A)/\mathbb{U}_L^{(d)}(A),\end{aligned}$$

is representable by an ind-k-scheme. Moreover, its restriction to the category of reduced k-algebras is representable by a reduced k-scheme.

According to Proposition II.3.4, it is sufficent to show that $(\mathbb{G}_{m,k} \times H)/\mathbb{U}^{(d)}_{k((t))}$ is representable by a reduced k-scheme. The case $d = 0$ is clear, while for $d \geq 1$, we have for any k-algebra A a bijection

$$A^\times \times A^{[\![1,d-1]\!]} \to (\mathbb{G}_{m,k} \times H)(A)/\mathbb{U}^{(d)}_{k((t))}(A)$$
$$(a_i)_{0 \leq i \leq d-1} \mapsto \sum_{i=0}^{d-1} a_i t^i,$$

hence the result.

II.3.7. From now on, we consider $\mathrm{Spec}(L)$, $\mathbb{L}^\times$ and $\mathbb{L}^\times/\mathbb{U}^{(d)}_L$ for each integer $d \geq 0$ as objects of the topos $\mathrm{Spec}(k)_{\text{Ét}}$. Let π be an uniformizer of L. We denote by Π the element of $\mathbb{L}(k)$ corresponding to π via the canonical identification $L \simeq \mathbb{L}(k)$. Thus the functor $\mathbb{L}^\times$ is given by

$$\mathbb{L}^\times : A \in \mathrm{Alg}_k \mapsto A((\Pi))^\times.$$

In particular, the Laurent series $(\Pi - \pi)^{-1}\Pi = -\sum_{n\geq 1} \pi^{-n}\Pi^n$ defines an L-point of $\mathbb{L}^\times$. We denote by $\varphi : \mathrm{Spec}(L) \to \mathbb{L}^\times$ the corresponding morphism. We follow here Contou-Carrère's convention; in [**Su13**], the morphism φ corresponds to the point $(\Pi - \pi)\Pi^{-1}$ instead. This is harmless since the inversion is an automorphism of the abelian group $\mathbb{L}^\times$.

Theorem II.3.8 ([**Su13**], Th. A (1)). *Let G be a finite abelian group. The functor*

$$\mathrm{Tors}^{\otimes}(\mathbb{L}^\times, G) \to \mathrm{Tors}(\mathrm{Spec}(L), G)$$
$$P \to \varphi^{-1}P$$

is an equivalence of categories (cf. II.2.2, II.2.5).

In the case where k is algebraically closed, Serre constructed in [**Se61**] an equivalence

$$\mathrm{Tors}(\mathrm{Spec}(L), G) \to \mathrm{Tors}^{\otimes}(\mathbb{L}^\times, G).$$

More precisely, Serre considers étale isogenies over $\mathbb{L}^\times$ and the link with $\mathrm{Tors}^{\otimes}(\mathbb{L}^\times, G)$ is provided by II.2.4. In [**Su13**], Suzuki shows that the functor from Theorem II.3.8 is a quasi-inverse to Serre's functor when k is algebraically closed, and extends the result to arbitrary perfect residue fields. In particular, the equivalence from Theorem II.3.8 is canonical, even though its definition depends on the choice of π. Suzuki's proof of Theorem II.3.8 relies on the Albanese property of the morphism φ, previously established by Contou-Carrère.

Let L^{sep} be a separable closure of L, and let G_L be the Galois group of L^{sep} over L, so that the small étale topos of $\mathrm{Spec}(L)$ is isomorphic to the topos of sets with continuous left G_L-action. By II.2.13, the category of G-torsors over $\mathrm{Spec}(L)$ in $\mathrm{Spec}(k)_{\text{Ét}}$ is isomorphic to the category of G-torsors in the small étale topos $\mathrm{Spec}(L)_{\text{ét}}$. Correspondingly, for each finite abelian group G, the group of isomorphism classes of the category $\mathrm{Tors}(\mathrm{Spec}(L), G)$ is isomorphic to the group of continuous homomorphisms from G_L to G.

We denote by $(G_L^j)_{j \geq -1}$ the ramification filtration of G_L ([**Se68**], IV.3), so that $G_L^{-1} = G_L$ and G_L^0 is the inertia subgroup of G_L, while $G_L^{0+} = \cup_{j>0} G_L^j$ is the wild inertia subgroup of G_L.

Definition II.3.9. Let G be a finite abelian group and let $d \geq 0$ be a rational number. A G-torsor over $\mathrm{Spec}(L)$ (in $\mathrm{Spec}(k)_{\text{Ét}}$), corresponding to a continuous homomorphism $\rho : G_L \to G$, is said to have **ramification bounded by** d if $\rho(G_L^d) = \{1\}$. A G-torsor over $\mathrm{Spec}(L)$ with ramification bounded by 0 (resp. 1) is said to be unramified (resp. tamely ramified).

Remark II.3.10. If $P \to \mathrm{Spec}(L)$ is a G-torsor in $\mathrm{Spec}(k)_{\text{Ét}}$, then we have a finite decomposition

$$P = \coprod_{i \in I} \mathrm{Spec}(L_i),$$

where each L_i is a finite separable extension of L, and are pairwise isomorphic. The G-torsor P has ramification bounded by d if and only if for each i (or, equivalently, for some i) the extension L_i/L has ramification bounded by d, in the sense G_L^d acts trivially on the finite set $\mathrm{Hom}_L(L_i, L^{\mathrm{sep}})$.

Proposition II.3.11. *Let G be a finite abelian group, let $d \geq 0$ be an integer, and let P be a multiplicative G-torsor P over $\mathbb{L}^\times$ (cf. II.2.5). Assume that k is algebraically closed. Then $\varphi^{-1}P$ has ramification bounded by d (cf. II.3.9) if and only if P is the pullback of a multiplicative G-torsor over $\mathbb{L}^\times/\mathbb{U}_L^{(d)}$ (cf. II.3.6).*

This follows from ([**Se61**], 3.2 Th. 1) and from the compatibility of φ^{-1} with Serre's construction ([**Su13**], Th. A (2)).

II.3.12. Let π and φ be as in II.3.7. Let K be a closed sub-extension of k in L, such that $K \to L$ is a finite extension of degree d. Since L is a finite free K-algebra of rank d, we have a canonical morphism of K-schemes

$$\psi : \mathrm{Spec}(K) \to \mathrm{Sym}_K^d(\mathrm{Spec}(L))$$

by II.2.21.

Proposition II.3.13. *The composition*

$$\mathrm{Spec}(K) \xrightarrow{\psi} \mathrm{Sym}_K^d(\mathrm{Spec}(L)) \to \mathrm{Sym}_k^d(\mathrm{Spec}(L)) \xrightarrow{\mathrm{Sym}_k^d(\varphi)} \mathrm{Sym}_k^d(\mathbb{L}^\times) \to \mathbb{L}^\times,$$

where the last morphism is given by the multiplication, corresponds to the K-point $P_\pi(\Pi)^{-1}\Pi^d$ of $\mathbb{L}^\times$, where the polynomial P_π is the characteristic polynomial of the K-linear endomorphism $x \mapsto \pi x$ of L.

We first describe the morphism ψ. The scheme $\mathrm{Sym}_K^d(\mathrm{Spec}(L))$ is the spectrum of the k-algebra $\mathrm{TS}_K^d(L)$ of symmetric tensors of degree d in L, cf. II.2.27. The elements $e_i = \pi^{i-1}$ for $i = 1, \dots, d$ form a K-basis of L, so that we have a decomposition

$$\mathrm{TS}_K^d(L) = \bigoplus_{\substack{\alpha:[\![1,d]\!]\to\mathbb{N} \\ \sum_i \alpha(i)=d}} Ke_\alpha,$$

where we have set (cf. II.2.16)

$$e_\alpha = \sum_{\substack{\beta:[\![1,d]\!]\to[\![1,d]\!] \\ \forall i, |\beta^{-1}(\{i\})|=\alpha(i)}} e_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)}.$$

Let us write the norm polynomial as

$$N_{L/K}\left(\sum_{i=1}^d x_i e_i\right) = \sum_{\substack{\alpha:[\![1,d]\!]\to\mathbb{N} \\ \sum_i \alpha(i)=d}} f_\alpha x^\alpha,$$

where $x^\alpha = x_1^{\alpha(1)} \dots x_d^{\alpha(d)}$, and the f_α's are uniquely determined elements of K. The morphism $\mathrm{TS}_K^d(L) \to K$ corresponding to ψ is the unique K-linear homomorphism which sends e_α to f_α (cf. II.2.19 and its proof).

Next we describe the composition

$$\mathrm{Sym}_K^d(\mathrm{Spec}(L)) \to \mathrm{Sym}_k^d(\mathrm{Spec}(L)) \xrightarrow{\mathrm{Sym}_k^d(\varphi)} \mathrm{Sym}_k^d(\mathbb{L}^\times) \to \mathbb{L}^\times.$$

Its precomposition with the projection $\mathrm{Spec}(L)^{\times_K d} \to \mathrm{Sym}^d_K(\mathrm{Spec}(L))$ corresponds to the element of $L^{\otimes_K d}((\Pi))^\times$ given by the formula

$$\prod_{i=1}^{d}\Big((\Pi - 1^{\otimes(i-1)}\otimes\pi\otimes 1^{\otimes(d-i)})^{-1}\Pi\Big) = P(\Pi)^{-1}\Pi^d,$$

where the polynomial $P(\Pi)$ can be computed as follows:

$$\begin{aligned} P(\Pi) &= \prod_{i=1}^{d}(\Pi - 1^{\otimes(i-1)}\otimes\pi\otimes 1^{\otimes(d-i)}) \\ &= \sum_{r=0}^{d}(-1)^r\Pi^{d-r}\sum_{\substack{(i_1,\dots,i_d)\in\{0,1\}^d\\ |\{s|i_s=1\}|=r}}\pi^{i_1}\otimes\cdots\otimes\pi^{i_d} \\ &= \sum_{r=0}^{d}(-1)^r e_{\alpha_r}\Pi^{d-r}, \end{aligned}$$

where $\alpha_r : [\![1,d]\!] \to \mathbb{N}$ is the map which sends 1 and 2 to $d-r$ and r respectively, and any $i > 2$ to 0. The image of $P(\Pi)$ by ψ in $K[\Pi]$ is the polynomial

$$\sum_{r=0}^{d}(-1)^r f_{\alpha_r}\Pi^{d-r} = N_{L[\Pi]/K[\Pi]}\left(\Pi e_1 - e_2\right).$$

Since $e_1 = 1$ and $e_2 = \pi$, we obtain II.3.13.

Proposition II.3.14. *Let G be a finite abelian group, and let Q be a G-torsor over* $\mathrm{Spec}(L)$ *(in* $\mathrm{Spec}(k)_{\text{Ét}}$*) of ramification bounded by d (cf. II.3.9). Then $\psi^{-1}Q^{[d]}$ (cf. II.2.32) is tamely ramified on* $\mathrm{Spec}(K)$.

Let K' be the maximal unramified extension of K in a separable closure of K. The formation of $\mathrm{Sym}^d_K(\mathrm{Spec}(L))$ is compatible with any base change by Proposition II.2.26 or by Proposition II.2.29, and so is the formation of φ. Moreover, a G-torsor over $\mathrm{Spec}(K)$ is tamely ramified if and only if its restriction to $\mathrm{Spec}(K')$ is tamely ramified. By replacing K and L by K' and the components of $K'\otimes_K L$ respectively, we can assume that the residue field k is algebraically closed.

Let P be the multiplicative G-torsor on $\mathbb{L}^\times$ (cf. II.2.5) associated to Q (cf. II.3.8), so that Q is isomorphic to $\varphi^{-1}P$. Then $\psi^{-1}Q^{[d]}$ is isomorphic to the pullback of P along the composition

$$\mathrm{Spec}(K) \xrightarrow{\psi} \mathrm{Sym}^d_K(\mathrm{Spec}(L)) \to \mathrm{Sym}^d_k(\mathrm{Spec}(L)) \xrightarrow{\mathrm{Sym}^d_k(\varphi)} \mathrm{Sym}^d_k(\mathbb{L}^\times) \to \mathbb{L}^\times$$

considered in II.3.13. By II.3.13, this composition corresponds to the K-point of $\mathbb{L}^\times$ given by $P_\pi(\Pi)^{-1}\Pi^d$, where P_π is the characteristic polynomial of π acting K-linearly by multiplication on L. Let us consider the morphism of pointed sets

$$\begin{aligned} \rho : \mathbb{L}^\times(K) &\to H^1(\mathrm{Spec}(K)_{\text{Ét}}, G) \\ \nu &\to \nu^{-1}P \end{aligned}$$

where an element ν of $\mathbb{L}^\times(K)$ is identified to a morphism $\mathrm{Spec}(K) \to \mathbb{L}^\times$. If ν_1 and ν_2 are elements of $\mathbb{L}^\times(K)$, then using the isomorphism $\theta : p_1^{-1}P\otimes p_2^{-1}P \to m^{-1}P$ from II.2.5, we obtain isomorphisms

$$(\nu_1\nu_2)^{-1}P \leftarrow (\nu_1\times\nu_2)^{-1}m^{-1}P \xleftarrow{(\nu_1\times\nu_2)^{-1}\theta} (\nu_1\times\nu_2)^{-1}(p_1^{-1}P\otimes p_2^{-1}P) \leftarrow \nu_1^{-1}P\otimes\nu_2^{-1}P.$$

Thus ρ is an homomorphism of abelian groups.

We have to prove that $\rho(\nu)$ is the isomorphism class of a tamely ramified G-torsor over $\mathrm{Spec}(K)$, where $\nu = P_\pi(\Pi)^{-1}\Pi^d$. Since P_π is an Eisenstein polynomial, it can be written as $P_\pi(\Pi) = \Pi^d + cR(\Pi)$, where $c = P_\pi(0)$ is a uniformizer of K, and R is a polynomial of degree $< d$ with coefficients in $\mathcal{O}_K$, such that $R(0) = 1$. Thus we can write

$$\nu = c^{-1}\nu_1\nu_2,$$

where $\nu_1 = R(\Pi)^{-1}\Pi^d$ and $\nu_2 = (1 + c^{-1}\Pi^d R(\Pi)^{-1})^{-1}$, so that $\rho(\nu) = \rho(c)^{-1}\rho(\nu_1)\rho(\nu_2)$.

Since Q has ramification bounded by d (cf. II.3.9), the restriction of ρ to $\mathbb{U}_L^{(d)}(K)$ is trivial (cf. II.3.11). In particular, $\rho(\nu_2)$ is trivial since ν_2 belongs to $\mathbb{U}_L^{(d)}(K)$.

The element ν_1 belongs to $\mathbb{L}^\times(\mathcal{O}_K)$, so that the morphism $\nu_1 : \mathrm{Spec}(K) \to \mathbb{L}^\times$ factors through $\mathrm{Spec}(\mathcal{O}_K)$. This implies that $\rho(\nu_1)$ is the isomorphism class of an unramified G-torsor over $\mathrm{Spec}(K)$. It remains to prove that $\rho(c)$ is the isomorphism class of a tamely ramified G-torsor over $\mathrm{Spec}(K)$. Since c belongs to $K^\times = \mathbb{G}_{m,k}(K) \subseteq \mathbb{L}^\times(K)$, this is a consequence of the following lemma:

LEMMA II.3.15. *Let T be a multiplicative G-torsor over the k-group scheme $\mathbb{G}_{m,k}$ (cf. II.2.5). Then T is tamely ramified at 0 and ∞.*

Let G_k be the constant k-group scheme associated to k. By II.2.9, there is a structure of k-group scheme on T and an exact sequence

$$1 \to G_k \to T \to \mathbb{G}_{m,k} \to 1 \tag{9}$$

in $\mathrm{Spec}(k)_{\text{Ét}}$, such that the structure of G-torsor on T is given by the action of its subgroup G by translations. Since the fppf topology is finer than the étale topology on $\mathrm{Sch}_{/k}$, the sequence 9 remains exact in the topos $\mathrm{Spec}(k)_{\mathrm{Fppf}}$. In particular, we obtain a class in the group $\mathrm{Ext}^1_{\mathrm{Fppf}}(\mathbb{G}_{m,k}, G_k)$ of extensions of $\mathbb{G}_{m,k}$ by G_k in $\mathrm{Spec}(k)_{\mathrm{Fppf}}$.

Let $n = |G|$. In the topos $\mathrm{Spec}(k)_{\mathrm{Fppf}}$ we have an exact sequence

$$1 \to \mu_{n,k} \to \mathbb{G}_{m,k} \xrightarrow{n} \mathbb{G}_{m,k} \to 1, \tag{10}$$

where $\mu_{n,k}$ is the k-group scheme of n-th roots of unity. By applying the functor $\mathrm{Hom}(\cdot, G_k)$, we obtain an exact sequence

$$\mathrm{Hom}(\mu_{n,k}, G_k) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathrm{fppf}}(\mathbb{G}_{m,k}, G_k) \xrightarrow{n} \mathrm{Ext}^1_{\mathrm{fppf}}(\mathbb{G}_{m,k}, G_k).$$

Since $n = |G|$, the group $\mathrm{Ext}^1_{\mathrm{Fppf}}(\mathbb{G}_{m,k}, G_k)$ is annihilated by n, so that the homomorphism δ above is surjective. Thus the exact sequence (9) in $\mathrm{Spec}(k)_{\mathrm{Fppf}}$ is the pushout of (10) along an homomorphism $\mu_{n,k} \to G_k$. Let n' be the largest divisor of n which is invertible in k. Then the largest étale quotient of $\mu_{n,k}$ is the epimorphism $\mu_{n,k} \to \mu_{n',k}$ given by $x \mapsto x^{\frac{n}{n'}}$. In particular, the homomorphism $\mu_{n,k} \to G_k$ factors through $\mu_{n',k}$, so that (9) is the pushout of the extension

$$1 \to \mu_{n',k} \to \mathbb{G}_{m,k} \xrightarrow{n'} \mathbb{G}_{m,k} \to 1$$

along an homomorphism $\mu_{n',k} \to G_k$. Since the morphism $\mathbb{G}_{m,k} \xrightarrow{n'} \mathbb{G}_{m,k}$ is tamely ramified above 0 and ∞, so is the morphism $T \to \mathbb{G}_{m,k}$.

II.4. Rigidified Picard schemes of relative curves

II.4.1. Let $f : X \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is Zariski-locally projective over S.

PROPOSITION II.4.2. *The canonical homomorphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism.*

If S is locally noetherian, then $\mathcal{O}_X$ is cohomologically flat over S in dimension 0 by ([**EGA3**], 7.8.6). This means that for any quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$, the canonical homomorphism $f_*f^*\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{M} \to f_*f^*\mathcal{M}$ is an isomorphism. This implies that the formation of $f_*\mathcal{O}_X$ commutes with arbitrary base change: if $f' : X \times_S S' \to S'$ is the base change of f by a morphism of schemes $S' \to S$, then the canonical morphism $f_*\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to f'_*\mathcal{O}_{X\times_S S'}$ is an isomorphism, cf. ([**EGA3**], 7.7.5.3). By applying this result to the inclusion $\mathrm{Spec}(\kappa(s)) \to S$ of a point s of S, we obtain that $f_*(\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ is isomorphic to $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. Since $f_*(\mathcal{O}_X)$ is a coherent $\mathcal{O}_S$-module, Nakayama's lemma yield that the canonical morphism $\mathcal{O}_S \to f_*(\mathcal{O}_X)$ is an epimorphism. It is also injective since f is faithfully flat, hence the result.

In general one can assume that S is affine and that X is projective over S, in which case there is a noetherian scheme S_0, a morphism $S \to S_0$ and a smooth projective S_0-scheme X_0 with geometrically connected fibers such that X is isomorphic to the S-scheme $X_0 \times_{S_0} S$, cf. ([**EGA4**], 8.9.1, 8.10.5(xiii), 17.7.9). We have already seen that in this case the canonical homomorphism $\mathcal{O}_{S_0} \to f_*\mathcal{O}_{X_0}$ is an isomorphism, and that the formation of $f_*\mathcal{O}_{X_0}$ commutes with arbitrary base change. In particular, both morphisms in the sequence

$$\mathcal{O}_S \to f_*\mathcal{O}_{X_0} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_S \to f_*\mathcal{O}_X$$

are isomorphisms.

Proposition II.4.3. *Let $d \geq 2g-1$ be an integer, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with degree d on each fiber of f. Then, the $\mathcal{O}_S$-module $f_*\mathcal{L}$ is locally free of rank $d-g+1$, the higher direct images $R^j(f_*\mathcal{L})$ vanish for $j > 0$, and the formation of $f_*\mathcal{L}$ commutes with arbitrary base change: if $f' : X' \to S'$ is the base change of f by a morphism $S' \to S$, then the canonical homomorphism $f_*\mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to f'_*(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})$ is an isomorphism.*

We first assume that S is locally noetherian. For each point of s of S and for each integer i, the Riemann-Roch theorem for smooth projective curves implies that the $k(s)$-vector space $H^i(X_s, \mathcal{L}_s)$ is of dimension $d-g+1$ for $i = 0$, and vanishes otherwise. This implies that $R^jf_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N})$ vanishes for any integer $j > 0$ and any $\mathcal{O}_S$-module $\mathcal{N}$ by the proof of ([**EGA3**], 7.9.8). Let

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{P} \to 0$$

be an exact sequence of $\mathcal{O}_S$-modules. Since f is flat and since $\mathcal{L}$ is a flat $\mathcal{O}_X$-module, the sequence

$$0 \to \mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \to \mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{M} \to \mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{P} \to 0$$

is exact as well. Since $R^1f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N})$ vanishes, the sequence

$$0 \to f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N}) \to f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{M}) \to f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{P}) \to 0$$

is exact. The $\mathcal{O}_X$-module $\mathcal{L}$ is therefore cohomologically flat over S in dimension 0, cf. ([**EGA3**], 7.8.1). By ([**EGA3**], 7.8.4(d)) the $\mathcal{O}_S$-module $f_*\mathcal{L}$ is locally free, and the formation of $f_*\mathcal{L}$ commutes with arbitrary base change. By applying the latter result to the inclusion $\mathrm{Spec}(\kappa(s)) \to S$ of a point s of S and by using that $H^0(X_s, \mathcal{L}_s)$ is of dimension $d-g+1$ over $\kappa(s)$, we obtain that the locally free $\mathcal{O}_X$-module $f_*\mathcal{L}$ is of constant rank $d-g+1$.

In general one can assume that S is affine and that X is projective over S, in which case there is a noetherian scheme S_0, a morphism $S \to S_0$, a smooth projective S_0-scheme X_0, and an invertible $\mathcal{O}_{X_0}$-module $\mathcal{L}_0$ such that X is isomorphic to the S-scheme $X_0 \times_{S_0} S$ and $\mathcal{L}$ is isomorphic to the pullback of $\mathcal{L}_0$ by the canonical projection $X_0 \times_{S_0} S \to X_0$, cf. ([**EGA4**], 8.9.1, 8.10.5(xiii), 17.7.9). We have seen that the $\mathcal{O}_{S_0}$-module $f_{0*}\mathcal{L}$ is locally free of rank $d-g+1$, and that its formation commutes with arbitrary base change. By performing the base change by the morphism $S \to S_0$, we obtain that $f_*\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank $d-g+1$ and that the formation of $f_*\mathcal{L}$ commutes with arbitrary base change.

II.4.4. Let $f : X \to S$ be as in II.4.1. The **relative Picard functor** of f is the sheaf of abelian groups $\mathrm{Pic}_S(X) = R^1 f_{\mathrm{Fppf},*}\mathbb{G}_m$ in S_{Fppf}. Alternatively, $\mathrm{Pic}_S(X)$ is the sheaf of abelian groups on S associated to the presheaf which sends an S-scheme T to $\mathrm{Pic}(X \times_S T)$, the abelian group of isomorphism classes of invertible $\mathcal{O}_{X\times_S T}$-modules. For any S-scheme S', we have $(S_{\mathrm{Fppf}})_{/S'} = S'_{\mathrm{Fppf}}$, and we thus have:

PROPOSITION II.4.5. *For any S-scheme S', the canonical morphism*

$$\mathrm{Pic}_{S'}(X \times_S S') \to \mathrm{Pic}_S(X) \times_S S'$$

is an isomorphism in S'_{Fppf}.

The elements of $\mathrm{Pic}(X \times_S T)$ which are pulled back from an element of $\mathrm{Pic}(T)$ yield trivial classes in $\mathrm{Pic}_S(X)(T)$, since invertible $\mathcal{O}_T$-modules are locally trivial on T (for the Zariski topology, and thus for the fppf-topology). This yields a sequence

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X \times_S T) \to \mathrm{Pic}_S(X)(T) \to 0, \tag{11}$$

which is however not necessarily exact. The following is Proposition 4 from ([**BLR90**], 8.1), whose assumptions are satisfied by II.4.2:

PROPOSITION II.4.6. *If f has a section, then the sequence* (11) *is exact for any S-scheme T.*

By a theorem of Grothendieck ([**BLR90**], 8.2.1) the sheaf $\mathrm{Pic}_S(X)$ is representable by a separated S-scheme. By ([**BLR90**], 9.3.1) the S-scheme $\mathrm{Pic}_S(X)$ is smooth of relative dimension g, and there is a decomposition

$$\mathrm{Pic}_S(X) = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_S(X),$$

into open and closed subschemes, where $\mathrm{Pic}^d_S(X)$ is the fppf-sheaf associated to the presheaf

$$\begin{aligned} \mathrm{Sch}^{\mathrm{fp}}_{/S} &\to \mathrm{Sets} \\ T &\mapsto \{\mathcal{L} \in \mathrm{Pic}(X \times_S T) | \forall \bar{t} \to T, \deg_{X_{\bar{t}}}(\mathcal{L}_{\bar{t}}) = d\}. \end{aligned}$$

Here the condition $\deg_{X_{\bar{t}}}(\mathcal{L}_{\bar{t}}) = d$ runs over all geometric points $\bar{t} \to T$ of T.

II.4.7. Let $f : X \to S$ be as in II.4.1, and let $i : Y \hookrightarrow X$ be a closed subscheme of X, which is finite locally free over S of degree $N \geq 1$. A **Y-rigidified line bundle on X** is a pair $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $\alpha : \mathcal{O}_Y \to i^*\mathcal{L}$ is an isomorphism of $\mathcal{O}_Y$-modules. Two Y-rigidified line bundles $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ are **equivalent** if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $(i^*\beta)\alpha = \alpha'$. If such an isomorphism β exists, then it is unique. Indeed, any other such isomorphism would take the form $\gamma\beta$ for some global section γ of $\mathcal{O}_X^\times$ such that $i^*\gamma = 1$. Since $f_*\mathcal{O}_X = \mathcal{O}_S$ (cf. II.4.2), we have $\gamma = f^*\delta$ for some global section δ of $\mathcal{O}_S^\times$. Since the restriction of δ along the finite flat surjective morphism $Y \to S$ is trivial, one must have $\delta = 1$ as well, hence $\gamma = 1$.

PROPOSITION II.4.8. *Let $\mathrm{Pic}_S(X, Y)$ be the presheaf of abelian groups on $\mathrm{Sch}^{\mathrm{fp}}_{/S}$ which maps a finitely presented S-scheme T to the set of isomorphism classes of Y_T-rigidified line bundles on X_T. Then, the presheaf $\mathrm{Pic}_S(X, Y)$ is representable by a smooth separated S-scheme of relative dimension $N + g - 1$.*

We first consider the case where $N = 1$:

LEMMA II.4.9. *The conclusion of Proposition II.4.8 holds if $N = 1$.*

Indeed, if $N=1$ then Y is the image of a section $x : S \to X$ of f. For any finitely presented S-scheme T, we have a morphism

$$\begin{aligned}\mathrm{Pic}(X \times_S T) &\to \mathrm{Pic}_S(X,x)(T)\\ \mathcal{L} &\to (\mathcal{L} \otimes (f^*x^*\mathcal{L})^{-1}, \mathrm{id}).\end{aligned}$$

The kernel of this homomorphism consists of all invertible $\mathcal{O}_{X\times_S T}$-modules which are given by the pullback of an invertible $\mathcal{O}_T$-module. Moreover, any isomorphism class $(\mathcal{L},\alpha)$ in $\mathrm{Pic}_S(X,x)(T)$ is the image of $\mathcal{L}$ by this morphism, hence its surjectivity. We conclude by II.4.6 that the canonical projection morphism

$$\begin{aligned}\mathrm{Pic}_S(X,x) &\to \mathrm{Pic}_S(X)\\ (\mathcal{L},\alpha) &\to \mathcal{L},\end{aligned}$$

is an isomorphism of presheaves of abelian groups on $\mathrm{Sch}^{\mathrm{fp}}_{/S}$. This yields Lemma II.4.9 since $\mathrm{Pic}_S(X)$ is a smooth separated S-scheme of relative dimension g (cf. II.4.4).

We now prove Proposition II.4.8. Since $X \times_S Y \to Y$ has a section $x = (i \times \mathrm{id}_Y) \circ \Delta_Y$ where $\Delta_Y : Y \to Y \times_S Y$ is the diagonal morphism of Y, we deduce from Lemma II.4.9 and its proof that the canonical projection morphism

$$\mathrm{Pic}_Y(X \times_S Y, x) \to \mathrm{Pic}_Y(X \times_S Y) = \mathrm{Pic}_S(X) \times_S Y$$

sending a pair $(\mathcal{L},\alpha)$ to the class of $\mathcal{L}$ is an isomorphism. Let Z be the Y-scheme $\mathrm{Pic}_Y(X\times_S Y, x)$, and let $(\mathcal{L}_u, \alpha_u)$ be the universal x-rigidified line bundle on $X \times_S Z$. The morphism $Y \times_S Z \to Z$ is finite locally free of rank N, so that the pushforward $\mathcal{A}$ (resp. $\mathcal{M}$) of $\mathcal{O}_{Y\times_S Z}$ (resp. $i_Z^*\mathcal{L}_u$) is a locally free $\mathcal{O}_Z$-algebra of rank N (resp. a locally free $\mathcal{O}_Z$-module of rank N). Let $\lambda : \mathcal{M} \to \mathcal{O}_Z$ be the surjective $\mathcal{O}_Z$-linear homomorphism corresponding to $\alpha_u^{-1} : x_Z^*\mathcal{L}_u \to \mathcal{O}_Z$.

Let T be a Y-scheme, and let $(\mathcal{L},\beta)$ be a Y_T-rigidified line bundle on X_T. The section $x_T : T \to X_T$ uniquely factors through Y_T and we still denote by x_T the corresponding section of Y_T. The pair $(\mathcal{L}, x_T^*\beta)$ is then an x_T-rigidified line bundle on X_T, so that there is a unique morphism $z : T \to Z$ such that $(\mathcal{L}, x_T^*\beta)$ is equivalent to the pullback by z of $(\mathcal{L}_u, \alpha_u)$. Let us assume that $(\mathcal{L}, x_T^*\beta)$ is equal to this pullback. Then the global section β of $i_T^*\mathcal{L}$ over $Y \times_S T$ provides a global section of $z^*\mathcal{M}$ over T, which we still denote by β, such that $(z^*\lambda)(\beta) = 1$ and $z^*\mathcal{M} = (z^*\mathcal{A})\beta$. Conversely, any such section produces a Y_T-rigidification of $\mathcal{L}$ on X_T. The functor $\mathrm{Pic}_S(X,Y) \times_S Y = \mathrm{Pic}_Y(X \times_S Y, Y \times_S Y)$ is therefore isomorphic to the functor

$$\begin{aligned}&\mathrm{Sch}^{\mathrm{fp}}_{/S} \to \mathrm{Sets}\\ &\quad T \mapsto \{(z,\beta) \mid z \in Z(T), \beta \in \Gamma(T, z^*\mathcal{M}),\ \lambda(\beta) = 1 \text{ and } \mathcal{M}_T = \mathcal{A}_T\beta\}.\end{aligned}$$

This implies that $\mathrm{Pic}_S(X,Y) \times_S Y$ is representable by a relatively affine Z-scheme, smooth of relative dimension $N-1$ over Z. By fppf-descent of affine morphisms of schemes along the fppf-cover $\mathrm{Pic}_S(X) \times_S Y \to \mathrm{Pic}_S(X)$, this implies the representability of $\mathrm{Pic}_S(X,Y)$ by an S-scheme, which is relatively affine and smooth of relative dimension $N-1$ over $\mathrm{Pic}_S(X)$. Since $\mathrm{Pic}_S(X)$ is separated and smooth of relative dimension g over S (cf. II.4.1), the S-scheme $\mathrm{Pic}_S(X,Y)$ is separated and smooth of relative dimension $g+N-1$.

II.4.10. Let $f : X \to S$ be as in II.4.1, and let $i : Y \hookrightarrow X$ be a closed subscheme of X, which is finite locally free over S of degree $N \geq 1$, and let $U = X \setminus Y$ be its complement. A **Y-trivial effective Cartier divisor of degree** d **on** X is a pair $(\mathcal{L},\sigma)$ such that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $\sigma : \mathcal{O}_X \hookrightarrow \mathcal{L}$ is an injective homomorphism such that $i^*\sigma$ is an isomorphism and such that the closed subscheme $V(\sigma)$ of X defined by the vanishing of the ideal $\sigma\mathcal{L}^{-1}$ of $\mathcal{O}_X$ is finite locally free of rank d over S. Two Y-trivial effective divisors $(\mathcal{L},\sigma)$ and $(\mathcal{L}',\sigma')$ are **equivalent** if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $\beta\sigma = \sigma'$. As in II.4.7, if such an isomorphism exists then it is unique.

PROPOSITION II.4.11. *The map* $(\mathcal{L}, \sigma) \mapsto (V(\sigma) \hookrightarrow X)$ *is a bijection from the set of equivalence classes of* Y*-trivial effective Cartiers divisor of degree* d *on* X *onto the set of closed subschemes of* U *which are finite locally free of degree* d *over* S.

Let $(\mathcal{L}, \sigma)$ be a Y-trivial effective divisor of degree d on X. The ideal $\mathcal{I} = \sigma\mathcal{L}^{-1}$ is an invertible ideal of $\mathcal{O}_X$ such that the vanishing locus $V(\mathcal{I})$ is finite locally free of rank d over S and is contained in U. The pair $(\mathcal{L}, \sigma)$ is equivalent to $(\mathcal{I}^{-1}, 1)$, and $\mathcal{I}$ is uniquely determined by $V(\mathcal{I})$. Conversely for any closed subscheme Z of U which is finite locally free of rank d over S, the scheme Z is proper over S hence closed in X as well, and its defining ideal $\mathcal{I}$ in $\mathcal{O}_{X_T}$ is invertible by ([**BLR90**], 8.2.6(ii)). The pair $(\mathcal{I}^{-1}, 1)$ is then a Y-trivial effective Cartier divisor of degree d on X.

PROPOSITION II.4.12. *Let* d *be an integer and let* $\mathrm{Div}_S^{d,+}(X, Y)$ *be the functor which to an* S*-scheme* T *associates the set of equivalence classes of* Y_T*-trivial effective Cartier divisors of degree* d *on* X_T*. Then* $\mathrm{Div}_S^{d,+}(X, Y)$ *is representable by the* S*-scheme* $\mathrm{Sym}_S^d(U)$*, the* d*-th symmetric power of* $U = X \setminus Y$ *over* S *(cf. II.2.22). In particular* $\mathrm{Div}_S^{d,+}(X, Y)$ *is smooth of relative dimension* d *over* S.

By Proposition II.4.11, the functor $\mathrm{Div}_S^{d,+}(X, Y)$ is isomorphic to the functor which sends an S-scheme T to the set of closed subschemes of U_T which are finite locally free of rank d over T. In other words, $\mathrm{Div}_S^{d,+}(X, Y)$ is isomorphic to the Hilbert functor of d-points in the S-scheme U.

If x is a T-point of U, we denote by $\mathcal{O}(-x)$ the kernel of the homomorphism $\mathcal{O}_{X \times_S T} \to x_*\mathcal{O}_T$, which is an invertible ideal sheaf, and by $\mathcal{O}(x)$ its dual, which is endowed with a section $1_x : \mathcal{O}_T \hookrightarrow \mathcal{O}(x)$. The morphism

$$\begin{aligned} \mathrm{Sym}_S^d(U) &\to \mathrm{Div}_S^{d,+}(X, Y) \\ (x_1, \dots, x_d) &\to \left(\bigotimes_{i=1}^d \mathcal{O}(x_i), \prod_{i=1}^d 1_{x_i} \right) \end{aligned}$$

is then an isomorphism of fppf-sheaves by ([**SGA4**], XVII.6.3.9), hence Proposition II.4.12.

REMARK II.4.13. Let T be an S-scheme. Let Z be a closed subscheme of U_T which is finite locally free of rank d over T, therefore defining a T-point of $\mathrm{Div}_S^{d,+}(X, Y) = \mathrm{Sym}_S^d(U)$ by Proposition II.4.11. By ([**SGA4**], XVII.6.3.9), this T-point is given by the composition

$$T \to \mathrm{Sym}_T^d(Z) \to \mathrm{Sym}_T^d(U_T) \to \mathrm{Sym}_S^d(U),$$

where the first morphism is the canonical morphism from Proposition II.2.21.

PROPOSITION II.4.14. *Let* $d \geq N + 2g - 1$ *be an integer, and let* $\mathrm{Pic}_S^d(X, Y)$ *be the inverse image of* $\mathrm{Pic}_S^d(X)$ *by the natural morphism* $\mathrm{Pic}_S(X, Y) \to \mathrm{Pic}_S(X)$*. Then the Abel-Jacobi morphism*

$$\begin{aligned} \Phi_d : \mathrm{Div}_S^{d,+}(X, Y) &\to \mathrm{Pic}_S^d(X, Y) \\ (\mathcal{L}, \sigma) &\mapsto (\mathcal{L}, i^*\sigma) \end{aligned}$$

is surjective smooth of relative dimension $d - N - g + 1$ *and it has geometrically connected fibers.*

Let Z be the scheme $\mathrm{Pic}_S^d(X, Y)$, and let $(\mathcal{L}_u, \alpha_u)$ be the universal Y-rigidified line bundle of degree d on X_Z. By ([**BLR90**], 8.2.6(ii)), the closed subscheme Y_Z of X_Z is defined by an invertible ideal sheaf $\mathcal{I}$.

Let $\mathcal{E}$ be the pushforward of $\mathcal{M} = \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{I}$ by the morphism $f_Z : X_Z \to Z$. By II.4.3, the $\mathcal{O}_Z$-module $\mathcal{E}$ is locally free of rank $d - N - g + 1$, and for any morphism $T \to Z$ the canonical homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_T \to f_{T*}(\mathcal{M} \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{X_T})$$

is an isomorphism, where $f_T : X_T \to T$ is the base change of f by the morphism $T \to S$. We thus obtain an isomorphism

$$E \to E', \tag{12}$$

of functors on the category of Z-schemes, where E is the functor $T \mapsto \Gamma(T, \mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$ and E' is the functor $T \mapsto \Gamma(X_T, \mathcal{M} \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{X_T})$. Let $\mathcal{F}$ be the pushfoward of $\mathcal{L}_u$ by the morphism f_Z. By the same argument, we obtain that the $\mathcal{O}_Z$-module $\mathcal{F}$ is locally free of rank $d - g + 1$, and that we have an isomorphism

$$F \to F', \tag{13}$$

of functors on the category of Z-schemes, where F is the functor $T \mapsto \Gamma(T, \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$ and F' is the functor $T \mapsto \Gamma(X_T, \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{X_T})$. Let us consider the exact sequence

$$0 \to \mathcal{M} \to \mathcal{L}_u \to \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{Y_Z} \to 0.$$

Since $R^1 f_{Z*}\mathcal{M} = 0$ by II.4.3, we obtain an exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{G}$ is a locally free $\mathcal{O}_Z$-module of rank N. Together with (12) and (13), this yields an exact sequence

$$0 \to E' \to F' \xrightarrow{b} G \to 0,$$

of Z-group schemes in Z_{fppf}, where G is the functor $T \mapsto \Gamma(T, \mathcal{G}_T \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$. The section α_u of $\mathcal{G}$ over Z corresponds to a morphism $\alpha_u : Z \to G$, and we have a morphism

$$\begin{aligned} \mathrm{Div}_S^{d,+}(X, Y) &\to F' \times_{b,G,\alpha_u} Z \\ (\mathcal{L}, \sigma) &\mapsto (\sigma, (\mathcal{L}, i^*\sigma)), \end{aligned}$$

which is an isomorphism: indeed, if $(\sigma, (\mathcal{L}, i^*\sigma))$ is a T-point of $F' \times_{b,G,\alpha_u} Z$, then for any point t of T the restriction σ_t of σ to the fiber $X_t = X_T \times_T t$ is a global section of the line bundle $\mathcal{L}_t = \mathcal{L} \otimes_{\mathcal{O}_{X_T}} \mathcal{O}_{X_t}$, which is non zero since non vanishing on $Y_t = Y_T \times_T t$, so that $\sigma_t : \mathcal{O}_{X_t} \to \mathcal{L}_t$ is an injective homomorphism and ([**EGA4**], 11.3.7) ensures that $\sigma : \mathcal{O}_X \to \mathcal{L}$ is an effective Cartier divisor on the relative curve X_T, see also ([**BLR90**], 8.2.6(iii)). Since b is an E'-torsor over G in Z_{fppf}, we obtain that $\mathrm{Div}_S^{d,+}(X, Y)$ is an E'-torsor in Z_{fppf}. Since E' is isomorphic to E by (12), it is smooth of relative dimension $d - N - g + 1$ over Z with geometrically connected fibers, hence the conclusion of Proposition II.4.14.

II.5. Geometric global class Field Theory

II.5.1. Let $f : X \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is Zariski-locally projective over S, and let $i : Y \hookrightarrow X$ be a closed subscheme of X which is finite locally free over S of degree $N \geq 1$. Let $j : U \to X$ be the open complement of Y. Let Λ be a finite ring whose cardinality is invertible on S.

Definition II.5.2. A locally free Λ-module $\mathcal{F}$ of rank 1 in $U_{\text{Ét}}$ has **ramification bounded by Y over S** if for any geometric point $\bar{x}$ of Y with image $\bar{s}$ in S, the restriction of $\mathcal{F}$ to $\mathrm{Spec}(\widehat{\mathcal{O}_{X_{\bar{s}},\bar{x}}}) \times_{X_{\bar{s}}} U_{\bar{s}}$ has ramification bounded by the multiplicity of $Y_{\bar{s}}$ at $\bar{x}$ (cf. II.3.9).

THEOREM II.5.3. *Let $\mathcal{F}$ be a locally free Λ-module of rank 1 in $U_{\text{Ét}}$ with ramification bounded by Y over S (cf. II.5.2). Then, there is a unique (up to isomorphism) multiplicative locally free Λ-module $\mathcal{G}$ of rank 1 on the S-group scheme $\mathrm{Pic}_S(X,Y)$ (cf. II.2.6) such that the pullback of $\mathcal{G}$ by the Abel-Jacobi morphism*

$$U \to \mathrm{Pic}_S(X,Y),$$

which sends x to $(\mathcal{O}(x),1)$, is isomorphic to $\mathcal{F}$.

In Section II.5.4, we study the restriction of the locally free Λ-module $\mathcal{F}^{[d]}$ of rank 1 on $\mathrm{Div}_S^{d,+}(X,Y)$ (cf. II.2.32 and II.4.12) to a geometric fiber of the Abel-Jacobi morphism (cf. II.4.14)

$$\begin{aligned}\Phi_d : \mathrm{Div}_S^{d,+}(X,Y) &\to \mathrm{Pic}_S^d(X,Y)\\ (\mathcal{L},\sigma) &\mapsto (\mathcal{L}, i^*\sigma).\end{aligned}$$

This study will enable us to prove Theorem II.5.3 in Section II.5.10.

II.5.4. Let k be an algebraically closed field, let X be a smooth connected projective curve of genus g over k and let $i : Y \to X$ be an effective Cartier divisor of degree N with complement$U$ in X. Let $\mathcal{L}$ be a line bundle of degree $d \geq N + 2g - 1$ on X, and let V be the $(d-N-g+1)$-dimensional affine space over k associated to the k-vector space $\mathcal{V} = H^0(X, \mathcal{L}(-Y))$, i.e. V is the spectrum of the symmetric algebra of the k-module $\mathrm{Hom}_k(\mathcal{V},k)$. Let τ be a global section of $\mathcal{L}$ on X such that $i^*\tau : \mathcal{O}_Y \to i^*\mathcal{L}$ is an isomorphism.

PROPOSITION II.5.5. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be a locally free Λ-module of rank 1 in $U_{\text{Ét}}$, with ramification bounded by Y (cf. II.5.2). Then the pullback of $\mathcal{F}^{[d]}$ (cf. II.2.32) by the morphism*

$$V \to \mathrm{Div}_k^{d,+}(X,Y),$$

which sends a section s of V to $(\mathcal{L}, \tau - s)$, is a constant étale sheaf.

The morphism

$$V \to \mathrm{Div}_k^{d,+}(X,Y),$$

which sends a point σ of V to $(\mathcal{L}, \tau-\sigma)$, is an isomorphism from V to the fiber of Φ_d over the k-point $(\mathcal{L}, i^*\tau)$, cf. II.4.14. Proposition II.5.5 thus implies:

COROLLARY II.5.6. *Let $\mathcal{F}$ be as in Proposition II.5.5. Then the locally free Λ-module $\mathcal{F}^{[d]}$ on $\mathrm{Div}_k^{d,+}(X,Y)_{\text{Ét}}$ is constant on the fiber at $(\mathcal{L}, i^*\tau)$ of the morphism*

$$\Phi_d : \mathrm{Div}_k^{d,+}(X,Y) \to \mathrm{Pic}_k^d(X,Y)$$

from II.4.14.

We now prove Proposition II.5.5. To this end, we consider the morphism

$$\psi : \mathbb{A}_V^1 \to \mathrm{Div}_k^{d,+}(X,Y),$$

which sends a pair (t,σ), where t and σ are points of $\mathbb{A}_k^1$ and V respectively, to the point $(\mathcal{L}, \tau - t\sigma)$ of $\mathrm{Div}_k^{d,+}(X,Y)$. Let $\mathcal{F}$ be as in Propoosition II.5.5, and let $\mathcal{G}$ be the pullback by ψ of $\mathcal{F}^{[d]}$ (cf. II.2.32). Denoting by $\iota_t : V \to \mathbb{A}_V^1$ the section corresponding to an element t of $k = \mathbb{A}_k^1(k)$, we must prove that the sheaf $\iota_1^{-1}\mathcal{G}$ is constant. The sheaf $\iota_0^{-1}\mathcal{G}$ is constant, since $\psi\iota_0$ is a constant morphism, hence it is sufficent to prove that $\iota_1^{-1}\mathcal{G}$ and $\iota_0^{-1}\mathcal{G}$ are isomorphic. The latter fact follows from the following lemma:

LEMMA II.5.7. *The locally free Λ-module $\mathcal{G}$ is the pullback of an etale sheaf on V by the projection $\pi : \mathbb{A}_V^1 \to V$.*

We now prove Lemma II.5.7. We start by proving that $\mathcal{G}$ is constant on each geometric fiber of the projection π. Since the formation of ψ and $\mathcal{G}$ is compatible with the base change along any field extension of k, it is sufficient to show that $\mathcal{G}$ is constant on each fiber of the projection $\mathbb{A}^1_V \to V$ at a k-point σ of V. If $\sigma = 0$, then the restriction of ψ to the fiber of π above σ is constant, hence $\mathcal{G}$ is constant on this fiber.

We now assume that σ is non zero. Since σ vanishes on the non empty divisor Y and τ does not, the sections σ and τ are k-linearly independent in $H^0(X, \mathcal{L})$. Let D be the greatest divisor on X such that $D \leq \operatorname{div}(\sigma)$ and $D \leq \operatorname{div}(\tau)$. Since the divisor of τ is contained in U, so is D. We can then write $\sigma = \widetilde{\sigma} 1_D$ and $\tau = \widetilde{\tau} 1_D$, where 1_D is the canonical section of $\mathcal{O}(D)$ and $\widetilde{\sigma}, \widetilde{\tau}$ are global sections of $\mathcal{L}(-D)$ on X without common zeroes. Thus $f = [\widetilde{\tau} : \widetilde{\sigma}]$ is a well defined non constant morphism from X to $\mathbb{P}^1_k$. Thus, if W is the closed subscheme of $X \times_k \mathbb{A}^1_k$ defined by the vanishing of $\tau - t\sigma$, where t is the coordinate on $\mathbb{A}^1_k$, then we have

$$W = D \times_k \mathbb{A}^1_k \cup (\operatorname{Graph}(f) \cap X \times_k \mathbb{A}^1_k) \hookrightarrow U \times_k \mathbb{A}^1_k.$$

Moreover, the projection $W \to \mathbb{A}^1_k$ is finite flat of degree d, and the restriction of ψ to the fiber at σ factors as

$$\mathbb{A}^1_k \xrightarrow{\varphi} \operatorname{Sym}^d_{\mathbb{A}^1_k}(W) \to \operatorname{Sym}^d_{\mathbb{A}^1_k}(U \times_k \mathbb{A}^1_k) \to \operatorname{Sym}^d_k(U) \to \operatorname{Div}^{d,+}_k(X, Y),$$

where the first morphism φ is obtained from Proposition II.2.21, and the last morphism is the isomorphism from Proposition II.4.12. Moreover, the pullback of $\mathcal{F}^{[d]}$ to $\operatorname{Sym}^d_{\mathbb{A}^1_k}(W)$ coincides with $(p_1^{-1}\mathcal{F})^{[d]}$, where $p_1 : W \to U$ is the first projection. In particular, the sheaf $\mathcal{G}$ is isomorphic to $\varphi^{-1}(p_1^{-1}\mathcal{F})^{[d]}$.

Let $K = k((t^{-1}))$ and let $\eta = \operatorname{Spec}(K) \to \mathbb{A}^1_k$ be the corresponding punctured formal neighbourhood of ∞. Let us form the following commutative diagram:

$$\begin{tikzcd}
\mathbb{A}^1_k \arrow[r, "\varphi"] & \operatorname{Sym}^d_{\mathbb{A}^1_k}(W) \\
\eta \arrow[u] \arrow[r] & \operatorname{Sym}^d_\eta(W \times_{\mathbb{A}^1_k} \eta). \arrow[u]
\end{tikzcd}$$

We can then write

$$W \times_{\mathbb{A}^1_k} \eta = D \times_k \eta \cup \operatorname{Graph}(f) \times_{\mathbb{P}^1_k} \eta = D \times_k \eta \cup X \times_{f, \mathbb{P}^1_k} \eta.$$

The divisors $D \times_k \eta$ and $X \times_{f,\mathbb{P}^1_k} \eta$ of $X \times_k \eta$ are disjoint, since the former lies over closed points of X, while the latter lies over the generic point of X. We thus have a decomposition

$$W \times_{\mathbb{A}^1_k} \eta = D \times_k \eta \amalg X \times_{f,\mathbb{P}^1_k} \eta = \coprod_i \operatorname{Spec}(L_i)$$

where L_i is either of the form $K[T]/(T^{d_i})$ if $\operatorname{Spec}(L_i)$ is a connected component of $D \times_k \eta$, or a field extension of degree d_i of K if $\operatorname{Spec}(L_i)$ is a connected component of $X \times_{f,\mathbb{P}^1_k} \eta$. In the former case, the restriction of $p_1^{-1}\mathcal{F}$ to $\operatorname{Spec}(L_i)$ is constant, while in the latter case, we have the further information that the restriction of $p_1^{-1}\mathcal{F}$ to $\operatorname{Spec}(L_i)$ has ramification bounded by d_i (cf. II.3.9), since the ramification index of f at a point x above ∞ is greater than or equal to the multiplicity of Y at x, and $\mathcal{F}$ has ramification bounded by Y by assumption. Moreover, we have $\sum_i d_i = d$, and the morphism $\eta \to \operatorname{Sym}^d_\eta(W \times_{\mathbb{A}^1_k} \eta)$ factors through the canonical morphism

$$\prod_i \operatorname{Sym}^{d_i}_\eta(\operatorname{Spec}(L_i)) \to \operatorname{Sym}^d_\eta(W \times_{\mathbb{A}^1_k} \eta).$$

By II.3.14, we obtain that the restriction of $\mathcal{G}$ to η is tamely ramified. Since the tame fundamental group of $\mathbb{A}^1_k$ is trivial, we conclude that $\mathcal{G}$ is a constant étale Λ-module on the fiber of π

at σ. The conclusion of Lemma II.5.7 then follows from a descent result, namely Lemma II.5.9 below.

Remark II.5.8. While the proof of Proposition II.3.14, which constitutes the core of the proof of Lemma II.5.7 above, uses geometric local class field theory, it should be noticed that its statement does not refer to it. This explains why no form of local-global compatibility is required in the proof of Lemma II.5.7.

Lemma II.5.9. *Let $g : T' \to T$ be a quasi-compact smooth compactifiable morphism of schemes of relative dimension δ with geometrically connected fibers, and let $\mathcal{G}$ be an étale sheaf of Λ-modules on $T'_{\text{ét}}$ which is constant on each geometric fiber of g. Then $\mathcal{G}$ is isomorphic to the pullback by g of an étale sheaf of Λ-modules on $T_{\text{ét}}$.*

By ([**SGA4**], XVIII 3.2.5) the functor $Rg_!$ on the derived category of Λ-modules on T admits the functor $g^! : K \mapsto g^*K(\delta)[2\delta]$ as a right adjoint. Let us apply the functor $\mathcal{H}^0$ to the adjunction morphism $\mathcal{G} \to g^! Rg_! \mathcal{G}$. The morphism

$$\mathcal{G} \to \mathcal{H}^0(g^! Rg_! \mathcal{G}) = g^* R^{2\delta} g_! \mathcal{G}(\delta)$$

is an isomorphism, as can be seen by checking the stalks at geometric points with the proper base change theorem.

II.5.10. We now prove Theorem II.5.3. Let $\mathcal{F}$ be a locally free Λ-module of rank 1 over $U_{\text{Ét}}$. The family $(F^{[d]})_{d \geq 0}$ of locally free Λ-modules of rank 1 yields a multiplicative étale Λ-module of rank 1 over the S-semigroup scheme

$$\mathrm{Div}^+_S(X,Y) = \coprod_{d\geq 0} \mathrm{Div}^{d,+}_S(X,Y).$$

For each integer $d \geq N + 2g - 1$, Corollary II.5.6 implies that the locally free Λ-module $\mathcal{F}^{[d]}$ of rank 1 on $\mathrm{Div}^{d,+}_S(X,Y)$ (cf. II.2.32 and II.4.12) is constant on the geometric fibers of the smooth surjective morphism (cf. II.4.14)

$$\begin{aligned} \Phi_d : \mathrm{Div}^{d,+}_S(X,Y) &\to \mathrm{Pic}^d_S(X,Y) \\ (\mathcal{L},\sigma) &\mapsto (\mathcal{L}, i^*\sigma). \end{aligned}$$

This morphism satisfies the conditions of Lemma II.5.9 by Proposition II.4.14. We can therefore apply Lemma II.5.9, and we obtain a locally free Λ-module $\mathcal{G}_d$ of rank 1 over $\mathrm{Pic}^d_S(X,Y)$ such that $\Phi_d^{-1}\mathcal{G}_d$ is isomorphic to $\mathcal{F}^{[d]}$. By Proposition II.2.8, the family $(\mathcal{G}_d)_{d\geq N+2g-1}$ yields a multiplicative locally free Λ-module of rank 1 on the S-semigroup scheme

$$M = \coprod_{d \geq N+2g-1} \mathrm{Pic}^d_S(X,Y).$$

Since the morphism

$$\begin{aligned} \rho : M \times_S M &\to \mathrm{Pic}_S(X,Y) \\ (x,y) &\mapsto xy^{-1} \end{aligned}$$

is faithfully flat and quasi-compact, we can apply Proposition II.2.15, which yields a multiplicative locally free Λ-module $\mathcal{G}$ of rank 1 over $\mathrm{Pic}_S(X,Y)$ whose restriction to $\mathrm{Pic}^d_S(X,Y)$ coincides with $\mathcal{G}_d$ for $d \geq N+2g-1$. The families $(\mathcal{F}^{[d]})_{d\geq 0}$ and $(\Phi_d^{-1}\mathcal{G}_d)_{d\geq 0}$ yield multiplicative locally free Λ-modules of rank 1 on the S-semigroup scheme $\mathrm{Div}^+_S(X,Y) = \coprod_{d\geq 0}\mathrm{Div}^{d,+}_S(X,Y)$, whose restrictions to the ideal

$$I = \coprod_{d\geq N+2g-1} \mathrm{Div}^{d,+}_S(X,Y)$$

of $\mathrm{Div}^+_S(X,Y)$ are isomorphic. We obtain by Proposition II.2.7 an isomorphism from $\mathcal{F}^{[d]}$ to $\Phi_d^{-1}\mathcal{G}_d$ for each $d \geq 0$. In particular, the locally free Λ-module $\Phi_1^{-1}\mathcal{G}_1$ of rank 1 is isomorphic to $\mathcal{F}$.

Chapitre III

Facteurs locaux ℓ-adiques

Sommaire

III.1. Introduction

III.1.1. The theory of local ε-factors over local fields with finite residue fields originated from Tate's thesis in rank 1, and was brought to its current form by works of Dwork [**Dw56**], Langlands [**Lan**], Deligne [**De73**] and Laumon [**La87**]. Central motivations for these developments were the problem of decomposing the constants of functional equations of Artin's L-functions, or of Weil's L-functions, as a product of local contributions, and the applications of such a decomposition to Langlands program through Deligne's recurrence principle, cf. ([**La87**], 3.2.2). Inspired by the work of Laumon [**La87**] and by Deligne's 1974 letter to Serre ([**BE01**], Appendix), we provide in this text an explicit cohomological construction of ε-factors for ℓ-adic Galois representations over equicharacteristic henselian discrete valuation fields, with (not necessarily finite) perfect residue fields of positive characteristic, such as the field of Laurent series $k((t))$ for any perfect field k of positive characteristic p. As it turns out, these geometric local ε-factors fit into a product formula for the determinant of the cohomology of an ℓ-adic sheaf on a curve over a perfect field of characteristic p.

III.1.2. Let us first recall the classical theory for local fields with finite residue fields. We restrict to the equicharacteristic case, and we give a slightly non standard presentation as a preparation for our extension to the case of a general perfect residue field (cf. III.1.6). Let us fix an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Let ℓ be a prime number distinct from p and let $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ be a non trivial homomorphism. Let us consider quadruples $(T, \mathcal{F}, \omega, \overline{s})$ where T is a henselian trait of equicharacteristic p, whose closed point s is finite over $\mathbb{F}_p$, where $\mathcal{F}$ is a constructible étale $\overline{\mathbb{Q}}_\ell$-sheaf on T, where ω is a non zero meromorphic 1-form on T (cf. III.7.1), and where $\overline{s} : \mathrm{Spec}(\overline{\mathbb{F}}_p) \to T$ is a morphism of schemes.

A *theory of ℓ-adic local ε-factors over $\mathbb{F}_p$, with respect to ψ,* is a rule ε which assigns to any such quadruple $(T, \mathcal{F}, \omega, \overline{s})$ a homomorphism $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ from the Galois group $\mathrm{Gal}(\overline{s}/s)$ to $\overline{\mathbb{Q}}_\ell^\times$, and which satisfies the following axioms:

(1) the homomorphism $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ depends only on the isomorphism class of the quadruple $(T, \mathcal{F}, \omega, \overline{s})$;

(2) there exists a finite extension E of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$, depending on $\mathcal{F}$, such that $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ is a continuous homomorphism from $\mathrm{Gal}(\overline{s}/s)$ to $E^\times$;

(3) for any exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$
of constructible étale $\overline{\mathbb{Q}}_\ell$-sheaves on T, we have
$$\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega) = \varepsilon_{\overline{s}}(T, \mathcal{F}', \omega)\varepsilon_{\overline{s}}(T, \mathcal{F}'', \omega);$$

(4) if $\mathcal{F}$ supported on the closed point of T, then $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ is the ℓ-adic character of $\mathrm{Gal}(\overline{s}/s)$ corresponding to the 1-dimensional representation $\det(\mathcal{F}_{\overline{s}})^{-1}$;

(5) for each finite generically étale extension $f : T' \to T$ of henselian traits, there exists a homomorphism $\lambda_f(\omega)$ from the Galois group $\mathrm{Gal}(\overline{s}/s)$ to $\overline{\mathbb{Q}}_\ell^\times$ such that
$$\varepsilon_{\overline{s}}(T, f_*\mathcal{F}, \omega) = \lambda_f(\omega)^{\mathrm{rk}(\mathcal{F})}\delta_{s'/s}^{a(T',\mathcal{F},f^*\omega)}\mathrm{Ver}_{s'/s}\left(\varepsilon_{\overline{s}'}(T', \mathcal{F}, f^*\omega)\right),$$
for any constructible étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on T', of generic rank $\mathrm{rk}(\mathcal{F})$, where the verlagerung $\mathrm{Ver}_{s'/s}$ and the signature $\delta_{s'/s}$ are defined in III.3.23, and the conductor $a(T, \mathcal{F}, f^*\omega)$ is defined in III.7.2;

(6) if $j : \eta \to T$ is the inclusion of the generic point of T and if $\mathcal{F}$ is a lisse étale $\overline{\mathbb{Q}}_\ell$-sheaf of rank 1 on η, then we have
$$(-1)^{a(T,j_*\mathcal{F})}\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)(\mathrm{Frob}_s) = \varepsilon(\chi_{\mathcal{F}}, \Psi_\omega),$$
where Frob_s is the geometric Frobenius element of $\mathrm{Gal}(\overline{s}/s)$, where the conductor $a(T, j_*\mathcal{F})$ is defined in III.7.2, where $\Psi_\omega : k(\eta) \to \Lambda^\times$ is the additive character given by $z \mapsto \psi(\mathrm{Tr}_{k/\mathbb{F}_p}(z\omega))$, where $\chi_{\mathcal{F}}$ is the character of $k(\eta)^\times$ associated to $\mathcal{F}$ by local class field theory, and where $\varepsilon(\chi_{\mathcal{F}}, \Psi_\omega)$ is the automorphic ε-factor of the pair $(\chi_{\mathcal{F}}, \Psi_\omega)$, cf. ([**La87**], 3.1.3.2).

THEOREM III.1.3. *For any prime number p, any prime number ℓ distinct from p and any non trivial homomorphism $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$, there exists a unique theory of ℓ-adic local ε-factors over $\mathbb{F}_p$, with respect to ψ.*

Since the Galois group $\mathrm{Gal}(\overline{s}/s)$ is procyclic, the ℓ-adic character $\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)$ is completely determined by its value at the geometric Frobenius element of $\mathrm{Gal}(\overline{s}/s)$. Actually, the rule which associates the quantity
$$(-1)^{a(T,\mathcal{F})}\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(\mathrm{Frob}_s),$$
to a quadruple $(T, \mathcal{F}, \omega, \overline{s})$, where $a(T, \mathcal{F})$ is the conductor of the pair $(T, \mathcal{F})$ (cf. III.7.2), satisfies the properties listed in ([**La87**], 3.1.5.4). Thus Theorem III.1.3 is a reformulation of the theorem of Langlands [**Lan**] and Deligne [**De73**] regarding the existence and uniqueness of local ε-factors.

The proof of existence by Deligne and Langlands in the finite field case is somewhat indirect: starting with the prescribed values (6) of the local ε-factors in rank 1, local ε-factors are defined in arbitrary rank by Brauer's theory and by the induction property (5), and the main problem is then to prove that the resulting factors are independent of the choices made. Our approach is different: we first give a simple cohomological definition of local ε-factors in arbitrary rank (cf. III.9.2) using the theory of Gabber-Katz extensions (cf. III.1.6 below), and we use Brauer's theory only to establish the main properties of these local ε-factors.

Laumon gave a cohomological formula for local ε-factors over local fields with finite residue fields ([**La87**], 3.5.1.1). If $\mathcal{F}$ is supported on the generic point η of T, then Laumon's formula takes the following form:

$$\varepsilon_{\overline{s}}(T,\mathcal{F},d\pi) = \det(F_\pi^{(0,\infty')}(\mathcal{F}))\circ\sigma_\pi, \tag{14}$$

where π is a uniformizer of $k(\eta)$, where Laumon's *local Fourier transform* $F_\pi^{(0,\infty')}(\mathcal{F})$ is an ℓ-adic representation of $\mathrm{Gal}(\overline{\eta}/\eta)$, cf. ([**La87**], 2.4.1), and where $\sigma_\pi : \mathrm{Gal}(\overline{s}/s) \to \mathrm{Gal}(\overline{\eta}/\eta)^{\mathrm{ab}}$ is the section of the natural homomorphism $\mathrm{Gal}(\overline{\eta}/\eta)^{\mathrm{ab}} \to \mathrm{Gal}(\overline{s}/s)$ corresponding by local class field theory to the unique section of the valuation homomorphism $k(\eta)^\times \to \mathbb{Z}$ sending the element 1 of $\mathbb{Z}$ to π.

It is straightforward to extend (14) to a rule $\varepsilon_{\mathrm{Lau}}$ satisfying the properties (1), (2), (3) and (4) of a theory of ℓ-adic local ε-factors over $\mathbb{F}_p$. Moreover, the normalization in rank 1, namely property (6), can be proved directly for $\varepsilon_{\mathrm{Lau}}$ by using Laumon's ℓ-adic stationary phase method from [**La87**]. Unfortunately, there seems to be no direct proof that $\varepsilon_{\mathrm{Lau}}$ satisfies the property (5), namely the induction formula, and it is therefore not possible to take Laumon's formula as a definition of local ε-factors. However, the ℓ-adic stationary phase method yields that the rule $\varepsilon_{\mathrm{Lau}}$ produced from Laumon's formula (14) coincides with our own definition (cf. III.1.11) in the finite field case (cf. III.11.8). Thus our main Theorem III.1.7 below, in conjunction with the ℓ-adic stationary phase method, proves the induction formula for $\varepsilon_{\mathrm{Lau}}$.

By using (14) and the ℓ-adic stationary phase method, Laumon proved the following product formula:

THEOREM III.1.4 ([**La87**], Th. 3.2.1.1). *Let X be a connected smooth projective curve of genus g over a finite field k, let $\overline{k}$ be an algebraic closure of k, let ω be a non zero global meromorphic differential 1-form on X and let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X of generic rank* $\mathrm{rk}(\mathcal{F})$*. The ℓ-adic character $\varepsilon_{\overline{k}}(X,\mathcal{F})$ of $\mathrm{Gal}(\overline{k}/k)$ associated to the 1-dimensional representation* $\det(R\Gamma(X_{\overline{k}},\mathcal{F}))^{-1}$ *(cf. III.8.2) admits the following decomposition:*

$$\varepsilon_{\overline{k}}(X,\mathcal{F}) = \chi_{\mathrm{cyc}}^{N(g-1)\mathrm{rk}(\mathcal{F})}\prod_{x\in|X|}\delta_{x/k}^{a(X_{(x)},\mathcal{F}_{|X_{(x)}})}\mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}(X_{(x)},\mathcal{F}_{|X_{(x)}},\omega_{|X_{(x)}})\right),$$

where N is the number of connected components of $X_{\overline{k}}$, where $|X|$ is the set of closed points of X, where $X_{(x)}$ is the henselization of X at a closed point x, and where χ_{cyc} is the ℓ-adic cyclotomic character of k. All but finitely many terms in this product are identically equal to 1.

The formulation of the product formula in Theorem III.1.4 differs from Laumon's ([**La87**], Th. 3.2.1.1), but yields an equivalent formula. Indeed, if k is of cardinality q, then evaluating the product formula in Theorem III.1.4 at the geometric Frobenius Frob_k yields that the determinant $\det\left(\mathrm{Frob}_k \mid R\Gamma(X_{\overline{k}},\mathcal{F})\right)^{-1}$ is equal to

$$q^{N(1-g(X))\mathrm{rk}(\mathcal{F})}\prod_{x\in|X|}(-1)^{([k(x):k]-1)a(X_{(x)},\mathcal{F}_{|X_{(x)}})}\varepsilon_{\overline{x}}(X_{(x)},\mathcal{F}_{|X_{(x)}},\omega_{|X_{(x)}})(\mathrm{Frob}_x),$$

and $\sum_{x\in X}[k(x):k]a(X_{(x)},\mathcal{F}_{|X_{(x)}})$ has the same parity as the Euler characteristic $-\chi(X_{\overline{k}},\mathcal{F})$ by the Grothendieck-Ogg-Shafarevich formula, hence the product formula asserts that the quantity

$$q^{N(1-g(X))\mathrm{rk}(\mathcal{F})}\prod_{x\in|X|}(-1)^{a(X_{(x)},\mathcal{F}_{|X_{(x)}})}\varepsilon_{\overline{x}}(X_{(x)},\mathcal{F}_{|X_{(x)}},\omega_{|X_{(x)}})(\mathrm{Frob}_x),$$

coincides with the determinant $\det\left(-\mathrm{Frob}_k \mid R\Gamma(X_{\overline{k}},\mathcal{F})\right)^{-1}$, as in Laumon's formulation ([**La87**], Th. 3.2.1.1).

For ℓ-adic sheaves with finite geometric monodromy, the product formula in Theorem III.1.4 reduces by Brauer's induction theorem to the rank 1 case, and the latter follows from Tate's thesis, cf. ([**La87**], 3.2.1.7). A geometric proof of the product formula in rank 1 was given by Deligne in his 1974 letter to Serre ([**BE01**], Appendix), using geometric class field theory. Deligne's proof in the rank 1 case, which we review in Section III.8, extends to the case of an arbitrary perfect base field k, and constitutes an important ingredient in the proof of the main theorem III.1.7 below.

III.1.5. Let us consider a quadruple $(T, \mathcal{F}, \omega, \overline{s})$, where T is an equicharacteristic henselian trait, with perfect residue field of positive characteristic p, equipped with an algebraic closure $\overline{s}$ of its closed point s, where ω is a non zero meromorphic 1-form on T (cf. III.7.1) and where $\mathcal{F}$ is a constructible étale $\overline{\mathbb{Q}}_\ell$-sheaf on T.

Let us assume that $\mathcal{F}$ is irreducible, with vanishing fiber at s, and that $k(s)$ is the perfection of a finitely generated extension of $\mathbb{F}_p$, so that $\mathcal{F}$ has finite geometric monodromy by Grothendieck's local monodromy theorem (cf. III.9.9). If one wishes to construct an ε-factor $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ by using Brauer's theorem from finite group theory, in order to reduce through additivity and induction to the rank 1 case, we need $\mathcal{F}$ to have finite monodromy, rather than merely having finite geometric monodromy. When k is finite, the Galois group $\mathrm{Gal}(\overline{s}/s)$ is procyclic, hence some twist of $\mathcal{F}$ by a geometrically constant $\overline{\mathbb{Q}}_\ell$-sheaf of rank 1 has finite monodromy, cf. ([**De73**], 4.10), and this allows Deligne and Langlands to reduce to the finite monodromy case.

In the general case, the Galois group $\mathrm{Gal}(\overline{s}/s)$ is not procyclic, nor abelian, and twisting by geometrically constant $\overline{\mathbb{Q}}_\ell$-sheaves of rank 1 is not enough to reduce to the finite monodromy case from the finite geometric monodromy case. However, it is possible to allow for such a reduction by considering more general twists. More precisely, we can reduce to the finite monodromy case at the following costs (cf. III.2.39, III.2.38):

(a) considering $\overline{\mathbb{Q}}_\ell$-sheaves on T twisted by a $\overline{\mathbb{Q}}_\ell^\times$-valued 2-cocycle on $\mathrm{Gal}(\overline{s}/s)$, rather than merely $\overline{\mathbb{Q}}_\ell$-sheaves,

(b) allowing twists by higher rank (twisted) geometrically constant sheaves, rather than rank 1 such sheaves.

The notion of twisted sheaf is recalled in III.3.7. Let us simply describe here the corresponding notion of twisted $\overline{\mathbb{Q}}_\ell$-representation. If η is the generic point of T and if $\overline{\eta}$ is a separable closure of $\eta_{\overline{s}}$, with Galois group $\mathrm{Gal}(\overline{\eta}/\eta)$ endowed with the natural homomorphism $r : \mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{Gal}(\overline{s}/s)$, then a $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{Gal}(\overline{\eta}/\eta)$ twisted by a $\overline{\mathbb{Q}}_\ell^\times$-valued 2-cocycle μ on $\mathrm{Gal}(\overline{s}/s)$, is a continuous map

$$\rho : \mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{GL}(V),$$

where V is a finite dimensional vector space over some finite extension of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$, which satisfies

$$\rho(g)\rho(h) = \mu(r(g), r(h))\rho(gh),$$

for all g, h in $\mathrm{Gal}(\overline{\eta}/\eta)$. When $\mu = 1$ is the trivial cocycle, a twisted $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{Gal}(\overline{\eta}/\eta)$ is simply a $\overline{\mathbb{Q}}_\ell$-Galois representation over η. The preliminary section III.2 is devoted to a more thorough discussion of twisted representations.

III.1.6. Let k be a perfect field of positive characteristic p, with algebraic closure $\overline{k}$, and let ℓ, ψ be as in III.1.1. Let us consider quadruples $(T, \mathcal{F}, \omega, \overline{s})$ where T is a henselian trait over k, whose closed point s is finite over k, equipped with a k-morphism $\overline{s} : \mathrm{Spec}(\overline{k}) \to T$, where ω is a non zero meromorphic 1-form on T (cf. III.7.1) and where $\mathcal{F}$ is a potentially unipotent (cf. III.9.8) constructible étale $\overline{\mathbb{Q}}_\ell$-sheaf on T twisted (cf. III.3.7) by some *unitary* 2-cocycle on $\mathrm{Gal}(\overline{s}/s)$, i.e. a 2-cocycle which is continuous with values in a finite subgroup of $\overline{\mathbb{Q}}_\ell^\times$ (cf. III.2.10).

Here, the potential unipotency assumption means that some open subgroup of the inertia group of T acts unipotently on the ℓ-adic (twisted) Galois representation associated to $\mathcal{F}$. If k is finite, or more generally if k is the perfection of a finitely generated field of characteristic p, then Grothendieck's local monodromy theorem (cf. III.9.9) asserts that any constructible étale $\overline{\mathbb{Q}}_\ell$-sheaf on T has potentially unipotent restriction to the generic point of T, hence in these cases any reference to potential unipotency can be dropped in what follows.

A *theory of twisted ℓ-adic local ε-factors over k, with respect to ψ,* is a rule ε which assigns to any such quadruple $(T, \mathcal{F}, \omega, \overline{s})$ a map $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ from the Galois group $\mathrm{Gal}(\overline{s}/s)$ to $\overline{\mathbb{Q}}_\ell^\times$, and which satisfies the following axioms:

(i) the map $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ depends only on the isomorphism class of the quadruple $(T, \mathcal{F}, \omega, \overline{s})$;

(ii) there exists a finite extension E of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$, depending on $\mathcal{F}$, such that $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ is a continuous map from $\mathrm{Gal}(\overline{s}/s)$ to $E^\times$;

(iii) (cf. III.9.3) for any exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

of potentially unipotent constructible étale $\overline{\mathbb{Q}}_\ell$-sheaves on T twisted by the same unitary 2-cocycle, we have

$$\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega) = \varepsilon_{\overline{s}}(T, \mathcal{F}', \omega)\varepsilon_{\overline{s}}(T, \mathcal{F}'', \omega);$$

(iv) (cf. III.9.6) if $\mathcal{F}$ supported on the closed point of T, then the value of $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ at an element g of $\mathrm{Gal}(\overline{s}/s)$ is given by

$$\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(g) = \det\left(g \mid \mathcal{F}_{\overline{s}}\right)^{-1};$$

(v) (cf. III.9.18) for each finite generically étale extension $f : T' \to T$ of henselian traits T' and T over k, there exists a homomorphism $\lambda_f(\omega)$ from the Galois group $\mathrm{Gal}(\overline{s}/s)$ to $\overline{\mathbb{Q}}_\ell^\times$ such that

$$\varepsilon_{\overline{s}}(T, f_*\mathcal{F}, \omega) = \lambda_f(\omega)^{\mathrm{rk}(\mathcal{F})}\delta_{s'/s}^{a(T', \mathcal{F}, f^*\omega)}\mathrm{Ver}_{s'/s}\left(\varepsilon_{\overline{s}'}(T', \mathcal{F}, f^*\omega)\right),$$

for any potentially unipotent constructible étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on T', of generic rank $\mathrm{rk}(\mathcal{F})$, twisted by some unitary 2-cocycle on $\mathrm{Gal}(\overline{s}/s)$, where the signature $\delta_{s'/s}$ and the *verlagerung* or *transfer* $\mathrm{Ver}_{s'/s}$ are defined in III.3.22;

(vi) if the fiber of $\mathcal{F}$ at the closed point s of T vanishes and if $\mathcal{F}$ is generically of rank 1, with Swan conductor $\nu - 1$, then the value of $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ at an element g of $\mathrm{Gal}(\overline{s}/s)$ is prescribed as follows:

$$\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(g) = \det\left(g \mid H_c^\nu\left(\mathrm{Pic}^{\nu+v(\omega)}(T, \nu s)_{\overline{s}}, \chi_{\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}(-v(\omega))\right)\right),$$

where $v(\omega)$ is the valuation of ω (cf. III.7.1), where $\mathrm{Pic}^{\nu+v(\omega)}(T, \nu s)$ is the component of degree $\nu + v(\omega)$ of the local Picard group (cf. III.5.20), where $\chi_{\mathcal{F}}$ is the multiplicative local system on the group $\mathrm{Pic}(T, \nu s)$ naturally associated to $\mathcal{F}$ by twisted local geometric class field theory (cf. III.5.45), and where $\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ is the Artin-Schreier local system associated to the residue morphism Res_ω, cf III.7.5 for details;

(vii) (cf. III.9.5) if $\mathcal{G}$ is a geometrically constant $\overline{\mathbb{Q}}_\ell$-sheaf on T, twisted by some unitary 2-cocycle on $\mathrm{Gal}(\overline{s}/s)$ (possibly different from the 2-cocycle by which $\mathcal{F}$ is twisted), then we have

$$\varepsilon_{\overline{s}}(T, \mathcal{F} \otimes \mathcal{G}, \omega) = \det(\mathcal{G}_{\overline{s}})^{a(T, \mathcal{F}, \omega)}\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)^{\mathrm{rk}(\mathcal{G})},$$

where the conductor $a(T, \mathcal{F}, \omega)$ is defined in III.7.2.

Our main result can then be stated as follows:

THEOREM III.1.7 (cf. III.9.20). *Let k be a perfect field of positive characteristic $p > 0$. Then for any prime number ℓ distinct from p and any non trivial homomorphism $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$, there exists a unique theory of twisted ℓ-adic local ε-factors over k, with respect to ψ. Moreover, we have the following properties:*

$(viii)$ the map $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ does not depend on the subfield k of $k(s)$.

(ix) (cf. III.9.4) for any quadruple $(T, \mathcal{F}, \omega, \overline{s})$ over k, where $\mathcal{F}$ is twisted by a unitary 2-cocyle μ on $\mathrm{Gal}(\overline{s}/s)$, we have

$$\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(g)\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(h) = \mu(g, h)^{a(T,\mathcal{F},\omega)}\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(gh)$$

for any elements g, h of $\mathrm{Gal}(\overline{s}/s)$, where s is the closed point of T and where $a(T, \mathcal{F}, \omega)$ is the conductor defined in III.7.2. In particular the 2-cocycle $\mu^{a(T,\mathcal{F},\omega)}$ is a coboundary (cf. III.2.6).

In the untwisted case, the property (vii) is a consequence of $(i)-(vi)$, since one can assume that the twist $\mathcal{G}$ is of rank 1, and one can then use Brauer's induction theorem together with $(i) - (v)$ to reduce to the case where $\mathcal{F}$ is also of rank 1, so that (vii) then follows from (vi), cf. ([**La87**], 3.1.5.6).

When k is finite, then for any finite extension $k(s)$ of k contained in $\overline{k}$, any unitary 2-cocycle on $\mathrm{Gal}(\overline{k}/k(s))$ is a coboundary, and thus the theory of twisted ℓ-adic local ε-factors over k is not more general than the classical theory of Deligne and Langlands. Actually, we have:

THEOREM III.1.8 (cf. III.9.21). *Let $(T, \mathcal{F}, \omega, \overline{s})$ be a quadruple as in III.1.6 over a finite field k, where $\mathcal{F}$ is untwisted, i.e. twisted by the trivial cocycle. Then the quantity*

$$(-1)^{a(T,\mathcal{F})}\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(\mathrm{Frob}_s),$$

*Frob_s is the geometric Frobenius in $\mathrm{Gal}(\overline{s}/s)$, where s is the closed point of T and where $a(T, \mathcal{F})$ is the conductor of $(T, \mathcal{F})$ (cf. III.7.2), coincides with the classical local ε-factor, normalized as in ([**La87**], Th. 3.1.5.4).*

This result will be deduced from the normalization (vi) of geometric local ε-factors and from the Grothendieck-Lefschetz trace formula (cf. III.7.22).

As in the case of a finite base field (cf. III.1.5), we have a product formula:

THEOREM III.1.9 (cf. 60). *Let X be a connected smooth projective curve of genus $g(X)$ over a perfect field k, let ω be a non zero global meromorphic differential 1-form on X and let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_\ell$-sheaf on X of generic rank $\mathrm{rk}(\mathcal{F})$, twisted by some unitary 2-cocycle on $\mathrm{Gal}(\overline{k}/k)$ (cf. III.3.7), such that for any closed point x of X the restriction of $\mathcal{F}$ to the henselization $X_{(x)}$ of X at x is potentially unipotent. Then the trace function $\varepsilon_{\overline{k}}(X, \mathcal{F})$ on $\mathrm{Gal}(\overline{k}/k)$ associated to the twisted 1-dimensional representation $\det(R\Gamma(X_{\overline{k}}, \mathcal{F}))^{-1}$ (cf. III.8.2) admits the following decomposition:*

$$\varepsilon_{\overline{k}}(X, \mathcal{F}) = \chi_{\mathrm{cyc}}^{N(g(X)-1)\mathrm{rk}(\mathcal{F})} \prod_{x \in |X|} \delta_{x/k}^{a(X_{(x)}, \mathcal{F}_{|X_{(x)}})} \mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})\right),$$

where N is the number of connected components of $X_{\overline{k}}$, where $|X|$ is the set of closed points of X and χ_{cyc} is the ℓ-adic cyclotomic character of k. All but finitely many terms in this product are identically equal to 1.

We first prove Theorem III.1.9 in the case of (twisted) $\overline{\mathbb{Q}}_\ell$-sheaves with finite geometric monodromy, cf. III.10.3, and we then prove the general case in Section III.11 by using Laumon's ℓ-adic stationary phase method. An important ingredient of the proof is the following extension of Laumon's formula (14):

THEOREM III.1.10 (cf. III.11.8). *Let T be an henselian trait with closed point s, such that $k(s)$ is a perfect field of positive characteristic p, and let $\mathcal{F}$ be a potentially unipotent $\overline{\mathbb{Q}}_\ell$-sheaf on T with vanishing fiber at s, twisted by some unitary 2-cocycle on $\mathrm{Gal}(\overline{k}/k)$ (cf. III.3.7). Let π be a uniformizer on T, and let $\chi_{\det(\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F}))}$ be the multiplicative $\overline{\mathbb{Q}}_\ell$-local system associated to $\det(\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F}))$ by geometric class field theory (cf. III.5.45), where $\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F})$ is Laumon's local Fourier transform, cf. ([**La87**], 2.4.1). Then the trace map of the stalk of $\chi_{\det(\mathrm{F}_\pi^{(0,\infty')}(\mathcal{F}))}$ at π^{-1} coincides with $\varepsilon_{\overline{k}}(T,\mathcal{F},d\pi)$.*

Laumon's proof of this result when k is finite starts with a reduction to the tamely ramified case ([**La87**], 3.5.3.1), and then resort to a computation in the latter case ([**La87**], 2.5.3.1). Instead of adapting Laumon's proof to the general case, we choose to avoid these steps in our treatment of Theorem III.1.10 : we give a direct proof by using the ℓ-adic stationary phase method (cf. III.11.5) and by specializing Theorem III.1.7 to the case where the base field k is (the perfection of) a henselian discretely valued field of equicharacteristic p.

III.1.11. Let us briefly describe our definition of twisted ℓ-adic local ε-factors over a perfect field k of positive characteristic p (cf. III.1.6). Let $(T,\mathcal{F},\omega,\overline{s})$ be a quadruple over k as in III.1.6, and let s be the closed point of T. We fix a uniformizer π of $\mathcal{O}_T$, and denote by π as well the morphism

$$\pi : T \to \mathbb{A}^1_s,$$

corresponding to the unique morphism $k(s)[t] \to \mathcal{O}_T$ of $k(s)$-algebras which sends t to π. The theory of Gabber-Katz extensions, originating from [**Ka86**] and reviewed in Section III.4, ensures the existence of a (twisted) $\overline{\mathbb{Q}}_\ell$-sheaf $\pi_\diamond\mathcal{F}$ on $\mathbb{A}^1_s$, unique up to isomorphism, such that:

(1) the pullback $\pi^{-1}\pi_\diamond\mathcal{F}$ is isomorphic to $\mathcal{F}$;

(2) the $\overline{\mathbb{Q}}_\ell$-sheaf $\pi_\diamond\mathcal{F}$ is tamely ramified at infinity;

(3) the restriction of $\pi_\diamond\mathcal{F}$ to $\mathbb{G}_{m,s}$ is a local system whose geometric monodromy group has a unique p-Sylow.

We then simply define

$$\varepsilon_{\overline{s}}(T,\mathcal{F},d\pi) = \det\left(R\Gamma_c(\mathbb{A}^1_{\overline{s}},\pi_\diamond\mathcal{F}\otimes\mathcal{L}_\psi^{-1})\right)^{-1},$$

where $\mathcal{L}_\psi$ is the Artin-Schreier sheaf on the affine line associated to ψ. Using geometric class field theory, we then define $\varepsilon_{\overline{s}}(T,\mathcal{F},\omega)$ for arbitrary meromorphic 1-forms ω on T, cf. III.9.2.

We then show in Section III.9, using a variant of Brauer's induction theorem (cf. III.2.36), that the resulting local ε-factor is independent of the choice of π (cf. III.9.16) and that it satisfies the properties $(i)-(ix)$ listed in III.1.6 and in Theorem III.1.7. The most notable of these properties is the induction formula (v), which is proved using generalized Gabber-Katz extensions (cf. III.4.18) and the product formula (cf. III.1.7) in generic rank 1, proved by Deligne in his 1974 letter to Serre, the latter being published as an appendix in [**BE01**] and reviewed in Section III.8.

III.1.12. We now describe the organization of this paper. Section III.2 contains preliminary definitions and results on representations of groups twisted by a 2-cocycle. It notably includes an extension of Brauer's induction theorem to this context, namely Theorem III.2.36, and a useful decomposition of a twisted representation according to its restriction to a finite normal subgroup in Proposition III.2.39.

Section III.3 is devoted to basic definitions and results regarding ℓ-adic sheaves and their twisted counterparts.

We review in Section III.4 the theory of Gabber-Katz extensions, following the exposition by Katz in [**Ka86**]. We provide mild generalizations of the results found in the latter article, namely an extension to twisted ℓ-adic sheaves on arbitrary Gabber-Katz curves.

Section III.5 is devoted to geometric class field theory, in both of its global and local incarnations. Since this topic is of independent interest, we choose to present more material than what is strictly necessary in order to prove the main results of this text. We discuss in particular the relations between different formulations of geometric local class field theory, namely those of Serre [**Se61**], of Contou-Carrère [**CC13**] and Suzuki [**Su13**], or of Gaitsgory. We also prove local-global compatibility in geometric class field theory (cf. III.5.36), as well as functoriality with respect to the norm homomorphism (cf. III.5.37).

In Section III.6, we perform a series of computations aiming at describing multiplicative local systems, namely the geometric analog of characters of abelian groups, on certain groups schemes, such as the additive group $\mathbb{G}_a$ or the group of Witt vectors of length 2 over $\mathbb{F}_2$. All of these computation can be considered as being part of the proof of the main proposition III.7.6 in Section III.7, which describes the cohomology groups appearing in our definition III.7.7 of geometric local ε-factors in generic rank 1.

In Section III.8, we review Deligne's 1974 letter to Serre on ε-factors, where the product formula III.1.9 is proved in generic rank 1 by using geometric class field theory. We also provide a mild generalization to the context of twisted ℓ-adic sheaves.

Section III.9 is devoted to the proofs of the main results of this text, namely that the twisted ℓ-adic ε-factors defined with Gabber-Katz extensions as in III.1.11 are independent of the choice of uniformizer and satisfy the properties $(i) - (ix)$ listed in III.1.6 and in Theorem III.1.7. Our main tools are the reduction to the rank 1 case allowed by the results of Section III.2, and the product formula in generic rank 1 from Section III.8. We also prove Theorem III.1.8 in this section.

Finally, we prove the product formula for (twisted) ℓ-adic sheaves of arbitrary rank, first for ℓ-adic sheaves with finite geometric monodromy in section III.10, by using the results of Section III.2 to reduce to the rank 1 case handled in Section III.8, and then in the general case in Section III.11, by following closely Laumon's proof in the finite field case.

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III.1.13. Conventions and notation. We fix a perfect field k of positive characteristic p, and we denote by $\overline{k}$ a fixed algebraic closure of k. We denote by $G_k = \mathrm{Gal}(\overline{k}/k)$ the Galois group of the extension $\overline{k}/k$. For any k-scheme X, and for any k-algebra k', we denote by $X_{k'}$ the fiber product of X and $\mathrm{Spec}(k')$ over $\mathrm{Spec}(k)$. The group G_k acts on the left on $\overline{k}$, and thus acts on the right on $X_{\overline{k}}$.

We fix as well a prime number ℓ different from p, and we denote by C an algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, endowed with the topology induced by the ℓ-adic valuation. We denote by $\mathbb{Z}_\ell(1)$ the invertible $\mathbb{Z}_\ell$-module consisting of sequences $(\zeta_n)_{n\geq 0}$ of elements of $\overline{k}$ such that $\zeta_0 = 1$ and $\zeta_{n+1}^\ell = \zeta_n$ for each n, endowed with the natural action of G_k. For each integer ν, we denote by $\mathbb{Z}_\ell(\nu)$ the invertible $\mathbb{Z}_\ell$-module $\mathbb{Z}_\ell(1)^{\otimes\nu}$. More generally, for any ℓ-adic sheaf $\mathcal{F}$ on a k-scheme (cf. III.3.5) we denote by $\mathcal{F}(\nu)$ the tensor product of $\mathcal{F}$ and $\mathbb{Z}_\ell(\nu)$ over $\mathbb{Z}_\ell$. We also denote by

$$\begin{aligned} \chi_{\mathrm{cyc}} : G_k &\to \mathbb{Z}_\ell^\times \\ g &\to \mathrm{Tr}(g \mid \mathbb{Z}_\ell(1)), \end{aligned}$$

the character associated to the ℓ-adic representation $\mathbb{Z}_\ell(1)$ of G_k.

III.2. Preliminaries on representations of twisted groups

III.2.1. Let G be a profinite topological group, and let $f : G \to X$ be a continuous map onto a finite set X endowed with the discrete topology. The open normal subgroups of G form a basis of open neighbourhoods at the unit element of G. Hence, for each element g of G, there exists an open normal subgroup I_g such that the coset gI_g is contained in the open subset $f^{-1}(f(g))$. Since G is compact, there exists a finite family $(g_j)_{j\in J}$ of elements of G such that the open subsets $(g_jI_{g_j})_{j\in J}$ form a cover of G. Thus, if I is the intersection of the open normal subgroups $(I_{g_j})_{j\in J}$, then I is itself an open normal subgroup of G and f is both left and right I-invariant.

III.2.2. An *admissible ℓ-adic ring* is a commutative topological ring which is isomorphic to one of the following:

(1) a finite local $\mathbb{Z}/\ell^n$-algebra for some integer n, endowed with the discrete topology,

(2) the ring of integers in a finite extension of $\mathbb{Q}_\ell$, endowed with the topology defined by the ℓ-adic valuation,

(3) a finite extension of $\mathbb{Q}_\ell$, endowed with the topology defined by the ℓ-adic valuation.

An *ℓ-adic coefficient ring* is a commutative topological ring Λ such that any finite subset of Λ is contained in a subring of Λ, which is an admissible ℓ-adic ring for the subspace topology. In particular, any admissible ℓ-adic ring is an ℓ-adic coefficient ring as well.

Example III.2.3. The ring C (cf. III.1.13) endowed with the topology induced by the ℓ-adic valuation, is an ℓ-adic coefficient ring.

Remark III.2.4. Any admissible ℓ-adic ring is a finitely presented $\mathbb{Z}_\ell$-agebra. In particular, the set of admissible ℓ-adic subrings of an ℓ-adic coefficient ring is filtered when ordered by inclusion.

III.2.5. Let T be a topological space and let Λ be an ℓ-adic coefficient ring (cf. III.2.2). A map $f : T \to \Lambda$ is said to be Λ-*admissible* if it is continuous and if its image is contained in an admissible ℓ-adic subring of Λ.

Similarly, if V is a free Λ-module of finite rank, a map $f : T \to \mathrm{Aut}_\Lambda(V)$ (resp. $f : T \to \mathrm{Aut}_\Lambda(V)/\Lambda^\times$) is said to be Λ-*admissible* if it is continuous and if there is an admissible ℓ-adic subring Λ_0 of Λ and a Λ_0-form V_0 of V such that f factors through $\mathrm{Aut}_{\Lambda_0}(V_0)$ (resp. $\mathrm{Aut}_{\Lambda_0}(V_0)/\Lambda_0^\times$).

III.2.6. Let G be a topological group, and let Λ be an ℓ-adic coefficient ring (cf. III.2.2). For each integer j, let $C^j(G,\Lambda^\times)$ be the group of Λ-admissible maps from G^j to $\Lambda^\times$. We define a complex

$$C^1(G,\Lambda^\times) \xrightarrow{d^1} C^2(G,\Lambda^\times) \xrightarrow{d^2} C^3(G,\Lambda^\times),$$

as follows: if λ is an element of $C^1(G,\Lambda^\times)$, we set

$$\begin{aligned} d^1(\lambda) : G^2 &\to \Lambda^\times \\ (x,y) &\to \lambda(x)\lambda(y)\lambda(xy)^{-1}, \end{aligned} \tag{15}$$

which is indeed Λ-admissible, and if μ is an element of $C^2(G,\Lambda^\times)$, we set

$$\begin{aligned} d^2(\mu) : G^3 &\to \Lambda^\times \\ (x,y,z) &\to \mu(x,y)\mu(xy,z)\mu(x,yz)^{-1}\mu(y,z)^{-1}, \end{aligned} \tag{16}$$

which is Λ-admissible as well. If λ is an element of $C^1(G,\Lambda^\times)$, then we have

$$d^1(\lambda)(x,y)d^1(\lambda)(xy,z) = \lambda(x)\lambda(y)\lambda(z)\lambda(xyz)^{-1} = d^1(\lambda)(x,yz)d^1(\lambda)(y,z),$$

and thus $d^2 \circ d^1$ vanishes.

Definition III.2.7. An admissible 2-*cocycle* (resp. 2-*boundary*) on G with values in $\Lambda^\times$ is an element of the kernel of d^2 (resp. of the image of d^1). The *second admissible cohomology group* of G with coefficients in $\Lambda^\times$, denoted $H^2_{\mathrm{adm}}(G,\Lambda^\times)$, is the quotient of the group of admissible 2-cocycles on G with values in $\Lambda^\times$, by the subgroup of admissible 2-boundaries.

We also have

$$H^2_{\mathrm{adm}}(G,\Lambda^\times) = \mathrm{colim}_{\Lambda_0} H^2_{\mathrm{adm}}(G,\Lambda_0^\times),$$

where Λ_0 runs over the filtered set of admissible ℓ-adic subrings of Λ (cf. III.2.4), and the group $H^2_{\mathrm{adm}}(G,\Lambda_0^\times)$ coincides with the second continuous cohomology group of G with coefficients in $\Lambda_0^\times$.

Remark III.2.8. If G is finite, then any map from G^j to $\Lambda^\times$ is Λ-admissible. Thus the group $H^2_{\mathrm{adm}}(G,\Lambda^\times)$ coincides with the second cohomology group $H^2(G,\Lambda^\times)$ of G with coefficients in Λ.

III.2.9. Let Λ be an ℓ-adic coefficient ring (cf. III.2.2). A Λ-*admissible multiplier* on a topological group G is an admissible 2-cocycle μ with values in $\Lambda^\times$ (cf. III.2.7), such that $\mu(1,1)=1$. In particular, a multiplier μ satisfies the cocycle relation

$$\mu(x,y)\mu(xy,z) = \mu(x,yz)\mu(y,z), \tag{17}$$

for all x,y,z in G. By specializing this relation to $x=y=1$, we obtain $\mu(1,z)=1$ for any z in G. Likewise, we have $\mu(x,1)=1$ for any x in G.

Definition III.2.10. A Λ-admissible multiplier μ on a topological group G is said to be *unitary* if there is an integer $r \geq 1$ such that $\mu^r = 1$.

Since the group of r-th roots of unity in Λ is a discrete subgroup of $\Lambda^\times$, any unitary multiplier on a topological group G must be locally constant. If moreover G is profinite, then any Λ-admissible multiplier on G must be left and right I-invariant for some open normal subgroup I of G (cf. III.2.1).

Definition III.2.11. A Λ-*twisted topological group* (resp. a Λ-*twisted group*) is a pair (G,μ), where G is a topological group (resp. a discrete group) and μ is a Λ-admissible multiplier on G. A morphism of Λ-twisted topological groups from (G,μ) to (G',μ') is a continuous group homomorphism $f: G \to G'$ such that $\mu(x,y) = \mu'(f(x),f(y))$ for all x,y in G.

Remark III.2.12. Let G be a topological group, and let $\lambda: G \to \Lambda^\times$ be a Λ-admissible map (cf. III.2.5) such that $\lambda(1)=1$. Then the 2-boundary $d^1(\lambda)$ (cf. 15) is a multiplier on G. The quotient of the group of Λ-admissible multipliers on G by the group of 2-coboundaries $d^1(\lambda)$ such that $\lambda(1)=1$, is isomorphic to the second admissible cohomology group $H^2_{\mathrm{adm}}(G,\Lambda^\times)$, since any Λ-admissible 2-cocycle μ on G factors as $\mu = c\mu'$, where μ' is a Λ-admissible multiplier on G and $c = \mu(1,1)$ is a unit of Λ, and we have $c = d^1(c)$.

III.2.13. Let (G,μ) be a Λ-twisted topological group (cf. III.2.11). A Λ-*admissible representation of* (G,μ) is a pair (V,ρ), where V is a free Λ-module of finite rank, and $\rho: G \to \mathrm{Aut}_\Lambda(V)$ is a Λ-admissible map (cf. III.2.5) such that

$$\rho(x)\rho(y) = \mu(x,y)\rho(xy),$$

for all x,y in G. Since $\mu(1,1)=1$, this relation implies $\rho(1) = \mathrm{id}_V$.

If (V,ρ) and (V',ρ') are both Λ-admissible representations of (G,μ), we define a morphism from (V,ρ) to (V',ρ') to be a homomorphism $f : V \to V'$ of Λ-modules such that $f \circ \rho(x) = \rho'(x) \circ f$ for all x in G.

We will denote by $\mathrm{Rep}_\Lambda(G,\mu)$ the category of Λ-admissible representations of (G,μ).

Remark III.2.14. If (V,ρ) is a Λ-admissible representation of (G,μ), then the composition of ρ with the projection $\mathrm{Aut}_\Lambda(V) \to \mathrm{Aut}_\Lambda(V)/\Lambda^\times$ to the projective general linear group of V is a genuine group homomorphism, thereby defining a projective representation of G. Moreover, any Λ-admissible projective representation of a discrete group G is obtained in this way from a Λ-admissible representation of (G,μ), for some multiplier μ. However, the category $\mathrm{Rep}_\Lambda(G,\mu)$ is additive, unlike the category of projective representations of G.

Remark III.2.15. If $\lambda : G \to \Lambda^\times$ is a Λ-admissible map with $\lambda(1) = 1$, then the functor $(V,\rho) \mapsto (V,\lambda\rho)$ is an equivalence of categories from $\mathrm{Rep}_\Lambda(G,\mu)$ to $\mathrm{Rep}_\Lambda(G,\mu d^1(\lambda))$ (cf. III.2.12). These categories are thus equivalent, although non canonically, since the isomorphism just constructed depends on λ. In particular, $\mathrm{Rep}_\Lambda(G,\mu)$ depends only on the cohomology class of μ, up to non unique equivalence.

Proposition III.2.16. *Let H be an open subgroup of finite index in a Λ-twisted topological group (G,μ). Let V be a free Λ-module of finite rank, and let $\rho : G \to \mathrm{Aut}_\Lambda(V)$ be a map such that $\rho(x)\rho(y) = \mu(x,y)\rho(xy)$ for all x,y in G. If $(V,\rho_{|H})$ is a Λ-admissible representation of $(H,\mu_{|H})$, then (V,ρ) is a Λ-admissible representation of (G,μ).*

Indeed, there exists an admissible ℓ-adic subring $\Lambda_0 \subseteq \Lambda$ and a Λ_0-form V_0 of V such that $\rho_{|H}$ factors through $\mathrm{Aut}_{\Lambda_0}(V_0)$ and such that the induced map $\rho_{|H} : H \to \mathrm{Aut}_{\Lambda_0}(V_0)$ is continuous. Let $(g_i)_{i\in I}$ be a finite family of left H-cosets representatives. Up to replacing Λ_0 with a larger admissible ℓ-adic subring of Λ, we can assume (and we do) that each $\rho(g_i)$ belongs to $\mathrm{Aut}_{\Lambda_0}(V_0)$, and that μ takes its values in $\Lambda_0^\times$. The restriction of ρ to the open subset Hg_i is then given by the formula

$$\rho(hg_i) = \mu(h,g_i)^{-1}\rho(h)\rho(g_i).$$

Thus the restriction $\rho_{|Hg_i}$ take its values in $\mathrm{Aut}_{\Lambda_0}(V_0)$ and is continuous. Therefore ρ takes its values in $\mathrm{Aut}_{\Lambda_0}(V_0)$ and is continuous, and consequently (V,ρ) is a Λ-admissible representation of (G,μ).

III.2.17. Non zero Λ-admissible representations of (G,μ) may not exist for every μ. Indeed, if Λ is an algebraically closed field, then the cohomology class associated to μ (cf. III.2.12) must have finite order for such a representation to exist, by the following proposition.

Proposition III.2.18. *Assume that the ℓ-adic coefficient ring Λ is an algebraically closed field. If a Λ-twisted topological group (G,μ) admits a Λ-admissible representation of rank $r \geq 1$, then there exists a Λ-admissible map $\lambda : G \to \Lambda^\times$ (cf. III.2.5) such that $\lambda(1) = 1$, and such that $\mu d^1(\lambda)$ (cf. III.2.12) is unitary (cf. III.2.10), with values in the group of r-th roots of unity in $\Lambda^\times$.*

Indeed, let $\Lambda_0 \subset \Lambda$ be an admissible ℓ-adic subring of Λ such that μ takes its values in $\Lambda_0^\times$, and such that there exists a Λ_0-admissible representation (V,ρ) of (G,μ), of rank $r \geq 1$. We have

$$\det(\rho(g))\det(\rho(h)) = \mu(g,h)^r \det(\rho(gh)),$$

for all g,h in G. Thus, if $\lambda : G \to \Lambda^\times$ is a Λ-admissible map such that $\lambda(1) = 1$ and $\lambda^r = \det\circ\rho$, then the multiplier $\mu d^1(\lambda)$ (cf. III.2.6) is unitary, with values in r-th roots of unity. It remains to show the existence of such a continuous map λ.

If Λ_0 is finite, then there exists a map f from $\Lambda_0^\times$ to $\Lambda^\times$ such that $f(w)^r = w$ for any w in $\Lambda_0^\times$. We can then choose $\lambda = f \circ \det\circ\rho$.

Otherwise, we can assume that Λ_0 is a finite extension E of $\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}_E$. The subgroup $1 + r\ell^2\mathcal{O}_E$ of $E^\times$ is open, and the map

$$1 + r\ell^2\mathcal{O}_E \to 1 + \ell\mathcal{O}_E$$
$$1 + x \to (1+x)^{\frac{1}{r}} = \sum_{n\geq 0} \frac{r^{-1}(r^{-1}-1)\cdots(r^{-1}-n+1)}{n!} x^n,$$

is continuous. Let H be an open subgroup of G such that $\det\circ\rho(H)$ is contained in $1 + r\ell^2\mathcal{O}_E$. Let $(g_c)_{c\in G/H}$ be a set of representatives for the right cosets of H in G, such that $g_1 = 1$, and let $(\lambda_c)_{c\in G/H}$ be elements of Λ such that $\lambda_c^r = \det(\rho(g_c))$ and $\lambda_1 = 1$. Then, setting $\lambda(hg_c) = \det(\rho(h))^{\frac{1}{r}}\mu(h, g_c)^{-1}\lambda_c$ for h and c in H and G/H respectively yields a Λ-admissible map from G to $\Lambda^\times$ such that $\lambda(1) = 1$ and $\lambda^r = \det\circ\rho$, hence the result.

III.2.19. Let $f : (G, \mu) \to (G', \mu')$ be a morphism of Λ-twisted topological groups (cf. III.2.11), and let (V, ρ) be a Λ-admissible representation of (G', μ') (cf. III.2.13). Then $(V, \rho f)$ is a Λ-admissible representation of (G, μ), which we denote by f^*V. This yields a functor f^* from $\mathrm{Rep}_\Lambda(G', \mu')$ to $\mathrm{Rep}_\Lambda(G, \mu)$.

III.2.20. Let (V_1, ρ_1) and (V_2, ρ_2) be Λ-admissible representations of Λ-twisted topological groups (G_1, μ_1) and (G_2, μ_2) respectively. Then the formula

$$(\mu_1 \otimes \mu_2)((x_1, x_2), (y_1, y_2)) = \mu_1(x_1, y_1)\mu_2(x_2, y_2)$$

for x_1, y_1 in G_1 and x_2, y_2 in G_2, defines a multiplier on the topological group $G_1 \times G_2$. The free Λ-module of finite rank $V_1 \otimes_\Lambda V_2$ is then endowed with a structure of continuous linear representation of $(G_1 \times G_2, \mu_1 \otimes \mu_2)$ over Λ, by defining $\rho(x_1, x_2) = \rho_1(x_1) \otimes \rho_2(x_2)$ for (x_1, x_2) in $G_1 \times G_2$.

If $G_1 = G_2$, then the diagonal morphism $G \to G \times G$ is a morphism of Λ-twisted topological groups from $(G, \mu_1\mu_2)$ to $(G_1 \times G_2, \mu_1 \otimes \mu_2)$. The restriction of the Λ-admissible representation $V_1 \otimes_\Lambda V_2$ of $(G_1 \times G_2, \mu_1 \otimes \mu_2)$ through this diagonal morphism then defines a Λ-admissible representation of $(G, \mu_1\mu_2)$, still denoted by $V_1 \otimes_\Lambda V_2$.

III.2.21. Let (G, μ) be a Λ-twisted group (cf. III.2.11). The **twisted group algebra** $\Lambda[G, \mu]$ of (G, μ) over Λ is given by a free Λ-module with basis $([x])_{x\in G}$, endowed with the Λ-bilinear product defined by

$$[x][y] = \mu(x, y)[xy],$$

for all x, y in G. The cocycle relation 17 is equivalent to the associativity of this product. Moreover, recall from III.2.9 that $\mu(x, 1) = \mu(1, z) = 1$ for all x, z in G, hence $[1]$ is a (left and right) neutral element, and thus $\Lambda[G, \mu]$ is a unital associative Λ-algebra.

PROPOSITION III.2.22. *Let (G, μ) be a finite discrete Λ-twisted group. The functor which sends a left $\Lambda[G, \mu]$-module V which is free of finite rank over Λ to the Λ-admissible representation of (G, μ) on V defined by the formula $\rho(x)(v) = [x]v$ for x in G and v in V is a Λ-linear equivalence of categories, from the category of left $\Lambda[G, \mu]$-modules which are free of finite rank over Λ to the category Λ-admissible representations of G over Λ (cf.* III.2.13*).*

In particular, if G is finite, then $\Lambda[G, \mu]$ is itself a non zero Λ-admissible representation of (G, μ). By combining this obervation with Proposition III.2.18, we recover the well-known result that any multiplier on G is cohomologous to a unitary multiplier with values in $|G|$-th roots of unity, and thus that the abelian group $H^2_{\mathrm{adm}}(G, \Lambda^\times) = H^2(G, \Lambda^\times)$ (cf. III.2.8) is finite with exponent dividing $|G|$ whenever Λ is an algebraically closed field. This observation admits the following reformulation in terms of central extensions:

Proposition III.2.23. *Assume that the ℓ-adic coefficient ring Λ is an algebraically closed field. Let G^* be a central extension of a finite group G by $\Lambda^\times$. Then there exists a section $\sigma : G \to G^*$ with $\sigma(1) = 1$ such that the 2-cocycle*

$$\begin{aligned}\mu : G^2 &\to \Lambda^\times\\ (x, y) &\to \sigma(x)\sigma(y)\sigma(xy)^{-1},\end{aligned}$$

is unitary.

Indeed, if $\sigma_0 : G \to G^*$ is an arbitrary section such that $\sigma_0(1) = 1$, with associated 2-cocycle μ_0, then we observed that there exists a map $\lambda : G \to \Lambda^\times$ such that $\lambda(1) = 1$ and such that $\mu_0 d^1(\lambda)$ is unitary. Since $\mu_0 d^1(\lambda)$ is the 2-cocycle associated to the section $\lambda\sigma_0 : G \to G^*$, we can take $\sigma = \lambda\sigma_0$ in Proposition III.2.23.

III.2.24. Let H be an open subgroup of finite index in a Λ-twisted topological group (G, μ). Then the restriction $\mu_{|H}$ of μ to $H \times H$ endows H with a structure of Λ-twisted topological group, such that the inclusion $\iota : H \to G$ is a morphism of twisted topological groups. The functor ι^* (cf. III.2.19) from $\mathrm{Rep}_\Lambda(G, \mu)$ to $\mathrm{Rep}_\Lambda(H, \mu_{|H})$ admits a left adjoint Ind_H^G, given by

$$\mathrm{Ind}_H^G(V) = \Lambda[G, \mu] \otimes_{\Lambda[H, \mu_{|H}]} V,$$

cf. III.2.22. In order to verify that $\mathrm{Ind}_H^G(V)$ is indeed a Λ-admissible representation of (G, μ), it is sufficient by Proposition III.2.16 to check that the restriction of $\mathrm{Ind}_H^G(V)$ to the finite index open subgroup $K = \cap_{g \in G/H} gHg^{-1}$ is a Λ-admissible representation of $(K, \mu_{|K})$. However, if $(g_c)_{c \in G/H}$ are left H-cosets representatives, then we have a decomposition

$$\Lambda[G, \mu] = \bigoplus_{c \in G/H} [g_c]\Lambda[H, \mu_{|H}],$$

as a right $\Lambda[H, \mu_{|H}]$-module, which yields in turn a decomposition

$$\mathrm{Ind}_H^G(V) = \bigoplus_{c \in G/H} [g_c]V,$$

where $[g_c]V$ is a Λ-admissible representation of $(K, \mu_{|K})$, since the action of K on this Λ-module is given by

$$\begin{aligned}[k][g_c]v &= \mu(k, g_c)[kg_c]v\\ &= \mu(k, g_c)\mu(g_c, g_c^{-1}kg_c)^{-1}[g_c][g_c^{-1}kg_c]v,\end{aligned}$$

for k in K and v in V, so that $[g_c]V$ is isomorphic to the Λ-module V, endowed with the Λ-admissible map $\rho_c : K \to \mathrm{Aut}_\Lambda(V)$ given by $\rho_c(k) = \mu(k, g_c)\mu(g_c, g_c^{-1}kg_c)^{-1}\rho(g_c^{-1}kg_c)$.

III.2.25. Let (G, μ) be a Λ-twisted topological group (cf. III.2.11), and let Z be a subgroup of $\Lambda^\times$ which contains the image of μ, and which is contained in an admissible ℓ-adic subring of Λ. Let us consider the central extension

$$1 \to Z \xrightarrow{\iota} G^* \xrightarrow{\pi} G \to 1,$$

associated to the 2-cocycle μ. The underlying topological space of G^* is the product $Z \times G$, the group law is given by

$$(\lambda_1, g_1) \cdot (\lambda_2, g_2) = (\lambda_1\lambda_2\mu(g_1, g_2), g_1g_2),$$

and we have $\iota(\lambda) = (\lambda, 1)$ and $\pi(\lambda, g) = g$ for (λ, g) in $Z \times G$. The continuous map π admits a distinguished continuous section, namely $\sigma : g \mapsto (1, g)$, which is a group homomorphism if and only if μ is trivial, i.e. $\mu = 1$.

If (V, ρ) is a Λ-admissible representation of (G, μ) (cf. III.2.13), then the continuous map

$$\rho^* : G^* \to \mathrm{Aut}_\Lambda(V),$$
$$(\lambda, g) \to \lambda\rho(g),$$

is a group homomorphism, hence (V, ρ^*) is a Λ-admissible representation of the topological group G^*. If $\zeta : Z \to \Lambda^\times$ is the character of Z given by the inclusion, then the restriction of (V, ρ^*) to Z is ζ-isotypical, i.e. $\rho^* \circ \iota(\lambda)(v) = \zeta(\lambda)v$ for all (λ, v) in $Z \times V$. Conversely, any Λ-admissible representation of G^* with ζ-isotypical restriction to Z yields a Λ-admissible representation of (G, μ) by composition with the section σ, and these two constructions are quasi-inverse to each other. We have obtained:

PROPOSITION III.2.26. *The functor $(V, \rho) \mapsto (V, \rho^*)$ is an equivalence of categories from $\mathrm{Rep}_\Lambda(G, \mu)$ to the category of Λ-admissible representations of $(G^*, 1)$ whose restriction to Z is ζ-isotypical.*

If H is a subgroup of G, endowed with the restriction of μ to H, then the corresponding group H^* is the inverse image of H by π. Let $\iota : H \to G$ be the inclusion, which is a morphism of Λ-twisted topological groups. Under the equivalence of Proposition III.2.26, the functor ι^* (cf. III.2.19) corresponds to the restriction functor from representations of G^* to representations of H^*. By taking left adjoints when available (cf. III.2.24), we obtain:

PROPOSITION III.2.27. *Let H be an open subgroup of finite index in G, with inverse image H^* in G^*. Under the equivalence of Proposition III.2.26, the functor Ind_H^G corresponds to $\mathrm{Ind}_{H^*}^{G^*}$.*

III.2.28. Let us recall that if (V, ρ) is a Λ-admissible representation of a Λ-twisted topological group (G, μ), then the composition

$$G \xrightarrow{\rho} \mathrm{Aut}_\Lambda(V) \to \mathrm{Aut}_\Lambda(V)/\Lambda^\times,$$

is a genuine group homomorphism. In particular, its image is a subgroup of $\mathrm{Aut}_\Lambda(V)/\Lambda^\times$.

DEFINITION III.2.29. A Λ-admissible representation (V, ρ) of a twisted topological group (G, μ) is said to have *finite projective image* if the composition of ρ with the projection from $\mathrm{Aut}_\Lambda(V)$ to $\mathrm{Aut}_\Lambda(V)/\Lambda^\times$ has finite image. We denote by $\mathrm{Rep}_\Lambda^{\mathrm{fin}}(G, \mu)$ the full subcategory of $\mathrm{Rep}_\Lambda(G, \mu)$ whose objects are the Λ-admissible representations of (G, μ) with finite projective image.

Under the equivalence of Proposition III.2.26, the subcategory $\mathrm{Rep}_\Lambda^{\mathrm{fin}}(G, \mu)$ of $\mathrm{Rep}_\Lambda(G, \mu)$ is equivalent to the category of continuous linear representations of G^* with finite projective image and with ζ-isotypical restriction to Z (with notation from III.2.26).

III.2.30. Let (V, ρ) be an object of $\mathrm{Rep}_\Lambda^{\mathrm{fin}}(G, \mu)$, where Λ is an ℓ-adic coefficient ring *in which ℓ is invertible*. Then the Λ-module $\mathrm{End}(V)$ is a Λ-admissible representation of $(G, 1)$ under the action given by $g \cdot u = \rho(g) \circ u \circ \rho(g)^{-1}$. Moreover, this action factors through the image G' of G in $\mathrm{Aut}_\Lambda(V)/\Lambda^\times$, which is a finite group. Thus we can form the projector

$$P : \mathrm{End}(V) \to \mathrm{End}(V)$$
$$u \mapsto \frac{1}{|G'|} \sum_{g \in G'} g \cdot u,$$

whose image is the space of endomorphisms of (V, ρ). If an element π of $\mathrm{End}(V)$ is a projector onto a G-stable subspace W, then $g \cdot \pi$ is a projector onto W as well for each g in G', hence $P(\pi)$ is a projector onto W which commutes with the action of G. Thus W is a direct summand of V in the additive category $\mathrm{Rep}_\Lambda(G, \mu)$. We thus obtain the following extension of Maschke's theorem:

Proposition III.2.31. *Let (G, μ) be a Λ-twisted topological group, where Λ is an ℓ-adic coefficient ring in which ℓ is invertible. Then any object of $\mathrm{Rep}_\Lambda^{\mathrm{fin}}(G, \mu)$ is semisimple.*

In particular, the indecomposable objects of $\mathrm{Rep}_\Lambda^{\mathrm{fin}}(G, \mu)$ are irreducible. Let us introduce the Grothendieck group of the category $\mathrm{Rep}_\Lambda^{\mathrm{fin}}(G, \mu)$:

Definition III.2.32. Let (G, μ) be a Λ-twisted topological group. The Grothendieck group $K_0^{\mathrm{fin}}(G, \mu, \Lambda)$ is the quotient of the free abelian group with basis $([V])_V$ indexed by all Λ-admissible representations of (G, μ) with finite projective image and whose underlying Λ-module is Λ^n for some integer n, by the relations

$$[V'] + [V''] - [V],$$

whenever V is an extension of V' by V'' in $\mathrm{Rep}_\Lambda^{\mathrm{fin}}(G, \mu)$. If G is finite, the group $K_0^{\mathrm{fin}}(G, \mu, \Lambda)$ is simply denoted by $K_0(G, \mu, \Lambda)$.

Remark III.2.33. The class of Λ-admissible representations of (G, μ) may not form a set, hence the unnatural restriction to Λ-modules which *are* Λ^n for some n, rather than being merely isomorphic to Λ^n for some n.

If V is a Λ-admissible representation of (G, μ) with finite projective image, then V is isomorphic to a representation V' whose underlying Λ-module is Λ^n for some integer n, and the class of $[V']$ in $K_0^{\mathrm{fin}}(G, \mu, \Lambda)$ depends only on V. This class will be simply denoted by $[V]$. By Corollary III.2.31, if ℓ is invertible in Λ then the group $K_0^{\mathrm{fin}}(G, \mu, \Lambda)$ is a free abelian group, with basis given by the classes $[V]$ where V is an (isomorphism class of) irreducible Λ-admissible representation of (G, μ) with finite projective image.

Before proceeding further, let us recall Brauer's induction theorem for finite groups:

Theorem III.2.34 ([**Se98**], 10.5 Th. 20). *If G is a finite group, then the abelian group $K_0(G, 1, C)$ is generated by the classes of representations of the form $\mathrm{Ind}_H^G V$, where H is a subgroup of G and V is a one-dimensional C-linear representation of H.*

Brauer's induction theorem III.2.34 is a consequence of the following two results:

(1) If G is a finite group, then $K_0(G, 1, C)$ is generated by the classes of representations of the form $\mathrm{Ind}_H^G V$, where H is a nilpotent subgroup of G, and where V is an irreducible representation of H.
(2) Any irreducible C-linear representation of a finite nilpotent group G is isomorphic to $\mathrm{Ind}_H^G V$ for some subgroup H and some one-dimensional representation V of H.

Moreover, it is sufficient for the first of these results to prove that the class of the trivial representation of G is a linear combination in $K_0(G, 1)$ of representations of the form $\mathrm{Ind}_H^G V$, where H is a nilpotent subgroup of G. We refer to ([**Se98**], 10) for proofs of these results, and for a more complete discussion of Brauer's theorem. We will also need the following variant of Brauer's theorem:

Theorem III.2.35 ([**De73**], Prop. 1.5). *If G is a finite group, then the abelian group $K_0(G, 1, C)$ is generated by the class $[C]$ of the trivial representation of G and by the classes of the form $[\mathrm{Ind}_H^G V] - [\mathrm{Ind}_H^G C]$, where H is a subgroup of G and V is a one-dimensional C-linear representation of H.*

We will need an extension of Brauer's theorem to continous representations of twisted groups with finite projective image:

Theorem III.2.36. *Let (G, μ) be a C-twisted topological group. Then the group $K_0^{\mathrm{fin}}(G, \mu, C)$ (cf. III.2.32) is generated by the classes of C-admissible representations of the form $\mathrm{Ind}_H^G V$, where H is an open subgroup of finite index in G and V is a one-dimensional C-admissible representation of $(H, \mu_{|H})$.*

Indeed, let (V,ρ) be an object of $\mathrm{Rep}_C^{\mathrm{fin}}(G,\mu)$, and let us consider the subgroup $G' = C^\times\rho(G)$ of $\mathrm{Aut}_C(V)$ generated by the image of ρ and by homotheties. The topological group G' is a central extension of the finite group $G'/C^\times$ by $C^\times$. By Proposition III.2.23, there exists a set-theoretic section σ from $G'/C^\times$ to G' such that $\sigma(1)=1$, whose associated 2-cocycle is unitary. There exists a unique continuous map $\lambda : G \to C^\times$ such that

$$\sigma(\rho(g)C^\times) = \lambda(g)\rho(g),$$

for all g in G, and we have $\lambda(1)=1$. By replacing (V,ρ) and (G,μ) by $(V,\lambda\rho)$ and $(G,d(\lambda)\mu)$ respectively (cf. III.2.15), we can therefore assume (and we do) that μ is unitary and that the set $\rho(G)$ is finite.

Since μ is now assumed to be unitary, we can choose Z to be a finite subroup of $C^\times$ in III.2.25. Moreover, the representation (V,ρ^*) of G^* corresponding to (V,ρ) by Proposition III.2.26 has finite image, namely $Z\rho(G)$. By applying Brauer's induction theorem III.2.34 to the finite group $\rho^*(G^*)$, we obtain a decomposition

$$[V] = \sum_{i\in I} n_i[\mathrm{Ind}_{H_i}^{G^*}\chi_i], \tag{18}$$

in $K_0^{\mathrm{fin}}(G^*,1)$, where n_i is a (possibly negative) integer, H_i is an open subgroup of finite index in G^* and χ_i is a continuous character of H_i with finite image. Since Z is central in G^*, the decomposition of a representation into the isotypical components of its restriction to Z yields a splitting

$$K_0^{\mathrm{fin}}(G^*,1) = \bigoplus_{\zeta' : Z\to C^\times} K_0^{\mathrm{fin}}(G^*,1)[\zeta'],$$

where ζ' runs through the set of characters of Z, and $K_0^{\mathrm{fin}}(G^*,1)[\zeta']$ is generated by continuous linear representations of G^* with finite projective image and ζ'-isotypical restriction to Z. By projecting the relation 18 onto the factor corresponding to ζ, we obtain

$$[V] = \sum_{i\in I} n_i[\mathrm{Ind}_{H_i}^{G^*}\chi_i[\zeta]],$$

in the abelian group $K_0^{\mathrm{fin}}(G^*,1)[\zeta]$, where we denoted by $[\zeta]$ the ζ-isotypical component.

For each i in I, we can identify the representation $\mathrm{Ind}_{H_i}^{G^*}\chi_i$ with the C-vector space

$$\{f : G^* \to C \mid \forall h\in H_i, g\in G^*, f(hg) = \chi_i(h)f(g)\},$$

endowed with the action $(g\cdot f)(x) = f(xg)$ for g,x in G^*. The ζ-isotypical component is then the subspace of functions $f : G^*\to C$ such that $f(gh) = \zeta(h)f(g)$ for all h in Z and all g in G^*. Thus the ζ-isotypical component vanishes whenever χ_i and ζ do not coincide on $H_i\cap Z$. On the other hand, if χ_i and ζ do coincide on $H_i\cap Z$, then there exists a unique character ψ_i of ZH_i such that $\psi_{i|Z} = \zeta$ and $\psi_{i|H_i} = \chi_i$, in which case

$$\begin{aligned}\mathrm{Ind}_{H_i}^{G^*}\chi_i[\zeta] &= \{f : G^*\to C \mid \forall h\in ZH_i, g\in G^*, f(hg) = \psi_i(h)f(g)\},\\ &= \mathrm{Ind}_{ZH_i}^{G^*}\psi_i.\end{aligned}$$

If G_i is the image in G of ZH_i, then G_i is an open subgroup of finite index in G, and ψ_i yields a continuous one-dimensional linear representation of $(G_i,\mu_{|G_i})$, such that the relation

$$V = \sum_{\substack{i\in I\\ \chi_{i|H_i\cap Z} = \zeta_{|H_i\cap Z}}} n_i[\mathrm{Ind}_{G_i}^{G}\psi_i]$$

holds in $K_0^{\mathrm{fin}}(G,\mu)$ (cf. III.2.26, III.2.27), hence the conclusion of Theorem III.2.36.

III.2.37. Let (G,μ) be a C-twisted topological group (cf. III.2.11), and let us consider an extension

$$1 \to I \xrightarrow{\iota} Q \xrightarrow{\pi} G \to 1,$$

of topological groups, where I is a finite discrete group. It is assumed that G carries the quotient topology from Q. Then the formula $(\pi^*\mu)(x,y) = \mu(\pi(x),\pi(y))$ for x,y in Q defines a multiplier on Q, so that $(Q,\pi^*\mu)$ is a C-twisted topological group, and π is a morphism of C-twisted topological groups (cf. III.2.11).

PROPOSITION III.2.38. *If a C-admissible representation of $(Q,\pi^*\mu)$ over C is irreducible as a representation of I, then it has finite projective image (cf. III.2.29).*

Indeed, if (V,ρ) is such a representation, then Schur's lemma implies that any element of the centralizer $C_Q(I)$ of I in Q must act on V as a homothety. The subgroup $C_Q(I)$ is the kernel of the homomorphism $\varphi : Q \to \mathrm{Aut}(I)$ given by the action of Q on I by conjugation, i.e. sending an element q of Q to $(t \mapsto qtq^{-1})$, and must therefore have finite index in Q, since $\mathrm{Aut}(I)$ is a finite group. The projective image of ρ is a quotient of the finite group $Q/C_Q(I)$, and is thus finite as well.

PROPOSITION III.2.39 (cp. [**CR62**], 51.7). *Let (G,μ) be a C-twisted topological group, and let*

$$1 \to I \xrightarrow{\iota} Q \xrightarrow{\pi} G \to 1,$$

an extension of topological groups,where G (resp. I) carries the quotient topology (resp. induced topology) from Q. Assume that the continuous map π has a section, and let (V,ρ) be a C-admissible representation of $(Q,\pi^*\mu)$ such that $\rho(I)$ is a finite group. Then there exists a finite family $(G_j)_{j\in J}$ of open subgroups of finite index in G, and for each j, a C-admissible multiplier ν_j on G_j, a C-admissible representation W_j of $(G_j,\mu\nu_j^{-1})$, and a C-admissible representation E_j of $(Q_j,\pi^*\nu_j)$, where $Q_j=\pi^{-1}(G_j)$, such that:

(1) for each j in J, the restriction of E_j to I is irreducible and the action of I on E_j factors through $\rho(I)$;

(2) the C-admissible representation (V,ρ) of Q is isomorphic to the direct sum (cf. III.2.19, III.2.20, III.2.24)

$$\bigoplus_{j\in J} \mathrm{Ind}_{Q_j}^{Q}(E_j \otimes_C \pi^* W_j).$$

REMARK III.2.40. The assumption of Proposition III.2.39 is fulfilled whenever Q is discrete, or whenever it is a profinite group. Indeed, any epimorphism of profinite groups admits a section in the category of topological spaces, cf. ([**Se94**], I.1.2, Prop. 1).

REMARK III.2.41. Even if we start with a genuine representation of Q, namely if $\mu=1$, then the cocycles ν_j appearing in Proposition III.2.39 are usually not trivial, nor cohomologically trivial.

REMARK III.2.42. If μ is unitary, then the multipliers ν_j which appear in Proposition III.2.39 can be taken to be unitary as well. Indeed, one can assume that each W_j is non zero, and thus by Proposition III.2.18, there exists for each j a C-admissible map $\lambda_j : G_j \to C^\times$ such that $\lambda_j(1)=1$ and such that $\nu_j d(\lambda_j)$ is unitary. Twisting E_j and W_j by $\lambda_j \circ \pi$ and λ_j^{-1} respectively (cf. III.2.15) then leaves the tensor product $E_j \otimes_C \pi^* W_j$ unaltered.

Let us prove Proposition III.2.39. Let (V,ρ) be a C-admissible representation of $(G,\pi^*\mu)$ such that $\rho(I)$ is a finite group. The subgroup $K = I \cap \ker(\rho)$ of Q is normal, and is open of finite index in I. By replacing (Q,I) with $(Q/K, I/K)$, we can assume (and we do) that I is finite. Since the restriction of $\pi^*\mu$ to I is trivial, the restriction ι^*V is a honest linear representation of the finite group I. For each irreducible character χ of I, let $V[\chi]$ be the

χ-isotypical component of $(V, \rho\iota)$. Let X be the set of irreducible characters χ such that $V[\chi]$ is non zero, so that we have a decomposition

$$V = \bigoplus_{\chi \in X} V[\chi].$$

Recall that we denoted by $\varphi : Q \to \mathrm{Aut}(I)$ the action of Q on I by automorphisms given by conjugation by an element of Q, i.e. sending an element q of Q to $(t \mapsto qtq^{-1})$. The homomorphism φ is continuous when $\mathrm{Aut}(I)$ is endowed with the discrete topology, since I is a finite discrete subgroup of Q. The group Q acts continuously on the left on X by $q \cdot \chi = \chi \circ \varphi(q)^{-1}$, and $\rho(q)$ sends $V[\chi]$ onto $V[q \cdot \chi]$. Since χ is a central function on I, this action factors through π, so that we obtain a continuous action of G on X. Let J be the set of G-orbits in X and for each j in J, let χ_j be a member of the orbit j. Let G_j be the stabilizer of χ_j in G, and let Q_j be its inverse image in Q. Then the action of Q on V restricts to an action of Q_j on $V[\chi_j]$, and the homomorphism

$$\begin{aligned} \mathrm{Ind}_{Q_j}^{Q}(V[\chi_j]) &\to \bigoplus_{q \in Q/Q_j} V[q\chi_j] = \bigoplus_{\chi \in j} V[\chi] \\ [q] \otimes v &\mapsto \rho(q)v \end{aligned}$$

is an isomorphism. Thus V is isomorphic to the direct sum

$$\bigoplus_{j \in J} \mathrm{Ind}_{Q_j}^{Q}(V[\chi_j]).$$

Let (E_j, ρ_j) be an irreducible representation of I with character χ_j, and let

$$W_j = \mathrm{Hom}_{C[I]}(E_j, V) = \mathrm{Hom}_{C[I]}(E_j, V[\chi_j]),$$

considered as a trivial representation of I, so that the C-linear homomorphism

$$\begin{aligned} E_j \otimes_C W_j &\to V[\chi_j] \\ e \otimes w &\mapsto w(e), \end{aligned} \tag{19}$$

is an isomorphism of representations of I.

For each ψ in $\mathrm{Aut}(I)$ such that $\chi_j \circ \psi^{-1} = \chi_j$, the representations (E_j, ρ_j) and $(E_j, \rho_j \circ \psi^{-1})$ are isomorphic, hence there exists a C-linear automorphism $\lambda_j(\psi)$ of E_j such that

$$\rho_j \circ \psi^{-1}(t) = \lambda_j(\psi)^{-1} \rho_j(t) \lambda_j(\psi),$$

for all t in I. We can take $\lambda_j(\mathrm{id}_I) = \mathrm{id}_{E_j}$. Note that Schur's lemma implies that $\lambda_j(\psi)$ is uniquely determined up to multiplication by an invertible scalar, and that if we denote by $\mathrm{Aut}(I)_{\chi_j}$ the subgroup of $\mathrm{Aut}(I)$ formed by the elements ψ of $\mathrm{Aut}(I)$ such that $\chi_j \circ \psi^{-1} = \chi_j$, then the composition

$$\mathrm{Aut}(I)_{\chi_j} \xrightarrow{\lambda_j} \mathrm{Aut}_C(E_j) \to \mathrm{Aut}_C(E_j)/C^{\times},$$

is a group homomorphism.

Let σ be a continuous section of the continuous map π, such that $\sigma(1) = 1$. For $q = \sigma(g)t$ in Q_j, with t in I and g in G_j (so that $g = \pi(q)$), we set

$$\widetilde{\rho}_j(q) = (\lambda_j \circ \varphi \circ \sigma)(g) \rho_j(t).$$

For t in I, the automorphism $\lambda_j(\varphi(t))$ differs from $\rho_j(t)$ only by an invertible scalar, hence for each q in Q_j, the automorphism $\widetilde{\rho}_j(q)$ differs from $\lambda_j \circ \varphi(q)$ by an invertible scalar.

Lemma III.2.43. *For each j in J, there exists a unique multiplier ν_j on G_j such that $(E_j, \widetilde{\rho}_j)$ is a continuous linear representation of $(Q_j, \pi^* \nu_j)$.*

Assuming the conclusion of Lemma III.2.43, we can conclude the proof of Proposition III.2.39 as follows. We first notice that the continuous map

$$\begin{aligned} Q_j &\to \mathrm{Aut}_C(W_j) \\ q &\mapsto (w \mapsto \rho(q) \circ w \circ \widetilde{\rho}_j(q)^{-1}). \end{aligned}$$

is right I-invariant, hence uniquely factors as $\tau_j \circ \pi_{|Q_j}$, where τ_j is a continuous map from G_j to $\mathrm{Aut}_C(W_j)$. One then checks that (W_j, τ_j) is a C-admissible representation of $(G_j, \mu\nu_j^{-1})$, so that (19) is an isomorphism of C-admissible representations of $(Q_j, \pi^*\mu)$. Thus the decomposition

$$V \simeq \bigoplus_{j \in J} \mathrm{Ind}_{Q_j}^{Q}(E_j \otimes_C \pi^* W_j).$$

provides the wanted conclusion.

III.2.44. Let us now prove Lemma III.2.43. The map λ_j factors through the continuous homomorphism from Q_j to the stabilizer of χ_j in $\mathrm{Aut}(I)$, which is a finite discrete group, hence λ_j is continuous, and so is $\widetilde{\rho}_j$. Let us write

$$c(g_1, g_2) = \sigma(g_2)^{-1}\sigma(g_1)^{-1}\sigma(g_1 g_2) \in I.$$

If $q_1 = \sigma(g_1)t_1$ and $q_2 = \sigma(g_2)t_2$ are elements of G_j, with t_1, t_2 in I, then we have

$$q_1 q_2 = \sigma(g_1 g_2)(c(g_1, g_2)^{-1}(\sigma(g_2)^{-1} t_1 \sigma(g_2))t_2),$$

so that we have

$$\begin{aligned} \widetilde{\rho}_j(q_1)\widetilde{\rho}_j(q_2) &= (\lambda_j \circ \varphi \circ \sigma)(g_1)\rho_j(t_1)(\lambda_j \circ \varphi \circ \sigma)(g_2)\rho_j(t_2) \\ &= (\lambda_j \circ \varphi \circ \sigma)(g_1)(\lambda_j \circ \varphi \circ \sigma)(g_2)\rho_j(\sigma(g_2)^{-1}t_1\sigma(g_2)t_2) \\ &= (\lambda_j \circ \varphi \circ \sigma)(g_1)(\lambda_j \circ \varphi \circ \sigma)(g_2)(\rho_j \circ c)(g_1, g_2)(\lambda_j \circ \varphi \circ \sigma)(g_1 g_2)^{-1}\widetilde{\rho}_j(q_1 q_2). \end{aligned}$$

Let us define

$$\nu_j(g_1, g_2) = (\lambda_j \circ \varphi \circ \sigma)(g_1)(\lambda_j \circ \varphi \circ \sigma)(g_2)(\rho_j \circ c)(g_1, g_2)(\lambda_j \circ \varphi \circ \sigma)(g_1 g_2)^{-1},$$

so that we have

$$\widetilde{\rho}_j(q_1)\widetilde{\rho}_j(q_2) = \nu_j(g_1, g_2)\widetilde{\rho}_j(q_1 q_2). \tag{20}$$

It remains to show that ν_j defines a multiplier on G_j, i.e. that $\nu_j(g_1, g_2)$ takes values in $C^\times$ and that it satisfies the cocycle formula (17). We noticed that for each q in Q_j, the automorphism $\widetilde{\rho}_j(q)$ differs from $\lambda_j \circ \varphi(q)$ by an invertible scalar (cf. the discussion before Lemma III.2.43). Thus, we have $r \circ \widetilde{\rho}_j = r \circ \lambda_j \circ \varphi$, where r is the projection from $\mathrm{Aut}_C(E_j)$ to $\mathrm{Aut}_C(E_j)/C^\times$. In particular, since $r \circ \lambda_j$ is a group homomorphism, so is $r \circ \widetilde{\rho}_j$. This implies that $r \circ \nu_j$ is identically equal to 1. Thus ν_j takes values in $C^\times$, and the formula (20) then implies that ν_j satisfies the cocycle condition (17). Consequently, $(E_j, \widetilde{\rho}_j)$ is a C-admissible representation of $(Q_j, \pi^*\nu_j)$.

III.2.45. Let (G, μ) be a C-twisted profinite topological group (cf. III.2.11), and let us consider an extension

$$1 \to I \xrightarrow{\iota} Q \xrightarrow{\pi} G \to 1,$$

of profinite topological groups, where G (resp. I) carries the quotient topology (resp. induced topology) from Q. Unlike the situation considered in III.2.37 we do not assume here that I is finite.

LEMMA III.2.46. *Let V be a C-admissible representation of $(Q, \pi^*\mu)$. The following properties are equivalent:*

(i) there exists an open subgroup Q' of Q such that $I \cap Q'$ acts unipotently on V,

(ii) there exists a normal open subgroup Q' of Q such that $I \cap Q'$ acts unipotently on V,

(iii) *there exists an open subgroup I' of I such that I' acts unipotently on V,*
(iv) *there exists a normal open subgroup I' of I such that I' acts unipotently on V.*

Indeed, if an open subgroup Q' of Q is such that $I \cap Q'$ acts unipotently on V, then the normal closure $Q'' = \bigcap_{q \in Q/Q'} qQ'q^{-1}$ is open as well in Q, and $I \cap Q''$ acts unipotently on V, hence (i) implies (ii). By replacing (Q, G) by $(I, 1)$, this also proves that (iii) implies (iv). If Q' is an open subgroup (resp. a normal open subgroup) of Q such that $I \cap Q'$ acts unipotently on V, then $I' = I \cap Q'$ is an open subgroup (resp. a normal open subgroup) of I which acts unipotently on V, hence (i) implies (iii) (resp. (ii) implies (iv)). It remains to prove that (iv) implies (i). Let I' be a normal open subgroup of I such that I' acts unipotently on V. Since I carries the induced topology from Q, there exists an open neighbourhood U of the neutral element in Q such that $I \cap U$ is contained in I'. Since Q is profinite, there exists an open subgroup Q' of Q such that Q' is contained in U. Then $I \cap Q'$ is contained in I', hence acts unipotently on V.

Definition III.2.47. A C-admissible representation V of $(Q, \pi^*\mu)$ is *potentially unipotent* if it satisfies the equivalent properties of Lemma III.2.46.

Proposition III.2.48. *If a C-admissible representation V of $(Q, \pi^*\mu)$ is potentially unipotent and irreducible, then I acts on V through a finite quotient.*

Let Q' be a normal open subgroup of Q such that $I \cap Q'$ acts unipotently on V. By a theorem of Kolchin ([**Ko48**], 1. Lemma), the sub-C-vector space $V^{I \cap Q'}$ of V consisting of $I \cap Q'$-invariant vectors in V is non zero. Since $I \cap Q'$ is a normal subgroup of Q, the sub-C-vector space $V^{I \cap Q'}$ is a sub-C-admissible representation of $(Q, \pi^*\mu)$, hence it must coincide with V by irreducibility. Thus $I \cap Q'$ acts trivially on V, hence the conclusion since $I \cap Q'$ has finite index in I.

III.2.49. Let (G, μ) be a C-twisted profinite topological group (cf. III.2.11). As in III.2.45, let us consider an extension

$$1 \to I \xrightarrow{\iota} Q \xrightarrow{\pi} G \to 1,$$

of profinite topological groups, where G (resp. I) carries the quotient topology (resp. induced topology) from Q. Let $K_0^u(Q, G, \mu, C)$ be the Grothendieck group of potentially unipotent C-admissible representations of $(Q, \pi^*\mu)$ (cf. III.2.47). Thus any potentially unipotent C-admissible representation V of $(Q, \pi^*\mu)$ has a well defined class $[V]$ in $K_0^u(Q, G, \mu, C)$, and the latter is generated by such classes with relations $[V] = [V'] + [V'']$ for each short exact sequence

$$0 \to V' \to V \to V'' \to 0,$$

of potentially unipotent C-admissible representations of $(Q, \pi^*\mu)$.

Proposition III.2.50. *Let us consider an extension*

$$1 \to I \xrightarrow{\iota} Q \xrightarrow{\pi} G \to 1,$$

of profinite topological groups, where G (resp. I) carries the quotient topology (resp. induced topology) from Q. Then the abelian group

$$\bigoplus_{\mu} K_0^u(Q, G, \mu, C),$$

where the sum runs over all unitary C-admissible multipliers μ on G, and $K_0^u(Q, G, \mu, C)$ is as in III.2.49, is generated by its subset of elements of the following two types:

(1) the class $[C]$ of the trivial representation in $K_0^u(Q, G, 1, C)$,

(2) *for any unitary C-admissible multiplier μ on G, any open subgroup Q' of Q, any unitary C-admissible multipliers μ_1 and μ_2 on the image G' of Q' in G such that $\mu_1\mu_2 = \mu_{|G'}$, any C-admissible representation V_1 of rank 1 of $(Q', \pi^*\mu_1)$, on which $I \cap Q'$ acts through a finite quotient, and any C-admissible representation V_2 of (G', μ_2), the class*

$$[\mathrm{Ind}_{Q'}^{Q}(V_1 \otimes \pi^* V_2)] - \mathrm{rk}(V_2)[\mathrm{Ind}_{Q'}^{Q} C],$$

cf. III.2.20, III.2.24, *in the sum of $K_0^u(Q, G, \mu, C)$ and $K_0^u(Q, G, 1, C)$.*

Let V be a potentially unipotent C-admissible representation of $(Q, \pi^*\mu)$, for some unitary C-admissible multiplier μ on G. We must prove that the class $[V]$ belongs to the group generated by the classes described in Proposition III.2.50. We can assume (and we do) that V is irreducible. By Proposition III.2.48, the kernel K of the representation $V_{|I}$ of I is an open subgroup of I, which is normal in Q. By replacing (I, Q) by $(I/K, Q/K)$, we can thus assume (and we do as well) that I is finite.

By Proposition III.2.39, and Remarks III.2.40, III.2.42, we can further assume (and we do) that V is of the form $\mathrm{Ind}_{Q'}^{Q}(E \otimes \pi^* W)$, where Q' is an open subgroup of Q, where E is a C-admissible representation of $(Q', \pi^*\mu_1)$, whose restriction to I is irreducible, and W is a C-admissible representation of (G', μ_2), for some unitary C-admissible multipliers μ_1 and μ_2 on the image G' of Q' in G, such that $\mu_1\mu_2 = \mu_{|G'}$.

By Proposition III.2.38 and Theorem III.2.36, we can assume (and we do) that E is of the form $\mathrm{Ind}_{Q''}^{Q'} V_1$, where Q'' is an open subgroup of Q', and where V_1 is a C-admissible representation of rank 1 of $(Q'', \pi^*\mu_1)$. We then have an isomorphism

$$V = \mathrm{Ind}_{Q'}^{Q}(\mathrm{Ind}_{Q''}^{Q'} V_1 \otimes \pi^* W) \cong \mathrm{Ind}_{Q''}^{Q}(V_1 \otimes \pi^* W),$$

hence a decomposition

$$[V] = \Big([\mathrm{Ind}_{Q''}^{Q}(V_1 \otimes \pi^* W)] - \mathrm{rk}(W)[\mathrm{Ind}_{Q''}^{Q} C]\Big) + \mathrm{rk}(W)[\mathrm{Ind}_{Q''}^{Q} C].$$

The first term in this decomposition is of the required type (2), while the last term $[\mathrm{Ind}_{Q''}^{Q} C]$ belongs to the subgroup of $K_0^u(Q, G, 1, C)$ generated by elements of type (1) or of type (2) (with trivial multipliers μ, μ_1, μ_2 and with trivial factor V_2): indeed, the group Q acts on $\mathrm{Ind}_{Q''}^{Q} C$ through a finite quotient, hence the result follows from Theorem III.2.35.

III.3. Twisted ℓ-adic sheaves

Let Λ be an ℓ-adic coefficient ring (cf. III.2.2). We fix a unitary Λ-admissible mutiplier μ on the topological group G_k (cf. III.1.13, III.2.9, III.2.10).

Definition III.3.1. A finite Galois extension k'/k contained in $\overline{k}$ is said to neutralize μ if μ is the pullback of a multiplier on the finite quotient $\mathrm{Gal}(k'/k)$ of G_k.

By III.2.1, the unitary multiplier μ is neutralized by some finite Galois extension of k.

III.3.2. Assume that Λ is a finite ℓ-adic coefficient ring, and let X be a k-scheme. We denote by $\mathrm{Loc}(X, \Lambda)$ the category of locally constant constructible Λ-modules on the small étale site of X. Moreover, we denote by $\mathrm{Sh}(X, \Lambda)$ the abelian category of constructible sheaves of Λ-modules on the small étale site of X. Thus an object $\mathcal{F}$ of $\mathrm{Sh}(X, \Lambda)$ is a Λ-module in the étale topos of X, such that for any affine open subset U of X, there exists a finite partition $U = \sqcup_{j \in J} U_j$ into constructible locally closed subschemes, such that $\mathcal{F}_{|U_j}$ belongs to $\mathrm{Loc}(X, \Lambda)$.

III.3.3. Assume that Λ is the ring of integers in a finite subextension of $\mathbb{Q}_\ell$ in C, and let X be a locally noetherian k-scheme. We denote by $\mathrm{Sh}(X, \Lambda)$ (resp. $\mathrm{Loc}(X, \Lambda)$) the inverse 2-limit of the categories $\mathrm{Sh}(X, \Lambda/\ell^n)$ (resp. $\mathrm{Loc}(X, \Lambda/\ell^n)$) where n ranges over all integers (cf. III.3.2). Thus the objects of $\mathrm{Loc}^{\otimes}(G, \Lambda)$ are projective systems $(\mathcal{F}_n)_{n\geq 1}$ where $\mathcal{F}_n$ is a multiplicative Λ/ℓ^n-local system on G, such that the transition maps $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}_{n+1} \otimes \Lambda/\ell^n \to \mathcal{F}_n$, for each integer n.

The category $\mathrm{Sh}(X, \Lambda)$ is abelian by ([**SGA5**], VI 1.1.3).

III.3.4. Assume that Λ is a finite subextension of $\mathbb{Q}_\ell$ in C, and let X be a locally noetherian k-scheme. Let Λ_0 be the ring of integers in Λ. We denote by $\mathrm{Sh}(X, \Lambda)$ the quotient of the abelian category $\mathrm{Sh}(X, \Lambda_0)$ by the thick subcategory of torsion Λ_0-sheaves, cf. ([**De80**], 1.1.1(c)), and by $\mathrm{Loc}(X, \Lambda)$ the essential image of $\mathrm{Loc}(X, \Lambda_0)$ in $\mathrm{Sh}(X, \Lambda)$. In particular, the category $\mathrm{Sh}(X, \Lambda)$ is abelian.

The natural functor $\mathrm{Sh}(X, \Lambda_0) \to \mathrm{Sh}(X, \Lambda)$ is essentially surjective, and will be denoted by $\otimes_{\Lambda_0}\Lambda$. If X is noetherian then the natural homomorphism

$$\mathrm{Hom}_{\Lambda_0}(\mathcal{F}, \mathcal{G}) \otimes_{\Lambda_0} \Lambda \to \mathrm{Hom}_{\Lambda}(\mathcal{F} \otimes_{\Lambda_0} \Lambda, \mathcal{G} \otimes_{\Lambda_0} \Lambda),$$

is an isomorphism for any objects $\mathcal{F}$ and $\mathcal{G}$ of $\mathrm{Sh}(X, \Lambda_0)$.

III.3.5. Let X be a locally noetherian k-scheme. We denote by $\mathrm{Sh}(X, \Lambda)$ (resp. $\mathrm{Loc}(X, \Lambda)$) the 2-colimit of the categories $\mathrm{Sh}(X, \Lambda_0)$, where Λ_0 ranges over admissible ℓ-adic subrings of Λ (cf. III.2.2). For $\Lambda = C$, this coincides with ([**De80**], 1.1.1(d)). We will simply refer to objects of $\mathrm{Sh}(X, \Lambda)$ (resp. $\mathrm{Loc}(X, \Lambda)$) as Λ-*sheaves on* X (resp. Λ-*local systems on* X). The category $\mathrm{Sh}(X, \Lambda)$ is abelian, since it is a filtered 2-colimit of abelian categories.

III.3.6. Let X be a locally noetherian k-scheme. Let $\Lambda \to \Lambda'$ be a continuous homomorphism of ℓ-adic coefficient rings. If Λ and Λ' are admissible, we define a functor

$$\otimes_{\Lambda}\Lambda' : \mathrm{Sh}(X, \Lambda) \to \mathrm{Sh}(X, \Lambda'),$$

as follows:

(1) if Λ is finite then so is Λ', and $\otimes_\Lambda\Lambda'$ is the functor which sends a Λ-sheaf $\mathcal{F}$ to the tensor product $\mathcal{F} \otimes_\Lambda \Lambda'$,

(2) if Λ and Λ' are rings of integers in finite extensions of $\mathbb{Q}_\ell$, then $\otimes_\Lambda\Lambda'$ is the functor which sends a projective system $(\mathcal{F}_n)_{n\geq 1}$ as in III.3.3 to $(\mathcal{F}_n \otimes_{\Lambda/\ell^n} \Lambda'/\ell^n)_{n\geq 1}$

(3) if Λ is a ring of integers in a finite extension of $\mathbb{Q}_\ell$ and if Λ' is finite, then $\otimes_\Lambda\Lambda'$ is the functor which sends a projective system $(\mathcal{F}_n)_{n\geq 1}$ as in III.3.3 to $\mathcal{F}_n \otimes_{\Lambda/\ell^n} \Lambda'$, where n is an integer such that ℓ^n vanishes in Λ'.

(4) if Λ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, and if Λ' is a finite extension of $\mathbb{Q}_\ell$ with ring of integers Λ'_0, then $\otimes_\Lambda\Lambda'$ is the functor which sends a Λ-sheaf $\mathcal{F}$ to $(\mathcal{F} \otimes_\Lambda \Lambda'_0) \otimes_{\Lambda'_0} \Lambda'$, cf. III.3.4.

(5) if Λ and Λ' are finite extensions of $\mathbb{Q}_\ell$, and if Λ_0 is the ring of integers in Λ, then $\otimes_\Lambda\Lambda'$ is a functor which sends $\mathcal{F} \otimes_{\Lambda_0} \Lambda$ to $\mathcal{F} \otimes_{\Lambda_0} \Lambda'$ for any Λ_0-sheaf $\mathcal{F}$.

In general we let $\otimes_\Lambda\Lambda'$ be the 2-colimit of the functors

$$\otimes_{\Lambda_0}\Lambda'_0 : \mathrm{Sh}(X, \Lambda_0) \to \mathrm{Sh}(X, \Lambda'_0),$$

where Λ_0 and Λ'_0 are admissible ℓ-adic subrings of Λ and Λ' respectively, such that Λ'_0 contains the image of Λ_0 in Λ'.

III.3.7. Let X be a locally noetherian k-scheme. Let k'/k be a finite Galois subextension of $\overline{k}$ neutralizing μ (cf. III.3.1). A **μ-twisted Λ-sheaf on** X, is a pair $\mathcal{F} = (\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g\in \mathrm{Gal}(k'/k)})$, where $\mathcal{F}_{k'}$ is Λ-sheaf on $X_{k'}$ (cf. III.3.5), and $\rho_{\mathcal{F}}(g) : g^{-1}\mathcal{F}_{k'} \to \mathcal{F}_{k'}$ is an isomorphism for each g in $\mathrm{Gal}(k'/k)$, such that the diagram

$$\begin{tikzcd} g^{-1}h^{-1}\mathcal{F}_{k'} \arrow[r, "g^{-1}\rho_{\mathcal{F}}(h)"] \arrow[rr, bend right, "\mu(g,h)\rho_{\mathcal{F}}(gh)"'] & g^{-1}\mathcal{F}_{k'} \arrow[r, "\rho_{\mathcal{F}}(g)"] & \mathcal{F}_{k'} \end{tikzcd}$$

is commutative for any g, h in $\mathrm{Gal}(k'/k)$. In particular, the endomorphism $\rho_{\mathcal{F}}(1)$ is the identity of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are μ-twisted Λ-sheaves on X, a morphism from $\mathcal{F}$ to $\mathcal{G}$ is a morphism $f : \mathcal{F}_{k'} \to \mathcal{G}_{k'}$ in $\mathrm{Sh}(X_{k'}, \Lambda)$ such that $f \circ \rho_{\mathcal{F}}(g) = \rho_{\mathcal{G}}(g) \circ (g^{-1}f)$ for any g in $\mathrm{Gal}(k'/k)$.

Remark III.3.8. Since the action of $\mathrm{Gal}(k'/k)$ on $X_{k'}$ is a right action, we have $(gh)^{-1} = g^{-1}h^{-1}$ on $\mathrm{Sh}(X_{k'}, \Lambda)$.

If k''/k' is a finite extension contained in $\overline{k}$ such that k'' is a Galois extension of k, then k''/k neutralizes μ as well. If π is the projection from $\mathrm{Gal}(k''/k)$ to $\mathrm{Gal}(k'/k)$, then, by descent along the $\mathrm{Gal}(k''/k')$-torsor $X_{k''} \to X_{k'}$, the functor

$$(\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g\in \mathrm{Gal}(k'/k)}) \mapsto (\mathcal{F}_{k'|X_{k''}}, (\rho_{\mathcal{F}}(\pi(g))_{g\in \mathrm{Gal}(k''/k)}),$$

is an equivalence between the corresponding categories of μ-twisted Λ-sheaves on X. We denote by $\mathrm{Sh}(X, \mu, \Lambda)$ the 2-limit of these categories along the filtered set of Galois extension of k contained in $\overline{k}$ which neutralizes μ. The category $\mathrm{Sh}(X, \mu, \Lambda)$ is abelian.

If $\Lambda \to \Lambda'$ is a continuous homomorphism of ℓ-adic coefficient rings, the natural functor

$$\mathrm{Sh}(X, \mu, \Lambda) \to \mathrm{Sh}(X, \mu, \Lambda'),$$

will be denoted $\otimes_\Lambda \Lambda'$.

III.3.9. Let k'' be a finite extension of k, and let X be a locally noetherian k''-scheme. Let Σ be the set of morphisms of k-algebras from k'' to $\overline{k}$. Let k'/k be a finite Galois subextension of $\overline{k}$ neutralizing μ (cf. III.3.1) and containing the image of any element of Σ. We thus have a decomposition

$$X \otimes_k k' = \coprod_{\iota\in\Sigma} X_\iota,$$

where X_ι is the k'-scheme $X \otimes_{k'',\iota} k'$. In particular, a Λ-sheaf on $X_{k'}$ can be considered as a collection $(\mathcal{F}_\iota)_{\iota\in\Sigma}$, where $\mathcal{F}_\iota$ is a Λ-sheaf on X_ι for each ι. Moreover, if $\mathcal{F} = (\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g\in \mathrm{Gal}(k'/k)})$ is a μ-twisted Λ-sheaf on the k-scheme X, then $\mathcal{F}_{k'}$ is a Λ-sheaf on $X_{k'}$, and its component $(\mathcal{F}_{k'})_\iota$ is a $\mu_{|\mathrm{Gal}(\overline{k}/\iota(k''))}$-twisted Λ-sheaf on the k''-scheme X.

Proposition III.3.10. *Let X and k'' be as in III.3.9. Let ι be an element of Σ, let $\mathrm{Gal}(\iota)$ be the Galois group of the extension $\iota : k'' \to \overline{k}$, and let $\mu_{k''}$ be the restriction of μ to $G_{k''}$. The functor which sends a μ-twisted Λ-sheaf $\mathcal{F} = (\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g\in \mathrm{Gal}(k'/k)})$ on the k-scheme X to the $\mu_{k''}$-twisted Λ-sheaf $((\mathcal{F}_{k'})_\iota, (\rho_{\mathcal{F}}(g)_\iota)_{g\in \mathrm{Gal}(\iota)})$ on the k''-scheme X, is an equivalence of categories.*

Indeed, a quasi-inverse to the functor from Proposition III.3.10 can be described as follows. If $(\mathcal{F}_\iota, \rho_{\mathcal{F}}(g))_{g\in \mathrm{Gal}(\iota)}$ is a $\mu_{k''}$-twisted Λ-sheaf on the k''-scheme X, then for each ι' in Σ, we define $\mathcal{F}_{\iota'}$ to be the sub-Λ-sheaf of

$$\prod_{\substack{g\in \mathrm{Gal}(k'/k)\\ g\iota = \iota'}} g^{-1}\mathcal{F}_\iota,$$

on $X_{\iota'}$, consisting of sections $(s_g)_g$ such that for any g in $\mathrm{Gal}(k'/k)$ with $g\iota = \iota'$ and any element h of $\mathrm{Gal}(\iota)$, we have

$$s_{gh} = \mu(g,h)\rho_{\mathcal{F}}(h)^{-1}s_g.$$

The collection $(\mathcal{F}_{\iota'})_{\iota'\in\Sigma}$ yields a Λ-sheaf on $X_{k'}$ which is naturally endowed with a structure $(\rho_{\mathcal{F}}(g))_{g\in\mathrm{Gal}(k'/k)}$ of μ-twisted Λ-sheaf on the k-scheme X. For each h in $\mathrm{Gal}(k'/k)$ and for each ι' in Σ, the morphism $\rho_{\mathcal{F}}(h)_{\iota'}$ sends a section $(s_g)_{g\iota=h^{-1}\iota'}$ of $h^{-1}\mathcal{F}_{h^{-1}\iota'}$ to the section $(\mu(h,h^{-1}g)s_{h^{-1}g})_{g\iota=\iota'}$ of $\mathcal{F}_{\iota'}$.

III.3.11. Assume that Λ is the ring of integers in a finite subextension of $\mathbb{Q}_\ell$ in C, let X be a locally noetherian k-scheme. Then the natural functor

$$\mathrm{Sh}(X,\mu,\Lambda)\to 2\text{-}\lim_n \mathrm{Sh}(X,\mu,\Lambda/\ell^n),$$

where n runs over all positive integers, is an equivalence of categories. Indeed, if k'/k is a finite Galois extension contained in $\overline{k}$ which neutralizes μ, then the natural functor

$$\mathrm{Sh}(X_{k'},\Lambda)\to 2\text{-}\lim_n \mathrm{Sh}(X_{k'},\Lambda/\ell^n),$$

is an equivalence of categories by definition, cf. III.3.3. Moreover, if $\mathcal{F}=(\mathcal{F}_n)_{n\geq 1}$ is an object of $\mathrm{Sh}(X_{k'},\Lambda)$, then we have

$$\mathrm{Hom}_\Lambda(g^{-1}\mathcal{F},\mathcal{F}) = \lim \mathrm{Hom}_{\Lambda/\ell^n}(g^{-1}\mathcal{F}_n,\mathcal{F}_n),$$

for any g in $\mathrm{Gal}(k'/k)$, hence a structure of μ-twisted Λ-sheaf on $\mathcal{F}$ amounts to a compatible system of structures of μ-twisted Λ-sheaves on each $\mathcal{F}_n$.

III.3.12. Assume that Λ is a finite subextension of $\mathbb{Q}_\ell$ in C, with ring of integers $\Lambda_0\subseteq\Lambda$, let X be a locally noetherian k-scheme. Then μ takes its values in Λ_0, and the natural functor

$$\mathrm{Sh}(X,\mu,\Lambda_0)\to\mathrm{Sh}(X,\mu,\Lambda) \tag{21}$$

induces an equivalence from the quotient of $\mathrm{Sh}(X,\mu,\Lambda_0)$ by its subcategory of torsion objects, to the category of μ-twisted Λ-sheaves on X. Unlike III.3.11, this statement is not entirely tautological.

By glueing, we can assume (and we do) that X is noetherian. Let k'/k be a finite Galois subextension of $\overline{k}$, with Galois group $G=\mathrm{Gal}(k'/k)$, which neutralizes μ (cf. III.3.1). For any objects $\mathcal{F}$ and $\mathcal{G}$ of $\mathrm{Sh}(X,\mu,\Lambda_0)$, the natural homomorphism

$$\mathrm{Hom}_{\Lambda_0}(\mathcal{F}_{k'},\mathcal{G}_{k'})\otimes_{\Lambda_0}\Lambda\to\mathrm{Hom}_\Lambda(\mathcal{F}_{k'}\otimes_{\Lambda_0}\Lambda,\mathcal{G}_{k'}\otimes_{\Lambda_0}\Lambda),$$

is an isomorphism, cf. III.3.4. Moreover, we have an exact sequence

$$0\to\mathrm{Hom}_{\Lambda_0}(\mathcal{F},\mathcal{G})\to\mathrm{Hom}_{\Lambda_0}(\mathcal{F}_{k'},\mathcal{G}_{k'})\xrightarrow{f\mapsto(f\rho_{\mathcal{F}}(g)-\rho_{\mathcal{G}}(g)(g^{-1}f))_g}\prod_{g\in\mathrm{Gal}(k'/k)}\mathrm{Hom}_{\Lambda_0}(g^{-1}\mathcal{F}_{k'},\mathcal{G}_{k'}).$$

By flatness of the ring homomorphism $\Lambda_0\to\Lambda$, we deduce that the natural homomorphism

$$\mathrm{Hom}_{\Lambda_0}(\mathcal{F},\mathcal{G})\otimes_{\Lambda_0}\Lambda\to\mathrm{Hom}_\Lambda(\mathcal{F}\otimes_{\Lambda_0}\Lambda,\mathcal{G}\otimes_{\Lambda_0}\Lambda),$$

is an isomorphism as well, hence the full faithfullness of the functor (21).

It remains to prove that the functor (21) is essentially surjective. Let $(\mathcal{F}_{k'},(\rho_{\mathcal{F}}(g))_{g\in G})$ be a pair representing an object $\mathcal{F}$ of $\mathrm{Sh}(X,\mu,\Lambda)$. Let $\mathcal{H}$ be a torsion free Λ_0-sheaf on $X_{k'}$ such that $\mathcal{F}_{k'}$ is isomorphic to $\mathcal{H}\otimes_{\Lambda_0}\Lambda$. For a sufficiently large integer n, we have for each element g of G a homomorphism $\widetilde{\rho}_{\mathcal{F}}(g)$ from $g^{-1}\mathcal{H}$ to $\mathcal{H}$, which induces the homomorphism $\ell^n\rho_{\mathcal{F}}(g)$ from $g^{-1}\mathcal{F}_{k'}$ to $\mathcal{F}_{k'}$. Let us consider the homomorphism

$$\theta:\bigoplus_{h\in G}h^{-1}\mathcal{H}\to\mathcal{H},$$

given by $\widetilde{\rho}_{\mathcal{F}}(h)$ on the component $h^{-1}\mathcal{H}$. We endow the source $\mathcal{H}' = \bigoplus_{h\in G} h^{-1}\mathcal{H}$ of θ with a structure of μ-twisted Λ_0-sheaf by equipping it for each g in G with the isomorphism

$$\rho_{\mathcal{H}'}(g) : g^{-1}\mathcal{H}' \to \mathcal{H}',$$

which sends a tuple $(x_h)_{h\in G}$, where x_h belongs to $g^{-1}h^{-1}\mathcal{H}$, to the tuple $(\mu(g, g^{-1}h)x_{g^{-1}h})_{h\in G}$. For each g in G, the diagram

$$\begin{tikzcd} g^{-1}\mathcal{H}' \arrow[r, "g^{-1}\theta"] \arrow[d, "\ell^n \rho_{\mathcal{H}'}(g)"] & g^{-1}\mathcal{H} \arrow[d, "\widetilde{\rho}_{\mathcal{F}}(g)"] \\ \mathcal{H}' \arrow[r, "\theta"] & \mathcal{H} \end{tikzcd}$$

is commutative. Since $\mathcal{H}$ is torsion free, this implies that $\rho_{\mathcal{H}'}(g)$ induces an isomorphism from the subobject $g^{-1}\ker(\theta)$ of $g^{-1}\mathcal{H}'$ to the subobject $\ker(\theta)$ of $\mathcal{H}'$. Thus the kernel of θ is a μ-twisted Λ_0-subsheaf of $\mathcal{H}'$. Consequently, the image of θ is a μ-twisted Λ_0-sheaf as well. Since $\mathrm{Im}(\theta) \otimes_{\Lambda_0} \Lambda$ is isomorphic to $\mathcal{F}$, this proves the essential surjectivity of the functor (21).

III.3.13. Let X be a locally noetherian k-scheme. Then the natural functor

$$2\text{-colim}_{\Lambda_0}\ \mathrm{Sh}(X, \mu, \Lambda_0) \to \mathrm{Sh}(X, \mu, \Lambda),$$

where Λ_0 runs over all admissible ℓ-adic subrings of Λ containing the image of μ, is an equivalence of categories. Indeed, if k'/k is a finite Galois extension contained in $\overline{k}$ which neutralizes μ, then the natural functor

$$2\text{-colim}_{\Lambda_0}\ \mathrm{Sh}(X_{k'}, \Lambda_0) \to \mathrm{Sh}(X_{k'}, \Lambda),$$

is an equivalence of categories by definition, cf. III.3.5. Moreover, if $\mathcal{F}$ is an object of $\mathrm{Sh}(X_{k'}, \Lambda_0)$, then the natural homomorphism

$$\mathrm{colim}_{\Lambda_1} \mathrm{Hom}_{\Lambda_1}(g^{-1}\mathcal{F} \otimes_{\Lambda_0} \Lambda_1, \mathcal{F} \otimes_{\Lambda_0} \Lambda_1) \to \mathrm{Hom}_{\Lambda}(g^{-1}\mathcal{F} \otimes_{\Lambda_0} \Lambda, \mathcal{F} \otimes_{\Lambda_0} \Lambda),$$

is an isomorphism for any g in $\mathrm{Gal}(k'/k)$, where Λ_1 runs over all admissible ℓ-adic subrings of Λ containing Λ_0 and the image of μ. Since $\mathrm{Gal}(k'/k)$ is finite, we obtain that a structure of μ-twisted Λ-sheaf on $\mathcal{F} \otimes_{\Lambda_0} \Lambda$ amounts to a structure of μ-twisted Λ_1-sheaf on $\mathcal{F} \otimes_{\Lambda_0} \Lambda_1$, for a large enough admissible ℓ-adic subring Λ_1 of Λ.

III.3.14. Let $f : X \to Y$ be a separated morphism of k-schemes of finite type. If $\mathcal{F}$ is a μ-twisted Λ-sheaf on X, given by a tuple $(\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g\in\mathrm{Gal}(k'/k)})$ (cf. III.3.7), then $R^\nu f_*\mathcal{F}_{k'}$ and $R^\nu f_!\mathcal{F}_{k'}$ are Λ-sheaves on $Y_{k'}$ for each integer ν by ([**SGA**$4\frac{1}{2}$], 7.1.1), and the isomorphisms

$$\rho_{\mathcal{F}}(g) : \mathcal{F}_{k'} \to g_*\mathcal{F}_{k'},$$

on $X_{k'}$ yield by functoriality isomorphisms

$$\begin{aligned} R^\nu f_*\mathcal{F}_{k'} &\to R^\nu f_* g_*\mathcal{F}_{k'} = g_* R^\nu f_*\mathcal{F}_{k'} \\ R^\nu f_!\mathcal{F}_{k'} &\to R^\nu f_! g_*\mathcal{F}_{k'} = g_* R^\nu f_!\mathcal{F}_{k'}, \end{aligned}$$

so that we obtain structures of μ-twisted Λ-sheaves on $R^\nu f_*\mathcal{F}_{k'}$ and $R^\nu f_!\mathcal{F}_{k'}$. We denote by $R^\nu f_*\mathcal{F}$ and $R^\nu f_!\mathcal{F}$ the resulting μ-twisted Λ-sheaves. In particular, by taking $Y = \mathrm{Spec}(k)$, we obtain a structure of μ-twisted representation of G_k on the cohomology groups $H^\nu_c(X_{\overline{k}}, \mathcal{F}_{\overline{k}})$.

III.3.15. Let X be a k-scheme. For any geometric point $\bar{x}$ of X, we denote by $\mathrm{ev}_{\bar{x}}$ the functor which to a finite étale X-scheme Y associates the set $Y(\bar{x})$ of $\bar{x}$-points of Y. For a couple $(\overline{x}_0, \overline{x}_1)$ of geometric points of X, we denote by $\pi_1(X, \overline{x}_0, \overline{x}_1)$ the set of isomorphisms from $\mathrm{ev}_{\overline{x}_0}$ to $\mathrm{ev}_{\overline{x}_1}$, endowed with the coarsest topology such that for any finite étale X-scheme Y, the map

$$\pi_1(X, \overline{x}_0, \overline{x}_1) \to \mathrm{Hom}(Y(\overline{x}_0), Y(\overline{x}_1)),$$

is continuous, where the target is endowed with the discrete topology. Thus $\pi_1(X, \overline{x}_0, \overline{x}_1)$ is a profinite topological space. For a triple $(\overline{x}_0, \overline{x}_1, \overline{x}_2)$ of geometric points of X, the composition induces a continuous map

$$\pi_1(X, \overline{x}_1, \overline{x}_2) \times \pi_1(X, \overline{x}_0, \overline{x}_1) \to \pi_1(X, \overline{x}_0, \overline{x}_2),$$

which satisfies the natural associativity condition. In particular, $\pi_1(X, \overline{x}_0, \overline{x}_0)$ is a profinite group, which will be simply denoted by $\pi_1(X, \overline{x}_0)$.

III.3.16. Let $f : X \to Y$ be a morphism of k-schemes, and let $(\overline{x}_0, \overline{x}_1)$ be a couple of geometric points of X. We have natural isomorphisms $\mathrm{ev}_{f(\overline{x}_0)} \cong \mathrm{ev}_{\overline{x}_0} \circ f^{-1}$ and $\mathrm{ev}_{f(\overline{x}_1)} \cong \mathrm{ev}_{\overline{x}_1} \circ f^{-1}$, whence precomposition with f^{-1} induces a continuous map

$$f_* : \pi_1(X, \overline{x}_0, \overline{x}_1) \to \pi_1(X, f(\overline{x}_0), f(\overline{x}_1)),$$

which is compatible with the composition laws. In particular, the formation of $\pi_1(X, \overline{x}_0, \overline{x}_1)$ (resp. $\pi_1(X, \overline{x}_0)$) is functorial in the triple $(X, \overline{x}_0, \overline{x}_1)$ (resp. in the couple $(X, \overline{x}_0)$).

III.3.17. Let X be a locally noetherian connected k-scheme, and let $\bar{x}$ be a geometric point of X. Then the fiber functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ is an equivalence of categories from $\mathrm{Loc}(X, \Lambda)$ to the category $\mathrm{Rep}_\Lambda(\pi_1(X, \overline{x}), 1)$ of Λ-admissible representations of the profinite group $\pi_1(X, \overline{x})$ (cf. III.2.13). This statement reduces to the case where Λ is finite, which in turn follows from ([**SGA1**], V.7).

III.3.18. Let X be a locally noetherian geometrically connected k-scheme, let k' be a finite Galois extension of k and let $\bar{x}$ be a geometric point of $X_{k'}$. Let $u : X_{k'} \to X$ be the natural projection. By ([**SGA1**], IX 6.1), we have an exact sequence

$$1 \to \pi_1(X_{k'}, \bar{x}) \xrightarrow{u_*} \pi_1(X, u(\bar{x})) \xrightarrow{r} \mathrm{Gal}(k'/k) \to 1, \tag{22}$$

which we now proceed to describe in greater details. For each g in $\mathrm{Gal}(k'/k)$, the geometric points $\bar{x}$ and $\bar{x}g$ have the same image by u, so that we obtain a continuous map

$$u_* : \pi_1(X_{k'}, \bar{x}, \bar{x}g) \to \pi_1(X, u(\bar{x}), u(\bar{x}g)) = \pi_1(X, u(\bar{x})).$$

The collection of these continuous maps yields a homeomorphism

$$\coprod_{g \in \mathrm{Gal}(k'/k)} \pi_1(X_{k'}, \bar{x}, \bar{x}g) \xrightarrow{u_*} \pi_1(X, u(\bar{x})), \tag{23}$$

whose composition with the homomorphism $r : \pi_1(X, u(\bar{x})) \to \mathrm{Gal}(k'/k)$ maps $\pi_1(X_{k'}, \bar{x}, \bar{x}g)$ to g for each g in $\mathrm{Gal}(k'/k)$. The homeomorphism above can be promoted to an isomorphism of profinite groups if we endow its source with the following group law: for any elements α and β of $\pi_1(X_{k'}, \bar{x}, \bar{x}g)$ and $\pi_1(X_{k'}, \bar{x}, \bar{x}h)$ respectively, we set $\alpha \cdot \beta = h_*\alpha \circ \beta$.

III.3.19. Let X be a locally noetherian geometrically connected k-scheme, let k'/k be a finite Galois subextension of $\overline{k}$ neutralizing μ (cf. III.3.1) and let $\bar{x}$ be a geometric point of $X_{k'}$. We abusively denote by μ the pullback of μ by the homomorphism $r : \pi_1(X, u(\bar{x})) \to \mathrm{Gal}(k'/k)$ from III.3.18.

Let $\mathrm{Loc}(X, \mu, \Lambda)$ be the full subcategory of $\mathrm{Sh}(X, \mu, \Lambda)$ consisting of objects $\mathcal{F}$ such that $\mathcal{F}_{k'}$ is a Λ-local system on $X_{k'}$. Then the fiber functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ is an equivalence of categories from $\mathrm{Loc}(X, \mu, \Lambda)$ to the category $\mathrm{Rep}_\Lambda(\pi_1(X, u(\overline{x})), \mu)$ of Λ-admissible representations of the Λ-twisted profinite group $(\pi_1(X, u(\overline{x})), \mu)$ (cf. III.2.11, III.2.13).

Indeed, by III.3.17 the functor $\mathcal{F}_{k'} \mapsto \mathcal{F}_{k',\bar{x}}$ realizes an equivalence of categories from $\mathrm{Loc}(X, \Lambda)$ to the category of Λ-admissible representations of the profinite group $(\pi_1(X_{k'}, \overline{x}), 1)$. Moreover, for each g in $\mathrm{Gal}(k'/k)$, the morphism

$$\begin{aligned} \mathrm{Hom}_\Lambda(g^{-1}\mathcal{F}_{k'}, \mathcal{F}_{k'}) &\to \mathrm{Hom}_{\pi_1(X_{k'},\bar{x})}(r^{-1}(g), \mathrm{Aut}_\Lambda(\mathcal{F}_{k',\bar{x}})) \\ \rho &\mapsto (\alpha \mapsto \rho_{\bar{x}} \circ u_*^{-1}(\alpha)), \end{aligned}$$

where u_*^{-1} is the inverse of the isomorphism from (23), realizes an isomorphism onto the set of left and right $\pi_1(X_{k'}, \bar{x})$-equivariant maps from $r^{-1}(g)$ to $\mathrm{Aut}_\Lambda(\mathcal{F}_{k',\bar{x}})$. Thus, structures of μ-twisted Λ-sheaf on $\mathcal{F}_{k'}$ correspond bijectively to structures of Λ-admissible representations of $(\pi_1(X, u(\overline{x})), \mu)$ on $\mathcal{F}_{k',\bar{x}}$, hence the result.

Example III.3.20. For $X = \mathrm{Spec}(k)$, the category $\mathrm{Sh}(X, \mu, \Lambda)$ is equivalent to the category of Λ-admissible representations of (G_k, μ) (cf. III.2.13).

III.3.21. Let Y be a locally noetherian geometrically connected k-scheme, and let $f : X \to Y$ be a finite étale morphism, where X is a geometrically connected k''-scheme for some finite extension k'' of k contained in $\overline{k}$. Let k'/k be a finite Galois subextension of $\overline{k}$ neutralizing μ and containing k'' (cf. III.3.1), let $\bar{x}$ be a geometric point of $X_{k'}$ and let $u : X_{k'} \to X$ be the natural projection. The homomorphism

$$f_* : \pi_1(X, u(\overline{x})) \to \pi_1(Y, fu(\overline{x})),$$

is injective and its image is an open subgroup of finite index in $\pi_1(Y, fu(\overline{x}))$. Then the following diagram is commutative.

$$\begin{tikzcd} \mathrm{Loc}(X, \mu, \Lambda) \arrow[r, "\mathcal{F} \mapsto \mathcal{F}_{u(\overline{x})}"] \arrow[d, "f_*"] & \mathrm{Rep}_\Lambda(\pi_1(X, u(\overline{x})), \mu) \arrow[d, "\mathrm{Ind}_{\pi_1(X,u(\overline{x}))}^{\pi_1(Y,fu(\overline{x}))}"] \\ \mathrm{Loc}(Y, \mu, \Lambda) \arrow[r, "\mathcal{F} \mapsto \mathcal{F}_{fu(\overline{x})}"] & \mathrm{Rep}_\Lambda(\pi_1(Y, fu(\overline{x})), \mu) \end{tikzcd}$$

In this diagram, the left vertical arrow is defined in III.3.15, while the right vertical arrow is defined in III.2.24. The horizontal arrows are equivalences of categories by III.3.19.

III.3.22. Let $s \to \mathrm{Spec}(k)$ be a finite extension of k, and let us fix a k-morphism $\overline{s} : \mathrm{Spec}(\overline{k}) \to s$, corresponding to a k-linear embedding of $k(s)$ in $\overline{k}$, so that the Galois group $G_s = \mathrm{Gal}(\overline{k}/k(s))$ can be considered as an open subgroup of finite index in G_k. We still denote by μ the restriction of μ to G_s. We denote by $\delta_{s/k}$ the Λ-admissible character of rank 1 of G_k given by

$$\delta_{s/k} = \det\left(\mathrm{Ind}_{G_s}^{G_k}\Lambda\right).$$

Thus $\delta_{s/k}$ is the signature character associated the left action of G_k on the finite set G_k/G_s.

DEFINITION III.3.23. Let V be a Λ-admissible representation of (G_s,μ). The *transfer* or *verlagerung* of V with respect to the extension $k(s)/k$ is the Λ-admissible map $\mathrm{Ver}_{s/k}(V)$ from G_k to $\Lambda^\times$ defined by

$$\mathrm{Ver}_{s/k}(V)=\delta_{s/k}^{-\mathrm{rk}(V)}\det\left(\mathrm{Ind}_{G_s}^{G_k}V\right),$$

where the induction is defined in III.2.24.

PROPOSITION III.3.24. *Let V be a Λ-admissible representation of (G_s,μ). Then the determinant $\det(V)$ is a Λ-admissible representation of $(G_s,\mu^{\mathrm{rk}(V)})$ and we have*

$$\mathrm{Ver}_{s/k}(V)=\mathrm{Ver}_{s/k}(\det(V)).$$

Moreover, if χ_1 and χ_2 are Λ-admissible representations of rank 1 of (G_s,μ_1) and (G_s,μ_2) respectively, for some multipliers ν_1,ν_2 on G_k, then

$$\mathrm{Ver}_{s/k}(\chi_1\chi_2)=\mathrm{Ver}_{s/k}(\chi_1)\mathrm{Ver}_{s/k}(\chi_2).$$

Indeed, let $(t_i)_{i\in I}$ be a family of G_s-left cosets representatives in G_k. Then we have a splitting (cf. III.2.24)

$$\mathrm{Ind}_{G_s}^{G_k}V=\bigoplus_{i\in I}[t_i]V.$$

Let g be an element of G_k and let us write $gt_i=t_{\sigma(g)(i)}h_{g,i}$ for some bijection $\sigma(g)$ of I onto itself, and some element $h_{g,i}$ of G_s. For any element v of V, we have

$$\begin{aligned}[g][t_i]v&=\mu(g,t_i)[gt_i]v\\&=\mu(g,t_i)\mu(t_{\sigma(g)(i)},h_{g,i})^{-1}[t_{\sigma(g)(i)}][h_{g,i}]v.\end{aligned}$$

Consequently, we have

$$\det(g\mid\mathrm{Ind}_{G_s}^{G_k}V)=\mathrm{sgn}(\sigma(g))^{\mathrm{rk}(V)}\prod_{i\in I}\mu(g,t_i)^{\mathrm{rk}(V)}\mu(t_{\sigma(g)(i)},h_{g,i})^{-\mathrm{rk}(V)}\det(h_{g,i}\mid V),$$

where sgn is the signature homomorphism. The sign $\mathrm{sgn}(\sigma(g))$ is equal to $\delta_{s/k}(g)$ and thus

$$\mathrm{Ver}_{s/k}(V)(g)=\prod_{i\in I}\mu(g,t_i)^{\mathrm{rk}(V)}\mu(t_{\sigma(g)(i)},h_{g,i})^{-\mathrm{rk}(V)}\det(h_{g,i}\mid V),$$

hence the conclusion of Proposition III.3.24.

REMARK III.3.25. If $\mu=1$ then $\det(V)$ is a group homomorphism from G_s to $\Lambda^\times$ and the computation above yields

$$\mathrm{Ver}_{s/k}(V)=\det(V)\circ\mathrm{ver}_{s/k},$$

where $\mathrm{ver}_{s/k}:G_k^{\mathrm{ab}}\to G_s^{\mathrm{ab}}$ is the usual transfer homomorphism, cf. ([**Se68**], VII.8 p.122) or ([**De73**], Prop. 1.2).

COROLLARY III.3.26. *Let $s'\to s$ be a finite extension. Then we have*

$$\begin{aligned}\delta_{s'/k}&=\delta_{s/k}^{[k(s'):k(s)]}\mathrm{Ver}_{s/k}(\delta_{s'/s}),\\ \mathrm{Ver}_{s'/k}&=\mathrm{Ver}_{s/k}\circ\mathrm{Ver}_{s'/s}.\end{aligned}$$

This follows from Proposition III.3.24 and from the existence of a natural isomorphism

$$\mathrm{Ind}_{G_{s'}}^{G_k}V\cong\mathrm{Ind}_{G_s}^{G_k}\mathrm{Ind}_{G_{s'}}^{G_s}V.$$

III.3.27. Let $s \to \mathrm{Spec}(k)$ be a finite extension of k, and let us fix a k-morphism $\overline{s}$: $\mathrm{Spec}(\overline{k}) \to s$, corresponding to a k-linear embedding of $k(s)$ in $\overline{k}$, so that the Galois group $G_s = \mathrm{Gal}(\overline{k}/k(s))$ can be considered as an open subgroup of finite index in G_k. We still denote by μ the restriction of μ to G_s.

Let X be a separated s-scheme of finite type. For any element t of G_k/G_s, we denote by $t(\overline{s})$ the composition of $\overline{s}$ with the k-automorphism of $\mathrm{Spec}(\overline{k})$ induced by some element of G_k lifting t, and by ${}^tX$ the $\overline{k}$-scheme $X \times_{s,t(\overline{s})} \mathrm{Spec}(\overline{k})$. Let us consider the product

$$\overline{X} = \prod_{t \in G_k/G_s} {}^tX.$$

For each t in G_k/G_s and each g in G_k, the k-automorphism of $\mathrm{Spec}(\overline{k})$ induced by g yields an isomorphism

$${}^{gt}X \to {}^tX.$$

By taking the product over t, we obtain an automorphism of $\overline{X}$, and this produces a right action of G_k on $\overline{X}$. We denote by p_1 be projection from $\overline{X}$ onto X through its factor $X_1 = X_{\overline{s}}$. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on X. We construct a $\mu^{[k(s):k]}$-twisted Λ-sheaf $\overline{\mathcal{F}}$ on $\overline{X}$ as follows. For each choice $g_\bullet = (g_t)_{t\in G_k/G_s}$ of left G_s-cosets representatives in G_k, we set

$$\overline{\mathcal{F}}_{g_\bullet} = \bigotimes_{t \in G_k/G_s} g_t^{-1}p_1^{-1}\mathcal{F}.$$

If $\tilde{g}_\bullet = (\tilde{g}_t)_{t\in G_k/G_s}$ is another set of left G_s-cosets representatives in G_k, then we can write $\tilde{g}_t = g_t h_t$ for some h_t in G_s, hence an isomorphism

$$g_t^{-1}p_1^{-1}\rho_{\mathcal{F}}(h_t) : \tilde{g}_t^{-1}p_1^{-1}\mathcal{F} \cong g_t^{-1}h_t^{-1}p_1^{-1}\mathcal{F} \to g_t^{-1}p_1^{-1}\mathcal{F},$$

which yields an isomorphism $\overline{\mathcal{F}}_{\tilde{g}_\bullet} \to \overline{\mathcal{F}}_{g_\bullet}$ by taking the tensor product over t. We then define $\overline{\mathcal{F}}$ to be the inverse limit of $\overline{\mathcal{F}}_{g_\bullet}$ over all choices $g_\bullet$ of left G_s-cosets representatives in G_k, with respect to these transition maps.

If g is an element of G_k, then $gg_\bullet$ is also a set of left G_s-cosets representatives in G_k, hence the transition maps

$$g^{-1}\overline{\mathcal{F}}_{g_\bullet} \cong \overline{\mathcal{F}}_{gg_\bullet} \to \overline{\mathcal{F}}_{g_\bullet},$$

yield an isomorphism $\rho_{\overline{\mathcal{F}}}(g) : g^{-1}\overline{\mathcal{F}} \to \overline{\mathcal{F}}$, which endows $\overline{\mathcal{F}}$ with a structure of $\mu^{[k(s):k]}$-twisted Λ-sheaf on $\overline{X}$.

Proposition III.3.28. *Let $s, \overline{s}, X, \overline{X}$ be as in III.3.27. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on X, and let us assume that $R\Gamma_c(X_{\overline{s}}, \mathcal{F})$ is concentrated in a single cohomological degree ν, and that $H_c^\nu(X_{\overline{s}}, \mathcal{F})$ is a free Λ-module of rank 1. Let $\overline{\mathcal{F}}$ be the $\mu^{[k(s):k]}$-twisted Λ-sheaf on $\overline{X}$ constructed in III.3.27. Then $R\Gamma_c(\overline{X}, \overline{\mathcal{F}})$ is concentrated in degree $[k(s):k]\nu$, and the Λ-admissible representation*

$$H_c^{[k(s):k]\nu}(\overline{X}, \overline{\mathcal{F}}),$$

is of rank 1, isomorphic to $\delta_{s/k}^{\nu}\mathrm{Ver}_{s/k}H_c^\nu(X_{\overline{s}}, \mathcal{F})$ (cf. III.3.22).

Indeed, for any choice $g_\bullet = (g_t)_{t\in G_k/G_s}$ of left G_s-cosets representatives in G_k, the canonical isomorphism $\overline{\mathcal{F}} \to \overline{\mathcal{F}}_{g_\bullet}$, together with Künneth's formula ([**SGA4**], XVII 5.4.3), yields that $R\Gamma_c(\overline{X}, \overline{\mathcal{F}})$ is concentrated in degree $[k(s):k]\nu$ and that we have a canonical isomorphism

$$\bigotimes_{t \in G_k/G_s} H_c^\nu({}^tX, g_t^{-1}\mathcal{F}) \to H_c^{[k(s):k]\nu}(\overline{X}, \overline{\mathcal{F}}). \tag{24}$$

It remains to understand how the action of G_k on the target of this isomorphism translates on its source. Let g be an element of G_k. Then we can write $gg_t = g_{\sigma(g)(t)}h_{g,t}$ for some bijection $\sigma(g)$ of G_k/G_s onto itself and $h_{g,t}$ is some element of G_s. Since (24) is an isomorphism of graded vector spaces (cf. ([**SGA4**], XVII 1.1.4), the element g of G_k acts on the source of 24 by $\mathrm{sgn}(\sigma(g))^\nu = \delta^\nu_{s/k}(g)$ multiplied by

$$\prod_{t\in G_k/G_s} \mu(g,g_t)\mu(g_{\sigma(g)(t)},h_{g,t})^{-1}\mathrm{Tr}(h_{g,t} \mid H^\nu_c({}^tX, g_t^{-1}p_1^{-1}\mathcal{F})).$$

The latter is exactly $\mathrm{Ver}_{s/k}H^\nu_c(X_{\overline{s}},\mathcal{F})$ evaluated at g, hence the conclusion of Proposition III.3.28.

III.4. Gabber-Katz extensions

In this section, we review the theory of Gabber-Katz extensions from [**Ka86**], which will constitute our main tool in order to define geometric local ε-factors in Section III.9.

Let T be the spectrum of a k-algebra, which is a henselian discrete valuation ring $\mathcal{O}_T$, with maximal ideal $\mathfrak{m}$, and whose residue field $\mathcal{O}_T/\mathfrak{m}$ is a finite extension of k of degree $\deg(s)$. Let $j:\eta\to T$ be the generic point of T, and let $i:s\to T$ be its closed point, so that T is canonically an s-scheme. We fix a uniformizer π of $\mathcal{O}_T$, and we abusively denote by π as well the morphism

$$\pi : T \to \mathbb{A}^1_s,$$

corresponding to the unique morphism $k(s)[t]\to\mathcal{O}_T$ of $k(s)$-algebras which sends t to π. We fix a k-morphism $\overline{s}$ from $\mathrm{Spec}(\overline{k})$ to s, and a separable closure $\overline{\eta}$ of $\eta_{\overline{s}}=\eta\times_s\overline{s}$. We consider $\overline{\eta}$ as a geometric point of $\mathbb{A}^1_s$ through the morphism π. We denote by G_s the Galois group of the extension $\overline{k}/k(s)$, and by G_η the Galois group of the extension $k(\overline{\eta})/k(\eta)$.

III.4.1. Let $\widehat{T}$ be the spectrum of the $\mathfrak{m}$-adic completion of $\mathcal{O}_T$, and let $\left(\mathbb{A}^1_s\right)_{(0)}$ (resp. $\widehat{\mathbb{A}^1_{s}}_{(0)}$) be the henselization (resp. spectrum of the completion) of $\mathbb{A}^1_s$ at 0. Then we have the following commutative diagram.

$$\begin{tikzcd} \widehat{T} \arrow[r] \arrow[d, "\pi"'] & T \arrow[d, "\pi"] \\ \widehat{\mathbb{A}^1_s}_{(0)} \arrow[r] & \left(\mathbb{A}^1_s\right)_{(0)} \end{tikzcd}$$

The left vertical morphism is an isomorphism, while the two horizontal morphisms induce isomorphisms on the corresponding étale sites, as it follows for example from Krasner's lemma, cf. ([**SP**], 09EJ). Thus the right vertical morphism induces an isomorphism on the corresponding étale sites.

III.4.2. Following ([**Ka86**], 1.3.1), we define a category of special coverings of $\mathbb{G}_{m,s}$. For any scheme X, let $\mathrm{Fét}(X)$ be the category of finite étale X-schemes. By ([**SGA1**], V.7), the functor which sends an object $U\to\mathbb{G}_{m,s}$ of $\mathrm{Fét}(\mathbb{G}_{m,s})$ to the fiber $U_{\overline{\eta}}$, endowed with the natural action of $\pi_1(\mathbb{G}_{m,s},\overline{\eta})$, realizes an equivalence of categories from $\mathrm{Fét}(\mathbb{G}_{m,s})$ to the category of finite sets endowed with a continuous left action of $\pi_1(\mathbb{G}_{m,s},\overline{\eta})$.

Definition III.4.3. ([**Ka86**], 1.3.1) The category $\mathrm{Fét}^\diamondsuit(\mathbb{G}_{m,s})$ of *special* finite étale $\mathbb{G}_{m,s}$-schemes is the full subcategory of $\mathrm{Fét}(\mathbb{G}_{m,s})$ (cf. III.4.2) whose objects are the finite étale morphisms $f:U\to\mathbb{G}_{m,s}$ such that:

(1) the morphism f is tamely ramified above ∞, i.e. the fiber product

$$U\times_{\mathbb{G}_{m,s}}\mathrm{Spec}(k(s)((t^{-1})))\to\mathrm{Spec}(k(s)((t^{-1}))),$$

is a finite disjoint union of spectra of tamely ramified extensions of $k(s)((t^{-1}))$.

(2) the geometric monodromy group of f, namely the image of the composition

$$\pi_1(\mathbb{G}_{m,\overline{s}},\overline{\eta}) \to \pi_1(\mathbb{G}_{m,s},\overline{\eta}) \to \mathrm{Aut}(U_{\overline{\eta}}),$$

has a unique p-Sylow subgroup.

Theorem III.4.4 ([**Ka86**], 1.4.1). *Let* $\mathrm{Fét}^{\diamondsuit}(\mathbb{G}_{m,s})$ *be as in* III.4.3. *The functor*

$$\pi_{|\eta}^{-1} : \mathrm{Fét}^{\diamondsuit}(\mathbb{G}_{m,s}) \to \mathrm{Fét}(\eta),$$

induced by the morphism $\pi_{|\eta} : \eta \to \mathbb{G}_{m,s}$*, is an equivalence of categories.*

III.4.5. For example, if h is an element of $k(\eta)$ then Theorem III.4.4 implies that the $\mathbb{F}_p$-torsor over η defined by the equation

$$x - x^p = h, \tag{25}$$

must extend to an $\mathbb{F}_p$-torsor over $\mathbb{G}_{m,s}$, which is tamely ramified above ∞, hence unramified since $\mathbb{F}_p$ is a p-group. Let us write h as $h_\pi(\pi) + r_\pi$, for some polynomial h_π in $k(s)[t^{-1}]$ and some element r_π of $\mathfrak{m}$. Since T is henselian, there exists a unique element u_π of $\mathfrak{m}$ such that $r_\pi = u_\pi - u_\pi^p$. Then the $\mathbb{F}_p$-torsor over $\mathbb{G}_{m,s}$ defined by the equation

$$y - y^p = h_\pi(t), \tag{26}$$

is unramified at ∞ and its pullback to η by $\pi_{|\eta}$ is isomorphic to (25). Therefore (26) is (up to isomorphism) the $\mathbb{F}_p$-torsor over $\mathbb{G}_{m,s}$ associated to (25) by the equivalence in Theorem III.4.4.

The Laurent polynomial $h_\pi(t)$ admits the following alternative description. Let $\nu \geq 1$ be an integer such that h is of valuation strictly larger than $-\nu$, and let u be the generator $1 \otimes 1 - t \otimes \pi^{-1}$ of the $\mathcal{O}_T/\mathfrak{m}^\nu[t,t^{-1}]$-module $k(s)[t,t^{-1}] \otimes_{k(s)} \mathfrak{m}^{-1}/\mathfrak{m}^{-1+\nu}$. Then we have

$$h_\pi(t) = -\mathrm{Res}\left(h\frac{du}{u}\right), \tag{27}$$

in $k(s)[t,t^{-1}]$. This formula should be understood as follows: we first take a lift $\tilde{u}$ of u in $\pi^{-1}A[[\pi]]$, where $A = k(s)[t,t^{-1}]$, so that $\tilde{u}$ is invertible in $A((\pi))$, and we then set

$$\mathrm{Res}\left(h\frac{du}{u}\right) = \mathrm{Res}\left(h\frac{d\tilde{u}}{\tilde{u}}\right),$$

where the right hand side is the specialization to $A = k(s)[t,t^{-1}]$ and $r = 1$ of the following definition:

Definition III.4.6. For any $k(s)$-algebra A, any non negative integer r and any element $w = \sum_{n<r} w_n \otimes \pi^n$ of $A \otimes k(\eta)/\mathfrak{m}^r$, we define

$$dw = \sum_{n<r} w_n \otimes n\pi^{n-1} d\pi,$$

in $A \otimes (k(\eta)/\mathfrak{m}^{r-1})d\pi$, and

$$\mathrm{Res}(wd\pi) = w_{-1}.$$

Let us prove (27). We consider the lift $\tilde{u} = 1 - t\pi^{-1}$ of u in $\pi^{-1}A[[\pi]]$. We have

$$\begin{aligned}\frac{d\tilde{u}}{\tilde{u}} &= \frac{td\pi}{\pi(\pi - t)} \\ &= -\frac{d\pi}{\pi(1 - t^{-1}\pi)} \\ &= -\sum_{n\geq 0} t^{-n}\pi^n \frac{d\pi}{\pi},\end{aligned}$$

in $A((\pi))d\pi$. If we write the image of h in $\mathfrak{m}^{1-\nu}/\mathfrak{m}$ as $\sum_{n=0}^{\nu-1} h_n\pi^{-n}$ for some elements $(h_n)_{0\leq n<\nu}$ of $k(s)$, then this yields

$$\begin{aligned}\mathrm{Res}\left(h\frac{du}{u}\right) &= -\sum_{0\leq n<\nu} t^{-n}\otimes \mathrm{Res}\left(h\pi^n\frac{d\pi}{\pi}\right)\\ &= -\sum_{0\leq n<\nu} h_n t^{-n},\end{aligned}$$

and the latter Laurent polynomial is exactly $-h_\pi(t)$, hence (27).

III.4.7. Let $f : U \to \mathbb{G}_{m,s}$ be a connected special finite étale cover of $\mathbb{G}_{m,s}$ (cf. III.4.3). Let $\eta' \to \eta$ be the pullback of U to η by π, and let us fix an η-morphism $\overline{\eta} \to \eta'$, so that we can consider $\overline{\eta}$ as a geometric point of U, henceforth denoted $\overline{\eta}'$. Similarly, let s' be a finite étale extension of s such that U is a geometrically connected s'-scheme, and let us fix an s-morphism $\overline{s} \to s'$, so that we can consider $\overline{s}$ as a geometric point, denoted $\overline{s}'$, of the normalization T' of T in η', or as a geometric point, also denoted $\overline{s}'$, of the normalization X of $\mathbb{A}^1_s$ in U. We have a natural morphism

$$\pi' : T' \to X,$$

whose restriction to η' is the natural morphism from η' to U.

Definition III.4.8. A finite étale morphism $V \to U$ is *f-special* if the composition

$$V \to U \xrightarrow{f} \mathbb{G}_{m,s}$$

is special (cf. III.4.3). We denote by $\mathrm{F\acute{e}t}^{\diamondsuit}(U, f)$ the category of f-special finite étale U-schemes, or equivalently the category of U-objects in $\mathrm{F\acute{e}t}^{\diamondsuit}(\mathbb{G}_{m,s})$.

Let us consider the fiber functor

$$\begin{aligned}\mathrm{F\acute{e}t}^{\diamondsuit}(U, f) &\to \mathrm{Sets}\\ (V \to U) &\mapsto V_{\overline{\eta}'} = \mathrm{Hom}_{\eta'}(\overline{\eta}', V).\end{aligned} \tag{28}$$

Let $\pi_1(U, f, \overline{\eta}')^{\diamondsuit}$ be the group of automorphisms of this functor, endowed with the coarsest topology such that for any object $V \to U$ of $\mathrm{F\acute{e}t}^{\diamondsuit}(U, f)$, the natural group homomorphism

$$\pi_1(U, f, \overline{\eta}')^{\diamondsuit} \to \mathrm{Aut}(V_{\overline{\eta}'}),$$

is continuous, when the finite set $\mathrm{Aut}(V_{\overline{\eta}'})$ is endowed with the discrete topology.

The topological group $\pi_1(U, f, \overline{\eta}')^{\diamondsuit}$ is profinite, and we have a natural surjective homomorphism

$$\pi_1(U, \overline{\eta}') \to \pi_1(U, f, \overline{\eta}')^{\diamondsuit}.$$

Moreover, the fiber functor (28) realizes an equivalence from $\mathrm{F\acute{e}t}^{\diamondsuit}(U, f)$ to the category of finite sets endowed with a continuous left action of $\pi_1(U, f, \overline{\eta}')^{\diamondsuit}$, cf. ([**Ka86**], 1.3.3). The natural surjective homomorphism

$$\pi_1(U, \overline{\eta}') \to G_{s'},$$

factors through $\pi_1(U, f, \overline{\eta}')^{\diamondsuit}$, and we denote by $\pi_1(U_{\overline{s}'}, f, \overline{\eta}')^{\diamondsuit}$ the kernel of the resulting surjective homorphism from $\pi_1(U, f, \overline{\eta}')^{\diamondsuit}$ to $G_{s'}$.

By Theorem III.4.4, the homomorphism

$$(\pi'_{|\eta'})_* : G_{\eta'} \to \pi_1(U, f, \overline{\eta}')^{\diamondsuit}, \tag{29}$$

is an isomorphism of profinite topological groups.

III.4.9. Let U, f, X be as in III.4.7. Let Λ be a finite admissible ℓ-adic ring (cf. III.2.2), and let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9). The category $\mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda)$ of *f-special* μ-twisted Λ-sheaves on X is the full subcategory of $\mathrm{Sh}(X, \mu, \Lambda)$ (cf. III.3.7) whose objects are the μ-twisted Λ-sheaves $\mathcal{F} = (\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g \in \mathrm{Gal}(k'/k)})$ on X such that:

(1) The restriction of $\mathcal{F}_{k'}$ to $U_{k'}$ is a Λ-local system (cf. III.3.2).

(2) The restriction of $\mathcal{F}_{k'}$ to any (equivalently, some) connected component of $U_{k'}$ is trivialized on a finite étale cover which is f-special, cf. III.4.8. Equivalently, the Λ-admissible representation $\mathcal{F}_{\overline{\eta}'}$ of $(\pi_1(U, \overline{\eta}'), \mu)$ (cf. III.3.19) factors through a Λ-admissible representation of $(\pi_1(U, f, \overline{\eta}')^{\diamondsuit}, \mu)$ (cf. III.4.7).

III.4.10. Let U, f, X be as in III.4.7. Let Λ be the ring of integers in a finite subextension of $\mathbb{Q}_\ell$ in C, and let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9). The category $\mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda)$ of *f-special* μ-twisted Λ-sheaves on X is the full subcategory of $\mathrm{Sh}(X, \mu, \Lambda)$ (cf. III.3.11) consisting of its objects $(\mathcal{F}_n)_n$, where $\mathcal{F}_n$ is a μ-twisted Λ/ℓ^n-sheaf on X, such that $\mathcal{F}_n$ is special for each n (cf. III.4.9).

III.4.11. Let U, f, X be as in III.4.7. Let Λ be a finite subextension of $\mathbb{Q}_\ell$ in C, with ring of integers Λ_0, and let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9). Then μ takes its values in $\Lambda_0^\times$. We define the category $\mathrm{Sh}^{\diamondsuit}(X, \mu, \Lambda)$ of *f-special* μ-twisted Λ-sheaves on X to be the essential image of $\mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda_0)$ (cf. III.4.10) in $\mathrm{Sh}(X, \mu, \Lambda)$ (cf. III.3.12).

III.4.12. Let U, f, X be as in III.4.7. Let Λ be an ℓ-adic coefficient ring (cf. III.2.2), and let μ be a Λ-admissible unitary multiplier on G_k. We define the category $\mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda)$ of *f-special* μ-twisted Λ-sheaves on X to be the full subcategory of $\mathrm{Sh}(X, f, \mu, \Lambda)$ (cf. III.3.13) whose objects belong to the essential image of $\mathrm{Sh}(X, f, \mu, \Lambda_0)^{\diamondsuit}$ (cf. III.4.9, III.4.10, III.4.11), for some admissible ℓ-adic subring Λ_0 of Λ containing the image of μ.

We define the category $\mathrm{Sh}^{\diamondsuit}(U, f, \mu, \Lambda)$ of *f-special* μ-twisted Λ-sheaves on U to be the full subcategory of $\mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda)$ consisting f-special μ-twisted Λ-sheaves supported on the open subscheme U of X.

III.4.13. Let U, f, X, η', s' be as in III.4.7, and let Λ and μ be as in III.4.12. By III.3.19 and III.4.7, the fiber functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{\eta}'}$ realizes an equivalence from the category of f-special μ-twisted Λ-sheaves on U (cf. III.4.12) to the category of Λ-admissible representations of $(\pi_1(U, f, \overline{\eta}')^{\diamondsuit}, \mu)$.

More generally, for any object $\mathcal{F}$ of the category $\mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda)$, we have a cospecialization homomorphism

$$c_{\mathcal{F}, \overline{s}', \overline{\eta}} : \mathcal{F}_{\overline{s}'} \to \mathcal{F}_{\overline{\eta}'},$$

and the functor $\mathcal{F} \mapsto (\mathcal{F}_{\overline{s}'}, \mathcal{F}_{\overline{\eta}'}, c_{\mathcal{F}, \overline{s}', \overline{\eta}'})$ realizes an equivalence of categories from $\mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda)$ to the category of triples $(V_{s'}, V_{\eta'}, c)$, where $V_{\eta'}$ is a Λ-admissible representation of $(\pi_1(U, f, \overline{\eta}')^{\diamondsuit}, \mu)$, where $V_{s'}$ is a Λ-admissible representation of $(G_{s'}, \mu)$, and where

$$c : V_{s'} \to V_{\eta'}^{\pi_1(U_{\overline{s}'}, f, \overline{\eta}')^{\diamondsuit}},$$

is a homomorphism of Λ-admissible representations of $(G_{s'}, \mu)$ from $V_{s'}$ to the subrepresentation of $V_{\eta'}$ consisting of its $\pi_1(U_{\overline{s}'}, f, \overline{\eta}')^{\diamondsuit}$-invariant elements (cf. III.4.7).

III.4.14. Let U, f, X, T', η', s' be as in III.4.7, and let Λ and μ be as in III.4.12. By III.3.19, the fiber functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{\eta}'}$ realizes an equivalence from the category of μ-twisted Λ-sheaves on η' (cf. III.3.7) to the category of Λ-admissible representations of $(G_{\eta'}, \mu)$.

More generally, the functor $\mathcal{F} \mapsto (\mathcal{F}_{\overline{s}'}, \mathcal{F}_{\overline{\eta}'}, c_{\mathcal{F}, \overline{s}', \overline{\eta}'})$, where $c_{\mathcal{F}, \overline{s}', \overline{\eta}'} : \mathcal{F}_{\overline{s}'} \to \mathcal{F}_{\overline{\eta}'}$ is the cospecialization homomorphism, realizes an isomorphism from $\mathrm{Sh}(T', \mu, \Lambda)$ to the category of triples $(V_{s'}, V_{\eta'}, c)$, where $V_{\eta'}$ is a Λ-admissible representation of $(G_{\eta'}, \mu)$, where $V_{s'}$ is a Λ-admissible representation of $(G_{s'}, \mu)$, and where

$$c : V_{s'} \to V_{\eta'}^{G_{\eta_{\overline{s}'}}},$$

is a homomorphism of Λ-admissible representations of $(G_{s'}, \mu)$ from $V_{s'}$ to the subrepresentation of $V_{\eta'}$ consisting of its $G_{\eta_{\overline{s}'}}$-invariant elements.

This remark, combined with the Galoisian description of f-special μ-twisted Λ-sheaves on X (cf. III.4.13) and with the isomorphism (29), yields:

Theorem III.4.15. *Let U, f, X, T', π' be as in III.4.7, let Λ be an ℓ-adic coefficient ring (cf. III.2.2), and let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9). The pullback functor*

$$(\pi')^{-1} : \mathrm{Sh}^{\diamondsuit}(X, f, \mu, \Lambda) \to \mathrm{Sh}(T', \mu, \Lambda),$$

is an equivalence from the category of f-special μ-twisted Λ-sheaves on X (cf. III.4.12) to the category of μ-twisted Λ-sheaves on T'.

By restricting to μ-twisted Λ-sheaves with vanishing fiber at s', we similarly obtain:

Theorem III.4.16. *Let U, f, η' be as in III.4.7, let Λ be an ℓ-adic coefficient ring (cf. III.2.2), and let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9). The pullback functor*

$$(\pi'_{|\eta})^{-1} : \mathrm{Sh}^{\diamondsuit}(U, f, \mu, \Lambda) \to \mathrm{Sh}(\eta', \mu, \Lambda),$$

is an equivalence from the category of f-special μ-twisted Λ-sheaves on U (cf. III.4.12) to the category of μ-twisted Λ-sheaves on η'.

III.4.17. For $U = \mathbb{G}_{m,s}$ and $f = \mathrm{id}$, let us simply refer to f-special sheaves as special sheaves. This agrees with the terminology in ([**Ka86**], 1.5). As a particular case of Theorem III.4.15, we have the following extension result:

Theorem III.4.18. *Let Λ be an ℓ-adic coefficient ring (cf. III.2.2), and let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9). The pullback functor*

$$\pi^{-1} : \mathrm{Sh}^{\diamondsuit}(\mathbb{A}^1_s, \mu, \Lambda) \to \mathrm{Sh}(T, \mu, \Lambda),$$

is an equivalence from the category of special μ-twisted Λ-sheaves on $\mathbb{A}^1_s$ (cf. III.4.12) to the category of μ-twisted Λ-sheaves on T.

By restricting to μ-twisted Λ-sheaves with vanishing special fiber we similarly obtain:

Theorem III.4.19. *Let Λ be an ℓ-adic coefficient ring (cf. III.2.2), and let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9). The pullback functor*

$$\pi_{|\eta}^{-1} : \mathrm{Sh}^{\diamondsuit}(\mathbb{G}_{m,s}, \mu, \Lambda) \to \mathrm{Sh}(\eta, \mu, \Lambda),$$

is an equivalence from the category of special μ-twisted Λ-sheaves on $\mathbb{G}_{m,s}$ (cf. III.4.12) to the category of μ-twisted Λ-sheaves on η.

When $\mu = 1$, the latter theorem matches ([**Ka86**], Th. 1.5.6).

III.4.20. Let Λ be an ℓ-adic coefficient ring (cf. III.2.2), let μ be a Λ-admissible unitary multiplier on G_k (cf. III.2.9), and let U, f, η' be as in III.4.7.

Lemma III.4.21. *If $\mathcal{F}$ is an object of $\mathrm{Sh}^{\diamondsuit}(U, f, \mu, \Lambda)$, then its pushforward $f_*\mathcal{F}$ (cf. III.3.14) belongs to $\mathrm{Sh}^{\diamondsuit}(\mathbb{G}_{m,s}, \mu, \Lambda)$.*

We can assume (and we do) that Λ is finite, in which case the Λ-local system $\mathcal{F}$ is represented by a finite étale morphism $g : V \to U$ which is f-special (cf. III.4.8). The Λ-local system $f_*\mathcal{F}$ is then represented by the finite étale morphism $fg : V \to \mathbb{G}_{m,s}$, which is special, hence the conclusion of Lemma III.4.21.

It follows from Lemma III.4.21 that we have the following commutative diagram (up to natural isomorphisms).

$$\begin{tikzcd}
\mathrm{Sh}(\eta', \mu, \Lambda) \arrow[d, "(f_{|\eta'})_*"] & \mathrm{Sh}^{\diamondsuit}(U, f, \mu, \Lambda) \arrow[l, "(\pi'_{|\eta})^{-1}"'] \arrow[d, "f_*"] \\
\mathrm{Sh}(\eta, \mu, \Lambda) & \mathrm{Sh}^{\diamondsuit}(\mathbb{G}_{m,s}, \mu, \Lambda) \arrow[l, "\pi_{|\eta}^{-1}"']
\end{tikzcd}$$

The rows of this diagram are equivalences of categories by Theorems III.4.16 and III.4.19.

III.5. Geometric class field theory

We review in this section global and local geometric class field theory. Let Λ be an ℓ-adic coefficient ring (cf. III.1.13, III.2.2). The purpose of geometric class field theory is to establish equivalences between groupoids of Λ-local systems of rank 1 on curves over k, or over germs of curves, and groupoids of multiplicative local systems over certain group schemes. The notion of multiplicative local system, which geometrizes the notion of character, is reviewed in III.5.1, III.5.4, III.5.5 and III.5.6 below.

III.5.1. Let us assume that Λ is a finite ℓ-adic coefficient ring (cf. III.2.2). Let S be a k-scheme and let G be a commutative S-group scheme, with multiplication $m : G \times_S G \to G$. A *multiplicative* Λ*-local system on* G is a Λ-local system $\mathcal{L}$ on G, of rank 1, together with an isomorphism $\theta : p_1^{-1}\mathcal{L} \otimes p_2^{-1}\mathcal{L} \to m^{-1}\mathcal{L}$ of Λ-local systems on $G \times G$ where p_1 and p_2 are the canonical projections, which satisfy the following two properties, cp. ([**Gu18**], Def. 2.5).

(1) Symmetry: if σ is the involution of $G \times G$ which switches the two factors, then the isomorphism
$$p_2^{-1}\mathcal{L} \otimes p_1^{-1}\mathcal{L} \to \sigma^{-1}(p_1^{-1}\mathcal{L} \otimes p_2^{-1}\mathcal{L}) \xrightarrow{\sigma^{-1}\theta} \sigma^{-1}m^{-1}\mathcal{L} \to m^{-1}\mathcal{L}$$
is the composition of θ with the canonical isomorphism $p_2^{-1}\mathcal{L} \otimes p_1^{-1}\mathcal{L} \to p_1^{-1}\mathcal{L} \otimes p_2^{-1}\mathcal{L}$.

(2) Associativity: if $q_i : G \times G \times G \to G$ (resp. $q_{ij} : G \times G \times G \to G \times G$) is the projection on the i-th factor for $i \in [\![1,3]\!]$ (resp. on the i-th and j-th factors for $(i,j) \in [\![1,3]\!]^2$ such that $i < j$) and if $m_3 : G \times G \times G \to G$ is the multiplication morphism, then the diagram of Λ-local systems on $G \times G \times G$

$$\begin{tikzcd}
 & q_1^{-1}\mathcal{L} \otimes (mq_{23})^{-1}\mathcal{L} \arrow[dr, "(q_1 \times mq_{23})^{-1}\theta"] & \\
q_1^{-1}\mathcal{L} \otimes q_2^{-1}\mathcal{L} \otimes q_3^{-1}\mathcal{L} \arrow[ur, "id \otimes q_{23}^{-1}\theta"] \arrow[dr, "q_{12}^{-1}\theta \otimes id"'] & & m_3^{-1}\mathcal{L} \\
 & (mq_{12})^{-1}\mathcal{L} \otimes q_3^{-1}\mathcal{L}. \arrow[ur, "(mq_{12} \times q_3)^{-1}\theta"'] &
\end{tikzcd}$$

is commutative.

A morphism between multiplicative Λ-local systems $(\mathcal{L}_1, \theta_1)$ and $(\mathcal{L}_2, \theta_2)$ on G is an isomorphism $\alpha : \mathcal{L}_1 \to \mathcal{L}_2$ of Λ-local systems such that the diagram

$$\begin{tikzcd} p_1^{-1}\mathcal{L}_2 \otimes p_2^{-1}\mathcal{L}_2 \arrow[r, "\theta_2"] & m^{-1}\mathcal{L}_2 \\ p_1^{-1}\mathcal{L}_1 \otimes p_2^{-1}\mathcal{L}_1 \arrow[r, "\theta_1"] \arrow[u, "p_1^{-1}\alpha \otimes p_2^{-1}\alpha"] & m^{-1}\mathcal{L}_1. \arrow[u, "m^{-1}\alpha"'] \end{tikzcd}$$

is commutative.

We denote by $\mathrm{Loc}^{\otimes}(G, \Lambda)$ the groupoid of multiplicative Λ-local systems on G. The group of automorphisms of an object of $\mathrm{Loc}^{\otimes}(G, \Lambda)$ is given by

$$\mathrm{Aut}_{\mathrm{Loc}^{\otimes}(G,\Lambda)}(\Lambda) = \mathrm{Hom}_{\mathrm{Grp}/S}(G, \Lambda_S^{\times}),$$

where $\Lambda_S^{\times}$ is the constant S-group scheme associated to $\Lambda^{\times}$. If S is connected and if G is an extension of a constant S-group scheme, associated to a discrete group $\pi_0(G)$, by an S-group scheme with connected geometric fibers, then we also have

$$\mathrm{Aut}_{\mathrm{Loc}^{\otimes}(G,\Lambda)}(\Lambda) = \mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G), \Lambda^{\times}).$$

Remark III.5.2. The functor $\mathcal{L} \mapsto \mathcal{I}som(\Lambda, \mathcal{L})$ which sends a multiplicative Λ-local system $\mathcal{L}$ on G to the $\Lambda^{\times}$-torsor of its local trivializations realizes an equivalence from the category $\mathrm{Loc}^{\otimes}(G, \Lambda)$, to the groupoid of multiplicative $\Lambda^{\times}$-torsors on G in the sense of ([**Gu18**], Def. 2.5). By ([**Gu18**], Def. 2.9), the latter groupoid is equivalent to the groupoid of extensions of G by $\Lambda^{\times}$ in the category of commutative S-group schemes. Namely, to an extension

$$0 \to \Lambda^{\times} \to E \to G \to 0,$$

of commutative S-group schemes, one associate the Λ-local system of rank 1 on G corresponding to the $\Lambda^{\times}$-torsor E over G, where $\Lambda^{\times}$ acts by left multiplication on E; this Λ-local system of rank 1 on G is naturally endowed with a structure of multiplicative Λ-local system on G, cf. ([**Gu18**], Def. 2.4).

Example III.5.3. Assume that k is finite of cardinality q, and that G is a connected commutative k-group scheme. The q-Frobenius morphism $F : G \to G$ is then an homomorphism of k-group schemes. Moreover, the sequence

$$0 \to G(k) \to G \xrightarrow{1-F} G \to 0,$$

is exact, cf. ([**La90**], 1.1.3). For any homomorphism $\chi : G(k) \to \Lambda^{\times}$, the pushout of this exact sequence provides an extension of G by $\Lambda^{\times}$, which yields in turn a multiplicative Λ-local system $\mathcal{L}_{\chi}$ on G by Remark III.5.2.

III.5.4. Let us assume that Λ is the ring of integers in a finite extension of $\mathbb{Q}_{\ell}$. Let S be a connected k-scheme and let G be a commutative S-group scheme, which is an extension of a constant S-group scheme, associated to a discrete group $\pi_0(G)$, by an S-group scheme with connected geometric fibers. We define the groupoid $\mathrm{Loc}^{\otimes}(G, \Lambda)$ of multiplicative Λ-local systems on G to be the 2-limit of the categories $\mathrm{Loc}^{\otimes}(G, \Lambda/\ell^n)$ (cf. III.5.1), where n runs over over all positive integers. Thus the objects of $\mathrm{Loc}^{\otimes}(G, \Lambda)$ are given by projective systems $(\mathcal{L}_n)_{n \geq 1}$ where $\mathcal{L}_n$ is a multiplicative Λ/ℓ^n-local system on G, such that the transition maps $\mathcal{L}_{n+1} \to \mathcal{L}_n$ induce isomorphisms $\mathcal{L}_{n+1} \otimes \Lambda/\ell^n \to \mathcal{L}_n$, for each integer n.

The group of automorphisms of an object of $\mathrm{Loc}^{\otimes}(G, \Lambda)$ is given by

$$\mathrm{Aut}_{\mathrm{Loc}^{\otimes}(G,\Lambda)}(\Lambda) = \lim_{n} \mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G), (\Lambda/\ell^n)^{\times}) = \mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G), \Lambda^{\times}).$$

III.5.5. Let us assume that Λ is a finite extension of $\mathbb{Q}_\ell$, with ring of integers $\Lambda_0 \subseteq \Lambda$. Let S be a connected k-scheme and let G be a commutative S-group scheme, which is an extension of a constant S-group scheme, associated to a discrete group $\pi_0(G)$, by an S-group scheme with connected geometric fibers. We define the groupoid $\mathrm{Loc}^{\otimes}(G,\Lambda)$ of multiplicative Λ-local systems on G to be the groupoid whose objects are those of $\mathrm{Loc}^{\otimes}(G,\Lambda_0)$ (cf. III.5.4), and whose morphisms are given by

$$\mathrm{Isom}_\Lambda(\mathcal{L}_1,\mathcal{L}_2) = \mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G),\Lambda^\times) \wedge \mathrm{Isom}_{\Lambda_0}(\mathcal{L}_1,\mathcal{L}_2),$$

namely the quotient of $\mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G),\Lambda^\times)\times\mathrm{Isom}_{\Lambda_0}(\mathcal{L}_1,\mathcal{L}_2)$ by the action of $\mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G),\Lambda_0^\times)$ given by $u(\lambda,\varphi) = (u^{-1}\lambda, u\varphi)$ for u in $\mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G),\Lambda_0^\times)$ and (λ,φ) in $\mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G),\Lambda^\times)\times \mathrm{Isom}_{\Lambda_0}(\mathcal{L}_1,\mathcal{L}_2)$. In particular, the group of automorphisms of an object of $\mathrm{Loc}^{\otimes}(G,\Lambda)$ is given by

$$\mathrm{Aut}_{\mathrm{Loc}^{\otimes}(G,\Lambda)}(\Lambda) = \mathrm{Hom}_{\mathrm{Grp}}(\pi_0(G),\Lambda^\times).$$

Isomorphisms between multiplicative Λ-local systems $\mathcal{L}_1$ and $\mathcal{L}_2$ on G can be alternatively described as isomorphisms $\alpha : \mathcal{L}_1 \to \mathcal{L}_2$ of Λ-local systems (cf. III.3.4) such that the diagram

$$\begin{tikzcd}
p_1^{-1}\mathcal{L}_2 \otimes p_2^{-1}\mathcal{L}_2 \arrow[r, "\theta_2"] & m^{-1}\mathcal{L}_2 \\
p_1^{-1}\mathcal{L}_1 \otimes p_2^{-1}\mathcal{L}_1 \arrow[r, "\theta_1"] \arrow[u, "p_1^{-1}\alpha\otimes p_2^{-1}\alpha"] & m^{-1}\mathcal{L}_1. \arrow[u, "m^{-1}\alpha"']
\end{tikzcd}$$

is commutative. Indeed, if π is a uniformizer of Λ, then such an isomorphism α must be of the form $\alpha = \pi^v\varphi$, where $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphism of Λ_0-local systems, and where v is a map from $\pi_0(G)$ to $\mathbb{Z}$. The commutativity of the diagram above implies that $d^1(\pi^v) = \pi^{d^1(v)}$ (cf. 15) takes its values in $\Lambda_0^\times$, hence is trivial. Thus $\lambda = \pi^v$ is a group homomorphism from $\pi_0(G)$ to $\Lambda^\times$ and φ is an isomorphism of multiplicative Λ_0-local systems.

III.5.6. We now consider an arbitrary ℓ-adic coefficient ring Λ. Let S be a connected k-scheme and let G be a commutative S-group scheme, which is an extension of a constant S-group scheme, associated to a discrete group $\pi_0(G)$, by an S-group scheme with connected geometric fibers. We define the groupoid $\mathrm{Loc}^{\otimes}(G,\Lambda)$ of multiplicative Λ-local systems on G to be the 2-colimit of the groupoids $\mathrm{Loc}^{\otimes}(G,\Lambda_0)$, where Λ_0 runs over all admissible ℓ-adic subrings of Λ (cf. III.5.1, III.5.4 and III.5.5). The group of automorphisms of an object of $\mathrm{Loc}^{\otimes}(G,\Lambda)$ is given by the group of Λ-*admissible* group homomorphisms from $\pi_0(G)$ to $\Lambda^\times$ (cf. III.2.5).

Remark III.5.7. If the discrete group $\pi_0(G)$ is finitely generated, then any group homomorphism from $\pi_0(G)$ to $\Lambda^\times$ is Λ-admissible.

Example III.5.8. Let us consider the discrete group scheme $G = \mathbb{Z}_S$. For each integer n, let $n : S \to \mathbb{Z}_S$ be the section corresponding to the element n of $\mathbb{Z}$. Then the pullback by the section 1 realizes an equivalence from $\mathrm{Loc}^{\otimes}(\mathbb{Z}_S,\Lambda)$ to the groupoid of Λ-local systems of rank 1 on S. A quasi-inverse to this functor is given by sending a Λ-local system $\mathcal{F}$ of rank 1 on S to the multiplicative Λ-local system on $\mathbb{Z}_S$ whose pullback by the section n is $\mathcal{F}^{\otimes n}$, for any integer n, together with the isomorphism $\theta : p_1^{-1}\mathcal{L}\otimes p_2^{-1}\mathcal{L} \to m^{-1}\mathcal{L}$, whose pullback by a section (n,m) of $\mathbb{Z}_S \times_S \mathbb{Z}_S$ is the canonical isomorphism

$$\mathcal{F}^{\otimes n}\otimes\mathcal{F}^{\otimes m} \to \mathcal{F}^{\otimes(n+m)}.$$

III.5.9. Let S be a connected k-scheme and let G_1, G_2 be commutative S-group schemes as in III.5.6. Then $G_1 \times_S G_2$ is an extension of the discrete group $\pi_0(G_1) \times \pi_0(G_2)$ by an S-group scheme with connected geometric fibers. Let p_1, p_2 be the natural projections from $G_1 \times_S G_2$ to G_1 and G_2 respectively. Then the functor

$$\begin{aligned} \mathrm{Loc}^{\otimes}(G_1, \Lambda) \times \mathrm{Loc}^{\otimes}(G_2, \Lambda) &\to \mathrm{Loc}^{\otimes}(G_1 \times_S G_2, \Lambda) \\ (\mathcal{L}_1, \mathcal{L}_2) &\mapsto p_1^{-1}\mathcal{L}_1 \otimes p_2^{-1}\mathcal{L}_2, \end{aligned}$$

is an equivalence of categories. Indeed, if ι_1, ι_2 are the inclusions of the factors G_1 and G_2 respectively in $G_1 \times_S G_2$, then for any multiplicative Λ-local system $(\mathcal{L}, \theta)$ on $G_1 \times_S G_2$, the isomorphism θ produces an isomorphism

$$\mathcal{L} \to p_1^{-1}\iota_1^{-1}\mathcal{L} \otimes p_2^{-1}\iota_2^{-1}\mathcal{L}.$$

III.5.10. Let G be a commutative k-group scheme as in III.5.6. Let $i : H \to G$ be a closed connected sub-k-group scheme of G, so that G/H is also an extension of the constant k-group scheme $\pi_0(G)$ by a quasi-compact connected k-group scheme. Let $r : G \to G/H$ be the canonical projection. Then the pullback functor r^{-1} induces an equivalence from the groupoid

$$\mathrm{Loc}^{\otimes}(G/H, \Lambda)$$

of multiplicative Λ-local systems on G/H (cf. III.5.6) to the groupoid of triples $(\mathcal{L}, \theta, \zeta)$, where $(\mathcal{L}, \theta)$ is a multiplicative Λ-local systems on G and $\zeta : \Lambda_H \to i^{-1}\mathcal{L}$ is an isomorphism of multiplicative Λ-local systems on H. Indeed, if $(\mathcal{L}, \theta, \zeta)$ is such a triple, if p_1, p_2, m are as in III.5.1, and if φ is the isomorphism

$$\begin{aligned} \varphi : G \times_k H &\to G \times_{G/H} G \\ (g, h) &\to (g, gh) \end{aligned}$$

then we have a sequence of isomorphisms

$$\varphi^{-1}p_1^{-1}\mathcal{L} \to p_1^{-1}\mathcal{L}_{|G\times_k H} \xrightarrow{\mathrm{id}\otimes\zeta} p_1^{-1}\mathcal{L} \otimes p_2^{-1}\mathcal{L}_{|G\times_k H} \xrightarrow{\theta_{|G\times_k H}} m^{-1}\mathcal{L}_{|G\times_k H} \to \varphi^{-1}p_2^{-1}\mathcal{L},$$

which yields a descent datum $p_1^{-1}\mathcal{L} \to p_2^{-1}\mathcal{L}$ on $G \times_{G/H} G$, with respect r, which is a morphism of effective descent for the fibered category of Λ-local systems, as well as for the fibered category of multiplicative Λ-local systems.

III.5.11. Let X be a smooth geometrically connected projective curve of genus g over k, let $i : D \to X$ be an effective divisor of degree $d \geq 1$ on X, and let U be the open complement of D in X. Our aim is to describe Λ-local systems of rank 1 on U. One first introduces a measure of the ramification at infinity of such a local system:

Definition III.5.12. A Λ-local system $\mathcal{F}$ of rank 1 on U has *ramification bounded by D* if for any point x of D, the Swan conductor of the restriction of F to the spectrum of the fraction field of the completed local ring of X at x is strictly less than the multiplicity of D at x.

The main theorem of geometric class field theory, namely Theorem III.5.15 below, states an equivalence between the groupoid of Λ-local systems of rank 1 on U with ramification bounded by D, and the groupoid of multiplicative Λ-local systems on a k-group scheme, the generalized Picard scheme, which we now introduce:

Definition III.5.13. The generalized Picard functor $\mathrm{Pic}_k(X, D)$ associated to (X, D) is the functor which to a k-scheme S associates the group of isomorphism classes of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_{X_S}$-module and $\alpha : \mathcal{O}_{D_S} \to i_S^*\mathcal{L}$ is an isomorphism of $\mathcal{O}_{D_S}$-modules. Here, by X_S, D_S, i_S we denote the base change of X, D, i along $S \to \mathrm{Spec}(k)$.

If the effective divisor D is given by a single k-point x of X with multiplicity 1, then the morphism

$$\begin{aligned} \mathrm{Pic}_k(X,x) &\to \mathrm{Pic}_k(X) \\ (\mathcal{L},\alpha) &\to \mathcal{L}, \end{aligned}$$

is an isomorphism, where $\mathrm{Pic}_k(X)$ is the Picard functor of X. The latter is well-known to be representable by a k-group scheme, namely an extension of $\mathbb{Z}_k$ by the Jacobian scheme of X, which is an abelian k-scheme of dimension g. In general, we have:

Proposition III.5.14 ([**Gu18**], Prop. 4.8). *The generalized Picard functor* $\mathrm{Pic}_k(X,D)$ *is representable by a smooth separated k-group scheme of dimension $d+g-1$.*

Let us consider the Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_k(X,D), \tag{30}$$

which sends a section x of U to the pair $(\mathcal{O}(x),1)$, where $1:\mathcal{O}_D \to \mathcal{O}(x)\otimes_{\mathcal{O}_X}\mathcal{O}_D$ is the trivialization of $\mathcal{O}(x)$ on D induced by the canonical section $1:\mathcal{O}_X \hookrightarrow \mathcal{O}(x)$. Global geometric class field theory can then be stated as follows:

Theorem III.5.15 (Global geometric class field theory). *Let $\mathcal{F}$ be a Λ-local system of rank 1 on U, with ramification bounded by D (cf. III.5.12). Then, there exists a unique (up to unique isomorphism) pair $(\chi_{\mathcal{F}},\beta)$, where $\chi_{\mathcal{F}}$ is a multiplicative Λ-local system on* $\mathrm{Pic}_k(X,D)$ *(cf. III.5.6), and $\beta:\Phi^{-1}\chi_{\mathcal{F}}\to\mathcal{F}$ is an isomorphism. The functor $\mathcal{F}\mapsto\chi_{\mathcal{F}}$ is an equivalence from the groupoid of Λ-local systems of rank 1 on U, with ramification bounded by D, to the groupoid of multiplicative Λ-local systems on* $\mathrm{Pic}_k(X,D)$.

This theorem reduces to the case where Λ is finite, which was originally proved by Serre and Lang, cf. ([**La56**], 6) and [**Se59**], by using the Albanese property of Rosenlicht's generalized Picard schemes [**Ro54**]. Deligne gave another proof in the tamely ramified case. An exposition of Deligne's proof in the unramified case over a finite field can be found in [**La90**]. Deligne's approach was later extended to allow arbitrary ramification simultaneously by the author ([**Gu18**], Th. 1.1) and by Takeuchi ([**Ta18**], Th. 1.1).

III.5.16. Let T be the spectrum of a k-algebra, which is a henselian discrete valuation ring whose residue field is a finite extension of k. Let η be the generic point of T, and let s be its closed point, so that $k(\eta)$ is a henselian discrete valuation field, with valuation subring $\mathcal{O}_{T,s}$, and with residue field $k(s)$ which is a finite extension of k. By Hensel's lemma, there exists a unique morphism $T\to s$ of k-schemes whose composition with the immersion $s\to T$ is the identity. We can thus consider T as an s-scheme.

Definition III.5.17. Let D be a closed subscheme of T supported on s. A Λ-local system $\mathcal{F}$ of rank 1 on η has *ramification bounded by D* if its Swan conductor is strictly less than the multiplicity of D at s, namely the length of $\mathcal{O}_{D,s}$ as an $\mathcal{O}_{T,s}$-module.

Before proceeding further, we need the following result:

Proposition III.5.18. *Let D be a closed subscheme of T supported on s. The kernel $\mathcal{I}$ of the homomorphism*

$$\begin{aligned} \mathcal{O}_T\otimes_{k(s)}\mathcal{O}_D &\to \mathcal{O}_D \\ f_1\otimes f_2 &\to f_1f_2, \end{aligned}$$

is an invertible ideal of $\mathcal{O}_T\otimes_{k(s)}\mathcal{O}_D$, which generates the unit ideal of $k(\eta)\otimes_{k(s)}\mathcal{O}_D$.

Indeed, if π is a uniformizer of the discrete valuation field $k(\eta)$, then the kernel of the multiplication from $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D$ to $\mathcal{O}_D$ is generated by $\pi \otimes 1 - 1 \otimes \pi$. Since $1 \otimes \pi$ is nilpotent in $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D$, the section $\pi \otimes 1 - 1 \otimes \pi$ becomes a unit in $k(\eta) \otimes_{k(s)} \mathcal{O}_D$. Since the natural homomorphism from $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D$ to $k(\eta) \otimes_{k(s)} \mathcal{O}_D$ is injective, this proves that $\pi \otimes 1 - 1 \otimes \pi$ generates an invertible ideal of $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D$, and this concludes the proof of Proposition III.5.18.

III.5.19. We aim at describing the Λ-local systems $\mathcal{F}$ of rank 1 on η with ramification bounded by D in terms of multiplicative Λ-local systems on a certain group scheme (cf. III.5.6), the local Picard group, which we now introduce.

Definition III.5.20. Let D be a closed subscheme of T supported on s, and let $\mathcal{I}$ be the invertible ideal of $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D$ from Proposition III.5.18. The local Picard group $\mathrm{Pic}(T, D)$ associated to the pair (T, D) is the functor which sends a T-scheme S to the group of pairs (d, u), where d is a locally constant $\mathbb{Z}$-valued map on S, and

$$u : \mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D \to \mathcal{O}_S \otimes_{\mathcal{O}_T} \mathcal{I}^{-d}$$

is an isomorphism of $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$-modules.

One can informally think of a section (d, u) of $\mathrm{Pic}(T, D)$ over a T-scheme S as a trivialization of the line bundle $\mathcal{O}(d\Delta)$ along the effective Cartier divisor $S \times_s D$ on the germ of S-curve $S \times_s T$, where $\Delta : S \to S \times_s T$ is the diagonal embedding.

Sending a section (d, u) of $\mathrm{Pic}(T, D)$ to d defines a homomorphism

$$\mathrm{Pic}(T, D) \to \mathbb{Z}_T,$$

of group valued functors. We denote by $\mathrm{Pic}^0(T, D)$ the kernel of this homomorphism. The special fiber $\mathrm{Pic}^0(T, D)_s$ is the functor which sends an s-scheme S to the group of units in $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$. The natural homomorphism

$$\mathrm{Pic}^0(T, D)_s \times_s T \to \mathrm{Pic}^0(T, D),$$

which sends a unit u of $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$ to the pair $(0, u)$ is an isomorphism.

Proposition III.5.21. *Let D be a closed subscheme of T supported on s. The functor* $\mathrm{Pic}(T, D)$ *is representable by a T-group scheme, which fits into a (split) exact sequence*

$$1 \to \mathrm{Pic}^0(T, D) \to \mathrm{Pic}(T, D) \to \mathbb{Z}_T \to 0,$$

where $\mathrm{Pic}^0(T, D)$ *is representable by a smooth separated affine T-group scheme, with geometrically connected fibers of dimension equal to the multiplicity of D at s.*

Indeed, if π is a uniformizer of the discrete valuation field $k(\eta)$, then the ideal $\mathcal{I}$ of $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D$ is generated $1 \otimes \pi - \pi \otimes 1$, hence for any S-point (d, u) of $\mathrm{Pic}(T, D)$ the isomorphism u can be uniquely written as a sum

$$u = \left(\sum_{0 \le n < \nu} u_n \pi^n \right) (1 \otimes \pi - \pi \otimes 1)^{-d},$$

where ν is the multiplicity of D at s, and $(u_n)_{0 \le n < \nu}$ are sections of $\mathcal{O}_S$, such that u_0 is invertible. Thus $\mathrm{Pic}(T, D)$ is representable by a product $\mathbb{Z}_T \times_T \mathbb{G}_{m,T} \times_T \mathbb{A}_T^{\nu-1}$, and this concludes the proof of Proposition III.5.21.

Remark III.5.22. The exact sequence

$$1 \to \mathrm{Pic}^0(T, D) \to \mathrm{Pic}(T, D) \to \mathbb{Z}_T \to 0,$$

is split, but the splitting constructed in the proof of Proposition III.5.21 depends on a choice of uniformizer, and is therefore non canonical.

III.5.23. By Proposition III.5.18, the natural homomorphism $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D \to \mathcal{I}^{-1}$ induces an isomorphism

$$u_{\text{can}} : k(\eta) \otimes_{k(s)} \mathcal{O}_D \to k(\eta) \otimes_{\mathcal{O}_T} \mathcal{I}^{-1}.$$

The pair $(1, u_{\text{can}})$ yields a $k(\eta)$-point of $\text{Pic}(T, D)$, corresponding to a morphism

$$\Phi_\eta : \eta \to \text{Pic}(T, D),$$

of T-schemes, which plays the role of a local Abel-Jacobi morphism in Theorem III.5.25. Recall that we have an isomorphism

$$\alpha : \text{Pic}^0(T, D)_s \times_s T \to \text{Pic}^0(T, D),$$

which sends a unit u of $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$ to the pair $(0, u)$. Let us also denote by p_1 the projection of $\text{Pic}^0(T, D)_s \times_s T$ onto the first factor.

Definition III.5.24. We define the groupoid $\text{Trip}(T, D, \Lambda)$ to be the category of triples $(\chi, \widetilde{\chi}, \beta)$, consisting of

(1) a multiplicative Λ-local system χ on the s-group scheme $\text{Pic}^0(T, D)_s$ (cf. III.5.6),
(2) a multiplicative Λ-local system $\widetilde{\chi}$ on the T-group scheme $\text{Pic}(T, D)$ (cf. III.5.6),
(3) an isomorphism $\beta : \alpha^{-1}\widetilde{\chi} \to p_1^{-1}\chi$ on $\text{Pic}^0(T, D)_s \times_s T$.

With this definition at hand, the main theorem of local geometric class field theory can be stated as follows:

Theorem III.5.25 (Local geometric class field theory). *Let D be a closed subscheme of T supported on s. Then, the functor Φ_η^{-1}, which sends an object $(\chi, \widetilde{\chi}, \beta)$ of* $\text{Trip}(T, D)$ *to the pullback $\Phi_\eta^{-1}\widetilde{\chi}$, is an equivalence from the groupoid* $\text{Trip}(T, D, \Lambda)$ *to the groupoid of Λ-local systems of rank* 1 *on η, with ramification bounded by D.*

We postpone the proof of Theorem III.5.25 to the paragraph III.5.34 below. We now provide an equivalent version of Theorem III.5.25, whose formulation is somewhat simpler, although non canonical, as it depends on a choice of uniformizer. Let π be a uniformizer of $k(\eta)$. Then $1 \otimes \pi - \pi \otimes 1$ is a generator of $\mathcal{I}$, and $1 \otimes \pi$ is a generator of $k(s) \otimes_{\mathcal{O}_T} \mathcal{I}$ as an $\mathcal{O}_D$-module. Thus we obtain an isomorphism

$$\begin{aligned} \alpha_\pi : \text{Pic}(T, D)_s \times_s T &\to \text{Pic}(T, D) \\ (d, u) &\mapsto (d, u(1 \otimes \pi)^d (1 \otimes \pi - \pi \otimes 1)^{-d}), \end{aligned}$$

whose restriction to $\text{Pic}^0(T, D)_s \times_s T$ coincides with α. Here, a section u over an s-scheme S is considered as an isomorphism

$$\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D \to \mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D \otimes_{\mathcal{O}_T} \mathfrak{m}^{-d},$$

where $\mathfrak{m}$ denotes the defining ideal of s in T. If $(\chi, \widetilde{\chi}, \beta)$ is a triple as in Theorem III.5.25, then we have a splitting

$$\text{Pic}(T, D)_s \times_s T \cong (\text{Pic}^0(T, D)_s \times_s T) \times_T \mathbb{Z}_T,$$

of T-group schemes (cf. III.5.21), and a corresponding decomposition of $\alpha_\pi^{-1}\widetilde{\chi}$ as $\alpha^{-1}\widetilde{\chi} \boxtimes \gamma$, where γ is a multiplicative Λ-local system on $\mathbb{Z}_T$ (cf. III.5.9). The first factor $\alpha^{-1}\widetilde{\chi}$ is isomorphic to $p_1^{-1}\chi$ by β. Since T is henselian, the pullback by the morphism $T \to s$ is an equivalence from the groupoid of multiplicative Λ-local systems on $\mathbb{Z}_s$ to the groupoid of multiplicative Λ-local systems on $\mathbb{Z}_T$, hence γ descends to $\mathbb{Z}_s$ (cf. III.5.8). Thus $\alpha_\pi^{-1}\widetilde{\chi}$ canonically descends to a multiplicative Λ-local system on $\text{Pic}(T, D)_s$. If we further note that $\alpha_\pi^{-1} \circ \Phi_\eta$ is given by the $k(\eta)$-point $(1, u)$ of $\text{Pic}(T, D)_s \times_s T$, where

$$u = (1 \otimes \pi)^{-1}(1 \otimes \pi - \pi \otimes 1) = 1 - \pi \otimes \pi^{-1},$$

then we obtain that Theorem III.5.25 is equivalent to the following:

THEOREM III.5.26 (Local geometric class field theory, second version). *Let D be a closed subscheme of T supported on s, and let π be a uniformizer of $k(\eta)$. Let $\Phi_{\eta,\pi} : \eta \to \mathrm{Pic}(T, D)_s$ be the morphism corresponding to the $k(\eta)$-point $(1, 1 - \pi \otimes \pi^{-1})$ of $\mathrm{Pic}(T, D)_s$. Then, the functor $\Phi_{\eta,\pi}^{-1}$ is an equivalence from the groupoid of multiplicative Λ-local systems on $\mathrm{Pic}(T, D)_s$ to the groupoid of Λ-local systems of rank 1 on η, with ramification bounded by D.*

III.5.27. Our deduction of Theorem III.5.26 from Theorem III.5.25 also shows that the functor

$$\mathrm{Trip}(T, D, \Lambda) \to \mathrm{Loc}^{\otimes}(\mathrm{Pic}(T, D)_s, \Lambda),$$

which sends a triple $(\chi, \widetilde{\chi}, \beta)$ to the restriction of $\widetilde{\chi}$ to the special fiber $\mathrm{Pic}(T, D)_s$ of $\mathrm{Pic}(T, D)$, is an equivalence of groupoids. Moreover, the composition of $\Phi_{\eta,\pi}^{-1}$ with this restriction functor coincides with the functor Φ_{η}^{-1} from Theorem III.5.25. In particular, the equivalence $\Phi_{\eta,\pi}^{-1}$ from Theorem III.5.26 does not depend on π, up to natural isomorphism. We denote by $\mathcal{F} \mapsto \chi_{\mathcal{F}}$ a quasi-inverse to $\Phi_{\eta,\pi}^{-1}$, which is well-defined up to natural isomorphism.

III.5.28. If $(\chi_1, \widetilde{\chi}_1, \beta_1)$ and $(\chi_2, \widetilde{\chi}_2, \beta_2)$ are objects of $\mathrm{Trip}(T, D, \Lambda)$ such that $\chi_1 = \chi_2$, then $\Phi_{\eta}^{-1}\widetilde{\chi}_2$ is isomorphic to $\Phi_{\eta}^{-1}\widetilde{\chi}_1 \otimes \mathcal{G}$, where $\mathcal{G}$ is the pullback to η of a Λ-local systems of rank 1 on s. We thus obtain a simpler (although weaker) version of Theorem III.5.25 by ignoring twists by unramified Λ-local systems of rank 1 on η:

THEOREM III.5.29 (Local geometric class field theory, third version). *Let D be a closed subscheme of T supported on s. If $\mathcal{F}$ is a Λ-local system of rank 1 on η, with ramification bounded by D, then there exists a unique (up to isomorphism) multiplicative Λ-local system χ on the s-group scheme $\mathrm{Pic}^0(T, D)_s$, such that the Λ-local system $\chi \boxtimes \mathcal{F}$ on the product*

$$\mathrm{Pic}^0(T, D)_s \times_s \eta \xrightarrow[\sim]{\alpha \cdot \Phi_\eta} \mathrm{Pic}^1(T, D)_\eta,$$

extends to a Λ-local system on $\mathrm{Pic}^1(T, D)$. This provides a bijection from the group of isomorphism classes of Λ-local systems of rank 1 on η with ramification bounded by D, up to twist by unramified Λ-local systems of rank 1 on η, to the group of isomorphism classes of multiplicative Λ-local systems on the s-group scheme $\mathrm{Pic}^0(T, D)_s$.

One should note that the restriction functor from $\mathrm{Pic}^1(T, D)$ to $\mathrm{Pic}^1(T, D)_\eta$ realizes an equivalence between the groupoid of Λ-local systems on $\mathrm{Pic}^1(T, D)$ and a full subcategory of the groupoid of Λ-local systems on $\mathrm{Pic}^1(T, D)_\eta$. The formulation of Theorem III.5.29 is due to Gaitsgory, and can also be found in Bhatt's Oberwolfach report ([**Bh16**], Th. 11).

III.5.30. We now describe the relation between our version of local geometric class field theory, namely Theorem III.5.25, and Contou-Carrere's theory of the local Jacobian. Let $\mathfrak{m}$ be the defining ideal of s, so that D is defined by $\mathfrak{m}^\nu$ for some nonnegative integer ν. Then $k(s) \otimes_{\mathcal{O}_T} \mathcal{I}^{-d}$ is naturally isomorphic to $\mathfrak{m}^{-d}/\mathfrak{m}^{-d+\nu}$ as a module over $\mathcal{O}_D = \mathcal{O}_T/\mathfrak{m}^\nu$, and we can identify $\mathrm{Pic}(T, D)_s$ with the functor which sends an s-scheme S to the group of pairs (d, u), where d is a locally constant $\mathbb{Z}$-valued map on S, and

$$u : \mathcal{O}_S \otimes_{k(s)} \mathcal{O}_T/\mathfrak{m}^\nu \to \mathcal{O}_S \otimes_{k(s)} \mathfrak{m}^{-d}/\mathfrak{m}^{-d+\nu}$$

is an isomorphism of $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_T/\mathfrak{m}^\nu$-modules. The inverse limit over ν of these s-group schemes can then be identified with the functor $J(\eta)$ which sends a $k(s)$-algebra A to the group of units in $A\widehat{\otimes}_{k(s)}k(\eta) = (A\widehat{\otimes}_{k(s)}\mathcal{O}_T) \otimes_{\mathcal{O}_T} k(\eta)$, where $A\widehat{\otimes}_{k(s)}\mathcal{O}_T$ is the $\mathfrak{m}$-adic completion of $A \otimes_{k(s)} \mathcal{O}_T$, which generate a sub-$(A\widehat{\otimes}_{k(s)}\mathcal{O}_T)$-module of the form $\mathfrak{m}^{-d}(A\widehat{\otimes}_{k(s)}\mathcal{O}_T)$ for some locally constant function d on $\mathrm{Spec}(A)$.

Remark III.5.31. A unit of $A\widehat{\otimes}_{k(s)}k(\eta)$ may not necessarily belong to $J(\eta)(A)$. For example, if a is a nilpotent element of A and if u is an element of $k(\eta)$ which does not belong to $\mathcal{O}_T$, then $1 - a \otimes u$ is a unit of $A\widehat{\otimes}_{k(s)}k(\eta)$ which does not belong to $J(\eta)(A)$. However, if A is reduced, then $J(\eta)(A)$ is simply the group of units in $A\widehat{\otimes}_{k(s)}k(\eta)$ by ([**CC13**], 0.8) or ([**Gu18**], Prop. 3.4).

Proposition III.5.32. *The functor $J(\eta)$ is representable by an s-group scheme, which fits into an exact sequence*

$$1 \to J(\eta)^0 \to J(\eta) \to \mathbb{Z}_s \to 0,$$

where $J(\eta)^0$ is representable by a geometrically connected affine s-group scheme.

Indeed, $J(\eta)^0$ is the limit of the inverse system $(\mathrm{Pic}^0(T, \mathfrak{m}^\nu)_s)_{\nu \geq 0}$ of affine s-group schemes.

Theorem III.5.33 (Local geometric class field theory, fourth version). *Let π be a uniformizer of $k(\eta)$. Let $\Psi_{\eta,\pi} : \eta \to J(\eta)$ be the morphism corresponding to the $k(\eta)$-point $(1, 1 - \pi \otimes \pi^{-1})$ of $J(\eta)$. Then, the functor $\Psi_{\eta,\pi}^{-1}$ is an equivalence from the groupoid of multiplicative Λ-local systems on $J(\eta)$ to the groupoid of Λ-local systems of rank 1 on η.*

If Λ is finite, then Theorem III.5.33 follows from Theorem III.5.26, since the category of finite etale $J(\eta)$-schemes is the 2-colimit of the categories of finite etale $\mathrm{Pic}(T, \mathfrak{m}^\nu)_s$-schemes when ν ranges over all integers, and thus the groupoid $\mathrm{Loc}^{\otimes}(J(\eta), \Lambda)$ is the 2-limit of the groupoids $\mathrm{Loc}^{\otimes}(\mathrm{Pic}(T, \mathfrak{m}^\nu)_s, \Lambda)$. If Λ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, then the conclusion of Theorem III.5.33 holds for the finite ℓ-adic coefficient rings Λ/ℓ^n for each n, and thus for Λ as well by taking 2-limits. This implies the validity of Theorem III.5.33 when Λ is a finite extension of $\mathbb{Q}_\ell$, and by taking 2-colimits this yields the result when Λ is an arbitrary ℓ-adic coefficient ring.

The s-group scheme $J(\eta)$ coincides Contou-Carrere's local Jacobian, and the morphism $\Psi_{\eta,\pi}$ in Theorem III.5.29 is the morphism studied by Contou-Carrere or considered by Deligne in his 1974 letter to Serre ([**BE01**], p.74). Contou-Carrere established an Albanese property for the morphism $\Psi_{\eta,\pi}$, which was used by Suzuki ([**Su13**], Th. A (1)) in order to give a different proof of Theorem III.5.33. Moreover, Suzuki (op. cit.) showed that the equivalence constructed by Serre in [**Se61**] when k is algebraically closed, is a quasi-inverse to the equivalence in Theorem III.5.33.

III.5.34. We now prove Theorem III.5.25, by combining the Gabber-Katz extension theorem III.4.14 with global geometric class field theory, namely Theorem III.5.15. More precisely, we prove its equivalent version III.5.26. Let D be a closed subscheme of T supported on s, and let π be a uniformizer of $k(\eta)$. The uniformizer π provides a morphism $k(s)[t, t^{-1}] \to k(\eta)$ sending t to π, corresponding to a morphism

$$\pi : \eta \to \mathbb{G}_{m,s},$$

of s-schemes.

By Theorem III.4.19, the restriction of the pullback functor π^{-1} to the category of special Λ-sheaves on $\mathbb{A}^1_s$ vanishing at 0 is an equivalence with the category of Λ-sheaves on η (cf. III.4.14). Let $\pi_\diamondsuit$ be a quasi-inverse to this equivalence. Let $\mathcal{F}$ be a Λ-sheaf on η with ramification bounded by D. Then $\pi_\diamondsuit \mathcal{F}$ is a Λ-local system on the open subscheme $\mathbb{G}_{m,s}$ of $\mathbb{P}^1_s$, extended by zero at 0 and ∞, with ramification bounded by the divisor $D' = D + [\infty]$. Let us consider the Abel-Jacobi morphism

$$\Phi : \mathbb{G}_{m,s} \to \mathrm{Pic}_s(\mathbb{P}^1_s, D'),$$

which sends a section t of $\mathbb{G}_{m,s}$ to the pair $(\mathcal{O}(t), 1)$, cf. (30). Since the Picard scheme of $\mathbb{P}^1_s$ is the constant group scheme $\mathbb{Z}_s$, we can identify the connected component of degree d of $\mathrm{Pic}_s(\mathbb{P}^1_s, D')$ with the functor which to an s-scheme S associates the quotient by $\mathbb{G}_{m,s}(S)$ of the group of isomorphisms $\alpha : \mathcal{O}_{D'_S} \to i'^{*}_S\mathcal{O}(d[0])$, where $i' : D' \to \mathbb{P}^1_s$ is the inclusion. The latter functor can be further identified with the functor which to an s-scheme S associates the group of isomorphisms $\alpha : \mathcal{O}_{D_S} \to i^*_S\mathcal{O}(d[0])$, where $i : D \to \mathbb{P}^1_s$ is the inclusion. Using the uniformizer π, we obtain an isomorphism

$$\theta : \mathrm{Pic}_s(\mathbb{P}^1_s, D') \to \mathrm{Pic}(T, D)_s.$$

Consequently, if $\mathcal{F}$ has rank 1, then Theorem III.5.15 implies that $\pi_\diamond\mathcal{F}$ is isomorphic to the pullback by $\theta \circ \Phi$ of a multiplicative Λ-local system on $\mathrm{Pic}(T, D)_s$. We therefore deduce from Theorems III.4.15 and III.5.15 that the pullback by $\theta \circ \Phi \circ \pi$ induces an equivalence from the groupoid of multiplicative Λ-local systems on $\mathrm{Pic}(T, D)_s$ to the groupoid of Λ-local systems of rank 1 on η, with ramification bounded by D.

It remains to check that the composition $\theta \circ \Phi \circ \pi$ coincides with the morphism $\Phi_{\eta,\pi}$ in Theorem III.5.26. If t is a section of $\mathbb{G}_{m,s}$ over an s-scheme S, then the isomorphism $\mathcal{O}([t]) \to \mathcal{O}([0])$ given by multiplication by $1 - tx^{-1}$, where x is the coordinate on $\mathbb{G}_{m,s}$, sends the canonical trivialization $1 : \mathcal{O}_{D'_S} \to i'^*_S\mathcal{O}([t])$ to the trivialization $\alpha : \mathcal{O}_{D'_S} \to i'^*_S\mathcal{O}([0])$ corresponding to $1 - tx^{-1}$. Thus $\theta \circ \Phi$ sends t to the S-point $\mathrm{Pic}(T, D)_s$ corresponding to $1 - t \otimes \pi^{-1}$. By taking $S = \eta$ and $t = \pi$, we obtain that $\theta \circ \Phi \circ \pi$ coincides with $\Phi_{\eta,\pi}$. This concludes our proof of Theorem III.5.26, which in turn implies Theorems III.5.25, III.5.29 and III.5.33.

Remark III.5.35. This proof of the main theorem of local geometric class field theory III.5.25 uses global geometric class field theory. The latter admits geometric proofs which do not use the local theory, cf. for example [**Ta18**], hence the argument is not circular. Moreover, the use of local geometric class field theory in ([**Gu18**], Prop. 3.14) can be avoided by resorting to a computation with Artin-Schreier-Witt theory, as in [**Ta18**].

III.5.36. In this paragraph, we describe the compatibility between local and global geometric class field theory, namely Theorems III.5.15 and III.5.25. Let X be a smooth geometrically connected projective curve of genus g over k, let $i : D \to X$ be an effective Cartier divisor on X, and let U be the open complement of D in X. We introduced in III.5.13 the generalized Picard group scheme $\mathrm{Pic}_k(X, D)$, and in (30) the Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_k(X, D),$$

which sends a section x of U to the pair $(\mathcal{O}(x), 1)$, where $1 : \mathcal{O}_D \to \mathcal{O}(x) \otimes_{\mathcal{O}_X} \mathcal{O}_D$ is the canonical trivialization of $\mathcal{O}(x)$ on D, cf. (30).

Let x be a point of D, and let $X_{(x)}$ be the henselisation of X at x, with generic point η_x. We identify the closed point of $X_{(x)}$ with x, and we denote by D_x the pullback of D to $X_{(x)}$, which is a closed subscheme of $X_{(x)}$ supported on x. Let $\widetilde{x}$ be the $X_{(x)}$-point of $X \times_k X_{(x)}$ given by the diagonal embedding. The restriction of $\widetilde{x}$ to η_x factors through $U \times_k X_{(x)}$. We now define a morphism

$$\tau : \mathrm{Pic}(X_{(x)}, D_x) \to \mathrm{Pic}_k(X, D) \times_k X_{(x)}$$

of $X_{(x)}$-group schemes as follows. Let (d, u) be a point of $\mathrm{Pic}(X_{(x)}, D_x)$ over an $X_{(x)}$-scheme S (cf. III.5.20). The image in $\mathcal{O}_D \otimes_k \mathcal{O}_S$ of the kernel of the natural multiplication homomorphism

$$\mathcal{O}_X \otimes_k \mathcal{O}_S \to \mathcal{O}_S,$$

is the ideal $\mathcal{I}$ as in Proposition III.5.18. Thus the pullback to $D \times_k S$ of the line bundle $\mathcal{O}(d\widetilde{x})$ on the S-curve $X \times_k S$ is given by the invertible module $\mathcal{O}_S \otimes_{\mathcal{O}_{X_{(x)}}} \mathcal{I}^{-d}$. Consequently, u provides a trivialization of $\mathcal{O}(d\widetilde{x})$ on the divisor $D_x \times_k S$ of $X \times_k S$. Moreover, the canonical section $1 : \mathcal{O}_{X \times_k S} \to \mathcal{O}(d\widetilde{x})$ provides a trivialization of $\mathcal{O}(d\widetilde{x})$ on the divisor $(D \setminus D_x) \times_k S$. We thus obtain a trivialization $\beta : \mathcal{O}_{D \times_k S} \to (i \times \mathrm{id}_S)^*\mathcal{O}(d\widetilde{x})$, and the pair $(\mathcal{O}(d\widetilde{x}), \beta)$ defines an S-point of $\mathrm{Pic}_k(X,D)\times_k X_{(x)}$. This construction is functorial in S, and thus defines a morphism τ as above.

We also let $\tau_x:\mathrm{Pic}(X_{(x)},D_x)_x\to\mathrm{Pic}_k(X,D)$ be the restriction of τ to the special fiber. We then have the following commutative diagram.

$$\begin{tikzcd}
U\times_k X_{(x)} \arrow[r, "\Phi\times\mathrm{id}"] & \mathrm{Pic}_k(X,D)\times_k X_{(x)} \arrow[r, "p_1"] & \mathrm{Pic}_k(X,D) \\
\eta_x \arrow[u, "\widetilde{x}_{|\eta_x}"] \arrow[r, "\Phi_\eta"] & \mathrm{Pic}(X_{(x)},D_x) \arrow[u, "\tau"'] & \mathrm{Pic}^0(X_{(x)},D_x)_x \arrow[u, "\tau_x"'] \\
& \mathrm{Pic}^0(X_{(x)},D_x)_x\times_x X_{(x)} \arrow[u, "\alpha"] \arrow[ur, "p_1"'] &
\end{tikzcd}$$

The morphisms α and Φ_η in this diagram are defined in III.5.23, while p_1 always denotes the projection onto the first factor. Let $\mathrm{Loc}_1(U,D,\Lambda)$ (resp. $\mathrm{Loc}_1(\eta_x,D_x,\Lambda)$) be the groupoid of Λ-local systems of rank 1 on U with ramification bounded by D (resp. on η_x with ramification bounded by D_x). If $\mathcal{L}$ is an object of $\mathrm{Loc}^{\otimes}(\mathrm{Pic}_k(X,D),\Lambda)$, we obtain an object $(\widetilde{\chi},\chi,\theta)$ of $\mathrm{Trip}(X_{(x)},D_x,\Lambda)$ (cf. III.5.24) as follows: we set $\widetilde{\chi}=(p_1\circ\tau)^{-1}\mathcal{L}$ and $\chi=\tau_x^{-1}\mathcal{L}$, while $\theta:\alpha^{-1}\widetilde{\chi}\to p_1^{-1}\chi$ is the natural isomorphism resulting from the commutativity of diagram above. We thus obtain a functor from $\mathrm{Loc}^{\otimes}(\mathrm{Pic}_k(X,D),\Lambda)$ to $\mathrm{Trip}(X_{(x)},D_x,\Lambda)$, which we abusively denote by τ^{-1} for simplicity. This functor τ^{-1} fits into the following diagram, which is commutative up to natural isomorphism.

$$\begin{tikzcd}
\mathrm{Loc}_1(X,D,\Lambda) \arrow[d] & \mathrm{Loc}^{\otimes}(\mathrm{Pic}_k(X,D),\Lambda) \arrow[l, "\Phi^{-1}"'] \arrow[d, "\tau^{-1}"'] \\
\mathrm{Loc}_1(\eta_x,D_x,\Lambda) & \mathrm{Trip}(X_{(x)},D_x,\Lambda) \arrow[l, "\Phi_\eta^{-1}"']
\end{tikzcd}$$

The rows of this diagram are equivalences of groupoids by Theorem III.5.15 and III.5.25. Thus the restriction functor, which is the left vertical arrow in this diagram, corresponds to the functor τ^{-1} in terms of multiplicative Λ-local systems.

III.5.37. We now describe the functoriality property of geometric local class field theory. Let T (resp. T') be the spectrum of a k-algebra, which is a henselian discrete valuation ring $\mathcal{O}_T$ (resp. $\mathcal{O}_{T'}$), whose residue field is a finite extension of k. Let η (resp. η') be the generic point of T (resp. T'), and let s (resp. s') be its closed point, so that $k(\eta)$ (resp. $k(\eta')$) is a henselian discrete valuation field, with valuation subring $\mathcal{O}_{T,s}$ (resp. $\mathcal{O}_{T',s'}$), and with residue field $k(s)$ (resp. $k(s')$) which is a finite extension of k.

Let $f:T'\to T$ be a finite morphism, such that the restriction $f_{|\eta'}:\eta'\to\eta$ is étale, namely such that the finite extension $k(\eta)\to k(\eta')$ induced by f is separable. Let D be a closed subscheme of T supported on s, and let D' be its pullback to T'. Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be the kernel of the homomorphism

$$\begin{aligned}\mathcal{O}_T\otimes_{k(s)}\mathcal{O}_D&\to\mathcal{O}_D\\ f_1\otimes f_2&\to f_1f_2,\end{aligned}$$

(resp. of the homomorphism $\mathcal{O}_{T'} \otimes_{k(s')} \mathcal{O}_{D'} \to \mathcal{O}_{D'}$), which, by Proposition III.5.18, is a principal invertible ideal of $\mathcal{O}_T \otimes_{k(s)} \mathcal{O}_D$ (resp. $\mathcal{O}_{T'} \otimes_{k(s')} \mathcal{O}_{D'}$) generating the unit ideal of $k(\eta) \otimes_{k(s)} \mathcal{O}_D$ (resp. $k(\eta') \otimes_{k(s')} \mathcal{O}_{D'}$).

For any T'-scheme S, the $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$-algebra $\mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D'}$ is free of finite rank equal to the ramification index e_f of the extension $k(\eta')/k(\eta)$, hence we can consider the norm map

$$N_f : \mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D'} \to \mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D, \tag{31}$$

which sends a section u of $\mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D'}$ to the determinant of the $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$-linear endomorphism $x \mapsto ux$ of $\mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D'}$. The norm map N_f is homogeneous of degree e_f, and therefore the image by N_f of a principal ideal of $\mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D'}$ generates a principal ideal of $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$.

Lemma III.5.38. *The ideal of $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$ generated by the image by N_f of the ideal $\mathcal{O}_S \otimes_{\mathcal{O}_{T'}} \mathcal{I}'$ of $\mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D'}$ is $\mathcal{O}_S \otimes_{\mathcal{O}_T} \mathcal{I}$.*

Indeed, if π' is a uniformizer of $k(\eta')$ then the ideal $\mathcal{O}_S \otimes_{\mathcal{O}_{T'}} \mathcal{I}'$ of $\mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D'}$ is generated by $\pi' \otimes 1 - 1 \otimes \pi'$. If $P_{\pi'}(X) = X^{e_f} + a_1 X^{e_f - 1} + \cdots + a_{e_f}$ is the characteristic polynomial of π' in the totally ramified extension $k(\eta')/k(\eta_{s'})$, where $\eta_{s'} = \eta \times_s s'$, then the ideal generated by $N_f(\mathcal{O}_S \otimes_{\mathcal{O}_{T'}} \mathcal{I}')$ is generated by the element

$$(1 \otimes P_{\pi'})(\pi' \otimes 1) = \pi'^{e_f} \otimes 1 + \pi'^{e_f - 1} \otimes a_1 + \cdots + 1 \otimes a_{e_f},$$

of $\mathcal{O}_S \otimes_{k(s')} \mathcal{O}_{D_{s'}}$. Since $P_{\pi'}$ is an Eisenstein polynomial, the elements $(a_i)_{i \leq e_f}$ of $\mathcal{O}_{T_{s'}}$ belong to the maximal ideal, and a_{e_f} is a uniformizer of $k(\eta_{s'})$. In particular, we can write $a_i = b_i a_{e_f}$, for some elements $(b_i)_{i \leq e_f}$ of $\mathcal{O}_{T_{s'}}$. We obtain a decomposition

$$(1 \otimes P_{\pi'})(\pi' \otimes 1) = P_{\pi'}(\pi') \otimes 1 + (u \otimes 1)(1 \otimes a_{e_f} - a_{e_f} \otimes 1) + (1 \otimes a_{e_f})v,$$

where we have set

$$u = 1 + \sum_{j=1}^{e_f - 1} b_{e_f - j} \pi'^j$$

$$v = \sum_{j=1}^{e_f} (\pi'^{e_f - j} \otimes 1)(1 \otimes b_j - b_j \otimes 1).$$

The term $P_{\pi'}(\pi')$ vanishes by the Cayley-Hamilton theorem. The elements $1 \otimes a_{e_f} - a_{e_f} \otimes 1$ and v belong to the ideal $\mathcal{O}_S \otimes_{\mathcal{O}_T} \mathcal{I}$, while $1 \otimes a_{e_f} - a_{e_f} \otimes 1$ generates the latter. Moreover, the elements $u \otimes 1$ and $1 \otimes a_{e_f}$ are respectively invertible and nilpotent in $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$. Thus $(1 \otimes P_{\pi'})(\pi' \otimes 1)$ is a generator of $\mathcal{O}_S \otimes_{\mathcal{O}_T} \mathcal{I}$, hence the result.

Definition III.5.39. The *norm morphism* associated to f is the homomorphism

$$N_f : \mathrm{Pic}(T', D') \to \mathrm{Pic}(T, D) \times_T T',$$

of T'-group schemes (cf. III.5.20) which sends a section (d, u) of $\mathrm{Pic}(T', D')$ over a T'-scheme S to the S-point $(d, N_f(u))$ of $\mathrm{Pic}(T, D)$, where $N_f(u)$ is the trivialization of the invertible $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_D$-module generated by $N_f(\mathcal{O}_S \otimes_{\mathcal{O}_{T'}} \mathcal{I}'^{-d})$, which coincides with $\mathcal{O}_S \otimes_{\mathcal{O}_T} \mathcal{I}^{-d}$ by Lemma III.5.38, obtained by applying the norm map (31) to u.

Let Φ_η (resp. $\Phi_{\eta'}$) be the local Abel-Jacobi morphism for (T, D) (resp. (T, D')), cf. III.5.23. The following diagram is clearly commutative.

$$\begin{tikzcd}
\eta' \arrow[r, "\Phi_{\eta'}"] \arrow[d, "f_{|\eta'}"'] & \mathrm{Pic}(T', D') \arrow[d, "N_f"] \\
\eta \arrow[r, "\Phi_\eta"] & \mathrm{Pic}(T, D)
\end{tikzcd}$$

Correspondingly, the following diagram is commutative up to natural isomorphism.

$$\begin{tikzcd}
\mathrm{Loc}_1(\eta', D', \Lambda) & \mathrm{Trip}(T', D', \Lambda) \arrow[l, "\Phi_{\eta'}^{-1}"'] \\
\mathrm{Loc}_1(\eta, D, \Lambda) \arrow[u, "f_{|\eta'}^{-1}"] & \mathrm{Trip}(T, D, \Lambda) \arrow[l, "\Phi_\eta^{-1}"'] \arrow[u, "N_f^{-1}"']
\end{tikzcd}$$

The rows of this diagram are equivalences of groupoids by Theorem III.5.24.

III.5.40. Let $\overline{k}$ be an algebraic closure of k. We denote by $G_k = \mathrm{Gal}(\overline{k}/k)$ the Galois group of the extension $\overline{k}/k$ and by μ a unitary Λ-admissible mutiplier on the topological group G_k (cf. III.2.9, III.2.10). Let k'/k be a neutralizing extension of k contained in $\overline{k}$, cf. III.3.1.

Definition III.5.41. Let S be a connected k-scheme and let G be a commutative S-group scheme, which fits into an exact sequence

$$1 \to G^0 \to G \xrightarrow{d} \mathbb{Z}_S \to 0,$$

where G^0 is an S-group scheme with connected geometric fibers. A **μ-twisted multiplicative Λ-local system on** G, is a pair $\mathcal{L} = (\mathcal{L}_{k'}, (\rho_{\mathcal{L}}(g))_{g\in\mathrm{Gal}(k'/k)})$, where $\mathcal{L}_{k'}$ is a multiplicative Λ-local system on the $S_{k'}$-group scheme $G_{k'}$ (cf. III.3.5), and $\rho_{\mathcal{L}}(g) : g^{-1}\mathcal{L}_{k'} \to \mathcal{L}_{k'}$ is an isomorphism of multiplicative Λ-local systems for each g in $\mathrm{Gal}(k'/k)$, such that the diagram

$$\begin{tikzcd}
g^{-1}h^{-1}\mathcal{L}_{k'} \arrow[r, "g^{-1}\rho_{\mathcal{L}}(h)"] \arrow[rr, bend right, "\mu(g,h)^d\rho_{\mathcal{L}}(gh)"'] & g^{-1}\mathcal{L}_{k'} \arrow[r, "\rho_{\mathcal{L}}(g)"] & \mathcal{L}_{k'}
\end{tikzcd}$$

is commutative for any g, h in $\mathrm{Gal}(k'/k)$. Here $\mu(g,h)^d$ is the section of $\Lambda^\times$ on G which is constant equal to $\mu(g,h)^r$ on the inverse image of r by the given homomorphism $d : G \to \mathbb{Z}_S$, for each integer r.

If $\mathcal{L}$ and $\mathcal{M}$ are μ-twisted multiplicative Λ-local systems on G, a morphism from $\mathcal{L}$ to $\mathcal{M}$ is an isomorphism $f : \mathcal{L}_{k'} \to \mathcal{M}_{k'}$ of multiplicative Λ-local systems such that $f \circ \rho_{\mathcal{L}}(g) = \rho_{\mathcal{M}}(g) \circ (g^{-1}f)$ for any g in $\mathrm{Gal}(k'/k)$.

Remark III.5.42. If S is a also a k''-scheme, for some finite extension k'' of k, and if $\iota : k'' \to \overline{k}$ is a k-linear embedding, with Galois group $\mathrm{Gal}(\iota)$, then as in III.3.10, the groupoid of μ-twisted multiplicative Λ-local systems on G, where S is considered as a k-scheme, is equivalent to the groupoid of $\mu_{|\mathrm{Gal}(\iota)}$-twisted multiplicative Λ-local systems on the S-group scheme G, where S is now considered as a k''-scheme.

III.5.43. Let T, η and s be as in III.5.16, and let Σ be the set of k-linear embeddings of $k(s)$ in $\overline{k}$, and assume that k' contains the image of any element of Σ. We then have a decomposition

$$T_{k'} = \coprod_{\iota\in\Sigma} T_\iota,$$

where $T_\iota = T\otimes_{k(s),\iota} k'$ is the spectrum of an henselian discrete valuation k'-algebra with residue field k'. If $\mathcal{F} = (\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g\in \mathrm{Gal}(k'/k)})$ is a μ-twisted Λ-local system of rank 1 on η, then the Swan conductor of the restriction $\mathcal{F}_\iota$ of $\mathcal{F}_{k'}$ to T_ι is independent of ι.

Definition III.5.44. Let D be a closed subscheme of T supported on s. A μ-twisted Λ-local system $\mathcal{F} = (\mathcal{F}_{k'}, (\rho_{\mathcal{F}}(g))_{g\in \mathrm{Gal}(k'/k)})$ of rank 1 on η has *ramification bounded by* D if for each ι in Σ (or equivalently, some ι), the Swan conductor of $\mathcal{F}_\iota$ is strictly less than the multiplicity of D at s.

Theorem III.5.45 (Local geometric class field theory, twisted version). *Let D be a closed subscheme of T supported on s, and let π be a uniformizer of $k(\eta)$. Let $\Phi_{\eta,\pi} : \eta \to \mathrm{Pic}(T, D)_s$ be the morphism corresponding to the $k(\eta)$-point $(1, 1 - \pi \otimes \pi^{-1})$ of $\mathrm{Pic}(T, D)_s$. Then, the functor $\Phi_{\eta,\pi}^{-1}$ is an equivalence from the groupoid of μ-twisted multiplicative Λ-local systems on $\mathrm{Pic}(T, D)_s$ (cf. III.5.41) to the groupoid of μ-twisted Λ-local systems of rank 1 on η, with ramification bounded by D.*

This follows immediately from Theorem III.5.26, and from the functoriality of local geometric class field theory, cf. III.5.37.

III.5.46. Let X be a smooth geometrically connected projective curve over k, let $i : D \to X$ be an effective Cartier divisor on X, and let U be the open complement of D in X.

Definition III.5.47. A μ-twisted Λ-local system $\mathcal{F}$ of rank 1 on U has *ramification bounded by* D if for any point x of D, the restriction of $\mathcal{F}$ to the generic point of the henselization $X_{(x)}$ of X at x has ramification bounded by the restriction of D to $X_{(x)}$ (cf. III.5.44).

Let us consider the Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_k(X, D),$$

which sends a section x of U to the pair $(\mathcal{O}(x), 1)$, cf. 30. As in III.5.43, we have the following twisted version of the main theorem of global geometric class field theory:

Theorem III.5.48 (Global geometric class field theory, twisted version). *Let X, U, D be as in III.5.46. Then the pullback Φ^{-1} by the Abel-Jacobi morphism of (X, D) realizes an equivalence from the groupoid of μ-twisted multiplicative Λ-local systems on $\mathrm{Pic}_k(X, D)$ (cf. III.5.41) to the category of groupoid of μ-twisted Λ-local systems of rank 1 on U, with ramification bounded by D (cf. III.5.47).*

III.6. Extensions of additive groups

Let A be a perfect $\mathbb{F}_p$-algebra, and let S be its spectrum. In this section, we denote by S_{fppf} the topos of sheaves of sets on the small fppf site of S, cf. ([**SGA4**], VII.4.2), and by $\mathrm{Ab}(S_{\mathrm{fppf}})$ the category of commutative group objects in S_{fppf}. The purpose of this section is to study extensions of the group scheme $\mathbb{G}_{a,S}$ by a finite abelian group Γ, or equivalently to study short exact sequences

$$0 \to \Gamma \to E \to \mathbb{G}_{a,S} \to 0,$$

of abelian groups in $\mathrm{Ab}(S_{\mathrm{fppf}})$, where Γ is considered as a constant S-group scheme. In such an exact sequence, the action of Γ on E by left multiplication turns E into a left Γ-torsor over $\mathbb{G}_{a,S}$, hence E is representable by a finite étale $\mathbb{G}_{a,S}$-scheme.

III.6.1. Let Γ be a finite abelian group, and let G be a finitely presented S-group scheme with geometrically connected fibers. The extensions of G by Γ in $\mathrm{Ab}(S_{\mathrm{fppf}})$ are classified by the elements of the abelian group $\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma)$.

PROPOSITION III.6.2. *Let $i : \Gamma' \to \Gamma$ be an injective homomorphism of finite abelian groups and let G be a finitely presented S-group scheme with geometrically connected fibers. Then the natural homomorphism*

$$\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma') \xrightarrow{i} \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma),$$

is injective.

Indeed, we have an exact sequence

$$\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma/\Gamma') \to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma') \to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma),$$

whose first term vanishes since G has geometrically connected fibers over S.

PROPOSITION III.6.3. *Let G be a finitely presented S-group scheme with geometrically connected fibers, annihilated by an integer $n \geq 1$. Let Γ be a finite abelian group and let $\Gamma[n]$ be its subgroup of n-torsion elements. Then the natural homomorphism*

$$\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma[n]) \to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma),$$

is an isomorphism.

Indeed, if $\Gamma' \subseteq \Gamma$ is the image of the multiplication by n in Γ, then we have a short exact sequence

$$0 \to \Gamma[n] \to \Gamma \xrightarrow{n} \Gamma' \to 0,$$

which yields an exact sequence

$$\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma') \to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma[n]) \to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma) \xrightarrow{n} \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma'),$$

whose first term vanishes since G has geometrically connected fibers over S, and whose last homomorphism vanishes as well, since its composition with the injective homomorphism

$$\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma') \to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma),$$

cf. III.6.2, is the multiplication by n on $\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,\Gamma)$, which is zero since n annihilates G.

III.6.4. Let G be a finitely presented affine commutative S-group scheme, and let H be an object of $\mathrm{Ab}(S_{\mathrm{fppf}})$ such that any H-torsor over a finitely presented affine S-scheme is trivial. Let us consider an extension

$$0 \to H \xrightarrow{\iota} E \xrightarrow{r} G \to 0, \tag{32}$$

of G by H in $\mathrm{Ab}(S_{\mathrm{fppf}})$. The action of H by left multiplication on E turns the latter into a left H-torsor over G. Since G is affine and finitely presented over S, this torsor is trivial, and thus we can assume (and we do) that E is $H \times_S G$ as an object of S_{fppf}, that $r(h,g) = g$ for any local sections h and g of H and G, and that the left action of H on E is given by $(h',0) + (h,g) = (h'+h,g)$. The addition on E must then take the form

$$(h_1,g_1) + (h_2,g_2) = (h_1 + h_2 + c(g_1,g_2), g_1 + g_2), \tag{33}$$

for some morphism of sheaves c from $G \times_S G$ to H. Since E is a commutative group under the law (33), we have the relations

$$c(g_1,g_2) = c(g_2,g_1), \tag{34}$$

$$c(g_1,g_2+g_3) + c(g_2,g_3) = c(g_1,g_2) + c(g_1+g_2,g_3), \tag{35}$$

corresponding respectively to the commutativity and the associativity of the law (33).

Conversely, if c is a morphism from $G \times_S G$ to H which satisfies the relations (34) and (35), then the formula (33) defines an extension E_c of G by H in $\mathrm{Ab}(S_{\mathrm{fppf}})$, whose underlying S-scheme is $H \times_S G$.

PROPOSITION III.6.5. *Let G be a finitely presented affine commutative S-group scheme, and let H be an object of $\mathrm{Ab}(S_{\mathrm{fppf}})$ such that any H-torsor over a finitely presented affine S-scheme is trivial. Let $\mathcal{C}(G,H)$ be the group of morphisms from $G \times_S G$ to H in S_{fppf} which satisfy the relations (34) and (35). We then have an exact sequence*

$$0 \to \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,H) \to \mathrm{Hom}_{S_{\mathrm{fppf}}}(G,H) \xrightarrow{d} \mathcal{C}(G,H) \xrightarrow{c \mapsto E_c} \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,H) \to 0,$$

where $d : \mathrm{Hom}_{S_{\mathrm{fppf}}}(G,H) \to \mathcal{C}(G,H)$ is the homomorphism given on sections by

$$d(f)(g_1,g_2) = f(g_1+g_2) - f(g_1) - f(g_2).$$

Indeed, we already know that the map $c \mapsto E_c$ from $\mathcal{C}(G,H)$ to $\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,H)$ is surjective, and its kernel consists of the elements c of $\mathcal{C}(G,H)$ for which E_c is a trivial extension of G by H. If c is such a morphism, then the surjective homomorphism $E_c \to G$ has a section, which must take the form $g \mapsto (f(g),g)$ for some morphism f from G to H. We have

$$(f(g_1+g_2), g_1+g_2) = (f(g_1),g_1) + (f(g_2),g_2) = (f(g_1)+f(g_2)+c(g_1,g_2), g_1+g_2),$$

for any local sections g_1, g_2 of G, hence $d(f) = c$. Conversely, any element f of $\mathrm{Hom}_{S_{\mathrm{fppf}}}(G,H)$ such that $d(f) = c$ provides a section $g \mapsto (f(g),g)$ of the extension E_c. Thus the sequence in Proposition III.6.5 is exact at $\mathcal{C}(G,H)$. The result then follows from the description of homomorphisms from G to H as elements f of $\mathrm{Hom}_{S_{\mathrm{fppf}}}(G,H)$ such that $d(f)$ vanishes.

EXAMPLE III.6.6. Let c be the image in $\mathbb{F}_p[U_1,U_2]$ of the polynomial

$$\frac{U_1^p + U_2^p - (U_1+U_2)^p}{p} = -\sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} U_1^i U_2^{p-i} \in \mathbb{Z}[U_1,U_2].$$

Then the morphism from $\mathbb{G}_{a,\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{G}_{a,\mathbb{F}_p}$ to $\mathbb{G}_{a,\mathbb{F}_p}$ corresponding to c belongs to $\mathcal{C}(\mathbb{G}_{a,\mathbb{F}_p}, \mathbb{G}_{a,\mathbb{F}_p})$. The corresponding extension E_c of $\mathbb{G}_{a,\mathbb{F}_p}$ by itself is isomorphic to the $\mathbb{F}_p$-group scheme of Witt vectors of length 2.

III.6.7. The group $G = H = \mathbb{G}_{a,S}$ satisfy the assumptions of Proposition III.6.5, and thus satisfy its conclusion. We therefore have a homomorphism

$$\mathrm{Hom}_{S_{\mathrm{fppf}}}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \xrightarrow{d} \mathcal{C}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}),$$

with kernel $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ and with cokernel $\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$. Moreover, the group $\mathrm{Hom}_{S_{\mathrm{fppf}}}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ can be identified with the group $A[U]$ of polynomials in one variable over A, while $\mathcal{C}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ can be identified with the group $A[U_1,U_2]$ of polynomials c in two variables over A, which satisfy the relations

$$\begin{aligned} &c(U_1,U_2) = c(U_2,U_1), \\ &c(U_1,U_2+U_3) + c(U_2,U_3) = c(U_1,U_2) + c(U_1+U_2,U_3). \end{aligned}$$

PROPOSITION III.6.8 ([**Se68**], V.5). *A polynomial f in $A[U]$ belongs to $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ if and only if it is of the form*

$$f(U) = \sum_{r \geq 0} a_r U^{p^r},$$

for some elements $(a_r)_{r \geq 0}$ of A.

Indeed, a polynomial $f(U) = \sum_{m\geq 0} b_m U^m$ of $A[U]$ belongs to $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ if and only if the element

$$d(f) = \sum_{m\geq 0} b_m d(U^m),$$

of $A[U_1, U_2]$ vanishes. Since each $b_m d(U^m)$ is a homogeneous polynomial of degree m, it is the homogeneous part of degree m of $d(f)$. Consequently, $d(f)$ vanishes if and only if so does $b_m d(U^m)$ for each m. The conclusion of Proposition III.6.8 then follows from the fact that for each integer $m \geq 1$, we have $d(U^m) = 0$ if and only if m is a power of p. Indeed, if $m = p^v n$ with n prime to p, then $d(U^m) = d(U^n)^{p^v}$ vanishes if and only if $d(U^n)$ does. We have $d(U) = 0$, and if $n > 1$, then the coefficient of $U_1 U_2^{n-1}$ in $d(U^n)$ is n, whence $d(U^n)$ is non zero since n is prime to p.

Proposition III.6.9. *Let E be an extension of $\mathbb{G}_{a,S}$ by itself in $\mathrm{Ab}(S_{\mathrm{fppf}})$, whose pushout by the homomorphism $t \mapsto t - t^p$ from $\mathbb{G}_{a,S}$ to itself is a trivial extension of $\mathbb{G}_{a,S}$ by itself. Then E is a trivial extension of $\mathbb{G}_{a,S}$ by itself.*

Indeed, let c be an element of $\mathcal{C}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ such that E_{c-c^p} a trivial extension of $\mathbb{G}_{a,S}$ by itself. By Proposition III.6.5, there exists an element $f = \sum_{n\geq 0} b_n U^n$ of $A[U]$ such that we have $d(f) = c - c^p$. We thus have

$$c(U_1, U_2) = c(U_1, U_2)^p + \sum_{n\geq 0} b_n d(U^n).$$

In particular, we have $c(0,0) = c(0,0)^p - b_0$ since $d(1) = -1$, and thus

$$c(U_1, U_2) - c(0,0) = (c(U_1, U_2) - c(0,0))^p + \sum_{n\geq 1} b_n d(U^n).$$

By iterating this identity, we obtain a relation

$$c(U_1, U_2) - c(0,0) = \sum_{v\geq 0} \left(\sum_{n\geq 1} b_n d(U^n) \right)^{p^v} = \sum_{m\geq 1} c_m d(U^m),$$

in the power series ring $A[[U_1, U_2]]$, where $c_m = \sum_{m=p^v n} b_n^{p^v}$. For each integer m, the polynomial $d(U^m)$ is homogeneous of degree m. In particular, the polynomial $c_m d(U^m)$ is the homogeneous part of degree m of $c(U_1, U_2) - c(0,0)$. Since $c(U_1, U_2) - c(0,0)$ is a polynomial, we must have $c_m d(U^m) = 0$ for m large enough. Thus the power series

$$-c(0,0) + \sum_{\substack{m\geq 1 \\ d(U^m)\neq 0}} c_m U^m,$$

is a polynomial, whose image by d is c. Consequently, E_c is a trivial extension of $\mathbb{G}_{a,S}$ by itself.

III.6.10. Let $F : \mathbb{G}_{a,S} \to \mathbb{G}_{a,S}$ be the Frobenius homomorphism, given on sections by $t \mapsto t^p$. We then have the so-called Artin-Schreier exact sequence

$$0 \to \mathbb{F}_{p,S} \to \mathbb{G}_{a,S} \xrightarrow{1-F} \mathbb{G}_{a,S} \to 0. \tag{36}$$

By applying the functor $\mathrm{Hom}(\mathbb{G}_{a,S}, -)$ to this short exact sequence, we obtain a long exact sequence

$$\begin{array}{c} \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \xrightarrow{1-F} \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \\ \xrightarrow{\delta} \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \xrightarrow{1-F} \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \end{array}$$

whose first term vanishes since $\mathbb{G}_{a,S}$ has geometrically connected fibers over S, and whose last homomorphism is injective by Proposition III.6.9. We thus have a short exact sequence

(37)
$$0 \to \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \xrightarrow{1-F} \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \xrightarrow{\delta} \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{F}_p) \to 0.$$

Proposition III.6.11. *If we denote, for each element a of the ring $A = \Gamma(S, \mathcal{O}_S)$, by m_a the endomorphism of $\mathbb{G}_{a,S}$ which sends a section t to at, then the homomorphism*

$$\begin{aligned} A \oplus \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) &\to \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S}) \\ (a, u) &\mapsto m_a + F(u) - u, \end{aligned}$$

is an isomorphism.

Indeed, the group $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ is the group of polynomials v in $A[T]$ which are additive, in the sense that
$$v(T + S) = v(T) + v(S).$$
These are exactly the polynomials of the form $v(T) = \sum_{j \geq 0} a_j T^{p^j}$, where $(a_j)_{j \geq 0}$ is a finite family of elements of A, cf. Proposition III.6.8. By writing successively monomials of the form aT^{p^j} with $j \geq 1$ as $a^{p^{-1}} T^{p^{j-1}} + (F-1)(u)$, where $u = a^{p^{-1}} T^{p^{j-1}}$, we obtain that any such polynomial v can be decomposed as
$$v(T) = aT + (F-1)(u),$$
for some additive polynomial u, and with $a = \sum_{j \geq 0} a_j^{p^{-j}}$. The homomorphism in Proposition III.6.11 is thus surjective.

On the other hand, if an additive polynomial $u = \sum_{j \geq 0} u_j T^{p^j}$ is such that $(F-1)(u)$ is of the form aT for some element a of A, then we have
$$aT = \sum_{j \geq 0} u_j^p T^{p^{j+1}} - \sum_{j \geq 0} u_j T^{p^j} = -u_0 T + \sum_{j \geq 1} (u_{j-1}^p - u_j) T^{p^j},$$
so that $a = -u_0$ and $u_{j-1}^p = u_j$ for each $j \geq 1$. This implies that for each $j \geq 0$, there exists an integer N such that $u_j^{p^N} = 0$, and thus $u_j = 0$ since A is reduced. Thus such an additive polynomial u must be zero, which proves that the homomorphism in Proposition III.6.11 is injective.

Corollary III.6.12. *The homomorphism of abelian groups*
$$\begin{aligned} A &\to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S}, \mathbb{F}_p) \\ a &\mapsto \delta(m_a) \end{aligned}$$
is an isomorphism.

This follows immediately from Proposition III.6.11 and from the exact sequence (37). For each element a of A, the extension $\delta(m_a)$ can be explicitly described as the pullback of the extension (36) by the homomorphism m_a.

III.6.13. Let Λ be an ℓ-adic coefficient ring, and let $\psi : \mathbb{F}_p \to \Lambda^\times$ be a non trivial (hence injective) homomorphism. The pushout by ψ of the Artin-Schreier $\mathbb{F}_p$-torsor (cf. 36) yields a multiplicative Λ-local system on $\mathbb{G}_{a,S}$ (cf. III.5.6), which we denote by $\mathcal{L}_\psi$. More generally, if $f : X \to \mathbb{G}_{a,S}$ is a morphism of S-schemes, we denote by $\mathcal{L}_\psi\{f\}$ the Λ-local system of rank 1 on X given by the pullback of $\mathcal{L}_\psi$ by f. If moreover X is an S-group scheme and if f is a homomorphism of S-group schemes, then $\mathcal{L}_\psi\{f\}$ is a multiplicative Λ-local system on X.

PROPOSITION III.6.14. *Let V be a finitely generated projective A-module, and let $\mathcal{V}$ be the corresponding S-group scheme. Then any multiplicative Λ-local system on $\mathcal{V}$ is isomorphic to $\mathcal{L}_\psi\{v^*\}$, for a unique A-linear homomorphism $v^* : V \to A$, considered as a homomorphism from $\mathcal{V}$ to $\mathbb{G}_{a,S}$.*

By III.5.9, we can assume (and we do) that V is the A-module A, and thus that $\mathcal{V}$ is $\mathbb{G}_{a,S}$. Moreover, we can assume (and we do as well) that Λ is finite, in which case multiplicative Λ-local systems on $\mathbb{G}_{a,S}$ correspond to extensions in S_{fppf} of $\mathbb{G}_{a,S}$ by the finite abelian group $\Lambda^\times$. Since ψ realizes an isomorphism from $\mathbb{F}_p$ to the p-torsion subgroup of $\Lambda^\times$, we deduce from Proposition III.6.3 that the homomorphism

$$\operatorname{Ext}_{\operatorname{Ab}(S_{\text{fppf}})}(\mathbb{G}_{a,S}, \mathbb{F}_p) \xrightarrow{\psi} \operatorname{Ext}_{\operatorname{Ab}(S_{\text{fppf}})}(\mathbb{G}_{a,S}, \Lambda^\times),$$

induced by ψ is an isomorphism. The conclusion then follows from Corollary III.6.12.

COROLLARY III.6.15. *Assume that $A = k$ is an algebraically closed field. Let V be a k-vector space of finite dimension $r \geq 1$, let $\mathcal{V}$ be the corresponding k-group scheme and let $\mathcal{M}$ be a multiplicative Λ-local system on $\mathcal{V}$. Then the cohomology group*

$$H^\nu_c(\mathcal{V}, \mathcal{M})$$

vanishes for each integer ν, unless $\nu = 2r$ and $\mathcal{M}$ is trivial, in which case it is a free Λ-module of rank 1.

By Proposition III.6.14, we can assume that V is k^r and that $\mathcal{M}$ is $\mathcal{L}_\psi\{x_1\}$, where $x_1 : k^r \to k$ is the first coordinate. If $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$ in k, then for each integer ν the group

$$H^\nu_c(\mathcal{V}, \mathcal{M}) = H^\nu_c(\mathbb{G}^r_{a,k}, \mathcal{L}_\psi\{x_1\})$$

is isomorphic to $H^\nu_c(\mathbb{G}^r_{a,\overline{\mathbb{F}}_p}, \mathcal{L}_\psi\{x_1\})$, which vanishes by ([**SGA**$4\frac{1}{2}$], [Sommes trig.] Th. 2.7).

COROLLARY III.6.16. *Assume that $A = k$ is an algebraically closed field of characteristic $p \neq 2$. Let V be a k-vector space of finite dimension $r \geq 1$, let $\mathcal{V}$ be the corresponding k-group scheme, let $\gamma : V \to k$ be a non zero linear form and let $\mathcal{M}$ be a multiplicative Λ-local system on $\mathcal{V}$. Then the cohomology group*

$$H^\nu_c(\mathcal{V}, \mathcal{M} \otimes \mathcal{L}_\psi\{\gamma^2\})$$

vanishes for each integer ν, unless $\nu = 2r-1$ and $\mathcal{M}$ is isomorphic to $\mathcal{L}_\psi\{\alpha\gamma\}$ for some element α of k, in which case it is a free Λ-module of rank 1.

Indeed, if t is the coordinate on $\mathbb{G}_{a,S}$, then the projection formula yields an isomorphism

$$R\gamma_!(\mathcal{M} \otimes \mathcal{L}_\psi\{\gamma^2\}) \cong R\gamma_!(\mathcal{M}) \otimes \mathcal{L}_\psi\{t^2\}.$$

By Corollary III.6.15 and by multiplicativity of $\mathcal{M}$, the geometric fibers of this complex vanish unless the restriction of $\mathcal{M}$ to the kernel of γ is trivial. By Proposition III.6.14, the multiplicative Λ-local system $\mathcal{M}$ is isomorphic to $\mathcal{L}_\psi\{v^*\}$ for a unique linear form v^* on V. We thus obtain that $R\gamma_!(\mathcal{M} \otimes \mathcal{L}_\psi\{\gamma^2\})$ vanishes unless the restriction of v^* to the kernel of γ vanishes, in which case we have $v^* = \alpha\gamma$ for some element α of k.

When $\mathcal{M}$ is isomorphic to $\mathcal{L}_\psi\{\alpha\gamma\}$ for some element α of k, we have

$$\begin{aligned} R\gamma_!(\mathcal{M} \otimes \mathcal{L}_\psi\{\gamma^2\}) &\cong R\gamma_!(\Lambda) \otimes \mathcal{L}_\psi\{\alpha t + t^2\} \\ &\cong \mathcal{L}_\psi\{\alpha t + t^2\}(-r)[-2r], \end{aligned}$$

and the conclusion then follows from the fact that the cohomology group

$$H^\nu_c(\mathbb{G}_{a,k}, \mathcal{L}_\psi\{\alpha t + t^2\})$$

is of rank 1 for $\nu = 1$, and vanishes otherwise. Indeed, this group vanishes for $\nu = 0$ since $\mathcal{L}_\psi\{\alpha t + t^2\}$ has no punctual sections, it vanishes for $\nu = 2$ by Poincaré duality, and its Euler characteristic is -1 by the Grothendieck-Ogg-Shafarevich formula, since the Swan conductor of $\mathcal{L}_\psi\{\alpha t + t^2\}$ at ∞ is 2.

III.6.17. We now assume that S is of characteristic $p = 2$. The element $c = U_1U_2$ of $A[U_1, U_2]$ belongs to $\mathcal{C}(\mathbb{G}_{a,S}, \mathbb{G}_{a,S})$ (cf. III.6.5), and thus defines an extension $G = E_c$ of $\mathbb{G}_{a,S}$ by itself, cf. III.6.7. Thus G is $\mathbb{G}_{a,S} \times_S \mathbb{G}_{a,S}$ as an S-scheme, endowed with the multiplication

$$(t_1, u_1) + (t_2, u_2) = (t_1 + t_2 + u_1u_2, u_1 + u_2),$$

for sections t_1, u_1, t_2, u_2 of $\mathbb{G}_{a,S}$. Equivalently, the S-group scheme G is the pullback to S of the group of Witt vectors of length 2 over $\mathbb{F}_2$, cf. III.6.6.

The group G satisfies the assumptions of Proposition III.6.5, and thus satisfies its conclusion. We therefore have a homomorphism

$$\mathrm{Hom}_{S_{\mathrm{fppf}}}(G, G) \xrightarrow{d} \mathcal{C}(G, G),$$

with kernel $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G)$ and with cokernel $\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G)$. The group $\mathrm{Hom}_{S_{\mathrm{fppf}}}(G, G)$ can be identified with the group of couples $f = (f_0, f_1)$ of elements of $A[T, U]$.

Proposition III.6.18. *For $p = 2$, let G be the extension of $\mathbb{G}_{a,S}$ by itself defined by $c(U_1, U_2) = U_1U_2$ (cf. III.6.7). A couple $f = (f_0, f_1)$ of elements of $A[T, U]$ belongs to $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G)$ if and only if it is of the form*

$$\left(a\left(T^{\frac{1}{2}}\right)^2 + \widetilde{a}(U) + b(U), a(U)\right),$$

where $a(U) = \sum_{r \geq 0} a_r U^{2^r}$ and $b(U)$ are additive polynomials, and where

$$a\left(T^{\frac{1}{2}}\right)^2 = \sum_{r \geq 0} a_r^2 T^{2^r},$$

$$\widetilde{a}(U) = \sum_{r_1 > r_2 \geq 0} a_{r_1}a_{r_2}U^{2^{r_1}+2^{r_2}}.$$

Indeed, such a couple $f = (f_0, f_1)$ belongs to $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G)$ if and only if $d(f)$ vanishes, namely if and only if the relations

$$f_0(T_1 + T_2 + U_1U_2, U_1 + U_2) = f_0(T_1, U_1) + f_0(T_2, U_2) + f_1(T_1, U_1)f_1(T_2, U_2) \tag{38}$$

$$f_1(T_1 + T_2 + U_1U_2, U_1 + U_2) = f_1(T_1, U_1) + f_1(T_2, U_2), \tag{39}$$

hold in $A[T_1, U_1, T_2, U_2]$. Setting $U_1 = T_2 = 0$ in (39), we obtain

$$f_1(T_1, U_2) = f_1(T_1, 0) + f_1(0, U_2).$$

Setting $U_1 = U_2 = 0$ in (39), we obtain that $f_1(T, 0)$ is an additive polynomial, so that we can write

$$f_1(T, 0) = \sum_{r \geq 0} x_r T^{2^r},$$

for some elements $(x_r)_r$ of A (cf. Proposition III.6.8), while setting $T_1 = T_2 = 0$ in (39) yields

$$f_1(0, U_1 + U_2) + f_1(U_1U_2, 0) = f_1(0, U_1) + f_1(0, U_2).$$

Writing $f_1(0, U)$ as $\sum_{n \geq 0} y_n U^n$ for some elements $(y_n)_n$ of A, we obtain

$$\sum_{n \geq 0} y_n \left((U_1 + U_2)^n - U_1^n - U_2^n\right) + \sum_{r \geq 0} x_r U_1^{2^r} U_2^{2^r} = 0.$$

For each integer $r \geq 0$, the homogeneous part of degree 2^{r+1} in this relation yields the vanishing of $x_r U_1^{2^r} U_2^{2^r}$, hence $x_r = 0$. We thus have

$$f_1(T, U) = a(U),$$

and $a(U) = f_1(0, U)$ is an additive polynomial.

Setting $U_1 = T_2 = 0$ in (38), we obtain

$$f_0(T_1, U_2) = f_0(T_1, 0) + f_0(0, U_2) + f_1(T_1, 0) f_1(0, U_2) = f_0(T_1, 0) + f_0(0, U_2),$$

since $f_1(T_1, 0) = a(0)$ vanishes. Setting $U_1 = U_2 = 0$ in (38), we obtain that $f_0(T, 0)$ is an additive polynomial, so that we can write

$$f_0(T, 0) = \sum_{r \geq 0} c_r T^{2^r},$$

for some elements $(c_r)_r$ of A (cf. Proposition III.6.8), while setting $T_1 = T_2 = 0$ in (38) yields

$$f_0(0, U_1 + U_2) + f_0(U_1 U_2, 0) = f_0(0, U_1) + f_0(0, U_2) + a(U_1) a(U_2). \tag{40}$$

Let us write the additive polynomial $a(U)$ as $\sum_{r \geq 0} a_r U^{2^r}$, cf. Proposition III.6.8, and let us write the polynomial $f_0(0, U)$ as $\sum_{n \geq 0} b_n U^n$ for some elements $(b_n)_n$ of A, so that (40) can be written as

$$\sum_{n \geq 0} b_n \left((U_1 + U_2)^n - U_1^n - U_2^n \right) = \sum_{r_1, r_2 \geq 0} a_{r_1} a_{r_2} U_1^{2^{r_1}} U_2^{2^{r_2}} - \sum_{r \geq 0} c_r U_1^{2^r} U_2^{2^r}.$$

For each integer $r \geq 0$, the homogeneous part of degree 2^{r+1} in this relation yields the vanishing of $(a_r^2 - c_r) U_1^{2^r} U_2^{2^r}$, hence $c_r = a_r^2$. Furthermore, for each integer n which is not a power of 2, the homogeneous part of degree n in this relation yields $b_n = a_{r_1} a_{r_2}$ if $n = 2^{r_1} + 2^{r_2}$ for some pair of distinct integers (r_1, r_2), and $b_n = 0$ otherwise. We thus have

$$f_0(0, U) = \widetilde{a}(U) + b(U),$$

where $b(U) = \sum_{r \geq 0} b_{2^r} U^{2^r}$ is an additive polynomial and where $\widetilde{a}(U) = \sum_{r_1 > r_2 \geq 0} a_{r_1} a_{r_2} U^{2^{r_1} + 2^{r_2}}$. Moreover, the relation $c_r = a_r^2$ yields $f_0(T, 0) = a(T^{\frac{1}{2}})^2$, so that

$$f_0(T, U) = f_0(T, 0) + f_0(0, U) = a\left(T^{\frac{1}{2}}\right)^2 + \widetilde{a}(U) + b(U),$$

hence the conclusion of Proposition III.6.18.

Proposition III.6.19. *For $p = 2$, let G be the extension of $\mathbb{G}_{a,S}$ by itself defined by $c(U_1, U_2) = U_1 U_2$ (cf. III.6.7), and let F be the endomorphism $(t, u) \mapsto (t^2, u^2)$ of G. Let E be an extension of G by itself in* $\mathrm{Ab}(S_{\mathrm{fppf}})$, *whose pushout by the endomorphism $1 - F$ of G is a trivial extension of G by itself. Then E is a trivial extension of G by itself.*

We prove Proposition III.6.19 by an argument similar to the one we used to prove Proposition III.6.9. We endow $A[T, U]$ (resp. $A[T_1, U_1, T_2, U_2]$) with a structure of $\mathbb{N}$-graded A-algebra by assigning weight 2 to the variable T (resp. T_1, T_2), and weight 1 to the variable U (resp. U_1, U_2). If B is an $\mathbb{N}$-graded A-algebra, an element $f = (b_0, b_1)$ of $G(B)$ is said to be *homogeneous of degree n* if b_0 and b_1 are homogeneous elements of degrees n and $\frac{n}{2}$ respectively in B. In particular, we have $b_1 = 0$ if n is odd. One should note that for each integer n, the subset of $G(B)$ consisting of homogeneous elements of degree n is a subgroup of $G(B)$. Any element f of $G(B)$ can be uniquely written as a finite sum

$$f = \sum_{n \geq 0} f_n,$$

where f_n is a homogeneous element of degree n in $G(B)$. The element f_n of $G(B)$ will be referred to as the *homogeneous part of degree n* of f.

Let γ be an element of $\mathcal{C}(G,G)$ such that $E_{\gamma-F(\gamma)}$ is a trivial extension of G by itself. By Proposition III.6.5, there exists an element f of $G(A[T,U])$ such that $d(f)=\gamma-F(\gamma)$. Let us write $f=\sum_{n\geq 0} f_n$ as the sum of its homogeneous parts, as above. We have

$$\gamma(T_1,U_1,T_2,U_2)=F(\gamma(T_1,U_1,T_2,U_2))+\sum_{n\geq 0} d(f_n),$$

in $A[T_1,U_1,T_2,U_2]$. In particular, we have $\gamma(0)=F(\gamma(0))+d(f_0)$, and thus

$$\gamma(T_1,U_1,T_2,U_2)-\gamma(0)=F(\gamma(T_1,U_1,T_2,U_2)-\gamma(0))+\sum_{n\geq 1} d(f_n).$$

By iterating this identity, we obtain a relation

$$\gamma(T_1,U_1,T_2,U_2)-\gamma(0)=\sum_{v\geq 0} F^v\left(\sum_{n\geq 1} d(f_n)\right)=\sum_{m\geq 1} d(g_m),$$

in the group $G(A[[T_1,U_1,T_2,U_2]])$, where $g_m=\sum_{m=p^v n} F^v(f_n)$ is homogeneous of degree m. For each integer m, the element $d(g_m)$ of $G(A[T_1,U_1,T_2,U_2])$ is homogeneous of degree m. In particular, the homogeneous part of degree m of $\gamma(T_1,U_1,T_2,U_2)-\gamma(0)$ is $d(g_m)$. Since the element $\gamma(T_1,U_1,T_2,U_2)-\gamma(0)$ of $G(A[T_1,U_1,T_2,U_2])$ has a non zero homogeneous part of degree m for only finitely many integers m, we must have $d(g_m)=0$ for m large enough. Thus the element

$$-\gamma(0)+\sum_{\substack{m\geq 1\\ d(g_m)\neq 0}} g_m,$$

of $G(A[[T_1,U_1,T_2,U_2]])$ belongs to $G(A[T_1,U_1,T_2,U_2])$, and its image by d is γ. Consequently, E_γ is a trivial extension of G by itself.

III.6.20. For $p=2$, let G be the extension of $\mathbb{G}_{a,S}$ by itself defined by $c(U_1,U_2)=U_1U_2$ (cf. III.6.7), and let F be the endomorphism $(t,u)\mapsto(t^2,u^2)$ of G. We then have the Lang-Artin-Schreier exact sequence

$$0\to G(\mathbb{F}_2)\to G\xrightarrow{1-F} G\to 0. \tag{41}$$

By applying the functor $\mathrm{Hom}(G,-)$ to this short exact sequence, we obtain a long exact sequence

$$\begin{array}{c}\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G(\mathbb{F}_2))\longrightarrow \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G)\xrightarrow{1-F}\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G)\\ \xrightarrow{\delta}\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G(\mathbb{F}_2))\longrightarrow \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G)\xrightarrow{1-F}\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G)\end{array}$$

whose first term vanishes since G has geometrically connected fibers over S, and whose last homomorphism is injective by Proposition III.6.19. We thus have a short exact sequence

$$0\to \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G)\xrightarrow{1-F}\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G)\xrightarrow{\delta}\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G,G(\mathbb{F}_2))\to 0. \tag{42}$$

Remark III.6.21. There is a unique isomorphism of abelian groups

$$\mathbb{Z}/4\mathbb{Z}\to G(\mathbb{F}_2)$$

which sends 1 to $(0,1)$.

For any additive polynomials $a(U)=\sum_{r\geq 0} a_r U^{2^r}$ and $b(U)$ with coefficients in A, let us denote by $\langle a,b\rangle$ the element of $\mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(\mathbb{G}_{a,S},\mathbb{G}_{a,S})$ given by

$$\langle b,a\rangle=\left(a\left(T^{\frac{1}{2}}\right)^2+\widetilde{a}(U)+b(U),a(U)\right)$$

where we have set $\widetilde{a}(U) = \sum_{r_1 > r_2 \geq 0} a_{r_1} a_{r_2} U^{2^{r_1}+2^{r_2}}$. By Proposition III.6.18, any endomorphism of G is of the form $\langle b, a\rangle$ for a (necessarily unique) couple (a, b) of additive polynomials with coefficients in A.

Proposition III.6.22. *The map*

$$\begin{aligned} A^2 \times \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G) &\to \mathrm{Hom}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G) \\ ((\beta, \alpha), f) &\mapsto \langle \beta U, \alpha U\rangle + f - F \circ f, \end{aligned}$$

is bijective.

Let $\langle b, a\rangle$ be an endomorphism of G, where $a(U) = \sum_{r \geq 0} a_r U^{2^r}$ and $b(U)$ are additive polynomials with coefficients in A. By Proposition III.6.11, there exists a unique element α of A and a unique additive polynomial $g(U)$ with coefficients in A such that $a(U)$ is equal to $\alpha U + g(U) - g(U)^2$. We thus have

$$\langle b, a\rangle = \langle b', \alpha U\rangle + (1 - F)(\langle 0, g\rangle),$$

for a uniquely determined additive polynomial b'. By Proposition III.6.11 again, there exists a unique element β of A and a unique additive polynomial $h(U)$ with coefficients in A such that $b'(U)$ is equal to $\beta U + h(U) - h(U)^2$. We then have

$$\begin{aligned} \langle b, a\rangle &= \langle \beta U, \alpha U\rangle + (1 - F)(\langle h, 0\rangle) + (1 - F)(\langle 0, g\rangle) \\ &= \langle \beta U, \alpha U\rangle + (1 - F)(\langle h, g\rangle), \end{aligned}$$

hence the conclusion of Proposition III.6.22.

Corollary III.6.23. *For $p = 2$, let G be the extension of $\mathbb{G}_{a,S}$ by itself defined by $c(U_1, U_2) = U_1 U_2$ (cf. III.6.7). Then the map*

$$\begin{aligned} G(A) &\to \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G(\mathbb{F}_2)) \\ (\beta, \alpha) &\mapsto \delta(\langle \beta^{\frac{1}{2}} U, \alpha U\rangle) \end{aligned}$$

is an isomorphism of abelian groups.

The bijectivity of the map in Corollary III.6.23 follows immediately from Proposition III.6.22 and from the exact sequence (42). The fact that this map is a group homomorphism follows from the following computation: if (β_1, α_1) and (β_2, α_2) are elements of $G(A)$, then we have

$$\langle \beta_1^{\frac{1}{2}} U, \alpha_1 U\rangle + \langle \beta_2^{\frac{1}{2}} U, \alpha_2 U\rangle = \langle (\beta_1 + \beta_2)^{\frac{1}{2}} U + \alpha_1 \alpha_2 U^2, (\alpha_1 + \alpha_2) U\rangle,$$

and the right hand side can be decomposed as

$$\langle (\beta_1 + \beta_2)^{\frac{1}{2}} U + \alpha_1 \alpha_2 U^2, (\alpha_1 + \alpha_2) U\rangle = \langle (\beta_1 + \beta_2 + \alpha_1 \alpha_2)^{\frac{1}{2}} U, (\alpha_1 + \alpha_2) U\rangle - (1 - F)(\langle \alpha_1^{\frac{1}{2}} \alpha_2^{\frac{1}{2}} U, 0\rangle),$$

hence the conclusion.

III.6.24. For $p = 2$, let G be the extension of $\mathbb{G}_{a,S}$ by itself defined by $c(U_1, U_2) = U_1 U_2$ (cf. III.6.7). Let Λ be an ℓ-adic coefficient ring, and let $\xi : G(\mathbb{F}_2) \to \Lambda^\times$ be an injective homomorphism of abelian groups; this amounts to a choice of primitive fourth root of unity in Λ, cf. Remark III.6.21.

The pushout by ξ of the Lang-Artin-Schreier $G(\mathbb{F}_2)$-torsor (cf. 36) yields a multiplicative Λ-local system on G (cf. III.5.6), which we denote by $\mathcal{L}_\xi$. More generally, if $f = (f_0, f_1) : X \to G$ is a morphism of S-schemes, we denote by $\mathcal{L}_\xi\{f_0, f_1\}$ the Λ-local system of rank 1 on X given by the pullback of $\mathcal{L}_\xi$ by f. If moreover X is an S-group scheme and if $f = (f_0, f_1)$ is a homomorphism of S-group schemes, then $\mathcal{L}_\xi\{f_0, f_1\}$ is a multiplicative Λ-local system on X.

If $f = (f_0, f_1)$ and $f' = (f'_0, f'_1)$ are S-morphisms from an S-scheme X to G, then multiplicativity of $\mathcal{L}_\xi$ on G yields an isomorphism

$$\mathcal{L}_\xi\{f_0, f_1\} \otimes \mathcal{L}_\xi\{f'_0, f'_1\} \cong \mathcal{L}_\xi\{f_0 + f'_0 + f_1 f'_1, f_1 + f'_1\} \tag{43}$$

of Λ-local systems on X.

Remark III.6.25. For any morphism $f : X \to \mathbb{G}_{a,S}$ of S-schemes, the Λ-local system $\mathcal{L}_\xi\{f, 0\}$ is isomorphic to the Artin-Schreier local system $\mathcal{L}_\psi\{f\}$ from III.6.13, where ψ is the restriction of ξ to the subgroup $\mathbb{F}_2$ of $G(\mathbb{F}_2)$. Moreover, the composition of ψ with the surjective homomorphism $G(\mathbb{F}_2) \to \mathbb{F}_2$ is equal to ξ^2, and we have isomorphisms by 43

$$\mathcal{L}_\xi\{f_0, f_1\}^{\otimes 2} \cong \mathcal{L}_\xi\{f_1^2, 0\} \cong \mathcal{L}_\psi\{f_1^2\} \cong \mathcal{L}_\psi\{f_1\},$$

for any morphism $f = (f_0, f_1) : X \to G$ of S-schemes.

Proposition III.6.26. *Let V be a finitely generated projective A-module, and let $\mathcal{V}$ be the corresponding S-group scheme. Let $\gamma : V \to A$ be a surjective A-linear homomorphism, and let $\widetilde{\mathcal{V}}$ be the extension of $\mathcal{V}$ by $\mathbb{G}_{a,S}$ defined by the element $c : (v_1, v_2) \mapsto \gamma(v_1)\gamma(v_2)$ of $\mathcal{C}(\mathcal{V}, \mathbb{G}_{a,S})$, cf. III.6.5. We denote by $t : \widetilde{\mathcal{V}} \to \mathbb{G}_{a,S}$ and by $v : \widetilde{\mathcal{V}} \to \mathcal{V}$ the canonical projections.*

Then any multiplicative Λ-local system on $\widetilde{\mathcal{V}}$ is isomorphic to $\mathcal{L}_\xi\{\alpha^2 t + v^*(v), \alpha\gamma(v)\}$, for a unique A-linear homomorphism $v^* : V \to A$, considered as a homomorphism from $\mathcal{V}$ to $\mathbb{G}_{a,S}$, and a unique element α of A.

By III.5.9 and by Proposition III.6.14, we can assume (and we do) that V is the A-module A, that γ is the identity, and thus that $\widetilde{\mathcal{V}}$ is G. Moreover, we can assume (and we do as well) that Λ is finite, in which case multiplicative Λ-local systems on G correspond to extensions in S_{fppf} of G by the finite abelian group $\Lambda^\times$. Since ξ realizes an isomorphism from $G(\mathbb{F}_2)$ to the 4-torsion subgroup of $\Lambda^\times$, we deduce from Proposition III.6.3 that the homomorphism

$$\mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, G(\mathbb{F}_2)) \xrightarrow{\xi} \mathrm{Ext}_{\mathrm{Ab}(S_{\mathrm{fppf}})}(G, \Lambda^\times),$$

induced by ξ is an isomorphism. The conclusion then follows from Corollary III.6.23.

Corollary III.6.27. *Assume that $A = k$ is an algebraically closed field of characteristic $p = 2$. Let V a k-vector space of finite dimension $r \geq 1$, let $\mathcal{V}$ be the corresponding k-group scheme, let $\gamma : V \to k$ be a non zero linear form, and let $\widetilde{\mathcal{V}}$ be the extension of $\mathcal{V}$ by $\mathbb{G}_{a,S}$ defined by the element $c : (v_1, v_2) \mapsto \gamma(v_1)\gamma(v_2)$ of $\mathcal{C}(\mathcal{V}, \mathbb{G}_{a,S})$, cf. III.6.5. We denote by $t : \widetilde{\mathcal{V}} \to \mathbb{G}_{a,S}$ and by $v : \widetilde{\mathcal{V}} \to \mathcal{V}$ the canonical projections.*

Let $\mathcal{M}$ be a multiplicative Λ-local system on $\mathcal{V}$, and let α be an element of k. Then the cohomology group

$$H_c^\nu(\widetilde{\mathcal{V}}, \mathcal{M} \otimes \mathcal{L}_\xi\{\alpha^2 t, 0\})$$

vanishes for each integer ν, unless $\nu = 2r + 2$, $\alpha = 0$ and $\mathcal{M}$ is trivial, or $\nu = 2r + 1$, α is non zero and $\mathcal{M}$ is isomorphic to $\mathcal{L}_\xi\{\alpha^2 t + \delta\gamma(v), \alpha\gamma(v)\}$ for some element δ of k, in which case it is a free Λ-module of rank 1.

By Proposition III.6.26, the multiplicative Λ-local system $\mathcal{M}$ isomorphic to $\mathcal{L}_\xi\{\beta^2 t + v^*(v), \beta\gamma(v)\}$ for some k-linear form $v^* : V \to k$, and some element β of k. The projection formula yields an isomorphism

$$Rv_!(\mathcal{M} \otimes \mathcal{L}_\xi\{\alpha^2 t, 0\}) \cong Rv_!(\mathcal{L}_\xi\{(\beta^2 + \alpha^2)t, 0\}) \otimes \mathcal{L}_\xi\{v^*(v), \beta\gamma(v)\},$$

and this complex is quasi-isomorphic to 0 by Corollary III.6.15 and Remark III.6.25, unless $\beta = \alpha$, in which case it is quasi-isomorphic to $\mathcal{L}_\xi\{v^*(v), \beta\gamma(v)\}[-2]$. We can thus assume (and we do) that $\beta = \alpha$, and we must prove that

$$H_c^\nu(\mathcal{V}, \mathcal{L}_\xi\{v^*(v), \alpha\gamma(v)\})$$

vanishes for each integer ν, unless $\nu = 2r$ and $\mathcal{M}$ is trivial, or $\nu = 2r-1$ and $v^* = \delta\gamma$ for some element δ of k, in which case it is a free Λ-module of rank 1.

If $\alpha = 0$, this follows from Corollary III.6.15 and Remark III.6.25. We now assume that α is non zero. In this case, by Corollary III.6.15 and Remark III.6.25 again, the complex

$$R\gamma_!(\mathcal{L}_\xi\{v^*(v), \alpha\gamma(v)\}) \cong R\gamma_!(\mathcal{L}_\xi\{v^*(v), 0\}) \otimes \mathcal{L}_\xi\{0, \alpha x\},$$

where x is the coordinate on $\mathbb{G}_{a,k}$, vanishes unless the restriction of v^* to the kernel of γ vanishes, namely if and only $v^* = \delta\gamma$ for some element δ of k, in which case it is isomorphic by the projection formula to

$$R\gamma_!(\Lambda) \otimes \mathcal{L}_\xi\{\delta x, 0\} \otimes \mathcal{L}_\xi\{0, \alpha x\} \cong \mathcal{L}_\xi\{\delta x, \alpha x\}[2-2r].$$

It remains to prove that

$$H^\nu_c(\mathbb{G}_{a,k}, \mathcal{L}_\xi\{\delta x, \alpha x\})$$

vanishes for each integer ν, unless $\nu = 1$, in which case it is of rank 1. Since $\mathcal{L}_\xi\{\delta x, \alpha x\}$ has no punctual sections, this group vanishes for $\nu = 0$, and by Poincaré duality it vanishes as well for $\nu = 2$. In order to conclude, it remains to compute the Euler characteristic with compact supports of $\mathcal{L}_\xi\{\delta x, \alpha x\}$ on $\mathbb{G}_{a,k}$.

The Swan conductor of $\mathcal{L}_\xi\{\delta x, \alpha x\}$ at infinity is equal to the highest ramification jump of the extension $k((x^{-1}))[t,u]$ of $k((x^{-1}))$ where

$$\begin{aligned} u - u^2 &= \alpha x, \\ t - t^2 &= u^3 + \delta x, \end{aligned}$$

corresponding to the equation $(t,u) - (t^2, u^2) = (\delta x, \alpha x)$ in G. The only ramification jump of the extension $k((x^{-1}))[t,u]/k((x^{-1}))[u]$ (resp. $k((x^{-1}))[u]/k((x^{-1}))$), which is a degree 2 Galois extension, is 3 (resp. 1). Thus the extension $k((x^{-1}))[t,u]$ of $k((x^{-1}))$ has two ramification jumps, namely 1 and some rational number $j > 1$. The slope between 1 and 3 of the Herbrand function of the extension $k((x^{-1}))[t,u]/k((x^{-1}))$, cf. ([Se68], IV.3), is equal $\frac{1}{2}$, namely the inverse of the degree of the subextension $k((x^{-1}))[u]/k((x^{-1}))$. This slope is also equal to $\frac{j-1}{3-1} = \frac{j-1}{2}$ hence $j - 1 = 1$, and thus the second ramification jump of the extension $k((x^{-1}))[t,u]/k((x^{-1}))$ is equal to $j = 2$. Consequently, the Swan conductor of $\mathcal{L}_\xi\{\delta x, \alpha x\}$ at infinity is equal to 2, and the Grothendieck-Ogg-Shafarevich formula implies that the Euler characteristic with compact supports of $\mathcal{L}_\xi\{\delta x, \alpha x\}$ on $\mathbb{G}_{a,k}$ is equal to -1, which concludes the proof of Proposition III.6.27.

III.7. Geometric local ε-factors for sheaves of generic rank at most 1

Let Λ be an ℓ-adic coefficient ring (cf. III.1.13, III.2.2) which is a field, and let $\psi : \mathbb{F}_p \to \Lambda^\times$ be a non trivial homomorphism. We fix a unitary Λ-admissible mutiplier μ on the topological group G_k (cf. III.2.9, III.2.10).

Let T be the spectrum of a k-algebra, which is a henselian discrete valuation ring $\mathcal{O}_T$, with maximal ideal $\mathfrak{m}$, and whose residue field $\mathcal{O}_T/\mathfrak{m}$ is a finite extension of k. Let $j : \eta \to T$ be the generic point of T, and let $i : s \to T$ be its closed point, so that T is canonically an s-scheme, as in III.5.16. We fix a $\overline{k}$-point $\overline{s} : \mathrm{Spec}(\overline{k}) \to T$ of T above s, so that the Galois group $G_s = \mathrm{Gal}(\overline{k}/k(s))$ can be considered as a subgroup of G_k. We still denote by μ the restriction of μ to G_s. We also fix a geometric point $\overline{\eta}$ of T above η.

III.7.1. We denote by $\Omega^1_\eta = \Omega^1_{\eta/k}$ the one-dimensional $k(\eta)$-vector space of 1-forms over η; it is endowed with a differential $d : k(\eta) \to \Omega^1_\eta$, which is continuous for the valuation topology, and such that $d\pi$ is non zero for any uniformizer π of $k(\eta)$. We also denote by $\Omega^{1,\times}_\eta$ the $k(\eta)^\times$-torsor of non zero elements of Ω^1_η. If ω is an element of $\Omega^{1,\times}_\eta$, we denote by $v(\omega)$ the unique integer such that for any uniformizer π of $k(\eta)$, the element $\frac{\omega}{d\pi}$ of $k(\eta)^\times$ has valuation $v(\omega)$.

III.7.2. Let $\mathcal{F}$ be a Λ-sheaf on T. The *conductor* of the couple $(T, \mathcal{F})$ is the integer

$$a(T, \mathcal{F}) = \mathrm{rk}(\mathcal{F}_{\overline{\eta}}) + \mathrm{sw}(\mathcal{F}_{\overline{\eta}}) - \mathrm{rk}(\mathcal{F}_{\overline{s}}).$$

Following Laumon ([**La87**], 3.1.5.1), if ω is an element of $\Omega^{1,\times}_{\eta}$ (cf. III.7.1), we define the conductor of the triple $(T, \mathcal{F}, \omega)$ to be the integer

$$a(T, \mathcal{F}, \omega) = a(T, \mathcal{F}) + \mathrm{rk}(\mathcal{F}_{\overline{\eta}})v(\omega).$$

III.7.3. Let $f : T' \to T$ be a finite generically étale morphism, where T' is the spectrum of a henselian discrete valuation ring, with generic extension $\eta' \to \eta$ and residual extension $s' \to s$. Let $\mathcal{F}$ be a Λ-sheaf on T', and let ω be an element of $\Omega^{1,\times}_{\eta'}$. We then have

$$a(T, f_*\mathcal{F}) = [s' : s]a(T, \mathcal{F}) + v(\partial_{\eta'/\eta})\mathrm{rk}(\mathcal{F}_{\overline{\eta}}),$$

by ([**Se68**], IV.2 Prop. 4), where $\partial_{\eta'/\eta}$ denotes the discriminant of the separable extension $\eta' \to \eta$, cf. ([**Se68**], III.3). We have $v(\partial_{\eta'/\eta}) = [s' : s]v(\partial_{\eta'/\eta_{s'}})$, and

$$[\eta' : \eta_{s'}]v(\omega) + v(\partial_{\eta'/\eta_{s'}}) = v(f^*\omega),$$

hence the formula

$$a(T, f_*\mathcal{F}, \omega) = [s' : s]a(T, \mathcal{F}, f^*\omega).$$

III.7.4. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on T *supported on* s (cf. III.3.7), where T is considered either as a k-scheme or as an s-scheme (cf. III.3.10), and let ω be an element of $\Omega^{1,\times}_{\eta}$. For any element ω of $\Omega^{1,\times}_{\eta}$, we define the ε*-factor* of the triple $(T, \mathcal{F}, \omega)$ to be the Λ-admissible map

$$\begin{aligned} \varepsilon_{\overline{s}}(T, \mathcal{F}, \omega) : G_s &\to \Lambda^\times \\ g &\mapsto \det(g \mid \mathcal{F}_{\overline{s}})^{-1} \end{aligned}$$

The map $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ defines a Λ-admissible representation of rank 1 of $(G_s, \mu^{-\mathrm{rk}\mathcal{F}_{\overline{s}}})$ which is isomorphic to $\det(\mathcal{F}_{\overline{s}})^{-1}$. In particular, we have

$$d^1(\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)) = \mu^{-\mathrm{rk}\mathcal{F}_{\overline{s}}} = \mu^{a(T,\mathcal{F},\omega)},$$

cf. III.2.6 and III.7.2 for the notation.

III.7.5. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on T (cf. III.3.7, III.3.10), *supported on* η, such that $j^{-1}\mathcal{F}$ is of rank 1, and let ω be an element of $\Omega^{1,\times}_{\eta}$. Then $j^{-1}\mathcal{F}$ is a μ-twisted Λ-local system of rank 1 on η. Let $D = \nu s$ be an effective Cartier divisor on T such that $j^{-1}\mathcal{F}$ has ramification bounded by D (cf. III.5.44). Theorem III.5.45 then produces a μ-twisted multiplicative Λ-local system $\chi_{j^{-1}\mathcal{F}}$ (cf. III.5.41, III.5.42) on the s-group scheme $\mathrm{Pic}(T, D)_s$ (cf. III.5.20).

Recall from III.5.30 that $\mathrm{Pic}(T, D)_s$ is naturally isomorphic to the functor which sends an s-scheme S to the group of pairs (d, u), where d is a locally constant $\mathbb{Z}$-valued map on S, and

$$u : \mathcal{O}_S \otimes_{k(s)} \mathcal{O}_T/\mathfrak{m}^\nu \to \mathcal{O}_S \otimes_{k(s)} \mathfrak{m}^{-d}/\mathfrak{m}^{-d+\nu}$$

is an isomorphism of $\mathcal{O}_S \otimes_{k(s)} \mathcal{O}_T/\mathfrak{m}^\nu$-modules. Denoting by $\mathrm{Pic}^d(T, D)_s$ the component of degree d of $\mathrm{Pic}(T, D)_s$, we consider the morphism

$$\begin{aligned} \mathrm{Res}_\omega : \mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T, D)_s &\to \mathbb{G}_{a,s} \\ u &\mapsto \mathrm{Res}(u\omega), \end{aligned}$$

cf. III.4.6, which is well defined since $\nu - a(T, \mathcal{F}, \omega)$ is greater than or equal to $-v(\omega)$.

Proposition III.7.6. *The cohomology group*

$$H^j_c\left(\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T, D)_{\overline{s}}, \chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}\right),$$

vanishes for $j \neq 2\nu - a(T, \mathcal{F})$, and is a Λ-module of rank 1 if $j = 2\nu - a(T, \mathcal{F})$.

The Artin-Schreier sheaf $\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ is defined in the paragraph III.6.13. An equivalent version of Proposition III.7.6 appears with a lacunary proof in Section $g.(B)$ of Deligne's 1974 letter to Serre, published as an appendix in [**BE01**]. We postpone the proof of Proposition III.7.6 to the paragraphs III.7.14, III.7.15, III.7.16, III.7.18 and III.7.20 below.

Definition III.7.7. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on T, *supported on* η, such that $j^{-1}\mathcal{F}$ is of rank 1, and let D be an effective Cartier divisor on T such that $j^{-1}\mathcal{F}$ has ramification bounded by D (cf. III.5.17), namely $\mathrm{sw}(\mathcal{F}_{\overline{\eta}})$ is strictly less than the multiplicity ν of D at s. Let ω be an element of $\Omega^{1,\times}_\eta$. The *ε-factor* of the triple $(T,\mathcal{F},\omega)$ is the Λ-admissible map $\varepsilon_{\overline{s}}(T,\mathcal{F},\omega):G_s\to\Lambda^\times$ such that

$$\varepsilon_{\overline{s}}(T,\mathcal{F},\omega)(g)=\mathrm{Tr}\left(g\mid H_c^{2\nu-a(T,\mathcal{F})}\left(\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_{\overline{s}},\chi_{j^{-1}\mathcal{F}}\otimes\mathcal{L}_\psi\{\mathrm{Res}_\omega\}(\nu-a(T,\mathcal{F},\omega))\right)\right),$$

for any g in G_s.

Since $\chi_{j^{-1}\mathcal{F}}\otimes\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ is a $\mu^{a(T,\mathcal{F},\omega)}$-twisted Λ-sheaf on $\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_s$ (cf. III.5.41), and since the cohomology group in III.7.7 is of rank 1 by Proposition III.7.6 we obtain

$$d^1(\varepsilon_{\overline{s}}(T,\mathcal{F},\omega))=\mu^{a(T,\mathcal{F},\omega)},$$

cf. III.3.14.

The notation $\varepsilon_{\overline{s}}(T,\mathcal{F},\omega)$ suggests that the choice of D is irrelevant. We have indeed:

Proposition III.7.8. *Let $\mathcal{F},D,\omega$ be as in* III.7.7. *Then the map $\varepsilon_{\overline{s}}(T,\mathcal{F},\omega)$ is independent of the choice of D.*

Indeed, if δ is a positive integer, let us consider the projection morphism

$$\tau:\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D+\delta s)_s\to\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_s,$$

which sends for any k-scheme S a trivialization u of $\mathcal{O}_S\otimes_{k(s)}\mathfrak{m}^{-a(T,\mathcal{F},\omega)}/\mathfrak{m}^{-a(T,\mathcal{F},\omega)+\nu+\delta}$ to its image in $\mathcal{O}_S\otimes_{k(s)}\mathfrak{m}^{-a(T,\mathcal{F},\omega)}/\mathfrak{m}^{-a(T,\mathcal{F},\omega)+\nu}$. It is a (trivial) fibration in affine spaces of dimension δ. Let $\mathcal{G}$ be the Λ-sheaf $\chi_{j^{-1}\mathcal{F}}\otimes\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ on $\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_s$. The trace homomorphism

$$R\tau_!\tau^{-1}\mathcal{G}(\delta)[2\delta]\to\mathcal{G},$$

is an isomorphism, and thus the Leray spectral sequence for $(\tau,\mathcal{G})$ yields that the Λ-admissible representation given by

$$H_c^{2\nu+2\delta-a(T,\mathcal{F})}\left(\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D+\delta)_{\overline{s}},\tau^{-1}\mathcal{G}(\nu+\delta-a(T,\mathcal{F},\omega))\right),$$

is isomorphic to

$$H_c^{2\nu-a(T,\mathcal{F})}\left(\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_{\overline{s}},\mathcal{G}(\nu-a(T,\mathcal{F},\omega))\right),$$

hence the result.

III.7.9. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on T, such that $\mathcal{F}_{\overline{\eta}}$ is of rank at most 1 over Λ, and let ω be an element of $\Omega^{1,\times}_\eta$. If $\mathcal{F}$ is supported on a single point of T, then we defined the ε-factor of $(T,\mathcal{F},\omega)$ in III.7.4 and III.7.7. We combine these two definitions as follows:

Definition III.7.10. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on T, such that $j^{-1}\mathcal{F}$ is of rank at most 1 over Λ. The *ε-factor* of the triple $(T,\mathcal{F},\omega)$ is the Λ-admissible map from G_s to $\Lambda^\times$ given by

$$\varepsilon_{\overline{s}}(T,\mathcal{F},\omega)=\varepsilon_{\overline{s}}(T,j_!j^{-1}\mathcal{F},\omega)\varepsilon_{\overline{s}}(T,i_*i^{-1}\mathcal{F},\omega).$$

It follows from III.7.4 and III.7.7 that we have

$$d^1(\varepsilon_{\overline{s}}(T,\mathcal{F},\omega))=\mu^{a(T,\mathcal{F},\omega)}.$$

III.7.11. Let $\mathcal{F}$ be a μ-twisted Λ-sheaf of rank 1 on η, with ramification bounded by $D = \nu s$ (cf. III.5.44). If z is an element of valuation d in $k(\eta)^\times$, then the image of z^{-1} in $\mathfrak{m}^{-d}/\mathfrak{m}^{-d+\nu}$ yields an s-point of $\mathrm{Pic}^d(T,D)_s$. The restriction $\chi_{\mathcal{F}|z^{-1}}$ of $\chi_{\mathcal{F}}$ to this s-point is a Λ-local system of rank 1 on s, and we define

$$\begin{aligned}\langle\chi_{\mathcal{F}}\rangle(z) : G_s &\to \Lambda^\times\\ g &\mapsto \det\left(g \mid (\chi_{\mathcal{F}|z^{-1}})_{\overline{s}}\right),\end{aligned}$$

so that $d^1(\langle\chi_{\mathcal{F}}\rangle(z)) = \mu^d$. The map $\langle\chi_{\mathcal{F}}\rangle(z)$ depends only on $\mathcal{F}$ and z, and not on the choice of D. By multiplicativity of the local system $\chi_{\mathcal{F}}$ (cf. III.5.1), we have

$$\langle\chi_{\mathcal{F}}\rangle(z_1z_2) = \langle\chi_{\mathcal{F}}\rangle(z_1)\langle\chi_{\mathcal{F}}\rangle(z_2)$$

for any elements z_1, z_2 of $k(\eta)^\times$.

Proposition III.7.12. *Let $\mathcal{F}$ be a μ-twisted Λ-sheaf on T, such that $j^{-1}\mathcal{F}$ is of rank 1 over Λ. For any element α of $k(\eta)^\times$, of valuation $v(\alpha)$, we have*

$$\varepsilon_{\overline{s}}(T,\mathcal{F},\alpha\omega) = \langle\chi_{j^{-1}\mathcal{F}}\rangle(\alpha)\chi_{\mathrm{cyc}}^{-v(\alpha)}\varepsilon_{\overline{s}}(T,\mathcal{F},\omega),$$

where $\langle\chi_{j^{-1}\mathcal{F}}\rangle$ is as in III.7.11, and χ_{cyc} is the ℓ-adic cyclotomic character of k (cf. III.1.13).

Indeed, we have an isomorphism

$$\begin{aligned}\theta : \mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_s &\to \mathrm{Pic}^{a(T,\mathcal{F},\alpha\omega)}(T,D)_s\\ u &\mapsto \alpha^{-1}u,\end{aligned}$$

of s-schemes, such that the pullback of $\chi_{j^{-1}\mathcal{F}}\otimes\mathcal{L}_\psi\{\mathrm{Res}_{\alpha\omega}\}(\nu - a(T,\mathcal{F},\alpha\omega))$ by θ is isomorphic to the twist of $\chi_{j^{-1}\mathcal{F}}\otimes\mathcal{L}_\psi\{\mathrm{Res}_\omega\}(\nu - a(T,\mathcal{F},\omega))$ by $(\chi_{j^{-1}\mathcal{F}})_{|\alpha^{-1}}(-v(\alpha))$.

Proposition III.7.13. *Let $n \geq 1$ be an integer prime to p, and let h be an element of $k(\eta)$ of valuation $-n$, and let us consider the Artin-Schreier Λ-sheaf $\mathcal{L}_\psi\{h\}$ on η (cf. III.6.13). Let $\mathcal{M}$ be a μ-twisted Λ-sheaf on η of rank 1 with ramification bounded by the divisor $\lceil\frac{n}{2}\rceil s$, and let us consider the μ-twisted Λ-sheaf $\mathcal{F} = \mathcal{M}\otimes\mathcal{L}_\psi\{-h\}$. For any element ω of $\Omega^{1,\times}_\eta$, we have:*

— *if $n = 2n'-1$ is odd then*

$$\varepsilon_{\overline{s}}(T,j_!\mathcal{F},\omega) = \langle\chi_{\mathcal{F}}\rangle\left(\frac{\omega}{dh}\right)\chi_{\mathrm{cyc}}^{-v\left(\frac{\omega}{dh}\right)+n'},$$

— *if $n = 2n'$ is even then p is odd and we have*

$$\varepsilon_{\overline{s}}(T,j_!\mathcal{F},\omega) = \langle\chi_{\mathcal{F}}\rangle\left(\frac{\omega}{dh}\right)\chi_{\mathrm{cyc}}^{-v\left(\frac{\omega}{dh}\right)+n'+1}\gamma_\psi(-nh_0),$$

where h_0 is an element of $k(s)^\times$ such that $\frac{h}{h_0}$ is a square in $k(\eta)^\times$, and where, for any element c of $k(s)$, we have set

$$\begin{aligned}\gamma_\psi(c) : G_s &\to \Lambda^\times\\ g &\mapsto \det\left(g \mid H^1_c\left(\mathbb{G}_{a,\overline{s}}, \mathcal{L}_\psi\{\frac{ct^2}{2}\}\right)\right).\end{aligned}$$

When k is a finite field of odd characteristic, the conclusion of Proposition III.7.13 follows from ([**AS10**], Prop. 8.7) and from Proposition III.7.22 below.

We note that the Swan conductor of a μ-twisted Λ-sheaf $\mathcal{F}$ as in Proposition III.7.13 is n. By Proposition III.7.12, it is sufficient to prove Proposition III.7.13 when $\omega = \frac{d\pi}{\pi^{n+1}}$, for a fixed uniformizer π, in which case $\frac{\omega}{dh}$ has valuation 0. The conclusion then follows from the proof of Proposition III.7.6 below, cf. Porisms III.7.17 and III.7.19.

III.7.14. We now prove Proposition III.7.6. Let us first consider the projection morphism

$$\tau : \mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_s \to \mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,a(T,\mathcal{F})s)_s,$$

of relative dimension $\delta = \nu - a(T,\mathcal{F})$, which sends for any k-scheme S a trivialization u of $\mathcal{O}_S \otimes_{k(s)} \mathfrak{m}^{-a(T,\mathcal{F},\omega)}/\mathfrak{m}^{-a(T,\mathcal{F},\omega)+\nu}$ to its image in $\mathcal{O}_S \otimes_{k(s)} \mathfrak{m}^{-a(T,\mathcal{F},\omega)}/\mathfrak{m}^{-v(\omega)}$. Let $\mathcal{G}$ be the Λ-sheaf $\chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ on the s-group scheme $\mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,a(T,\mathcal{F})s)_s$. As in the proof of III.7.8, we have an isomorphism

$$R\tau_!\tau^{-1}\mathcal{G}(\delta)[2\delta] \to \mathcal{G},$$

hence the Leray spectral sequence for $(\tau,\mathcal{G})$ implies that it is enough to prove Proposition III.7.6 when the multiplicity ν of D at s is exactly $a(T,\mathcal{F})$.

Let us now assume that ν is equal to $a(T,\mathcal{F})$. Let π be a uniformizer of $k(\eta)$. Let us write $\omega = \alpha^{-1}\frac{d\pi}{\pi^\nu}$ for some element α of $k(\eta)^\times$ of valuation $-a(T,\mathcal{F},\omega) = -\nu - v(\omega)$, and let us consider the isomorphism

$$\begin{aligned}\theta : \mathrm{Pic}^0(T,D)_s &\to \mathrm{Pic}^{a(T,\mathcal{F},\omega)}(T,D)_s\\ u &\mapsto \alpha u.\end{aligned}$$

The pullback of $\chi_{j^{-1}\mathcal{F}}$ by θ coincides with $\chi_{j^{-1}\mathcal{F}}$ on $\mathrm{Pic}^0(T,D)_s$, up to twist by the fiber $\alpha^{-1}\chi_{j^{-1}\mathcal{F}}$ of $\chi_{j^{-1}\mathcal{F}}$ at the s-point α, while the pullback of $\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ is isomorphic to $\mathcal{L}_\psi\{\mathrm{Res}_{\pi^{-\nu}d\pi}\}$. Thus we can assume that ω is equal to $\frac{d\pi}{\pi^\nu}$, so that we have $a(T,\mathcal{F},\omega) = 0$.

III.7.15. Let us prove Proposition III.7.6 when ω is equal to $\frac{d\pi}{\pi^\nu}$ and when $\nu = a(T,\mathcal{F})$ is equal to 1. In this case, the s-group scheme $\mathrm{Pic}^0(T,D)_s$ is simply the multiplicative group $\mathbb{G}_{m,s}$, and $\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ coincides with $\mathcal{L}_\psi\{t\}$. Any multiplicative Λ-local system on $\mathbb{G}_{m,s}$, such as $\chi_{j^{-1}\mathcal{F}}$, is tamely ramified at 0 and ∞, cf. for example ([**Gu18**], 3.15). Thus the Λ-local system $\chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{t\}$ on $\mathbb{G}_{m,s}$ has Swan conductor 0 at 0, and 1 at infinity. By the Grothendieck-Ogg-Shafarevich formula, we have

$$\chi_c(\mathbb{G}_{m,\bar{s}}, \chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{t\}) = 2 - 1 - 2 = -1.$$

Moreover, the cohomology groups $H^0_c(\mathbb{G}_{m,\bar{s}}, \chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{t\})$ and $H^0(\mathbb{G}_{m,\bar{s}}, \chi^{-1}_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{-t\})$ both vanish, and so does the group

$$H^2_c(\mathbb{G}_{m,\bar{s}}, \chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{t\}),$$

by Poincaré duality. This proves that the cohomology group

$$H^j_c(\mathbb{G}_{m,\bar{s}}, \chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{t\}),$$

vanishes when $j \neq 1$, and is of rank 1 when j is equal to 1.

III.7.16. Let us prove Proposition III.7.6 when ω is equal to $\frac{d\pi}{\pi^\nu}$ and when $\nu = a(T,\mathcal{F})$ is even, hence of the form $2\nu'$ for some integer $\nu' \geq 1$. Let us consider the projection morphism

$$\sigma : \mathrm{Pic}^0(T,D)_s \to \mathrm{Pic}^0(T,\nu' s)_s,$$

which is a homomorphism of s-group schemes. Let $\mathcal{V}$ be the additive s-group scheme associated to the finite dimensional s-vector space $V = \mathfrak{m}^{\nu'}/\mathfrak{m}^\nu$. Then the morphism

$$\begin{aligned}r : \mathcal{V} &\to \mathrm{Pic}^0(T,D)_s\\ x &\mapsto 1 + x,\end{aligned}$$

realizes an isomorphism of s-group schemes from $\mathcal{V}$ onto the kernel of σ. By Proposition III.6.14, the multiplicative Λ-local system $r^{-1}\chi_{j^{-1}\mathcal{F}}$ is isomorphic to an Artin-Schreier sheaf $\mathcal{L}_\psi\{-v^*\}$, for some linear form v^* on V. The $k(s)$-linear map

$$\begin{aligned} \mathcal{O}_T/\mathfrak{m}^{\nu'} &\to \mathrm{Hom}_{k(s)}(V, k(s)) \\ y &\mapsto \mathrm{Res}_{y\omega} = (x \mapsto \mathrm{Res}(xy\omega)), \end{aligned} \tag{44}$$

is an isomorphism onto the $k(s)$-linear dual of $V = \mathfrak{m}^{\nu'}/\mathfrak{m}^\nu$. Thus there exists a unique element y of $\mathcal{O}_T/\mathfrak{m}^{\nu'}$ such that v^* is equal to $\mathrm{Res}_{y\omega}$.

Since the ramification of $\chi_{j^{-1}\mathcal{F}}$ is not bounded by the divisor $(\nu-1)s = \mathrm{sw}(\mathcal{F}_{\overline{\eta}})s$, it follows from Theorem III.5.45 and from III.5.10 that the restriction of $r^{-1}\chi_{j^{-1}\mathcal{F}}$ to the sub-s-group scheme corresponding to $\mathfrak{m}^{\nu-1}/\mathfrak{m}^\nu$ is non trivial. Thus the restriction of $v^* = \mathrm{Res}_{y\omega}$ to $\mathfrak{m}^{\nu-1}/\mathfrak{m}^\nu$ is non trivial, and consequently y is a unit of $\mathcal{O}_T/\mathfrak{m}^{\nu'}$. In particular, y defines an s-point of $\mathrm{Pic}^0(T, \nu' s)_s$.

Let $\overline{t}$ be the spectrum of an algebraically closed extension of $k(s)$, and let u be a $\overline{t}$-point of $\mathrm{Pic}^0(T, D)_{\overline{t}}$. The morphism

$$\begin{aligned} ur : \mathcal{V}_{\overline{t}} &\to \mathrm{Pic}^0(T, D)_{\overline{t}} \\ x &\mapsto u(1+x), \end{aligned}$$

realizes an isomorphism onto the fiber of σ above $\sigma(u)$. Let $\mathcal{G}$ be the Λ-sheaf $\chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ on the s-group scheme $\mathrm{Pic}^0(T, D)_s$. The pullback of $\mathcal{G}$ by ur is isomorphic to $\mathcal{L}_\psi\{\mathrm{Res}_{(u-y)\omega}\}$, with notation as in (44), up to twist by the stalk $\mathcal{G}_u$ of $\mathcal{G}$ at the geometric point u of $\mathrm{Pic}^0(T, D)_s$. Together with Proposition III.6.15, this implies that the complex

$$R\Gamma_c(\sigma^{-1}(\sigma(u)), \mathcal{G}),$$

vanishes unless $\sigma(u)$ is equal to y, in which case it is concentrated in degree $2\nu' = \nu$, and the cohomology group $H^\nu_c(\sigma^{-1}(\sigma(u)), \mathcal{G})$ is of rank 1. We obtain that $R\sigma_!\mathcal{G}$ is of the form $y_*\mathcal{L}[-\nu]$, where $\mathcal{L}$ is a Λ-sheaf of rank 1 on s, and thus that the complex

$$R\Gamma_c(\mathrm{Pic}^0(T, D)_{\overline{s}}, \mathcal{G}) \cong R\Gamma_c(\mathrm{Pic}^0(T, \nu' s)_{\overline{s}}, R\sigma_!\mathcal{G}),$$

is isomorphic to $\mathcal{L}_{\overline{s}}[-\nu]$, hence the conclusion of Proposition III.7.6.

Porism III.7.17. Let us assume that $\mathcal{F} = \mathcal{M} \otimes \mathcal{L}_\psi\{-h\}$ is as in Proposition III.7.13, so that $\nu = n+1$ and $n = 2\nu' - 1$. We keep the notation from III.7.16. Since $\mathcal{M}$ has ramification bounded by ν', the restriction $r^{-1}\chi_{\mathcal{M}}$ is trivial. Moreover, by III.4.5 and III.5.25, we have $\chi_{\mathcal{F}} = \chi_{\mathcal{M}} \otimes \mathcal{L}_\psi\{\mathrm{Res}(h\frac{du}{u})\}$, where u is the universal unit parametrized by $\mathrm{Pic}(T, D)_s$. For any section x of $\mathcal{V}$, we have

$$\begin{aligned} \mathrm{Res}\left(h\frac{d(1+x)}{1+x}\right) &= \mathrm{Res}(hdx) - \mathrm{Res}\left(xh\frac{dx}{1+x}\right) \\ &= -\mathrm{Res}(xdh) - \mathrm{Res}\left(xh\frac{dx}{1+x}\right), \end{aligned}$$

which is equal to $-\mathrm{Res}(xdh)$ since $\frac{xhdx}{1+x}$ has nonnegative valuation. Thus the element y of $\mathcal{O}_T/\mathfrak{m}^{\nu'}$ which appears in the proof above is equal to $\frac{dh}{\omega}$. Thus $R^\nu\sigma_!\mathcal{G}$ is concentrated on the s-point $\frac{dh}{\omega}$ of $\mathrm{Pic}^0(T, \nu' s)_s$, and its restriction to the s-point $\frac{dh}{\omega}$ is $\chi_{\mathcal{F}|\frac{dh}{\omega}}(-\nu')$, hence the conclusion of Proposition III.7.13 when n is odd.

III.7.18. Let us prove Proposition III.7.6 when ω is equal to $\frac{d\pi}{\pi^\nu}$, when $\nu = a(T, \mathcal{F})$ is of the form $2\nu' + 1$ for some integer $\nu' \geq 1$, and when the characteristic p of k is odd. As in III.7.16, let us consider the projection morphisms

$$\begin{tikzcd} \mathrm{Pic}^0(T, D)_s \arrow[r, "\sigma_1"] \arrow[rr, bend right, "\sigma"] & \mathrm{Pic}^0(T, (\nu'+1)s)_s \arrow[r, "\sigma_2"] & \mathrm{Pic}^0(T, \nu' s)_s, \end{tikzcd}$$

so that σ, σ_1 and σ_2 are all homomorphisms of s-group schemes. Let $\mathcal{V}$ be the additive s-group scheme associated to the finite dimensional s-vector space $V = \mathfrak{m}^{\nu'}/\mathfrak{m}^\nu$. Then the morphism

$$\begin{aligned} r : \mathcal{V} &\to \mathrm{Pic}^0(T, D)_s \\ x &\mapsto 1 + x + \frac{x^2}{2}, \end{aligned}$$

realizes an isomorphism of s-group schemes from $\mathcal{V}$ to the kernel of σ. By Proposition III.6.14, the multiplicative Λ-local system $r^{-1}\chi_{j^{-1}\mathcal{F}}$ is isomorphic to an Artin-Schreier sheaf $\mathcal{L}_\psi\{-v^*\}$, for some linear form v^* on V. Let y be the unique element of $\mathcal{O}_T/\mathfrak{m}^{\nu'+1}$ such that v^* coincides with the linear form

$$\begin{aligned} \mathrm{Res}_{y\omega} : \mathfrak{m}^{\nu'}/\mathfrak{m}^\nu &\to k(s) \\ x &\mapsto \mathrm{Res}(xy\omega). \end{aligned}$$

As in III.7.16, the element y is a unit of $\mathcal{O}_T/\mathfrak{m}^{\nu'+1}$, whence y defines an s-point of $\mathrm{Pic}^0(T, (\nu' + 1)s)_s$.

Let $\bar{t}$ be the spectrum of an algebraically closed extension of $k(s)$, and let u be a $\bar{t}$-point of $\mathrm{Pic}^0(T, D)_{\bar{t}}$. The morphism

$$\begin{aligned} ur : \mathcal{V}_{\bar{t}} &\to \mathrm{Pic}^0(T, D)_{\bar{t}} \\ x &\mapsto u(1 + x + \frac{x^2}{2}), \end{aligned}$$

realizes an isomorphism onto the fiber of σ above $\sigma(u)$. Let $\mathcal{G}$ be the Λ-sheaf $\chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ on the s-group scheme $\mathrm{Pic}^0(T, D)_s$. The pullback of $\mathcal{G}$ by ur is isomorphic to $\mathcal{L}_\psi\{\mathrm{Res}_{(u-y)\omega} + \alpha\gamma^2\}$, up to twist by the stalk $\mathcal{G}_u$ of $\mathcal{G}$ at the geometric point u of $\mathrm{Pic}^0(T, D)_s$, where α is the image of $\frac{u}{2}$ in $k(\bar{t})^\times$ and $\gamma : V \to k(s)$ is the linear form which sends an element x to the image of $\pi^{-\nu'}x$ in $k(s)$. Together with Proposition III.6.16, this implies that the complex

$$R\Gamma_c(\sigma^{-1}(\sigma(u)), \mathcal{G}),$$

vanishes unless the linear form $\mathrm{Res}_{(u-y)\omega}$ on V is proportional to γ, namely unless $\sigma(u) = \sigma_2(y)$, in which case this complex is concentrated in degree $2(\nu' + 1) - 1 = \nu$, and the cohomology group $H^\nu_c(\sigma^{-1}(\sigma(u)), \mathcal{G})$ is of rank 1. We obtain that $R\sigma_!\mathcal{G}$ is of the form $\sigma_2(y)_*\mathcal{L}[-\nu]$, where $\mathcal{L}$ is a Λ-sheaf of rank 1 on s, and thus that the complex

$$R\Gamma_c(\mathrm{Pic}^0(T, D)_{\bar{s}}, \mathcal{G}) \cong R\Gamma_c(\mathrm{Pic}^0(T, \nu' s)_{\bar{s}}, R\sigma_!\mathcal{G}),$$

is isomorphic to $\mathcal{L}_{\bar{s}}[-\nu]$, hence the conclusion of Proposition III.7.6.

Porism III.7.19. Let us assume that $\mathcal{F} = \mathcal{M} \otimes \mathcal{L}_\psi\{-h\}$ is as in Proposition III.7.13, so that $\nu = n + 1$ and $n = 2\nu'$. We keep the notation from III.7.18. Since $\mathcal{M}$ has ramification bounded by ν', the restriction $r^{-1}\chi_{\mathcal{M}}$ is trivial. Moreover, by III.4.5 and III.5.25, we have

$\chi_{\mathcal{F}} = \chi_{\mathcal{M}} \otimes \mathcal{L}_\psi\{\mathrm{Res}(h\frac{du}{u})\}$, where u is the universal unit parametrized by $\mathrm{Pic}(T,D)_s$ (cf. III.5.30). For any section x of $\mathcal{V}$, we have

$$\begin{aligned}\mathrm{Res}\left(h\frac{d(1+x+\frac{x^2}{2})}{1+x+\frac{x^2}{2}}\right) &= \mathrm{Res}(hdx) - \mathrm{Res}\left(x^2h\frac{dx}{2(1+x)}\right)\\ &= -\mathrm{Res}(xdh) - \mathrm{Res}\left(x^2h\frac{dx}{2(1+x)}\right),\end{aligned}$$

which is equal to $-\mathrm{Res}(xdh)$ since $x^2h\frac{dx}{2(1+x)}$ has nonnegative valuation. Thus the element y of $\mathcal{O}_T/\mathfrak{m}^{\nu'+1}$ which appears in the proof above is equal to $\frac{dh}{\omega}$. Thus $R^\nu\sigma_!\mathcal{G}$ is concentrated on the s-point $\frac{dh}{\omega}$ of $\mathrm{Pic}^0(T,\nu's)_s$. Moreover, choosing $u = y = \frac{dh}{\omega}$ in the computation above, the restriction $(ur)^{-1}\chi_{\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ is isomorphic to $\chi_{\mathcal{F}|\frac{dh}{\omega}} \otimes \mathcal{L}_\psi\{\alpha\gamma^2\}$, where α is the image of $\frac{dh}{2\omega}$ in $k(s)^\times$. If h_0 is the image in $k(s)^\times$ of $\pi^n h$, then we have $\alpha = -\frac{nh_0}{2}$. We thus have

$$\varepsilon_{\overline{s}}(T,j_!\mathcal{F},\omega)(g) = \langle\chi_{\mathcal{F}}\rangle\left(\frac{\omega}{dh}\right)(g)\chi_{\mathrm{cyc}}(g)^{\nu'+1}\mathrm{Tr}\left(g \mid H^\nu_c\left(\mathcal{V}_{\overline{s}},\mathcal{L}_\psi\{-\frac{nh_0\gamma^2}{2}\})\right)\right),$$

for any g in G_s, hence the conclusion of Proposition III.7.13 when n is even.

III.7.20. Let us prove Proposition III.7.6 when ω is equal to $\frac{d\pi}{\pi^\nu}$, when $\nu = a(T,\mathcal{F})$ is of the form $2\nu'+1$ for some integer $\nu' \geq 1$, and when k is of characteristic $p = 2$. As in III.7.18, let us consider the projection morphisms

$$\begin{tikzcd} \mathrm{Pic}^0(T,D)_s \arrow[r, "\sigma_1"] \arrow[rr, bend right, "\sigma"'] & \mathrm{Pic}^0(T,(\nu'+1)s)_s \arrow[r, "\sigma_2"] & \mathrm{Pic}^0(T,\nu's)_s, \end{tikzcd}$$

so that σ, σ_1 and σ_2 are all homomorphisms of s-group schemes. Let $\mathcal{V}$ be the additive s-group scheme associated to the finite dimensional $k(s)$-vector space $V = \mathfrak{m}^{\nu'}/\mathfrak{m}^\nu$. Let $\gamma : V \to k(s)$ be the $k(s)$-linear form which sends an element v of V to the image in $k(s)$ of the element $\pi^{-\nu'}v$ of $\mathcal{O}_T/\mathfrak{m}^{\nu'+1}$, and let $\widetilde{\mathcal{V}}$ be the extension of $\mathcal{V}$ by $\mathbb{G}_{a,s}$ defined by the element $c : (v_1,v_2) \mapsto \gamma(v_1)\gamma(v_2)$ of $\mathcal{C}(\mathcal{V},\mathbb{G}_{a,S})$, cf. III.6.5. For any sections (t_1,v_1) and (t_2,v_2) of the $k(s)$-scheme $\widetilde{\mathcal{V}} = \mathbb{G}_{a,s} \times_s \mathcal{V}$, we have

$$(t_1,v_1) + (t_2,v_2) = (t_1+t_2+\gamma(v_1)\gamma(v_2), v_1+v_2),$$

in $\widetilde{\mathcal{V}}$, and we have

$$(1+v_1+\pi^{2\nu'}t_1)(1+v_2+\pi^{2\nu'}t_2) = 1+v_1+v_2+\pi^{2\nu'}\left(t_1+t_2+(\pi^{-\nu'}v_1)(\pi^{-\nu'}v_2)\right),$$

in $\mathrm{Pic}^0(T,D)_s$. Thus the morphism

$$\begin{aligned} r : \widetilde{\mathcal{V}} &\to \mathrm{Pic}^0(T,D)_s\\ (t,v) &\mapsto 1+v+\pi^{2\nu'}t,\end{aligned}$$

of s-schemes is a homomorphism of s-group schemes. It surjects onto the kernel of σ, and its kernel is isomorphic to $\mathbb{G}_{a,s}$ (cf. 45 below).

Let G be the extension of $\mathbb{G}_{a,S}$ by itself defined by $c(U_1,U_2) = U_1U_2$ (cf. III.6.7), and let $\xi : G(\mathbb{F}_2) \to \Lambda^\times$ be an injective character (cf. III.6.24), whose restriction to the subgroup $\mathbb{F}_2$ of $G(\mathbb{F}_2)$ is ψ. By Proposition III.6.26, the multiplicative Λ-local system $r^{-1}\chi_{j^{-1}\mathcal{F}}$ is isomorphic to $\mathcal{L}_\xi\{\alpha^2t+v^*(v),\alpha\gamma(v)\}$ (cf. III.6.24 for the notation), for some linear form v^* on V and some element α of k. Let y be the unique element of $\mathcal{O}_T/\mathfrak{m}^{\nu'+1}$ such that v^* coincides with the linear form

$$\begin{aligned} \mathrm{Res}_{y\omega} : \mathfrak{m}^{\nu'}/\mathfrak{m}^{\nu} &\to k(s) \\ x &\mapsto \mathrm{Res}(xy\omega). \end{aligned}$$

Since the homomorphism of s-group schemes

$$\begin{aligned} \tau : \mathbb{G}_{a,s} &\to \widetilde{\mathcal{V}} \\ t &\mapsto (t, \pi^{2\nu'} t), \end{aligned} \tag{45}$$

realizes an isomorphism onto the kernel of r, the pullback $\mathcal{L}_\psi\{(\alpha^2 + v^*(\pi^{2\nu'}))t\}$ of $r^{-1}\chi_{j^{-1}\mathcal{F}}$ by τ is trivial, hence $\alpha^2 = v^*(\pi^{2\nu'})$ by Proposition III.6.14.

Since the ramification of $\chi_{j^{-1}\mathcal{F}}$ is not bounded by the divisor $(\nu-1)s = \mathrm{sw}(\mathcal{F}_{\overline{\eta}})s$, it follows from Theorem III.5.45 and from III.5.10 that the restriction of $r^{-1}\chi_{j^{-1}\mathcal{F}}$ to the sub-s-group scheme $\mathbb{G}_{a,s}$ of $\widetilde{\mathcal{V}}$ is non trivial, hence α is non zero. This implies that the scalar $v^*(\pi^{2\nu'}) = \alpha^2$ is non zero, and consequently that y is a unit of $\mathcal{O}_T/\mathfrak{m}^{\nu'+1}$. In particular, y defines an s-point of $\mathrm{Pic}^0(T, (\nu'+1)s)_s$.

Let $\overline{x}$ be the spectrum of an algebraically closed extension of $k(s)$, and let u be an $\overline{x}$-point of $\mathrm{Pic}^0(T, D)_{\overline{x}}$. The morphism

$$\begin{aligned} ur : \widetilde{\mathcal{V}}_{\overline{x}} &\to \mathrm{Pic}^0(T, D)_{\overline{x}} \\ (t, v) &\mapsto u(1 + v + \pi^{2\nu'} t), \end{aligned}$$

surjects onto the fiber of σ above $\sigma(u)$. Let $\mathcal{G}$ be the Λ-sheaf $\chi_{j^{-1}\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ on the s-group scheme $\mathrm{Pic}^0(T, D)_s$. The pullback of $\mathcal{G}$ by ur is isomorphic to $\mathcal{L}_\xi\{(\alpha^2+\beta)t + \mathrm{Res}_{(u-y)\omega}(v), \alpha\gamma(v)\}$, up to twist by the stalk $\mathcal{G}_u$ of $\mathcal{G}$ at the geometric point u of $\mathrm{Pic}^0(T, D)_s$, where β is the image in $k(\overline{x})$ of u. The trace morphism

$$R(ur)_!(ur)^{-1}\mathcal{G}[2](1) \to \mathcal{G}_{|\sigma^{-1}(\sigma(u))},$$

is an isomorphism, since ur is a $\mathbb{G}_a$-torsor over $\sigma^{-1}(\sigma(u))$. Together with Proposition III.6.27, this implies that the complex

$$R\Gamma_c(\sigma^{-1}(\sigma(u)), \mathcal{G}),$$

vanishes unless $\beta = \alpha^2$ and the linear form $\mathrm{Res}_{(u-y)\omega}$ on V is proportional to γ, namely unless $\sigma(u) = \sigma_2(y)$, in which case it is concentrated in degree $2(\nu'+1) - 1 = \nu$, and the cohomology group $H^\nu_c(\sigma^{-1}(\sigma(u)), \mathcal{G})$ is of rank 1. We obtain that $R\sigma_!\mathcal{G}$ is of the form $\sigma_2(y)_*\mathcal{L}[-\nu]$, where $\mathcal{L}$ is a Λ-sheaf of rank 1 on s, and thus that the complex

$$R\Gamma_c(\mathrm{Pic}^0(T, D)_{\overline{s}}, \mathcal{G}) \cong R\Gamma_c(\mathrm{Pic}^0(T, \nu' s)_{\overline{s}}, R\sigma_!\mathcal{G}),$$

is isomorphic to $\mathcal{L}_{\overline{s}}[-\nu]$, hence the conclusion of Proposition III.7.6.

This, together with the paragraphs III.7.14, III.7.15, III.7.16, and III.7.18 concludes the proof of Proposition III.7.6.

III.7.21. We now assume that k is a finite field of cardinality q, that Λ is C, and that $\mu = 1$ is the trivial C-admissible mutiplier on G_k. The k-automorphism $x \mapsto x^q$ is a topological generator of G_k, and we denote by Frob_k its inverse. Similarly, $\mathrm{Frob}_s = \mathrm{Frob}_k^f$ is a topological generator of the subgroup G_s of G_k, where f is the degree of the extension $k(s)/k$.

If $\mathcal{F}$ is C-local system of rank 1 on η, and let D be an effective Cartier divisor on T such that $\mathcal{F}$ has ramification bounded by D (cf. III.5.17), namely $\mathrm{sw}(\mathcal{F}_{\overline{\eta}})$ is strictly less than the multiplicity ν of D at s. Then Theorem III.5.26 produces a multiplicative C-local system $\chi_{\mathcal{F}}$ on the s-group scheme $\mathrm{Pic}(T,D)_s$. Moreover, the map

$$\begin{aligned}\chi_{\mathcal{F}} : k(\eta)^{\times} &\to C^{\times}\\ z &\mapsto \langle \chi_{\mathcal{F}}\rangle(z)(\mathrm{Frob}_s)\end{aligned}$$

is a group homomorphism, cf. III.7.11.

Proposition III.7.22. *Let $\mathcal{F}$ be a C-local system of rank 1 on η, and let c be an arbitrary element of $k(\eta)$ of valuation $a(T, j_*\mathcal{F}, \omega)$. Then we have*

$$(-1)^{a(T,j_*\mathcal{F})}\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)(\mathrm{Frob}_s) = \int_{c^{-1}\mathcal{O}_T^{\times}} \chi_{\mathcal{F}}^{-1}(z)\psi(\mathrm{Tr}_{k/\mathbb{F}_p}\mathrm{Res}(z\omega))dz,$$

if $\mathcal{F}$ is ramified, where dz is the Haar measure on $k(\eta)$ normalized so that $\int_{\mathcal{O}_T} dz = 1$. If $\mathcal{F}$ is unramified, then we have $a(T, j_*\mathcal{F}) = 0$ and

$$\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)(\mathrm{Frob}_s) = \chi_{\mathcal{F}}(c)q^{fv(\omega)}.$$

Let $\Psi_\omega : k(\eta) \to \Lambda^{\times}$ be the additive character given by $z \mapsto \psi(\mathrm{Tr}_{k/\mathbb{F}_p}(z\omega))$. Proposition III.7.22 can then be summarized as an equality

$$(-1)^{a(T,j_*\mathcal{F})}\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)(\mathrm{Frob}_s) = \varepsilon(\chi_{\mathcal{F}}, \Psi_\omega),$$

where $\varepsilon(\chi_{\mathcal{F}}, \Psi_\omega)$ is the automorphic ε-factor of the pair $(\chi_{\mathcal{F}}, \Psi_\omega)$, cf ([**La87**], 3.1.3.2).

III.7.23. We now prove Proposition III.7.22. Let us first assume that $\mathcal{F}$ is unramified, so that $\mathcal{F}$ is the pullback to η of a C-local system $\mathcal{G}$ of rank 1 on s. We can take $D = s$ and $\nu = 1$ above. For each integer d, the multiplicative C-local system $\chi_{\mathcal{F}}$ is given on the component $\mathrm{Pic}^d(T,s)_s$ by the pullback of $\mathcal{G}^{\otimes d}$. The C-admissible representations of rank 1 corresponding to $\varepsilon_k(T, i_*i^{-1}j_*\mathcal{F}, \omega)$ and $\varepsilon_k(T, j_!j^{-1}j_*\mathcal{F}, \omega)$ are then respectively isomorphic to $\mathcal{G}_{\overline{s}}^{-1}$ and to

$$\begin{aligned}&H_c^{2\nu - a(T,j_!\mathcal{F})}\left(\mathrm{Pic}^{a(T,j_!\mathcal{F},\omega)}(T,s)_{\overline{s}}, \mathcal{L}_\psi\{\mathrm{Res}_\omega\}(\nu - a(T,\mathcal{F},\omega))\right) \otimes \mathcal{G}_{\overline{s}}^{\otimes a(T,j_!\mathcal{F},\omega)}\\ =&H_c^{1}\left(\mathrm{Pic}^{1+v(\omega)}(T,s)_{\overline{s}}, \mathcal{L}_\psi\{\mathrm{Res}_\omega\}(-v(\omega))\right) \otimes \mathcal{G}_{\overline{s}}^{\otimes (1+v(\omega))}.\end{aligned}$$

Let π be a uniformizer of $k(\eta)$, and let us write ω as $\alpha\frac{d\pi}{\pi}$ for some element α of $k(\eta)^{\times}$ of valuation $1 + v(\omega)$. We have an isomorphism

$$\begin{aligned}\theta : \mathbb{G}_{m,s} &\to \mathrm{Pic}^{1+v(\omega)}(T,s)_s\\ t &\mapsto t\alpha^{-1},\end{aligned}$$

so that $\theta^{-1}\mathcal{L}_\psi\{\mathrm{Res}_\omega\}$ is isomorphic to $\mathcal{L}_\psi\{t\}$. By Proposition III.7.6 and by the Grothendieck-Lefschetz trace formula ([**Gr66**], éq. (25)), we have

$$\begin{aligned}\mathrm{Tr}\left(\mathrm{Frob}_s \mid H_c^1\left(\mathrm{Pic}^{1+v(\omega)}(T,s)_{\overline{s}}, \mathcal{L}_\psi\{\mathrm{Res}_\omega\}\right)\right) &= \mathrm{Tr}\left(\mathrm{Frob}_s \mid H_c^1\left(\mathbb{G}_{m,\overline{s}}, \mathcal{L}_\psi\{t\}\right)\right)\\ &= -\sum_{t\in k(s)^{\times}} \psi(\mathrm{Tr}_{k/\mathbb{F}_p}(t))\\ &= 1.\end{aligned}$$

We thus obtain that the quantity $\varepsilon_{\overline{s}}(T, j_!\mathcal{F}, \omega)(\mathrm{Frob}_s)$ is given by $\mathrm{Tr}\left(\mathrm{Frob}_s \mid \mathcal{G}_{\overline{s}}\right)^{1+v(\omega)} q^{fv(\omega)}$. This implies that the value of $\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)$ at Frob_k is given by $\mathrm{Tr}\left(\mathrm{Frob}_s \mid \mathcal{G}_{\overline{s}}\right)^{v(\omega)} q^{fv(\omega)}$, hence the result since we have

$$\chi_{\mathcal{F}}(c) = \mathrm{Tr}\left(\mathrm{Frob}_s \mid \mathcal{G}_{\overline{s}}\right)^{v(c)} = \mathrm{Tr}\left(\mathrm{Frob}_s \mid \mathcal{G}_{\overline{s}}\right)^{v(\omega)}.$$

III.7.24. We now prove Proposition III.7.22 in the case where $\mathcal{F}$ is ramified. In this situation, the C-sheaf $j_*\mathcal{F}$ is supported on η, and the quantity $(-1)^{a(T,j_*\mathcal{F})}\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)(\mathrm{Frob}_s)$ is thus equal to

$$(-1)^{a(T,j_!\mathcal{F})}q^{fa(T,j_!\mathcal{F},\omega)-f\nu}\mathrm{Tr}\left(\mathrm{Frob}_s \mid H_c^{2\nu-a(T,j_!\mathcal{F})}\left(\mathrm{Pic}^{a(T,j_!\mathcal{F},\omega)}(T,D)_{\overline{s}}, \chi_{\mathcal{F}} \otimes \mathcal{L}_\psi\{\mathrm{Res}_\omega\}\right)\right).$$

By Proposition III.7.6 and by the same Grothendieck-Lefschetz trace formula ([**Gr66**], éq. (25)), the latter quantity coincides with

$$q^{fa(T,j_!\mathcal{F},\omega)-f\nu}\sum_{u\in c^{-1}(\mathcal{O}_T/\mathfrak{m}^\nu)^\times}\chi_{\mathcal{F}}^{-1}(u)\psi(\mathrm{Tr}_{k/\mathbb{F}_p}(u\omega)).$$

The factor $q^{fa(T,j_!\mathcal{F},\omega)-f\nu}$ is equal to $\int_{c^{-1}(u+\mathfrak{m}^\nu\mathcal{O}_T)}dz$ for any element u of $(\mathcal{O}_T/\mathfrak{m}^\nu)^\times$, so that we obtain

$$\begin{aligned}(-1)^{a(T,j_*\mathcal{F})}\varepsilon_{\overline{s}}(T, j_*\mathcal{F}, \omega)(\mathrm{Frob}_s) &= \sum_{u\in(\mathcal{O}_T/\mathfrak{m}^\nu)^\times}\int_{c^{-1}(u+\mathfrak{m}^\nu\mathcal{O}_T)}\chi_{\mathcal{F}}^{-1}(z)\psi(\mathrm{Tr}_{k/\mathbb{F}_p}(z\omega))dz \\ &= \int_{c^{-1}\mathcal{O}_T^\times}\chi_{\mathcal{F}}^{-1}(z)\psi(\mathrm{Tr}_{k/\mathbb{F}_p}(z\omega))dz,\end{aligned}$$

hence the result.

III.8. The product formula for sheaves of generic rank at most 1 (after Deligne)

We review in this section Deligne's computation of the determinant of the cohomology of rank 1 local systems on curves, as exposed in his 1974 letter to Serre, which is published as an appendix in [**BE01**]. The material of this section is thus entirely due to Deligne, besides the terminology regarding twisted sheaves.

III.8.1. Let us recall that the base field k is assumed throughout to be a perfect field of characteristic p. Let $\psi : \mathbb{F}_p \to C^\times$ be a non trivial homomorphism. We fix a unitary C-admissible mutiplier μ on the topological group G_k (cf. III.2.9, III.2.10).

Definition III.8.2. Let X be a connected smooth curve over k, and let $\mathcal{F}$ be a μ-twisted C-sheaf on X. The global ε-factor of the pair $(X, \mathcal{F})$ is the Λ-admissible map $\varepsilon_{\overline{k}}(X, \mathcal{F}) : G_k \to C^\times$ defined by

$$\varepsilon_{\overline{k}}(X, \mathcal{F})(g) = \det(g, R\Gamma_c(X_{\overline{k}}, \mathcal{F}))^{-1},$$

for any g in G_k, cf. III.3.14.

For any smooth connected projective curve over k, we denote by $|X|$ the set of closed points of X and by $X_{(x)}$ the henselization of X at a closed point x.

Theorem III.8.3. *Let X be a smooth connected projective curve of genus g over k, let ω be a non zero global meromorphic differential 1-form on X and let $\mathcal{F}$ be a μ-twisted C-sheaf on X of generic rank $\mathrm{rk}(\mathcal{F})$ at most 1. Then, for all but finitely many closed points x of X the ε-factor of the triple $(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})$ is identically equal to 1, and we have*

$$\varepsilon_{\overline{k}}(X, \mathcal{F}) = \chi_{\mathrm{cyc}}^{N(g-1)\mathrm{rk}(\mathcal{F})}\prod_{x\in|X|}\delta_{x/k}^{a(X_{(x)},\mathcal{F}_{|X_{(x)}})}\mathrm{Ver}_{x/k}\varepsilon_{\overline{x}}(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}}),$$

where N is the number of connected components of $X_{\overline{k}}$, where $\overline{x}$ is a k-morphism from $\mathrm{Spec}(\overline{k})$ to x, and where $\delta_{x/k}$ and $\mathrm{Ver}_{x/k}$ are defined in III.3.22.

In Theorem III.8.3, the image by d^1 of the left hand side (cf. III.2.6) is

$$d^1(\varepsilon_{\overline{k}}(X,\mathcal{F})) = \mu^{-\chi(X_{\overline{k}},\mathcal{F})},$$

cf. III.3.14, while the image by d^1 of the right hand side is $\prod_{x\in|X|}\mu^{\deg(x)a(X_{(x)},\mathcal{F}_{|X_{(x)}},\omega_{|X_{(x)}})}$ (cf. III.7.2, III.7.9), where the conductor $a(X_{(x)},\mathcal{F}_{|X_{(x)}},\omega_{|X_{(x)}})$ vanishes for all but finitely many closed points x of X. Thus the conclusion of Theorem III.8.3 is consistent with the identity

$$-\chi(X_{\overline{k}},\mathcal{F}) = \sum_{x\in|X|}\deg(x)a(X_{(x)},\mathcal{F}_{|X_{(x)}},\omega_{|X_{(x)}}),$$

which results from the Grothendieck-Ogg-Shafarevich formula.

III.8.4. We now describe Deligne's proof of Theorem III.8.3. Let $X, g, N, \omega, \mathcal{F}$ be as in III.8.3. By replacing k with a finite extension if necessary, we can assume (and we do) that X is geometrically connected over k, so that $N = 1$. Let $j : U \to X$ be a non empty open subscheme such that $j^{-1}\mathcal{F}$ is a μ-twisted C-local system on U. We have an exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to \bigoplus_{x\in X\setminus U} i_{x*}i_x^{-1}\mathcal{F} \to 0,$$

where $i_x : x \to X$ is the inclusion of a closed point of X. The product formula III.8.3 holds for $i_{x*}i_x^{-1}\mathcal{F}$ for each x in $|X|$, hence we can assume (and we now do) that $\mathcal{F}$ vanishes outside U, i.e. $\mathcal{F} = j_!j^{-1}\mathcal{F}$. By replacing U with a smaller non empty open subscheme of X if necessary, we can further assume (and we do as well) that the complement $X \setminus U$ contains at least two closed points and that $j^{-1}\mathcal{F}$ is of rank $\mathrm{rk}(\mathcal{F}) = 1$.

Let us consider the effective Cartier divisor

$$D = \sum_{x\in X\setminus U}(1+\mathrm{sw}_x(\mathcal{F}))x,$$

where $\mathrm{sw}_x(\mathcal{F})$ is the Swan conductor of $\mathcal{F}$ at x. Equivalently, we have

$$D = \sum_{x\in|X|} a(X_{(x)},\mathcal{F}_{|X_{(x)}})x,$$

cf. III.7.2. The Grothendieck-Ogg-Shafarevich formula then implies that the Euler characteristic of $\mathcal{F}$, namely

$$\chi_c(U, j^{-1}\mathcal{F}) = \sum_{i\in\mathbb{Z}}(-1)^i \dim H^i_c(U, j^{-1}\mathcal{F}),$$

is equal to $-d$, where $d = \deg(D) - 2 + 2g$ is a nonnegative integer. Since the canonical line bundle ω_X of X has degree $2g-2$, the integer d is also the degree of the line bundle $\omega_X(D)$.

The μ-twisted C-local system $j^{-1}\mathcal{F}$ of rank 1 on U has ramification bounded by D, cf. III.5.47, and consequently there exists a μ-twisted multiplicative C-local system $\chi_{\mathcal{F}}$ (cf. III.5.41) on $\mathrm{Pic}_k(X,D)$ (cf. III.5.13) whose pullback by the Abel-Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_k(X,D),$$

cf. 30, is isomorphic to $j^{-1}\mathcal{F}$.

Let $\mathrm{Sym}^d_k(U)$ be the quotient of U^d by the group of bijection of $\{1,\dots,d\}$ onto itself, acting by permuting the d factors of U^d, cf. ([**Gu18**], Prop. 2.27). The C-local system $p_1^{-1}j^{-1}\mathcal{F}\otimes\cdots\otimes p_d^{-1}j^{-1}\mathcal{F}$ on U^d, where $(p_i)_{i=1}^d$ are the projections on each factor, descends to a C-local system $\mathcal{F}^{[d]}$ of rank 1 on $\mathrm{Sym}^d_k(U)$, cf. ([**Gu18**], Prop. 2.32). The symmetric Künneth formula ([**SGA4**], XVII 5.5.21) implies that we have a natural isomorphism

$$R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}},\mathcal{F}^{[d]})[d] \cong L\Gamma^d\left(R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})\right)[d], \tag{46}$$

where the functor $L\Gamma^d$ is defined in ([**Ill72**], I.4.2.2.6). By ([**Ill72**], I.4.3.2.1), we have Quillen's shift formula

$$L\Gamma^d\left(R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})\right)[d] \cong L\Lambda^d\left(R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})[1]\right), \tag{47}$$

where the derived d-th exterior power $L\Lambda^d$ is also defined in ([**Ill72**], I.4.2.2.6). The shifted complex $R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})[1]$ is of rank d, and the isomorphism

$$\det(\Lambda^{\mathrm{rk}(V)}V) \cong \det(V),$$

valid for any finite dimensional C-vector space V, extends to an isomorphism

$$\det\left(L\Lambda^d\left(R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})[1]\right)\right) \cong \det(R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})[1]). \tag{48}$$

By combining 46, 47 and 48, we obtain a natural isomorphism

$$\det R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})^{-1} \cong \det R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}}, \mathcal{F}^{[d]})^{(-1)^d}. \tag{49}$$

Moreover, if we denote by

$$\Phi_d : \mathrm{Sym}^d_k(U) \to \mathrm{Pic}_k(X, D), \tag{50}$$

the d-th Abel-Jacobi map, whose composition with the canonical projection $U^d \to \mathrm{Sym}^d_k(U)$ sends a section $(x_i)_{i=1}^d$ of U^d to $\prod_{i=1}^d \Phi(x_i)$, then the multiplicativity of $\chi_{\mathcal{F}}$ implies that the pullback of $\Phi_d^{-1}\chi_{\mathcal{F}}$ to U^d is isomorphic

$$((\Phi p_1)(\Phi p_2)\dots(\Phi p_d))^{-1}\chi_{\mathcal{F}} \cong p_1^{-1}\Phi^{-1}\chi_{\mathcal{F}} \otimes \dots \otimes p_d^{-1}\Phi^{-1}\chi_{\mathcal{F}} \cong p_1^{-1}j^{-1}\mathcal{F} \otimes \dots \otimes p_d^{-1}j^{-1}\mathcal{F},$$

and thus the pullback $\Phi_d^{-1}\chi_{\mathcal{F}}$ is isomorphic to $\mathcal{F}^{[d]}$.

The Leray spectral sequence for $(\Phi_d, \mathcal{F}^{[d]})$ and the projection formula then yield

$$\begin{aligned}\det R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}}, \mathcal{F}^{[d]}) &\cong \otimes_{q\in\mathbb{Z}} \det R\Gamma_c(\mathrm{Pic}_k(X,D)_{\overline{k}}, R^q\Phi_{d!}\Phi_d^{-1}\chi_{\mathcal{F}})^{(-1)^q} \\ &\cong \otimes_{q\in\mathbb{Z}} \det R\Gamma_c(\mathrm{Pic}_k(X,D)_{\overline{k}}, \chi_{\mathcal{F}} \otimes R^q\Phi_{d!}C)^{(-1)^q}.\end{aligned} \tag{51}$$

III.8.5. Let $i : D \to X$ be the closed immersion of D into X, and let J (resp. J^0) be the functor which associates to a k-scheme S the set of isomorphisms $\alpha : \mathcal{O}_{D_S} \to i_S^*\omega_X(D)$ of $\mathcal{O}_{D_S}$-modules (resp. of automorphisms of $\mathcal{O}_{D_S}$ as a module over itself). Then J^0 is representable by a smooth connected affine group scheme of dimension $\deg(D)$ over k, and J is a J^0-torsor. In particular, since the action of $\mathbb{G}_{m,k}$ by multiplication on $\mathcal{O}_D$ turns $\mathbb{G}_{m,k}$ into a sub-k-group scheme of J^0, the J^0-torsor J is naturally endowed with an action of $\mathbb{G}_{m,k}$ by left multiplication. Moreover, the morphism

$$\begin{aligned}f : J' &\to \mathrm{Pic}_k(X, D)\\ \alpha &\to (\omega_X(D), \alpha),\end{aligned}$$

where $J' = J/\mathbb{G}_{m,k}$, is a closed immersion, its image being the fiber of the canonical projection $\mathrm{Pic}_k(X, D) \to \mathrm{Pic}_k(X)$ at $\omega_X(D)$.

Let us denote by $\mathrm{Res}_D : i^*\omega_X(D) \to k$ the residue homomorphism, given by

$$\mathrm{Res}_D(\alpha) = \sum_{x\in|D|} \mathrm{Tr}_{k(x)/k}\mathrm{Res}_x(\alpha),$$

where Res_x denotes the residue homomorphism at a closed point x (cf. III.4.6). We denote by $g : \Sigma \to J$ the closed subscheme consisting of isomorphisms α such that $\mathrm{Res}_D(\alpha) = 0$, and by $g' : \Sigma' \to J'$ its quotient by $\mathbb{G}_{m,k}$, which is a closed immersion as well.

LEMMA III.8.6 ([**BE01**], p.82). *The C-sheaves $R^q\Phi_{d!}C$ (cf. 50) on $\mathrm{Pic}^d_k(X, D)$ admit the following description:*

(1) *for $q = 2g-2$, there exists a short exact sequence*

$$0 \to R^{2g-2}\Phi_{d!}C \to C(1-g) \to f_*C(1-g) \to 0,$$

of C-sheaves on $\mathrm{Pic}^d_k(X,D)$.

(2) *for $q = 2g$, the C-sheaf $R^{2g}\Phi_{d!}C$ is isomorphic to $(fg')_*C(-g)$,*

(3) *the C-sheaf $R^q\Phi_{d!}C$ vanishes if q is not equal to $2g$ or $2g-2$.*

Let $\bar{t}$ be the spectrum of an algebraically closed extension of k, and let $(\mathcal{L},\alpha)$ be a $\bar{t}$-point of $\mathrm{Pic}^d_k(X,D)$ (cf. III.5.13). The fiber of Φ_d above $(\mathcal{L},\alpha)$ parametrizes sections σ in $H^0(X_{\bar{t}},\mathcal{L})$ whose image in $H^0(D_{\bar{t}},\mathcal{L})$ is α, cf. ([**Gu18**], Prop. 4.12). The degree of the line bundle $\mathcal{L}(-D)$ is $2g-2$, and the Riemann-Roch theorem yields:

(1) if $\mathcal{L}$ is not isomorphic to $\omega_X(D)$, then the fiber of Φ_d above $(\mathcal{L},\alpha)$ is a torsor under the additive $\bar{t}$-group scheme associated to the $(g-1)$-dimensional $k(\bar{t})$-vector space $H^0(X_{\bar{t}},\mathcal{L}(-D))$.

(2) if $\mathcal{L} = \omega_X(D)$ and if $\mathrm{Res}_D(\alpha) = 0$ then the fiber of Φ_d above $(\mathcal{L},\alpha)$ is a torsor under the additive $\bar{t}$-group scheme associated to the g-dimensional $k(\bar{t})$-vector space $H^0(X_{\bar{t}},\mathcal{L}(-D))$.

(3) if $\mathcal{L} = \omega_X(D)$ and if $\mathrm{Res}_D(\alpha) \neq 0$ then the fiber of Φ_d above $(\mathcal{L},\alpha)$ is empty.

Let $w : W \to \mathrm{Pic}^d_k(X,D)$ be the open complement of the image of the closed immersion f. Then we have a distinguished triangle

$$w_!w^{-1}R\Phi_{d!}C \to R\Phi_{d!}C \to f_*f^{-1}R\Phi_{d!}C \xrightarrow{[1]} .$$

Above W, the morphism Φ_d is a fibration in affine spaces, of relative dimension $g-1$, hence $w^{-1}R\Phi_{d!}C$ is quasi-isomorphic to $C(1-g)[2-2g]$. Moreover, the description above of the fibers of Φ_d imply that $f^{-1}R\Phi_{d!}C$ is supported on Σ'. Above Σ', the morphism Φ_d is a fibration in affine spaces, of relative dimension g, hence $f^{-1}R\Phi_{d!}C$ is quasi-isomorphic to $g'_*C(-g)[-2g]$. Thus $R^q\Phi_{d!}C$ vanishes if q is not equal to $2g$ or $2g-2$, and we have isomorphisms

$$R^{2g-2}\Phi_{d!}C \cong w_!C(1-g),$$
$$R^{2g}\Phi_{d!}C \cong (fg')_*C(-g),$$

hence the conclusion of Lemma III.8.6.

III.8.7. By combining the formula 51 with Lemma III.8.6, we obtain that the determinant of the complex

$$R\Gamma_c(\mathrm{Sym}^d_k(U)_{\bar{k}},\mathcal{F}^{[d]}),$$

is isomorphic to

$$\frac{\det R\Gamma_c(\mathrm{Pic}^d_k(X,D)_{\bar{k}},\chi_{\mathcal{F}}(1-g))\det R\Gamma_c(\Sigma'_{\bar{k}},(fg')^{-1}\chi_{\mathcal{F}}(-g))}{\det R\Gamma_c(J'_{\bar{k}},f^{-1}\chi_{\mathcal{F}}(1-g))}. \tag{52}$$

Lemma III.8.8 ([**BE01**], p.82). *The factor* $\det R\Gamma_c(\mathrm{Pic}^d_k(X,D)_{\bar{k}},\chi_{\mathcal{F}}(1-g))$ *is isomorphic to C, as a C-admissible representation of G_k of rank 1.*

When $\mathcal{F}$ is everywhere tamely ramified, then $\mathrm{Pic}^d_k(X,D)_{\bar{k}}$ is a torsor under the $\bar{k}$-group scheme $\mathrm{Pic}^0_k(X,D)_{\bar{k}}$, which is an extension of an abelian scheme of dimension g by a torus of dimension $\deg(D)$. The result follows in this case from the fact that the determinant of the cohomology of a tame Λ-local system on an extension of an abelian k-scheme by a torus of dimension at least 2 is canonically trivial. We refer to ([**BE01**], Constr. 1 p.70) for a proof of the latter result.

When $\mathcal{F}$ is not everywhere tamely ramified, there exists a closed point x on D such that the multiplicity ν of D at x is at least 2. Since $\mathrm{Pic}^d_k(X,D)$ is a torsor over the k-group scheme $\mathrm{Pic}^0_k(X,D)$ and since $\chi_{\mathcal{F}}$ is multiplicative, it is sufficient to prove that the complex

$$R\Gamma_c(\mathrm{Pic}^0_k(X,D)_{\overline{k}}, \chi_{\mathcal{F}}),$$

is quasi-isomorphic to 0. The projection

$$\tau : \mathrm{Pic}^0_k(X,D) \to \mathrm{Pic}^0_k(X,D-x),$$

is a homomorphism of k-group schemes, whose kernel is the additive k-group scheme associated to the k-vector space $\mathfrak{m}_x^{\nu-1}/\mathfrak{m}_x^{\nu}$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. Since the ramification of $\mathcal{F}$ is not bounded by the divisor $D-x$, Theorem III.5.48 and III.5.10 imply that the restriction of $\chi_{\mathcal{F}}$ to the kernel of τ is non trivial. Together with Proposition III.6.15 and with the multiplicativity of $\mathcal{F}$, this implies the vanishing of $R\tau_!\chi_{\mathcal{F}}$, hence the result.

III.8.9. By combining the formula 52 with Lemma III.8.8, we obtain an isomorphism

$$\det R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}}, \mathcal{F}^{[d]}) \cong \det R\Gamma_c(\Sigma'_{\overline{k}}, (fg')^{-1}\chi_{\mathcal{F}}(-g)) \det R\Gamma_c(J'_{\overline{k}}, f^{-1}\chi_{\mathcal{F}}(1-g))^{-1}. \tag{53}$$

Let $\tau : J \to J'$ be the natural projection (cf. III.8.5).

LEMMA III.8.10 ([**BE01**], p.84). *The C-sheaves $R^q\tau_!\mathcal{L}_\psi\{\mathrm{Res}\}$ admits the following description:*

(1) the C-sheaf $R^1\tau_!\mathcal{L}_\psi\{\mathrm{Res}\}$ is isomorphic to the constant sheaf C on J',

(2) the C-sheaf $R^2\tau_!\mathcal{L}_\psi\{\mathrm{Res}\}$ is isomorphic to $g'_*C(-1)$, with g' as in III.8.5.

(3) the C-sheaf $R^q\tau_!\mathcal{L}_\psi\{\mathrm{Res}\}$ vanishes when q is not equal to 1 or 2.

Recall that τ is a $\mathbb{G}_m$-torsor. Let $\overline{J}$ be the quotient of $\mathbb{G}_{a,k} \times_k J$ by the action of $\mathbb{G}_{m,k}$ given by $t\cdot(y,\alpha) = (t^{-1}y, t\alpha)$, and let $\overline{\mathrm{Res}} : \overline{J} \to \mathbb{G}_{a,k}$ be the morphism which sends the class in $\overline{J}$ of a section (y,α) of J to $y\mathrm{Res}(\alpha)$. Let $u : J \to \overline{J}$ be the open immersion which sends a section α to the class of $(1,\alpha)$, and let $\overline{\tau} : \overline{J} \to J'$ be the morphism which sends the class of a section (y,α) to $\tau(\alpha)$. Let $i : J' \to \overline{J}$ be the section of $\overline{\tau}$ which sends $\tau(\alpha)$ to the class of $(0,\alpha)$ in $\overline{J}$, for any section α of J. We then have an exact sequence

$$0 \to u_!\mathcal{L}_\psi\{\mathrm{Res}\} \to \mathcal{L}_\psi\{\overline{\mathrm{Res}}\} \to i_*C \to 0.$$

By applying the funtor $R\overline{\tau}_!$, we obtain a distinguished triangle

$$R\tau_!\mathcal{L}_\psi\{\mathrm{Res}\} \to R\overline{\tau}_!\mathcal{L}_\psi\{\overline{\mathrm{Res}}\} \to C[0] \xrightarrow{[1]} . \tag{54}$$

Moreover, the fiber of $\overline{\tau}$ over a geometric point $\overline{\alpha}$ of J' is isomorphic to $\mathbb{G}_a$ and the restriction of $\mathcal{L}_\psi\{\overline{\mathrm{Res}}\}$ to this fiber is a multiplicative C-local system, which is trivial if and only if $\overline{\alpha}$ factors through g' (cf. III.8.5). Thus Proposition III.6.15 implies that $R\overline{\tau}_!\mathcal{L}_\psi\{\overline{\mathrm{Res}}\}$ is supported on the image Σ' of g', and consequently

$$R\overline{\tau}_!\mathcal{L}_\psi\{\overline{\mathrm{Res}}\} \cong g'_*g'^{-1}R\overline{\tau}_!\mathcal{L}_\psi\{\overline{\mathrm{Res}}\} \cong g'_*R(\overline{\tau}_{|\overline{\tau}^{-1}(\Sigma')})_!C.$$

Above Σ', the morphism $\overline{\tau}$ is a fibration in affine spaces, of relative dimension 1, hence $R\overline{\tau}_!\mathcal{L}_\psi\{\overline{\mathrm{Res}}\}$ is quasi-isomorphic to $g'_*C(-1)[-2]$. The conclusion of Lemma III.8.10 then follows from 54.

III.8.11. By combining the formula 53 with Lemma III.8.10, we obtain an isomorphism

$$\begin{aligned}\det R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}}, \mathcal{F}^{[d]}) &\cong \det R\Gamma_c(J'_{\overline{k}}, f^{-1}\chi_{\mathcal{F}}(1-g)\otimes R\tau_!\mathcal{L}_\psi\{\mathrm{Res}\}) \\ &\cong \det R\Gamma_c(J_{\overline{k}}, (f\tau)^{-1}\chi_{\mathcal{F}}(1-g)\otimes \mathcal{L}_\psi\{\mathrm{Res}\})\end{aligned} \tag{55}$$

By Proposition III.7.12 and Proposition III.3.24, the product formula III.8.3 for some non zero meromorphic 1-form ω on X implies the product formula for all such 1-forms. In particular, we can assume (and we do) that ω is a global section of $\omega_X(D)$ such that $i^*\omega : \mathcal{O}_D \to i^*\omega_X(D)$ is an isomorphism. For any closed point x of X, we denote by $i_x : D_x \to X_{(x)}$ the restriction of D to the henselization $X_{(x)}$ of X at x, by $\mathcal{F}_x$ and ω_x the restrictions of $\mathcal{F}$ and ω to $X_{(x)}$, and by Σ_x the set of k-linear embeddings of $k(x)$ into $\overline{k}$.

The sections of J over a $\overline{k}$-scheme S consist of all isomorphisms $\alpha : \mathcal{O}_{D_S} \to i_S^*\omega_X(D)$ of $\mathcal{O}_{D_S}$-modules. We have a decomposition

$$\mathcal{O}_{D_S} \cong (\mathcal{O}_D \otimes_k \overline{k}) \otimes_{\overline{k}} \mathcal{O}_S \cong \prod_{x\in D}\prod_{\iota\in\Sigma_x}(\mathcal{O}_{D_x}\otimes_{k(x),\iota}\mathcal{O}_S),$$

hence α can be identified with a tuple $(\alpha_{x,\iota})_{x\in D,\iota\in\Sigma_x}$, where each $\alpha_{x,\iota}$ is a trivialization of the $(\mathcal{O}_{D_x}\otimes_{k(x),\iota}\mathcal{O}_S)$-module $i_x^*\omega_X(D_x)\otimes_{k(x),\iota}\mathcal{O}_S$. In particular, the morphism

$$\begin{aligned}\delta : \prod_{x\in D}\prod_{\iota\in\Sigma_x}\mathrm{Pic}^0(X_{(x)}, D_x)_{\iota,\overline{k}} &\to J_{\overline{k}} \\ (u_x)_{x\in D,\iota\in\Sigma_x} &\to (u_x\omega_x)_{x\in D,\iota\in\Sigma_x},\end{aligned}$$

is an isomorphism of $\overline{k}$-schemes, which fits into a commutative diagram

$$\begin{tikzcd} \prod_{x\in D}\prod_{\iota\in\Sigma_x}\mathrm{Pic}^0(X_{(x)}, D_x)_{\iota,\overline{k}} \arrow[r, "\delta"] \arrow[dd] & J_{\overline{k}} \arrow[d, "\tau"] \\ & J'_{\overline{k}} \arrow[d, "f"] \\ \mathrm{Pic}^0_k(X, D)_{\overline{k}} \arrow[r] & \mathrm{Pic}^d_k(X, D)_{\overline{k}}, \end{tikzcd}$$

where the bottom horizontal arrow is the translation by the $\overline{k}$-point of $\mathrm{Pic}^d_k(X, D)$ which is the image by Φ_d of the $\overline{k}$-point of $\mathrm{Sym}^d(U)$ corresponding to $\sum_{x\notin D}\sum_{\iota\in\Sigma_x} v_x(\omega)\iota(x)$. If $(p_{x,\iota})_{x\in D,\iota\in\Sigma_x}$ are the natural projections from the source of δ onto each factor, then we have decompositions

$$\delta^{-1}\mathcal{L}_\psi\{\mathrm{Res}\} \cong \bigotimes_{x\in D}\bigotimes_{\iota\in\Sigma_x} p_{x,\iota}^{-1}\mathcal{L}_\psi\{\mathrm{Res}_{\omega_x}\},$$

$$(f\tau\delta)^{-1}\chi_{\mathcal{F}} \cong \bigotimes_{x\in D}\bigotimes_{\iota\in\Sigma_x} p_{x,\iota}^{-1}\chi_{\mathcal{F}_x} \otimes \bigotimes_{x\notin D}\bigotimes_{\iota\in\Sigma_x}\mathcal{F}_{x,\iota}^{\otimes v(\omega_x)},$$

by the compatibility of local and global geometric class field theory, cf. III.5.36. By Proposition III.7.6, each complex

$$R\Gamma_c(\mathrm{Pic}^0(X_{(x)}, D_x)_{\iota,\overline{k}}, \chi_{\mathcal{F}_x}\otimes\mathcal{L}_\psi\{\mathrm{Res}_{\omega_x}\})$$

is a one-dimensional C-vector space concentrated in degree $a_x = a(X_{(x)}, \mathcal{F}_{|X_{(x)}})$. By Künneth's formula ([**SGA4**], Thm. 5.4.3), and by Proposition III.3.28, we obtain that the complex

$$R\Gamma_c(J_{\overline{k}}, (f\tau)^{-1}\chi_{\mathcal{F}}\otimes\mathcal{L}_\psi\{\mathrm{Res}\})$$

is concentrated in degree $\sum_{x\in D} a_x = \deg(D)$, and that G_k acts on its cohomology group of degree $\deg(D)$ through the C-admissible map given by

$$\prod_{x\in D}\delta_{x/k}^{a_x}\mathrm{Ver}_{x/k}H_c^{a_x}(\mathrm{Pic}^0(X_{(x)}, D_x)_{\overline{x}}, \chi_{\mathcal{F}_x}\otimes\mathcal{L}_\psi\{\mathrm{Res}_{\omega_x}\})\prod_{x\notin D}\mathrm{Ver}_{x/k}\mathcal{F}_{\overline{x}}^{\otimes v(\omega_x)},$$

with notation as in III.3.22, and the latter is equal to

$$\prod_{x\in|X|} \delta_{x/k}^{a_x} \mathrm{Ver}_{x/k}\left(\chi_{\mathrm{cyc}}^{v(\omega_x)} \varepsilon_{\overline{x}}(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})\right).$$

Since d has the same parity as $\deg(D)$, the latter map is equal by (55) to the trace function on G_k of the C-admissible representation $\det R\Gamma_c(\mathrm{Sym}^d_k(U)_{\overline{k}}, \mathcal{F}^{[d]})^{(-1)^d}(g-1)$ of (G_k, μ^d), and thus to the trace function of $\det R\Gamma_c(U_{\overline{k}}, j^{-1}\mathcal{F})^{-1}(g-1)$ as well by (49), hence the conclusion of Theorem III.8.3, since we have

$$\prod_{x\in|X|} \mathrm{Ver}_{x/k}\left(\chi_{\mathrm{cyc}}^{v(\omega_x)}\right) = \prod_{x\in|X|} \chi_{\mathrm{cyc}}^{[k(x):k]v(\omega_x)} = \chi_{\mathrm{cyc}}^{2g-2}.$$

III.9. Geometric local ε-factors in arbitrary rank

Let $\psi : \mathbb{F}_p \to C^\times$ be a non trivial homomorphism, hence producing a multiplicative C-local system $\mathcal{L}_\psi$ on $\mathbb{G}_{a,k}$, cf. III.6.13.

As in III.7, let T be the spectrum of a k-algebra, which is a henselian discrete valuation ring $\mathcal{O}_T$, with maximal ideal $\mathfrak{m}$, and whose residue field $\mathcal{O}_T/\mathfrak{m}$ is a finite extension of k of degree $\deg(s)$. Let $j : \eta \to T$ be the generic point of T, and let $i : s \to T$ be its closed point, so that T is canonically an s-scheme, as in III.5.16. We fix a $\overline{k}$-point $\overline{s} : \mathrm{Spec}(\overline{k}) \to T$ of T above s, so that the Galois group $G_s = \mathrm{Gal}(\overline{k}/k(s))$ can be considered as a subgroup of G_k. We fix a unitary C-admissible mutiplier μ on the topological group G_s (cf. III.2.9, III.2.10).

III.9.1. Let π be a uniformizer of $\mathcal{O}_T$. We abusively denote by π as well the morphism

$$\pi : T \to \mathbb{A}^1_s,$$

corresponding to the unique morphism $k(s)[t] \to \mathcal{O}_T$ of $k(s)$-algebras which sends t to π. By Theorem III.4.18, the pullback functor π^{-1} realizes an equivalence from the category of special μ-twisted C-sheaves on $\mathbb{A}^1_s$ to the category of C-sheaves on T. We denote by $\pi_\diamondsuit$ a quasi-inverse to this equivalence.

Definition III.9.2. Let $\mathcal{F}$ be a μ-twisted C-sheaf on T, and let ω be an element of $\Omega^{1,\times}_\eta$ (cf. III.7.1). Then the *ε-factor* of the triple $(T, \mathcal{F}, \omega)$ is the C-admissible map $\varepsilon_{\pi,\overline{s}}(T, \mathcal{F}, \omega)$ from G_s to $C^\times$ defined by

$$\begin{aligned} &G_s \to C^\times \\ &g \mapsto \langle \chi_{\det(j^{-1}\mathcal{F})}\rangle(\frac{\omega}{d\pi})(g)\chi_{\mathrm{cyc}}(g)^{-v(\omega)\mathrm{rk}(j^{-1}\mathcal{F})} \det\left(g \mid R\Gamma_c\left(\mathbb{A}^1_{\overline{s}}, \pi_\diamondsuit\mathcal{F}\otimes\mathcal{L}_\psi^{-1}\right)\right)^{-1}, \end{aligned}$$

cf. III.7.11 and III.7.12, or equivalently by

$$\varepsilon_{\pi,\overline{s}}(T, \mathcal{F}, \omega) = \langle \chi_{\det(j^{-1}\mathcal{F})}\rangle(\frac{\omega}{d\pi})\chi_{\mathrm{cyc}}^{-v(\omega)\mathrm{rk}(j^{-1}\mathcal{F})} \varepsilon_{\overline{s}}(\mathbb{A}^1_s, \pi_\diamondsuit\mathcal{F}\otimes\mathcal{L}_\psi^{-1}),$$

cf. III.8.2.

It is clear from Definition III.9.2 that ε-factors are multiplicative in short exact sequences:

Proposition III.9.3. *For any exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

of C-sheaves on T, and for any element ω of $\Omega^{1,\times}_\eta$, we have

$$\varepsilon_{\pi,\overline{s}}(T, \mathcal{F}, \omega) = \varepsilon_{\pi,\overline{s}}(T, \mathcal{F}', \omega)\varepsilon_{\pi,\overline{s}}(T, \mathcal{F}'', \omega).$$

Proposition III.9.4. *Let $\mathcal{F}$ be a μ-twisted C-sheaf on T, and let ω be an element of $\Omega^{1,\times}_{\eta}$. The coboundary of $\varepsilon_{\pi,\overline{s}}(T,\mathcal{F},\omega)$ (cf. III.2.6) is then given by*

$$d^1\left(\varepsilon_{\pi,\overline{s}}(T,\mathcal{F},\omega)\right) = \mu^{a(T,\mathcal{F},\omega)},$$

cf. III.7.2 for the notation.

Indeed, $\det(j^{-1}\mathcal{F})$ is a $\mu^{\mathrm{rk}(j^{-1}\mathcal{F})}$-twisted C-sheaf of rank 1 on η, hence

$$d^1\left(\langle\chi_{\det(j^{-1}\mathcal{F})}\rangle(\frac{\omega}{d\pi})\right) = \mu^{v(\omega)\mathrm{rk}(j^{-1}\mathcal{F})}.$$

Moreover, $\pi_{\diamondsuit}\mathcal{F}\otimes\mathcal{L}_{\psi}^{-1}$ is a μ-twisted C-sheaf on $\mathbb{A}^1_s$, and consequently we have

$$d^1\left(\varepsilon_{\overline{s}}(\mathbb{A}^1_s,\pi_{\diamondsuit}\mathcal{F}\otimes\mathcal{L}_{\psi}^{-1})\right) = \mu^{-\chi_c(\mathbb{A}^1_{\overline{s}},\pi_{\diamondsuit}\mathcal{F}\otimes\mathcal{L}_{\psi}^{-1})}.$$

The Swan conductor of $\pi_{\diamondsuit}\mathcal{F}\otimes\mathcal{L}_{\psi}^{-1}$ at infinity is equal to $\mathrm{rk}(j^{-1}\mathcal{F})$, hence the conductor of the pair $(\pi_{\diamondsuit}\mathcal{F}\otimes\mathcal{L}_{\psi}^{-1},dt)$ at infinity is equal to $\mathrm{rk}(j^{-1}\mathcal{F})(2+v_{\infty}(dt))=0$, so that the Grothendieck-Ogg-Shafarevich formula yields

$$-\chi_c(\mathbb{A}^1_{\overline{s}},\pi_{\diamondsuit}\mathcal{F}\otimes\mathcal{L}_{\psi}^{-1}) = a(T,\mathcal{F},d\pi),$$

hence the conclusion of Proposition III.9.4.

Proposition III.9.5. *Let $\mathcal{F}$ be a μ-twisted C-sheaf on T, and let $\mathcal{G}$ be a ν-twisted C-local system on T, for some C-admissible multiplier ν on G_s. We then have*

$$\varepsilon_{\pi,\overline{s}}(T,\mathcal{F}\otimes\mathcal{G},\omega) = \det(\mathcal{G}_{\overline{s}})^{a(T,\mathcal{F},\omega)}\varepsilon_{\pi,\overline{s}}(T,\mathcal{F},\omega)^{\mathrm{rk}(\mathcal{G})},$$

where $\det(\mathcal{G}_{\overline{s}})$ is the C-admissible map on G_s which sends an element g of G_s to the determinant of its action on the stalk $\mathcal{G}_{\overline{s}}$ of $\mathcal{G}$ at $\overline{s}$.

We can assume (and we do) that ω is $d\pi$, in which case the conclusion of Proposition III.9.5 follows from the definition III.9.2 and from the fact that $\pi_{\diamondsuit}(\mathcal{F}\otimes\mathcal{G}) = \pi_{\diamondsuit}\mathcal{F}\otimes\pi_{\diamondsuit}\mathcal{G}$ is the twist of $\pi_{\diamondsuit}\mathcal{F}$ by $\pi_{\diamondsuit}\mathcal{G}$, the geometrically constant ν-twisted C-sheaf on $\mathbb{A}^1_s$ associated to the fiber $i^{-1}\mathcal{G}$.

Proposition III.9.6. *Let $\mathcal{F}$ be a μ-twisted C-sheaf on s. Then we have*

$$\varepsilon_{\pi,\overline{s}}(T,i_*\mathcal{F},\omega) = \det(\mathcal{F}_{\overline{s}})^{-1},$$

where $\det(\mathcal{F}_{\overline{s}})$ is the C-admissible map on G_s which sends an element g to the determinant of its action on the stalk $\mathcal{F}_{\overline{s}}$ of $\mathcal{F}$ at $\overline{s}$. In particular, $\varepsilon_{\pi,\overline{s}}(T,i_*\mathcal{F},\omega)$ agrees with the local ε-factor $\varepsilon_{\overline{s}}(T,i_*\mathcal{F},\omega)$ defined in III.7.4.

Indeed, the μ-twisted C-sheaf $\pi_{\diamondsuit}i_*\mathcal{F}$ is supported on the s-point 0 of $\mathbb{A}^1_s$, with restriction $\mathcal{F}$ to 0, and consequently we have

$$R\Gamma_c\left(\mathbb{A}^1_{\overline{s}},\pi_{\diamondsuit}\mathcal{F}\otimes\mathcal{L}_{\psi}^{-1}\right) \cong \mathcal{F}_{\overline{s}}[0],$$

hence the conclusion of Proposition III.9.6.

III.9.7. Let $k(\overline{\eta})$ be a separable closure of $k(\eta_{\overline{s}})$ and let $\overline{\eta}\to\eta_{\overline{s}}$ be the corresponding morphism of k-schemes. We then have an exact sequence

$$1\to I_{\eta}\to G_{\eta}\to G_s\to 1,$$

where G_s is the Galois group of the extension $\overline{k}/k(s)$ and $G_{\eta}=\pi_1(\eta,\overline{\eta})$ is the Galois group of the extension $k(\overline{\eta})/k(\eta)$, while $I_{\eta}=\pi_1(\eta_{\overline{s}},\overline{\eta})$ is the inertia group of this extension. The functor $\mathcal{F}\mapsto\mathcal{F}_{\overline{\eta}}$ is then an equivalence from the category of μ-twisted C-sheaves on η to the category of C-admissible representations of (G_{η},μ), cf. III.3.19.

Definition III.9.8. A μ-twisted C-sheaf $\mathcal{F}$ on η is *potentially unipotent* if so is the C-admissible representation $\mathcal{F}_{\overline{\eta}}$ of (G_{η},μ) (cf. III.2.47).

Proposition III.9.9. *(Grothendieck's ℓ-adic monodromy theorem) Assume that the ℓ-adic cyclotomic character $\chi_{\mathrm{cyc}} : G_s \to \mathbb{Z}_\ell^\times$ has infinite image. Then any μ-twisted C-sheaf $\mathcal{F}$ on η is potentially unipotent.*

By replacing η with $\eta_{k'}$, for some finite extension k' of $k(s)$ contained in $\overline{k}$, we can assume (and we do) that μ is the trivial multiplier 1. In this case, the conclusion follows from ([**ST68**], Appendix) or from ([**FO**], Th. 1.24).

Corollary III.9.10. *If the base field k is the perfection of a finitely generated field extension of $\mathbb{F}_p$, then any μ-twisted C-sheaf $\mathcal{F}$ on η is potentially unipotent.*

Indeed, the assumption in Proposition III.9.9 is stable by finite extensions, and holds for (perfections of) purely transcendental extensions of $\mathbb{F}_p$, since for such a field k, the natural homomorphism $G_k \to G_{\mathbb{F}_p}$, through which the cyclotomic character χ_{cyc} factors, is surjective, and the cyclotomic character for $k = \mathbb{F}_p$ has infinite image.

III.9.11. Let $K_0(T, \mu, C)$ be the Grothendieck group of the full subcategory $\mathrm{Sh}(T, \mu, C)$ consisting of μ-twisted C-sheaves on T with potentially unipotent restriction to η (cf. III.9.8). Thus any μ-twisted C-sheaf $\mathcal{F}$ on T, with potentially unipotent restriction to η, has a well defined class $[\mathcal{F}]$ in $K_0(T, \mu, C)$, and the latter is generated by such classes with relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ for each short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

of μ-twisted C-sheaves on T, with potentially unipotent restriction to η.

Proposition III.9.12. *Let $K_0(T, \mu, C)$ be as in III.9.11. The abelian group*

$$\bigoplus_{\nu} K_0(T, \nu, C),$$

where the sum runs over all unitary C-admissible multipliers on G_s, is generated by its subset of elements of the following three types:

(1) the class $[j_! C]$ in $K_0(T, 1, C)$,

(2) for any unitary C-admissible multiplier ν on G_s and any ν-twisted C-sheaf $\mathcal{G}$ on s, the class $[i_* \mathcal{G}]$ in $K_0(T, \nu, C)$,

(3) for any unitary C-admissible multiplier ν on G_s, any connected finite étale cover $\eta' \to \eta$ with normalization $f : T' \to T$, for any s-morphism $\overline{s}' : \mathrm{Spec}(\overline{k}) \to T'$ over the closed point s' of T' such that $f(\overline{s}') = \overline{s}$, any unitary C-admissible multipliers ν_1 and ν_2 on $G_{s'}$ such that $\nu_1\nu_2 = \nu_{|G_{s'}}$, any ν_1-twisted C-sheaf $\mathcal{F}_1$ of rank 1 over η' with finite geometric monodromy, and any unramified ν_2-twisted C-sheaf $\mathcal{F}_2$ over η', the class

$$[f_* j'_!(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[f_* j'_! C],$$

in the sum of $K_0(T, \nu, C)$ and $K_0(T, 1, C)$, where $j' : \eta' \to T'$ is the canonical open immersion.

Indeed, let us consider the class $[\mathcal{F}]$ of a μ-twisted C-sheaf $\mathcal{F}$ on T with potentially unipotent restriction to η. We have an exact sequence

$$0 \to j_! j^{-1} \mathcal{F} \to \mathcal{F} \to i_* i^{-1} \mathcal{F} \to 0,$$

hence it is sufficient to prove that $[j_! j^{-1} \mathcal{F}]$ belongs to the subgroup generated by classes of type (1) or (3). This follows from Proposition III.2.50 and from the dictionary between μ-twisted C-sheaves on η and C-admissible representations of (G_η, μ), cf. III.3.19, III.3.21.

PROPOSITION III.9.13. *Let $\eta' \to \eta$ be a connected finite étale cover of η, with normalization $f : T' \to T$. Let $\overline{s}' : \mathrm{Spec}(\overline{k}) \to T'$ be an s-morphism over the closed point s' of T' such that $f(\overline{s}') = \overline{s}$, and let $j' : \eta' \to T'$ be the natural inclusion. Let μ_1 and μ_2 be unitary C-admissible multipliers on $G_{s'}$ such that $\mu_1\mu_2 = \mu_{|G_{s'}}$, let $\mathcal{F}_1$ be a μ_1-twisted C-sheaf of rank 1 over η' and let $\mathcal{F}_2$ be a geometrically constant μ_2-twisted C-sheaf over T'. Then for any element ω of $\Omega^{1,\times}_{\eta}$, there exists a Λ-admissible homomorphism $\lambda_{f,\pi}(\omega)$ from G_s to $C^\times$, depending only on f, π and ω, such that the map $\varepsilon_{\pi,\overline{s}}(T, f_*(j'_!\mathcal{F}_1 \otimes \mathcal{F}_2), \omega)$ is equal to*

$$\lambda_{f,\pi}(\omega)^{\mathrm{rk}(\mathcal{F}_2)} \delta_{s'/s}^{a(T',j'_!\mathcal{F}_1,f^*\omega)\mathrm{rk}(\mathcal{F}_2)} \mathrm{Ver}_{s'/s}\left(\det\left(\mathcal{F}_{2,\overline{s'}}\right)^{a(T',j'_!\mathcal{F}_1,f^*\omega)} \varepsilon_{\overline{s}'}(T', j'_!\mathcal{F}_1, f^*\omega)^{\mathrm{rk}(\mathcal{F}_2)}\right),$$

where $\varepsilon_{\overline{s}'}(T', j'_!\mathcal{F}_1, f^*\omega)$ is the ε-factor defined in III.7.7 and where $\delta_{s'/s}, \mathrm{Ver}_{s'/s}$ are defined in III.3.22 and III.3.23 respectively. When $\eta' = \eta$, the factor $\lambda_{f,\pi}(\omega)$ is identically equal to 1.

By Proposition III.7.12 and Proposition III.3.24, we can further assume (and we do) that $\omega = d\pi$. Let us now consider the special cover $f' : U \to \mathbb{G}_{m,s}$ (cf. III.4.3) associated to π and to the extension η' of η, cf. Theorem III.4.4. Let $\mathcal{G}_2$ be a ν_2-twisted C-sheaf on s' whose pullback to η' is isomorphic to $\mathcal{F}_2$. Let $\mathcal{G}_1$ be the f'-special ν_1-twisted C-local system of rank 1 on U associated to $\mathcal{F}_1$ by Theorem III.4.16, so that the pullback of $\mathcal{G}_1$ to η' is isomorphic to $\mathcal{F}_1$. The ν_1-twisted C-local system $\pi_\diamondsuit j_! f_*(\mathcal{F}_1)$ vanishes at 0, and its restriction to $\mathbb{G}_{m,s}$ is isomorphic to $f'_*\mathcal{G}$, cf. III.4.20. We thus have

$$\begin{aligned}\varepsilon_{\pi,\overline{s}}(T, f_*j'_!(\mathcal{F}_1 \otimes \mathcal{F}_2), d\pi) &= \varepsilon_{\overline{s}}(\mathbb{G}_{m,s}, f'_*(\mathcal{G}_1 \otimes \mathcal{G}_2) \otimes \mathcal{L}_\psi\{-t\}) \\ &= \varepsilon_{\overline{s}}(U, \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{L}_\psi\{-f'\}) \\ &= \delta_{s'/s}^{-\chi_c(U_{\overline{s}'},\mathcal{G}_1\otimes\mathcal{G}_2\otimes\mathcal{L}_\psi\{-f'\})} \mathrm{Ver}_{s'/s}\left(\varepsilon_{\overline{s}'}(U, \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{L}_\psi\{-f'\})\right).\end{aligned}$$

Let $u : U \to X$ be a smooth compactification of U, so that X is a geometrically connected smooth projective curve over s. Let D be the closed complement in X of the union of U and the closed point of T'. For any point x of D, we have

$$\begin{aligned}a(X_{(x)}, u_!\mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{L}_\psi\{-f'\}, df') &= (1 + \mathrm{sw}_x(\mathcal{L}_\psi\{-f'\}) + v_x(df'))\mathrm{rk}(\mathcal{F}_2) \\ &= 0,\end{aligned}$$

since $\mathcal{G}_1\otimes\mathcal{G}_2$ is tamely ramified at x (cf. III.4.3) and since $\mathrm{sw}_x(\mathcal{L}_\psi\{-f'\})$ is equal to the valuation $v_x(f')$ of f' at x, the latter being an integer prime to p. Consequently, the Grothendieck-Ogg-Shafarevich formula ([**La87**], 3.1.5.3) yields

$$-\chi_c(U_{\overline{s}'}, \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{L}_\psi\{-f'\}) = \alpha\mathrm{rk}(\mathcal{F}_2),$$

where $\alpha = a(T', j'_!\mathcal{F}_1, f^{-1}d\pi)$. We also have

$$\varepsilon_{\overline{s}'}(U, \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{L}_\psi\{-f'\}) = \det(\mathcal{G}_{2,\overline{s}'})^{-\chi_c(U_{\overline{s}'},\mathcal{G}_1\otimes\mathcal{L}_\psi\{-f'\})} \varepsilon_{\overline{s}'}(U, \mathcal{G}_1 \otimes \mathcal{L}_\psi\{-f'\})^{\mathrm{rk}(\mathcal{F}_2)},$$

with $-\chi_c(U_{\overline{s}'}, \mathcal{G}_1 \otimes \mathcal{L}_\psi\{-f'\}) = \alpha$, hence

$$\varepsilon_{\pi,\overline{s}}(T, f_*j'_!(\mathcal{F}_1 \otimes \mathcal{F}_2), d\pi) = \delta_{s'/s}^{\alpha\mathrm{rk}(\mathcal{F}_2)} \mathrm{Ver}_{s'/s}\left(\det\left(\mathcal{F}_{2,\overline{s'}}\right)^{\alpha} \varepsilon_{\overline{s}'}(U, \mathcal{G}_1 \otimes \mathcal{L}_\psi\{-f'\})^{\mathrm{rk}(\mathcal{F}_2)}\right). \tag{56}$$

If we have a formula of the form

$$\varepsilon_{\pi,\overline{s'}}(T_{s'}, f_*j'_!\mathcal{F}_1, d\pi) = \lambda\varepsilon_{\overline{s}'}(T', j'_!\mathcal{F}_1, f^*\omega), \tag{57}$$

for some C-admissible homomorphism Λ from $G_{s'}$ to $C^\times$, then (56) applied to $s = s'$ and $\mathcal{F}_2 = C$ yields

$$\lambda\varepsilon_{\overline{s}'}(T', j'_!\mathcal{F}_1, f^*\omega) = \varepsilon_{\overline{s}'}(U, \mathcal{G}_1 \otimes \mathcal{L}_\psi\{-f'\}),$$

and inserting the latter formula in (56) yields the required result with $\lambda_{f,\pi}(d\pi) = \mathrm{Ver}_{s'/s}(\lambda)$. It is therefore sufficient to prove (57), so that we can assume that $s = s'$ and $\mathcal{F}_2 = C$ (and we henceforth do), in which case the formula to be proved is

$$\varepsilon_{\pi,\overline{s}}(T, f_* j'_! \mathcal{F}_1, d\pi) = \lambda_{f,\pi}(d\pi)\varepsilon_{\overline{s}}(T', j'_! \mathcal{F}_1, f^* d\pi),$$

for some C-admissible map $\lambda_{f,\pi}(d\pi)$ from G_s to $C^\times$. We now apply the product formula III.8.3 with $\omega = f'^* dt = df'$, according to which the map $\varepsilon_{\overline{s}}(U, \mathcal{G}_1 \otimes \mathcal{L}_\psi\{-f'\})$ is equal to

$$\lambda_{f,\pi}(d\pi)\varepsilon_{\overline{s}}(T', j'_! \mathcal{F}_1, f^* d\pi),$$

where we have set

$$\lambda_{f,\pi}(d\pi) = \chi_{\mathrm{cyc}}^{g-1} \prod_{x \in D} \delta_{x/s}^{1-v_x(f')} \mathrm{Ver}_{x/s}\left(\varepsilon_{\overline{x}}(X_{(x)}, u_! \mathcal{G}_1 \otimes \mathcal{L}_\psi\{-f'\}_{|X_{(x)}}, df')\right),$$

and g is the genus of X. Moreover, for each point x of D, the valuation $v_x(f')$ of f' at x is prime to p since f' is tamely ramified at this point (cf. III.4.3), and the restriction $u_! \mathcal{F}_{|X_{(x)}}$ is tamely ramified as well, since $\mathcal{F}$ is f'-special. Hence we can apply Proposition III.7.13, which implies

$$\varepsilon_{\overline{x}}(X_{(x)}, u_! \mathcal{G}_1 \otimes \mathcal{L}_\psi\{-f'\}_{|X_{(x)}}, df') = \varepsilon_{\overline{x}}(X_{(x)}, \mathcal{L}_\psi\{-f'\}_{|X_{(x)}}, df').$$

Thus we obtain the required formula with

$$\lambda_{f,\pi}(d\pi) = \chi_{\mathrm{cyc}}^{g-1} \prod_{x \in D} \delta_{x/s}^{1-v_x(f')} \mathrm{Ver}_{x/s}\left(\varepsilon_{\overline{x}}(X_{(x)}, \mathcal{L}_\psi\{-f'\}_{|X_{(x)}}, df')\right), \tag{58}$$

which depends indeed only on f and π. When $f = \mathrm{id}$, we have $X = \mathbb{P}^1_s$ and D is the closed point ∞, so that Proposition III.7.13 applies with $n = 1$ and yields

$$\varepsilon_{\overline{s}}(\mathbb{P}^1_{s,(\infty)}, \mathcal{L}_\psi\{-t\}_{|\mathbb{P}^1_{s,(\infty)}}, dt) = \chi_{\mathrm{cyc}}.$$

By inserting this identity in (58), in which $g = 0$, we obtain $\lambda_{\mathrm{id},\pi}(d\pi) = 1$, which in turn implies $\lambda_{\mathrm{id},\pi}(\omega) = 1$ for any ω; this concludes the proof of Proposition III.9.13.

Remark III.9.14. The formula (58), combined with Proposition III.7.13, yields an expression of $\lambda_{f,\pi}(d\pi)$ as a product of certain quadratic Gauss sums and of a power of the cyclotomic character. More precisely, with notation as in (58), let D^+ (resp. D^-) be the closed subset of X consisting of the points of D such that $v_x(f')$ is odd (resp. even), endowed with its reduced scheme structure. Then we have, for any totally ramified extension f,

$$\lambda_{f,\pi}(d\pi) = \chi_{\mathrm{cyc}}^{g-1+\deg(D^-)+\frac{1}{2}(\deg(f)+\deg(D^+))} \prod_{x \in D^-} \delta_{x/s} \mathrm{Ver}_{x/s}\left(\gamma_\psi(v_x(f')h_x)\right),$$

with notation as in III.7.13, where h_x is an element of $k(x)^\times$ such that $h_x f'$ is a square in the field of fractions of $\mathcal{O}_{X_{(x)}}$. This formula simplifies greatly when $p = 2$, since D^- is then empty.

Corollary III.9.15. *Let $\mathcal{F}$ be a μ-twisted C-sheaf on T such that $j^{-1}\mathcal{F}$ is of rank at most 1. Then for any element ω of $\Omega^{1,\times}_\eta$, we have*

$$\varepsilon_{\pi,\overline{s}}(T, \mathcal{F}, \omega) = \varepsilon_{\overline{s}}(T, \mathcal{F}, \omega),$$

where $\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)$ is the ε-factor defined in III.7.7.

Indeed, this results from Proposition III.9.6 if $\mathcal{F}$ is supported on s, or from Proposition III.9.13 with $\eta' = \eta$ and $s' = s$ if $\mathcal{F}$ is supported on η.

Corollary III.9.16. *Let π' be an other uniformizer of $k(\eta)$. Then for any element ω of $\Omega^{1,\times}_\eta$ and any μ-twisted C-sheaf $\mathcal{F}$ on T such that $j^{-1}\mathcal{F}$ is potentially unipotent (cf. III.9.8), we have*

$$\varepsilon_{\pi,\overline{s}}(T, \mathcal{F}, \omega) = \varepsilon_{\pi',\overline{s}}(T, \mathcal{F}, \omega).$$

Indeed, by multiplicativity of ε_π and $\varepsilon_{\pi'}$ in short exact sequences (cf. III.9.3), the maps $\mathcal{F} \mapsto \varepsilon_{\pi,\overline{s}}(T,\mathcal{F},\omega)$ and $\mathcal{F} \mapsto \varepsilon_{\pi',\overline{s}}(T,\mathcal{F},\omega)$ extend to homomorphisms $\varepsilon_{\pi,\overline{s}}(T,\omega)$ and $\varepsilon_{\pi',\overline{s}}(T,\omega)$ respectively from the abelian group

$$\bigoplus_\nu K_0(T,\nu,C),$$

where the sum runs over all unitary C-admissible multipliers on G_k (cf. III.9.11), to $C^\times$. It is therefore sufficient to prove that $\varepsilon_{\pi',k}(T,\omega)$ and $\varepsilon_{\pi',k}(T,\omega)$ agree on the three types of generators described in Proposition III.9.12. For generators of type (1) or (2) this follows from Corollary III.9.15, while this follows from Proposition III.9.13 for generators of type (3).

Notation III.9.17. Let $\mathcal{F}$ be a μ-twisted C-sheaf on T such that $j^{-1}\mathcal{F}$ is potentially unipotent III.9.8, and let ω be an element of $\Omega^{1,\times}_\eta$. We denote by $\varepsilon_{\overline{s}}(T,\mathcal{F},\omega)$ the C-admissible map $\varepsilon_{\pi,\overline{s}}(T,\mathcal{F},\omega)$ (cf. III.9.2), which does not depend on the uniformizer π by Corollary III.9.16.

By Corollary III.9.16, this definition does not depend on the choice of the uniformizer π, and by Corollary III.9.15 this does not conflict with the definition III.7.10 when $j^{-1}\mathcal{F}$ is of rank at most 1.

Proposition III.9.18. *Let $\eta' \to \eta$ be a connected finite étale cover of η, with normalization $f: T' \to T$. Let $\overline{s}' : \mathrm{Spec}(\overline{k}) \to T'$ be an s-morphism over the closed point s' of T' such that $f(\overline{s}') = \overline{s}$, and let $j' : \eta' \to T'$ be the natural inclusion. Then for any element ω of $\Omega^{1,\times}_\eta$, there exists a (unique) Λ-admissible homomorphism $\lambda_f(\omega)$ from $G_{\overline{s}}$ to $\Lambda^\times$, depending only on f and ω with the following property: for any μ-twisted C-sheaf $\mathcal{F}$ on T' such that $j'^{-1}\mathcal{F}$ is potentially unipotent (cf. III.9.8), we have*

$$\varepsilon_{\overline{s}}(T, f_*\mathcal{F},\omega) = \lambda_f(\omega)^{\mathrm{rk}(j'^{-1}\mathcal{F})}\delta_{s'/s}^{a(T',\mathcal{F},f^*\omega)}\mathrm{Ver}_{s'/s}\left(\varepsilon_{\overline{s}'}(T',\mathcal{F},f^*\omega)\right),$$

where $\delta_{s'/s}$ and $\mathrm{Ver}_{s'/s}$ are defined in III.3.22 and III.3.23 respectively.

Indeed, Corollary III.9.16 implies that the group homomorphism $\lambda_{f,\pi}(\omega)$ from Proposition III.9.13 does not depend on π. Let us denote it by $\lambda_f(\omega)$. The maps

$$\begin{aligned} &A : \mathcal{F} \mapsto \varepsilon_{\overline{s}}(T, f_*\mathcal{F},\omega) \\ &B : \mathcal{F} \mapsto \lambda_f(\omega)^{\mathrm{rk}(j'^{-1}\mathcal{F})}\delta_{s'/s}^{a(T',\mathcal{F},f^*\omega)}\mathrm{Ver}_{s'/s}\left(\varepsilon_{\overline{s}'}(T',\mathcal{F},f^*\omega)\right), \end{aligned}$$

both extend by multiplicativity (cf. III.9.3) to homomorphisms from the abelian group

$$\bigoplus_\nu K_0(T',\nu,C),$$

where the sum runs over all unitary C-admissible multipliers on G_s (cf. III.9.11), to $C^\times$. Consequently, it is sufficient to check that the homomorphisms A and B coincide on the three type of generators described in Proposition III.9.12. For the generator of type (1), this follows from Proposition III.9.13 with $\nu_1 = \nu_2 = 1$ and $\mathcal{G}_1 = \mathcal{G}_2 = C$, while it holds as well for generators of type (2) by Proposition III.9.6. It remains to handle generators of the third type described in Proposition III.9.12.

Let $\eta'' \to \eta'$ be a connected finite étale cover of η', with normalization $f' : T'' \to' T$. Let $\overline{s}'' : \mathrm{Spec}(\overline{k}) \to T''$ be an s'-morphism over the closed point s'' of T'' such that $f(\overline{s}'') = \overline{s}'$, and let $j'' : \eta' \to T'$ be the natural inclusion. Let μ_1 and μ_2 be unitary C-admissible multipliers on $G_{s''}$ such that $\mu_1\mu_2 = \mu_{|G_{s''}}$, let $\mathcal{F}_1$ be a μ_1-twisted C-sheaf of rank 1 on η'', with finite geometric monodromy, and let $\mathcal{F}_2$ be an unramified μ_2-twisted C-sheaf on η''. We must prove that the homomorphisms A and B associate the same map to the class

$$[f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[f'_*j''_!C].$$

By applying Proposition III.9.13 to the extension $\eta'' \to \eta'$, we obtain an equality

$$A(f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)) = \lambda_{ff'}(\omega)^{\mathrm{rk}(\mathcal{F}_2)}\delta^a_{s''/s}\mathrm{Ver}_{s''/s}(\mathcal{L}),$$

with $a = a(T'', j''_!\mathcal{F}_1, (ff')^*\omega)\mathrm{rk}(\mathcal{F}_2)$ and

$$\mathcal{L} = \det(\mathcal{F}_{2,\overline{s}''})^{a(T'', j''_!\mathcal{F}_1, (ff')^*\omega)}\,\varepsilon_{\overline{s}''}(T'', j''_!\mathcal{F}_1, (ff')^*\omega)^{\mathrm{rk}(\mathcal{F}_2)}.$$

A second application of Proposition III.9.13 yields, together with III.7.3,

$$B(f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)) = \lambda_f(\omega)^{\mathrm{rk}(\mathcal{F}_2)\deg(f')}\delta^{[s'':s']a}_{s'/s}\mathrm{Ver}_{s'/s}\left(\lambda_{f'}(f^*\omega)^{\mathrm{rk}(\mathcal{F}_2)}\delta^a_{s''/s'}\mathrm{Ver}_{s''/s'}(\mathcal{L})\right),$$

hence by Proposition III.3.24 we have

$$B(f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)) = \left(\lambda_f(\omega)\mathrm{Ver}_{s'/s}(\lambda_{f'}(f^*\omega))\right)^{\mathrm{rk}(\mathcal{F}_2)}\delta^{[s'':s']a}_{s'/s}\mathrm{Ver}_{s'/s}\left(\delta_{s''/s'}\right)^a\mathrm{Ver}_{s'/s}\mathrm{Ver}_{s''/s'}(\mathcal{L}),$$

By Corollary III.3.26, this yields

$$(A/B)(f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)) = \left(\lambda_{ff'}(\omega)\lambda_f(\omega)^{-1}\mathrm{Ver}_{s'/s}(\lambda_{f'}(f^*\omega))^{-1}\right)^{\mathrm{rk}(\mathcal{F}_2)}.$$

By applying this formula to $\mathcal{F}_1 = \mathcal{F}_2 = C$, we obtain

$$(A/B)(f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)) = (A/B)(f'_*j''_!C)^{\mathrm{rk}(\mathcal{F}_2)},$$

hence the equality

$$A([f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[f'_*j''_!C]) = B([f'_*j''_!(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[f'_*j''_!C]),$$

which concludes our proof of Corollary III.9.18.

Corollary III.9.19. *Let us consider a tower $\eta'' \to \eta' \to \eta$ of connected finite separable extensions, and let*

$$T'' \xrightarrow{f'} T' \xrightarrow{f} T,$$

be the normalizations of T in η' and η'', with closed points s'' and s'. Then for any element ω of $\Omega^{1,\times}_\eta$, we have

$$\lambda_{ff'}(\omega) = \lambda_f(\omega)\mathrm{Ver}_{s'/s}(\lambda_{f'}(f^*\omega)),$$

with notation as in Proposition III.9.18.

This is an immediate consequence of Propositions III.9.18 and III.3.26.

III.9.20. We can now prove Theorem III.1.7. The rule ε defined in III.9.17 and III.9.2 clearly satisfies the properties (i) and (ii) from III.1.6. It also satisfies the properties (iii), (iv), (v), (vi), (vii) from III.1.6 by III.9.3, III.9.6, III.9.18, III.9.15 and III.9.5 respectively. This proves the existence statement in Theorem III.1.7, while the uniqueness is an immediate consequence of Proposition III.9.12. The property $(viii)$ in Theorem III.1.7 follows from the uniqueness, while the property (ix) follows from Proposition III.9.4.

III.9.21. Let us now prove Theorem III.1.8. We assume that k is a finite field. By Theorem III.1.7 and by Proposition III.7.22, the rule which associates the quantity

$$(-1)^{a(T,\mathcal{F})}\varepsilon_{\overline{s}}(T, \mathcal{F}, \omega)(\mathrm{Frob}_s),$$

to any quadruple $(T, \mathcal{F}, \omega, \overline{s})$, where T is a henselian trait over k, with closed point s finite over k, where $\overline{s} : \mathrm{Spec}(\overline{k}) \to T$ is a k-morphism, where $\mathcal{F}$ is a C-sheaf on T and where ω is a non zero meromorphic 1-form on T, satisfies all the properties listed in ([**La87**], Th. 3.1.5.4), hence must coincide with the local ε-factor considered there.

III.10. The product formula for sheaves with finite geometric monodromy

Let $\psi : \mathbb{F}_p \to C^\times$ be a non trivial homomorphism. We fix a unitary C-admissible mutiplier μ on the topological group G_k (cf. III.2.9, III.2.10).

III.10.1. For any connected smooth projective curve X over k, let s be the spectum of a finite extension of k, such that X is a geometrically connected s-scheme, and let us fix a k-morphism $\overline{s} : \mathrm{Spec}(\overline{k}) \to s$. If η is the generic point of X and if $\overline{\eta}$ is the spectrum of a separable closure of $k(\eta)$, then for any non empty open subscheme U in X, we have an exact sequence

$$1 \to \pi_1(U_{\overline{s}}, \overline{\eta}) \to \pi_1(U, \overline{\eta}) \to G_s \to 1,$$

where G_s is the Galois group of the extension $\overline{k}/k(s)$.

Definition III.10.2. A μ-twisted C-sheaf $\mathcal{F}$ on a connected smooth projective curve X has *finite geometric monodromy* if there exists a non empty open subscheme U of X such that $\mathcal{F}_{|U}$ is a μ-twisted C-local system and such that the group $\pi_1(U_{\overline{s}}, \overline{\eta})$ acts through a finite quotient on the C-admissible representation $\mathcal{F}_{\overline{\eta}}$ of $(\pi_1(U, \overline{\eta}), \mu)$.

If X is a connected smooth projective curve X, we denote by $|X|$ the set of its closed points. For any closed point x of X, we denote by $X_{(x)}$ the henselization of X at x. We then have the following product formula:

Theorem III.10.3. *Let X be a connected smooth projective curve of genus g over k, let ω be a non zero global meromorphic differential 1-form on X and let $\mathcal{F}$ be a μ-twisted C-sheaf on X with finite geometric monodromy (cf. III.10.2), of generic rank $\mathrm{rk}(\mathcal{F})$. Then, for all but finitely many closed points x of X the ε-factor of the triple $(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})$ (cf. III.9.2, III.9.17) is identically equal to 1, and we have*

$$\varepsilon_{\overline{k}}(X, \mathcal{F}) = \chi_{\mathrm{cyc}}^{N(g-1)\mathrm{rk}(\mathcal{F})} \prod_{x \in |X|} \delta_{x/k}^{a(X_{(x)}, \mathcal{F}_{|X_{(x)}})} \mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})\right),$$

where N is the number of connected components of $X_{\overline{k}}$, and where $\delta_{x/k}$ and $\mathrm{Ver}_{x/k}$ are defined in III.3.22.

This theorem will be proved in section III.10.7. It is usually convenient to introduce the character

$$\delta_{X/k}(\omega) = \chi_{\mathrm{cyc}}^{N(g-1)} \prod_{x \in |X|} \delta_{x/k}^{v_x(\omega)}, \tag{59}$$

so that we have, by noting that $\delta_{x/k}^{v_x(\omega)} = \delta_{x/k}^{-v_x(\omega)}$, the following equivalent formulation of the product formula:

$$\varepsilon_{\overline{k}}(X, \mathcal{F}) = \delta_{X/k}(\omega)^{\mathrm{rk}(\mathcal{F})} \prod_{x \in |X|} \delta_{x/k}^{a(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})} \mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})\right). \tag{60}$$

Corollary III.10.4. *Let $f : X' \to X$ be a finite generically étale morphism of connected smooth projective curves over k, of respective genera g' and g. Then for any non zero global meromorphic differential 1-form ω on X, we have*

$$\prod_{x' \in |X'|} \mathrm{Ver}_{f(x')/k}\left(\lambda_{f_{x'}}(\omega_{|X_{(f(x'))}})\right) = \delta_{X'/k}(f^*\omega)\delta_{X/k}(\omega)^{-1},$$

where $f_{x'} : X'_{(x)} \to X_{(f(x'))}$ is for each closed point x' of X' the morphism induced by f on the henselizations of X' and X at x' and $f(x')$ respectively, where $\lambda_{f_{x'}}(\omega_{|X_{(f(x'))}})$ is the homomorphism defined in III.9.18, and where $\delta_{X'/k}(f^*\omega)$, $\delta_{X/k}(\omega)$ are as in 59.

Indeed, this follows from Proposition III.9.18 and from Theorem III.10.3 for the triples $(X', C, f^*\omega)$ and (X, f_*C, ω), since $\varepsilon(X, f_*C)$ is equal to $\varepsilon(X', C)$.

III.10.5. Let X be a connected smooth projective curve over k. Let s be the spectum of a finite extension of k, such that X is a geometrically connected s-scheme, and let us fix a k-morphism $\overline{s} : \mathrm{Spec}(\overline{k}) \to s$. Let $K_0(X, \mu, C)$ be the Grothendieck group of the full subcategory of $\mathrm{Sh}(X, \mu, C)$ consisting of μ-twisted C-sheaves on X with finite geometric monodromy (cf. III.10.2). Thus any μ-twisted C-sheaf $\mathcal{F}$ on X with finite geometric monodromy has a well defined class $[\mathcal{F}]$ in $K_0(X, \mu, C)$, and the latter is generated by such classes with relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ for each short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

of μ-twisted C-sheaves on X, with finite geometric monodromy.

Proposition III.10.6. *Let X be a connected smooth projective curve over k, let s, $\overline{s}$ and $K_0(X, \mu, C)$ be as in III.10.5. The abelian group*

$$\bigoplus_{\nu} K_0(X, \nu, C),$$

where the sum runs over all unitary C-admissible multipliers on G_k, is generated by its subset of elements of the following three types:

(1) the class $[C]$ in $K_0(X, 1, C)$,

(2) for any unitary C-admissible multiplier μ on G_k, any closed point $i_x : x \to X$ of X and any μ-twisted C-sheaf $\mathcal{G}$ on x, the class $[i_{x*}\mathcal{G}]$ in $K_0(T, \mu, C)$,

(3) for any finite extension $s' \to s$, together with an s-morphism $\overline{s}' : \mathrm{Spec}(\overline{k}) \to s'$ whose composition with $s' \to s$ is $\overline{s}$, for any unitary C-admissible multiplier μ on G_s, for any non empty open subscheme $j : U \to X$, any finite étale morphism $f : U' \to U$ such that U' is a geometrically connected s'-scheme, any unitary C-admissible multipliers μ_1 and μ_2 on $G_{s'}$ such that $\mu_1\mu_2 = \mu_{|G_{s'}}$, any μ_1-twisted C-sheaf $\mathcal{F}_1$ of rank 1 over U' with finite geometric monodromy, and any geometrically constant μ_2-twisted C-sheaf $\mathcal{F}_2$ over U', the class $[j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[j_! f_* C]$ in the sum of $K_0(X, \mu, C)$ and $K_0(X, 1, C)$.

Let $\mathcal{F}$ be a μ-twisted C-sheaf on X with finite geometric monodromy. Let $j : U \to X$ be a non empty open subscheme of X such that $j^{-1}\mathcal{F}$ is a μ-twisted C-local system. We have an exact sequence

$$0 \to j_! j^{-1}\mathcal{F} \to \mathcal{F} \to \bigoplus_{x \in |X \setminus U|} i_{x*} i_x^{-1}\mathcal{F} \to 0,$$

hence we can assume (and we do) that $\mathcal{F} = j_! j^{-1}\mathcal{F}$. The result then follows from Proposition III.2.50 by using the dictionary between μ-twisted C-local systems on U and C-admissible representations of $(\pi_1(U, \overline{\eta}), \mu)$, for a fixed geometric point $\overline{\eta}$ over the generic point of X, cf. III.3.19 and III.3.21.

III.10.7. We now prove Theorem III.10.3, in its equivalent form 60. Let $s, \overline{s}$ be as in III.10.5. The maps

$$A : \mathcal{F} \mapsto \varepsilon_{\overline{k}}(X, \mathcal{F}),$$

$$B : \mathcal{F} \mapsto \delta_{X/k}(\omega)^{\mathrm{rk}(\mathcal{F})} \prod_{x \in |X|} \delta_{x/k}^{a(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})} \mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})\right),$$

both extend by multiplicativity to homomorphisms from the abelian group

$$\bigoplus_{\nu} K_0(X, \nu, C),$$

where the sum runs over all unitary C-admissible multipliers on G_k (cf. III.10.5), to $C^\times$. It is therefore sufficient to prove that the homomorphisms A and B coincide on the three type of generators described in Proposition III.10.6. For the generator of type (1), this follows from the product formula III.8.3 in generic rank 1, while it holds for generators of type (2) by Proposition III.9.6.

We now consider a generator of the third type described in Proposition III.10.6. Let $s' \to s$ be a finite extension, together with an s-morphism $\overline{s}' : \mathrm{Spec}(\overline{k}) \to s'$ whose composition with $s' \to s$ is $\overline{s}$, let μ be a unitary C-admissible multiplier on G_s, let $j : U \to X$ be a non empty open subscheme of X, let $f : U' \to U$ be a finite étale morphism such that U' is a geometrically connected s'-scheme, let μ_1 and μ_2 be unitary C-admissible multipliers on $G_{s'}$ such that $\mu_1\mu_2 = \mu_{|G_{s'}}$, let $\mathcal{F}_1$ be a μ_1-twisted C-sheaf of rank 1 over U' with finite geometric monodromy, and let $\mathcal{F}_2$ be a μ_2-twisted C-sheaf over s'. We must prove that the homomorphisms A and B agree on the class $[j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[j_! f_* C]$.

First, we have

$$\begin{aligned}
A(j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)) &= \varepsilon_{\overline{k}}(U', \mathcal{F}_1 \otimes \mathcal{F}_2) \\
&= \det\left(R\Gamma(U'_{\overline{k}}, \mathcal{F}_1 \otimes \mathcal{F}_2)\right)^{-1} \\
&= \det\left(\mathrm{Ind}_{G_{s'}}^{G_k} R\Gamma(U'_{\overline{s}'}, \mathcal{F}_1) \otimes \mathcal{F}_{2,\overline{s}'}\right)^{-1} \\
&= \delta_{s'/k}^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1 \otimes \mathcal{F}_2)} \mathrm{Ver}_{s'/k}\left(R\Gamma(U'_{\overline{s}'}, \mathcal{F}_1) \otimes \mathcal{F}_{2,\overline{s}'}\right)^{-1} \\
&= \delta_{s'/k}^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1 \otimes \mathcal{F}_2)} \mathrm{Ver}_{s'/k}\left(\det(\mathcal{F}_{2,\overline{s}'})^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1)} \varepsilon_{\overline{s}'}(U', \mathcal{F}_1)^{\mathrm{rk}(\mathcal{F}_2)}\right).
\end{aligned}$$

Let $X' \to X$ be the extension of f to a smooth compactification X' of U', with an open immersion $j' : U' \to X'$, and let g' the genus of X'. For any closed point x' of X', let us write

$$\begin{aligned}
a_{x'} &= a(X'_{(x')}, j'_! \mathcal{F}_{1|X'_{(x')}}, f^*\omega_{|X'_{(x')}}) \\
\mathcal{L}_{x'} &= \mathrm{Ver}_{x'/s'}\left(\varepsilon_{\overline{x}'}\left(X'_{(x')}, j'_! \mathcal{F}_{1|X'_{(x')}}, f^*\omega_{|X'_{(x')}}\right)\right).
\end{aligned}$$

Then Proposition III.9.18, together with III.7.3, implies that $B(j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2))$ is equal to (with notation as in Corollary III.10.4)

$$\delta_{X/k}(\omega)^{\mathrm{rk}(\mathcal{F}_2)} \prod_{x \in |X|} \left(\prod_{\substack{x' \in |X'| \\ f(x') = x}} \delta_{x/k}^{[k(x'):k(x)] a_{x'} \mathrm{rk}(\mathcal{F}_2)} \right) \mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}\left(X_{(x)}, j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)_{|X_{(x)}}, \omega_{|X_{(x)}}\right)\right),$$

while the induction formula III.9.18 and Propositions III.9.5, III.3.26 yield

$$\begin{aligned}
&\mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}\left(X_{(x)}, j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)_{|X_{(x)}}, \omega_{|X_{(x)}}\right)\right) \\
&= \prod_{\substack{x' \in |X'| \\ f(x') = x}} \mathrm{Ver}_{x/k}\left(\delta_{x'/x}^{a_{x'} \mathrm{rk}(\mathcal{F}_2)} \lambda_{f_{x'}}(\omega_{|X'_{(x')}})^{\mathrm{rk}(\mathcal{F}_2)} \mathrm{Ver}_{x'/x}\left(\varepsilon_{\overline{x}'}\left(X'_{(x')}, j'_!(\mathcal{F}_1 \otimes \mathcal{F}_2)_{X'_{(x')}}, f^*\omega_{|X'_{(x')}}\right)\right)\right) \\
&= \prod_{\substack{x' \in |X'| \\ f(x') = x}} \mathrm{Ver}_{x/k}\left(\delta_{x'/x}^{a_{x'}} \lambda_{f_{x'}}(\omega_{|X'_{(x')}})\right)^{\mathrm{rk}(\mathcal{F}_2)} \mathrm{Ver}_{x'/k}\left(\varepsilon_{\overline{x}'}\left(X'_{(x')}, j'_!(\mathcal{F}_1 \otimes \mathcal{F}_2)_{X'_{(x')}}, f^*\omega_{|X'_{(x')}}\right)\right) \\
&= \prod_{\substack{x' \in |X'| \\ f(x') = x}} \mathrm{Ver}_{x/k}\left(\delta_{x'/x}^{a_{x'}} \lambda_{f_{x'}}(\omega_{|X'_{(x')}})\right)^{\mathrm{rk}(\mathcal{F}_2)} \mathrm{Ver}_{s'/k}\left(\det(\mathcal{F}_{2,\overline{s}'})^{[k(x'):s'] a_{x'}} \mathcal{L}_{x'}^{\mathrm{rk}(\mathcal{F}_2)}\right).
\end{aligned}$$

Thus we can write $B(j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2))$ as a product DE where D is given by

$$D = \delta_{X/k}(\omega)^{\mathrm{rk}(\mathcal{F}_2)} \prod_{x' \in |X'|} \mathrm{Ver}_{f(x')/k}(\lambda_{f_{x'}}(\omega_{|X'_{(x')}}))^{\mathrm{rk}(\mathcal{F}_2)}$$

and E is given by

$$\begin{aligned} E &= \prod_{x' \in |X'|} \left(\delta_{x/k}^{[k(x'):k(f(x'))]} \mathrm{Ver}_{f(x')/k}(\delta_{x'/f(x')})\right)^{a_{x'}\mathrm{rk}(\mathcal{F}_2)} \mathrm{Ver}_{s'/k}\left(\det(\mathcal{F}_{2,\overline{s}'})^{[k(x'):s']a_{x'}} \mathcal{L}_{x'}^{\mathrm{rk}(\mathcal{F}_2)}\right) \\ &= \delta_{s'/k}^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1 \otimes \mathcal{F}_2)} \mathrm{Ver}_{s'/k}\left(\prod_{x' \in |X'|} \delta_{x'/s'}^{a_{x'}\mathrm{rk}(\mathcal{F}_2)} \det(\mathcal{F}_{2,\overline{s}'})^{[k(x'):s']a_{x'}} \mathcal{L}_{x'}^{\mathrm{rk}(\mathcal{F}_2)}\right) \\ &= \delta_{s'/k}^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1 \otimes \mathcal{F}_2)} \mathrm{Ver}_{s'/k}\left(\det(\mathcal{F}_{2,\overline{s}'})^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1)} \prod_{x' \in |X'|} \delta_{x'/s'}^{a_{x'}\mathrm{rk}(\mathcal{F}_2)} \mathcal{L}_{x'}^{\mathrm{rk}(\mathcal{F}_2)}\right), \end{aligned}$$

by the Grothendieck-Ogg-Shafarevich formula. By Theorem III.8.3, we also have

$$\begin{aligned} E &= \mathrm{Ver}_{s'/k}\left(\delta_{X'/s'}(f^*\omega)\right)^{-\mathrm{rk}(\mathcal{F}_2)} \delta_{s'/k}^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1 \otimes \mathcal{F}_2)} \mathrm{Ver}_{s'/k}\left(\det(\mathcal{F}_{2,\overline{s}'})^{-\chi_c(U_{\overline{s}'}, \mathcal{F}_1)} \varepsilon_{\overline{s}'}(U', \mathcal{F}_1)^{\mathrm{rk}(\mathcal{F}_2)}\right) \\ &= \mathrm{Ver}_{s'/k}\left(\delta_{X'/s'}(f^*\omega)\right)^{-\mathrm{rk}(\mathcal{F}_2)} A(j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)). \end{aligned}$$

Consequently, we have

$$(B/A)(j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)) = \left(\delta_{X/k}(\omega) \mathrm{Ver}_{s'/k}\left(\delta_{X'/s'}(f^*\omega)\right)^{-1} \prod_{x' \in |X'|} \mathrm{Ver}_{f(x')/k}(\lambda_{f_{x'}}(\omega_{|X'_{(x')}}))\right)^{\mathrm{rk}(\mathcal{F}_2)}.$$

By applying this identity to $\mathcal{F}_1 = \mathcal{F}_2 = C$ we obtain

$$(B/A)(j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)) = (B/A)(j_! f_* C)^{\mathrm{rk}(\mathcal{F}_2)},$$

hence the equality

$$A([j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[j_! f_* C]) = B([j_! f_*(\mathcal{F}_1 \otimes \mathcal{F}_2)] - \mathrm{rk}(\mathcal{F}_2)[j_! f_* C]),$$

which concludes our proof of Theorem III.10.3.

III.11. The product formula : the general case

Let $\psi : \mathbb{F}_p \to C^\times$ be a non trivial homomorphism. We fix a unitary C-admissible mutiplier μ on the topological group G_k (cf. III.2.9, III.2.10). We prove in this section the following product formula.

Theorem III.11.1. *Let X be a connected smooth projective curve of genus g over k and let ω be a non zero global meromorphic differential 1-form on X. Let $\mathcal{F}$ be a μ-twisted C-sheaf on X of generic rank $\mathrm{rk}(\mathcal{F})$, such that for any closed point x of X the restriction of $\mathcal{F}$ to the henselization $X_{(x)}$ of X at x is potentially unipotent. The trace function $\varepsilon_{\overline{k}}(X, \mathcal{F})$ on $\mathrm{Gal}(\overline{k}/k)$ associated to the twisted 1-dimensional representation $\det(R\Gamma(X_{\overline{k}}, \mathcal{F}))^{-1}$ admits the following decomposition:*

$$\varepsilon_{\overline{k}}(X, \mathcal{F}) = \chi_{\mathrm{cyc}}^{N(g-1)\mathrm{rk}(\mathcal{F})} \prod_{x \in |X|} \delta_{x/k}^{a(X_{(x)}, \mathcal{F}_{|X_{(x)}})} \mathrm{Ver}_{x/k}\left(\varepsilon_{\overline{x}}(X_{(x)}, \mathcal{F}_{|X_{(x)}}, \omega_{|X_{(x)}})\right),$$

where N is the number of connected components of $X_{\overline{k}}$, where $|X|$ is the set of closed points of X, where $\delta_{x/k}$ and $\mathrm{Ver}_{x/k}$ are defined in III.3.22 and χ_{cyc} is the ℓ-adic cyclotomic character of k. All but finitely many terms in this product are identically equal to 1.

We postpone the proof of Theorem III.11.1 to III.11.11 below. When k is finite, this result is due to Laumon ([**La87**], 3.2.1.1). For the general case, we follow closely Laumon's proof, or rather its exposition by Katz in [**Ka88**]. The main ingredient we use is the ℓ-adic stationary phase method, of which we only use the special case stated in Theorem III.11.5 below, and which was already established by Laumon ([**La87**], 2.3.3.1) in the case of an arbitrary perfect base field of positive characteristic. The only innovation in our proof lies in the treatment of Theorem III.11.8 below. Laumon's proof of the latter result when k is finite ([**La87**], 3.5.1.1) starts with a reduction to the tamely ramified case ([**La87**], 3.5.3.1), and then resorts to a computation in the latter case ([**La87**], 2.5.3.1). We give instead a direct proof in the general case by using geometric local class field theory.

III.11.2. Throughout this section, we consider two copies $\mathbb{A} = \mathrm{Spec}(k[t])$ and $\mathbb{A}' = \mathrm{Spec}(k[t'])$ of the affine line over k, with natural compactifications $\mathbb{P}$ and $\mathbb{P}'$ respectively, and we denote by pr and pr$'$ the projections of $\mathbb{A}\times_k\mathbb{A}'$ onto its first and second factors respectively. For any μ-twisted C-sheaf $\mathcal{F}$ on $\mathbb{A}$, we denote by $\mathrm{F}_\psi(\mathcal{F})$ its Fourier transform defined as follows:

$$\mathrm{F}_\psi(\mathcal{F}) = R^1\mathrm{pr}'_!\left(\mathrm{pr}^{-1}\mathcal{F}\otimes\mathcal{L}_\psi\{tt'\}\right),$$

which is a μ-twisted C-sheaf on $\mathbb{A}'$. This functor F_ψ would be denoted $\mathcal{H}^0(\mathcal{F}_\psi)$ in Laumon's notation ([**La87**], 1.2.1.1), and is called the "naive Fourier transform" by Katz in ([**Ka88**], p.112).

III.11.3. For any closed points s, s' of $\mathbb{P}$ and $\mathbb{P}'$ respectively, with respective separable closures $\overline{s}$ and $\overline{s}'$, we denote by η_s (resp. $\eta_{s'}$) the generic point of the henselisation $\mathbb{P}_{(s)}$ (resp. $\mathbb{P}'_{(s')}$) of $\mathbb{P}$ at s (resp. of $\mathbb{P}'$ at s'), and by $\overline{\eta}_s$ (resp. $\overline{\eta}_{s'}$) a separable closure thereof. For any μ-twisted C-sheaf $\mathcal{F}$ on $\mathbb{A}_{(s)}$ with vanishing fiber at s, we denote by $\mathrm{F}_\psi^{(s,s')}(\mathcal{F})$ its local Fourier transform defined as follows:

$$\mathrm{F}_\psi^{(s,s')}(\mathcal{F}) = H^1\left(\left((\mathbb{P}\times_k\mathbb{P}')_{(\overline{s},\overline{s}')}\right)_{\overline{\eta}_{s'}}, u_!(\mathrm{pr}^{-1}\mathcal{F}\otimes\mathcal{L}_\psi\{tt'\})\right),$$

where u is the natural open immersion of $\mathbb{A}\times_k\mathbb{A}'$ into $\mathbb{P}\times_k\mathbb{P}'$. Thus $\mathrm{F}_\psi^{(s,s')}(\mathcal{F})$ is a C-admissible representation of $(G_{\eta_{s'}},\mu)$, where $G_{\eta_{s'}} = \mathrm{Gal}(\overline{\eta}_{s'}/\eta_{s'})$.

III.11.4. We now state a special case of Laumon's ℓ-adic stationary phase method.

Theorem III.11.5 (ℓ-adic stationary phase). *Let $\mathcal{F}$ be a μ-twisted C-sheaf on $\mathbb{P}$, whose fibers at 0 and ∞ vanish, whose restriction to $\mathbb{A}\backslash\{0\}$ is a C-local system, and whose ramification at ∞ is bounded by 1, i.e. the ramification slopes of $\mathcal{F}_{|\eta_\infty}$ are strictly less than 1. Then the C-sheaf $\mathrm{F}_\psi(\mathcal{F}_{|\mathbb{A}})$ on $\mathbb{A}'$ (cf. III.11.2) has the following properties:*

(*i*) *the restriction of $\mathrm{F}_\psi(\mathcal{F}_{|\mathbb{A}})$ to $\mathbb{A}'\setminus\{0\}$ is a C-local system;*

(*ii*) *there is a functorial isomorphism*

$$\mathrm{F}_\psi(\mathcal{F}_{|\mathbb{A}})_{|\overline{\eta}_{\infty'}} \cong \mathrm{F}_\psi^{(0,\infty')}(\mathcal{F}_{|\mathbb{A}_{(0)}}),$$

of C-admissible representations of $(G_{\eta_{\infty'}},\mu)$ (cf. III.11.3);

(*iii*) *the restriction of $\mathrm{F}_\psi(\mathcal{F})$ to $\eta_{0'}$ fits into a functorial exact sequence,*

$$0\to H^1(\mathbb{P}_{\overline{k}},\mathcal{F})\to\mathrm{F}_\psi(\mathcal{F}_{|\mathbb{A}})_{|\overline{\eta}_{0'}}\to\mathrm{F}_\psi^{(\infty,0')}(\mathcal{F}_{|\mathbb{P}_{(\infty)}})\to H^2(\mathbb{P}_{\overline{k}},\mathcal{F})\to 0,$$

of C-admissible representations of $(G_{\eta_{0'}},\mu)$ (cf. III.11.3), where $H^\nu(\mathbb{P}_{\overline{k}},\mathcal{F})$ is considered as an unramified representation of $(G_{\eta_{0'}},\mu)$ by the natural homomorphism $G_{\eta_{0'}}\to G_k$.

By functoriality, Theorem III.11.5 follows from its untwisted special case, i.e. when $\mu = 1$, which follows itself from ([**Ka88**], Th. 3 and 10) or from ([**La87**], 2.3.3.1, 2.3.2), the latter being applied to the perverse sheaf $\mathcal{F}[1]$.

COROLLARY III.11.6. *Let $\mathcal{F}$ be a μ-twisted C-sheaf on $\mathbb{P}$, whose fibers at 0 and ∞ vanish, whose restriction to $\mathbb{A} \setminus \{0\}$ is a C-local system, and whose ramification at ∞ is bounded by 1. Then we have an equality*

$$\varepsilon_{\overline{k}}(\mathbb{P}, \mathcal{F})\langle \chi_{\det(\mathrm{F}_{\psi}^{(\infty,0')}(\mathcal{F}_{|\mathbb{P}_{(\infty)}}))}\rangle(t') = \langle \chi_{\det(\mathrm{F}_{\psi}^{(0,\infty')}(\mathcal{F}_{|\mathbb{A}_{(0)}}))}\rangle(t'^{-1}),$$

of C-admissible maps on G_k, with notation as in III.7.11.

Indeed, Theorem III.11.5(iii) yields

$$\varepsilon_{\overline{k}}(\mathbb{P}, \mathcal{F})\langle \chi_{\det(\mathrm{F}_{\psi}^{(\infty,0')}(\mathcal{F}_{|\mathbb{P}_{(\infty)}}))}\rangle(t') = \langle \chi_{\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}})_{|\overline{\eta}_{0'}})}\rangle(t').$$

Let D be an effective Cartier divisor on $\mathbb{P}'$, supported on $0'$ and ∞', such that the μ-twisted C-local system $\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}}))_{|\mathbb{A}' \setminus \{0'\}}$ of rank 1 (cf. III.11.5(i)) has ramification bounded by D, and let $\chi_{\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}}))}$ be the μ-twisted multiplicative C-local system on $\mathrm{Pic}_k(\mathbb{P}', D)_k$ associated to $\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}}))$ by geometric class field theory, cf. III.5.48. If x_0 is the k-point of $\mathrm{Pic}_k(\mathbb{P}', D)_k$ corresponding to the line bundle $\mathcal{O}([0'])$ trivialized by t'^{-1} at $0'$ and by 1 at ∞', then $\langle \chi_{\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}})_{|\overline{\eta}_{0'}})}\rangle(t')$ is the trace function of the stalk of $\chi_{\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}})_{|\overline{\eta}_{0'}})}$ at t'^{-1} (cf. III.7.11), or alternatively of the stalk of $\chi_{\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}}))}$ at $\overline{x}_0$, by local-global compatibility (cf. III.5.36).

Likewise, Theorem III.11.5(ii) yields

$$\langle \chi_{\det(\mathrm{F}_{\psi}^{(0,\infty')}(\mathcal{F}_{|\mathbb{A}_{(0)}}))}\rangle(t'^{-1}) = \langle \chi_{\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}})_{|\overline{\eta}_{\infty'}})}\rangle(t'^{-1}),$$

and the latter coincides, by local-global compatibility (cf. III.5.36), with the trace function of the stalk of $\chi_{\det(\mathrm{F}_{\psi}(\mathcal{F}_{|\mathbb{A}}))}$ at $\overline{x}_{\infty}$, where x_{∞} is the k-point of $\mathrm{Pic}_k(\mathbb{P}', D)_k$ corresponding to the line bundle $\mathcal{O}([\infty'])$ trivialized by 1 at $0'$ and by t' at ∞'. The conclusion of Proposition III.11.6 then follows from the fact that $x_0 = x_{\infty}$ in $\mathrm{Pic}_k(\mathbb{P}', D)_k$.

III.11.7. Let T be the spectrum of a k-algebra, which is a henselian discrete valuation ring $\mathcal{O}_T$ with residue field k, and let $i : s \to T$ be its closed point. Let π be a uniformizer of $k(\eta)$, and let

$$\pi : T \to \mathbb{A}_{(0)},$$

be the k-morphism sending π to the t. Laumon's cohomological formula for local ε-factors ([**La87**], 3.5.1.1) admits the following extension to the case of an arbitrary perfect base field of positive characteristic p.

THEOREM III.11.8. *Let $\mathcal{F}$ be a potentially unipotent μ-twisted C-sheaf on T, with vanishing fiber at s. Then we have*

$$\varepsilon_{\overline{k}}(T, \mathcal{F}, d\pi) = \langle \chi_{\det(\mathrm{F}^{(0,\infty')}(\pi_*\mathcal{F}))}\rangle(t'^{-1}),$$

with notation as in III.7.11.

Let us prove Theorem III.11.8. We can assume (and we do) that T is the henselization $\mathbb{A}_{(0)}$ of $\mathbb{A}$ at 0, and that $\pi = t$ (cf. III.4.1). Let $\mathcal{F}$ be a potentially unipotent μ-twisted C-sheaf on $\mathbb{A}_{(0)}$, with vanishing fiber at 0, and let $t_{\diamondsuit}\mathcal{F}$ be its Gabber-Katz extension to $\mathbb{A}$ with respect to the uniformizer t (cf. III.4.18). Thus $t_{\diamondsuit}\mathcal{F}$ is tamely ramified at ∞, and its fiber at 0 vanishes. By Theorem III.11.5(ii), we have an isomorphism

$$\mathrm{F}_{\psi}^{(0,\infty')}(\mathcal{F}) \cong \mathrm{F}_{\psi}(t_{\diamondsuit}\mathcal{F})_{|\overline{\eta}_{\infty'}},$$

of C-admissible representations of $(G_{\eta_{\infty'}}, \mu)$. Let $\eta_{\infty'}^{\mathrm{perf}}$ (resp. $\overline{\eta}_{\infty'}^{\mathrm{perf}}$) be the perfection of $\eta_{\infty'}$ (resp. $\overline{\eta}_{\infty'}$), so that $\overline{\eta}_{\infty'}^{\mathrm{perf}}$ is an algebraic closure of $\eta_{\infty'}^{\mathrm{perf}}$. By the proper base change theorem, we have

$$\begin{aligned}\mathrm{F}_\psi(t_\diamondsuit\mathcal{F})_{|\overline{\eta}_{\infty'}} &\cong H^1_c(\mathbb{A}_{\overline{\eta}_{\infty'}}, t_\diamondsuit\mathcal{F}\otimes\mathcal{L}_\psi\{tt'\})\\ &\cong H^1_c(\mathbb{A}_{\overline{\eta}^{\mathrm{perf}}_{\infty'}}, t_\diamondsuit\mathcal{F}\otimes\mathcal{L}_\psi\{tt'\}).\end{aligned}$$

Since the complex $R\Gamma_c(\mathbb{A}_{\overline{\eta}^{\mathrm{perf}}_{\infty'}}, t_\diamondsuit\mathcal{F}\otimes\mathcal{L}_\psi\{tt'\})$ is concentrated in degree 1, we obtain

$$\det\left(\mathrm{F}^{(0,\infty')}_\psi(\mathcal{F})\right)\cong\det\left(\mathrm{F}_\psi(t_\diamondsuit\mathcal{F})_{|\overline{\eta}_{\infty'}}\right)\cong\det\left(R\Gamma_c(\mathbb{A}_{\overline{\eta}^{\mathrm{perf}}_{\infty'}}, t_\diamondsuit\mathcal{F}\otimes\mathcal{L}_\psi\{tt'\})\right)^{-1}. \tag{61}$$

Let us consider the uniformizer $\widetilde{t}=-t't$ on $(\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}})_{(0)}$, and the isomorphism θ from $\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}}$ to itself which sends t to $-t'^{-1}t$. The natural morphism $t:(\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}})_{(0)}\to\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}}$ factors as a composition

$$(\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}})_{(0)}\xrightarrow{\widetilde{t}}\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}}\xrightarrow{\theta}\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}},$$

hence we have isomorphisms

$$\begin{aligned}\theta^{-1}t_\diamondsuit\mathcal{F}&\cong\widetilde{t}_\diamondsuit\mathcal{F}\\ \theta^{-1}\mathcal{L}_\psi\{tt'\}&\cong\mathcal{L}_\psi^{-1},\end{aligned}$$

where $\widetilde{t}_\diamondsuit\mathcal{F}$ is the Gabber-Katz extension of (the restriction to $(\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}})_{(0)}$ of) $\mathcal{F}$ to $\mathbb{A}$ with respect to the uniformizer $\widetilde{t}$ (cf. III.4.18). Hence (61) yields

$$\det\left(\mathrm{F}^{(0,\infty')}_\psi(\mathcal{F})\right)\cong\det\left(R\Gamma_c(\mathbb{A}_{\overline{\eta}^{\mathrm{perf}}_{\infty'}}, \widetilde{t}_\diamondsuit\mathcal{F}\otimes\mathcal{L}_\psi^{-1})\right)^{-1}.$$

By Definition III.9.2 with $\pi=\widetilde{t}$, we obtain that the composition of $\varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{F},dt)$ with the canonical surjective homomorphism r from $G_{\eta^{\mathrm{perf}}_{\infty'}}=G_{\eta_{\infty'}}$ to G_k is given by

$$\begin{aligned}\varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{F},dt)\circ r&=\varepsilon_{\overline{\eta}^{\mathrm{perf}}_{\infty'}}((\mathbb{A}_{\eta^{\mathrm{perf}}_{\infty'}})_{(0)},\mathcal{F},dt)\\ &=\langle\chi_{\det(j^{-1}\mathcal{F})}\rangle\left(\frac{dt}{d\widetilde{t}}\right)\det\left(\mathrm{F}^{(0,\infty')}_\psi(\mathcal{F})\right)\\ &=\langle\chi_{\det(j^{-1}\mathcal{F})}\rangle\left(-t'^{-1}\right)\det\left(\mathrm{F}^{(0,\infty')}_\psi(\mathcal{F})\right),\end{aligned}\tag{62}$$

where $\chi_{\det(j^{-1}\mathcal{F})}$ is the multiplicative C-local system on $\mathrm{Pic}(\mathbb{A}_{(0)},\nu[0])_k$, for some integer ν such that $\det(j^{-1}\mathcal{F})$ has ramification bounded by ν, associated to $\det(j^{-1}\mathcal{F})$ by geometric local class field theory, cf. III.5.45, and $\langle\chi_{\det(j^{-1}\mathcal{F})}\rangle\left(-t'^{-1}\right)$ is the trace function of the stalk of $\chi_{\det(j^{-1}\mathcal{F})}$ at the $\eta_{\infty'}$-point of $\mathrm{Pic}^0(\mathbb{A}_{(0)},\nu[0])_k$ corresponding to the unit $-t'$ (cf. III.5.30). Let us rewrite (62) as

$$\det\left(\mathrm{F}^{(0,\infty')}_\psi(\mathcal{F})\right)=\langle\chi_{\det(j^{-1}\mathcal{F})}\rangle\left(-t'\right)\left(\varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{F},dt)\circ r\right). \tag{63}$$

We now use (63) in order to compute the multiplicative C-local system associated to the determinant of $\mathrm{F}^{(0,\infty')}_\psi(\mathcal{F})$ by geometric local class field theory, cf. III.5.45. Let us consider the local Abel-Jacobi morphism $\Phi_{\eta_{\infty'},t'^{-1}}$ corresponding to the divisor $[\infty']$ on $\mathbb{P}'_{(\infty')}$ (cf. III.5.45). The morphism $\Phi_{\eta_{\infty'},t'^{-1}}$ factors as the composition

$$\eta_{\infty'}\xrightarrow{t'^{-1}}\mathbb{G}_{m,k}\xrightarrow{t\mapsto 1-tt'}\mathrm{Pic}^1(\mathbb{P}'_{(\infty')},[\infty'])_k.$$

Let us consider the k-isomorphism τ from $\mathrm{Pic}^1(\mathbb{P}'_{(\infty')},[\infty'])_k$ to $\mathbb{G}_{m,k}$ which sends a section u (cf. III.5.30) to the section $t'^{-1}u$ of $\mathrm{Pic}^0(\mathbb{P}'_{(\infty')},[\infty'])_k=\mathbb{G}_{m,k}$. For any section t of $\mathbb{G}_{m,k}$, the sections $1-tt'$ and $-tt'$ of $\mathrm{Pic}^1(\mathbb{P}'_{(\infty')},[\infty'])_k$ coincides, hence the following commutative diagram.

$$\begin{tikzcd} \eta_{\infty'} \arrow[r, "\Phi_{\eta_{\infty'},t'^{-1}}"] \arrow[rr, bend right, "-t'^{-1}"'] & \mathrm{Pic}^1(\mathbb{P}'_{(\infty')},[\infty'])_k \arrow[r, "\tau"] & \mathbb{G}_{m,k} \end{tikzcd}$$

Let χ_0 be the restriction of $\chi_{\det(j^{-1}\mathcal{F})}$ to the subgroup $\mathbb{G}_{m,k}$ of $\mathrm{Pic}^0(\mathbb{A}_{(0)},\nu[0])_k$, and let $\widetilde{\chi}$ be the pullback of $\chi_0 \otimes \varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{F},dt)$ by τ, where $\varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{F},dt)$ is considered as a C-local system on $\mathrm{Spec}(k)$, pulled back to $\mathbb{G}_{m,k}$. Then the commutative diagram above, together with (63), shows that $\Phi^{-1}_{\eta_{\infty'},t'^{-1}}\widetilde{\chi}$ is isomorphic to $\det\left(\mathrm{F}_\psi^{(0,\infty')}(\mathcal{F})\right)$.

Moreover, $\widetilde{\chi}$ is the restriction to $\mathrm{Pic}^1(\mathbb{P}'_{(\infty')},[\infty'])_k$ of the unique (up to isomorphism) multiplicative C-local system on $\mathrm{Pic}(\mathbb{P}'_{(\infty')},[\infty'])_k$, still denoted by $\widetilde{\chi}$, whose restriction to $\mathrm{Pic}^0(\mathbb{P}'_{(\infty')},[\infty'])_k = \mathbb{G}_{m,k}$ is given by χ_0, and whose stalk at the k-point t' of $\mathrm{Pic}^1(\mathbb{P}'_{(\infty')},[\infty'])_k$ is given by $\varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{F},dt)$. Thus the multiplicative C-local system associated to $\det\left(\mathrm{F}_\psi^{(0,\infty')}(\mathcal{F})\right)$ by geometric local class field theory (cf. III.5.45) is $\widetilde{\chi}$ (up to isomorphism). We therefore obtain the equality

$$\langle \chi_{\det\left(\mathrm{F}_\psi^{(0,\infty')}(\mathcal{F})\right)}\rangle(t'^{-1}) = \langle\widetilde{\chi}\rangle(t'^{-1}) = \varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{F},dt),$$

which concludes the proof of Theorem III.11.8.

Remark III.11.9. This proof of Theorem III.11.8 incidentally shows that $\det\left(\mathrm{F}_\psi^{(0,\infty')}(\mathcal{F})\right)$ is tamely ramified. The latter fact alternatively follows from the Hasse-Arf theorem and from ([**La87**], 2.4.3(i)(b)), which asserts that the ramification breaks of $\mathrm{F}_\psi^{(0,\infty')}(\mathcal{F})$ are strictly less than 1.

Corollary III.11.10. *Let $\mathcal{F}$ be a potentially unipotent μ-twisted C-sheaf on $\mathbb{P}_{(\infty)}$, with vanishing fiber at ∞ and with ramification bounded by* 1. *Then we have*

$$\varepsilon_{\overline{k}}(\mathbb{P}_{(\infty)},\mathcal{F},dt)\langle\chi_{\det(\mathrm{F}_\psi^{(\infty,0')}(\mathcal{F}))}\rangle(t') = \chi_{\mathrm{cyc}}^{\mathrm{rk}(\mathcal{F})}.$$

We can assume (and we do) that $j^{-1}\mathcal{F}$ is irreducible, so that its geometric monodromy is finite (cf. III.2.48). By Theorem III.4.18 there exists a C-sheaf $\mathcal{G}$ on $\mathbb{A}$, with vanishing fiber at 0, with finite geometric monodromy, which is tamely ramified at 0, and such that $\mathcal{G}_{|\mathbb{P}_{(\infty)}}$ is isomorphic to $\mathcal{F}$. Then Corollary III.11.6 and Theorem III.11.8 yield

$$\varepsilon_{\overline{k}}(\mathbb{A},\mathcal{G})\langle\chi_{\det(\mathrm{F}_\psi^{(\infty,0')}(\mathcal{F}))}\rangle(t') = \varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{G},dt), \tag{64}$$

while the product formula for μ-twisted C-sheaves with finite geometric monodromy (cf. III.10.3) yields

$$\varepsilon_{\overline{k}}(\mathbb{A},\mathcal{G}) = \chi_{\mathrm{cyc}}^{-\mathrm{rk}(\mathcal{F})}\varepsilon_{\overline{k}}(\mathbb{A}_{(0)},\mathcal{G},dt)\varepsilon_{\overline{k}}(\mathbb{P}_{(\infty)},\mathcal{F},dt). \tag{65}$$

The conclusion of Corollary III.11.10 then follows by combining (64) with (65).

III.11.11. We now prove Theorem III.11.1. Its conclusion holds when $\mathcal{F}$ has finite geometric monodromy by III.10.3. In particular, it holds for the constant sheaf C. Thus Theorem III.11.1 is equivalent to the formula

$$\frac{\varepsilon_{\overline{k}}(X,\mathcal{F}_1)}{\varepsilon_{\overline{k}}(X,\mathcal{F}_2)} = \prod_{x\in|X|}\delta_{x/k}^{a(X_{(x)},\mathcal{F}_{1|X_{(x)}})-a(X_{(x)},\mathcal{F}_{2|X_{(x)}})}\mathrm{Ver}_{x/k}\left(\frac{\varepsilon_{\overline{x}}(X_{(x)},\mathcal{F}_{1|X_{(x)}},\omega_{|X_{(x)}})}{\varepsilon_{\overline{x}}(X_{(x)},\mathcal{F}_{2|X_{(x)}},\omega_{|X_{(x)}})}\right),$$

for any $\mathcal{F}_1,\mathcal{F}_2$ satisfying the assumptions of Theorem III.11.1 (twisted by possibly different cocycles), with the same generic rank. If $f : X \to \mathbb{P}$ is a finite generically étale k-morphism, then the latter formula holds for $(X, \mathcal{F}_1, \mathcal{F}_2)$ if and only if it does for $(\mathbb{P}, f_*\mathcal{F}_1, f_*\mathcal{F}_2)$. Thus the conclusion of Theorem III.11.1 holds in general if and only if it holds for $X = \mathbb{P}$. If $X = \mathbb{P}$, then there exists a non empty open subscheme U of $\mathbb{A}$ such that $\mathcal{F}_{1|U}$ and $\mathcal{F}_{2|U}$ are C-local systems, and we can find a polynomial h in $k[t]$ whose vanishing locus in $\mathbb{A}$ is the complement of U in $\mathbb{A}$. The k-morphism $\theta : \mathbb{P} \to \mathbb{P}$ which sends t to $(t - h(t)^{-p})^{-1}$ is finite, and induces a finite étale morphism from U onto $\mathbb{P} \setminus \{0\}$. By replacing $(\mathbb{P}, \mathcal{F}_1, \mathcal{F}_2)$ with $(\mathbb{P}, \theta_*\mathcal{F}_1, \theta_*\mathcal{F}_2)$, we can thus assume that the restrictions of $\mathcal{F}_1$ and $\mathcal{F}_2$ to $\mathbb{P} \setminus \{0\}$ are C-local systems.

Remark III.11.12. The last reduction to C-sheaves on $\mathbb{P}$ with ramification concentrated on a single point is due to Katz, cf. ([**Ka88**], Lemma 16).

In order to prove Theorem III.11.1, we can thus assume (and we do) that $X = \mathbb{P}$ and that $\mathcal{F}$ is a C-sheaf on $\mathbb{P}$, whose restriction to $\mathbb{A} \setminus \{0\}$ is a C-local system, which is unramified at ∞, and which is potentially unipotent at 0. We can further assume (and we do) that the fibers of $\mathcal{F}$ at 0 and ∞ vanish. The formula to be proved is then

$$\varepsilon_{\overline{k}}(\mathbb{P}, \mathcal{F}) = \chi_{\mathrm{cyc}}^{-\mathrm{rk}(\mathcal{F})} \varepsilon_{\overline{k}}(\mathbb{A}_{(0)}, \mathcal{F}, dt) \varepsilon_{\overline{k}}(\mathbb{P}_{(\infty)}, \mathcal{F}, dt),$$

By Corollary III.11.6, we have

$$\varepsilon_{\overline{k}}(\mathbb{P}, \mathcal{F}) = \langle \chi_{\det(\mathrm{F}_\psi^{(0,\infty')}(\mathcal{F}_{|\mathbb{A}_{(0)}}))} \rangle (t'^{-1}) \langle \chi_{\det(\mathrm{F}_\psi^{(\infty,0')}(\mathcal{F}_{|\mathbb{P}_{(\infty)}}))} \rangle (t')^{-1},$$

and the conclusion then results from the formulas

$$\langle \chi_{\det(\mathrm{F}_\psi^{(0,\infty')}(\mathcal{F}_{|\mathbb{A}_{(0)}}))} \rangle (t'^{-1}) = \varepsilon_{\overline{k}}(\mathbb{A}_{(0)}, \mathcal{F}, dt),$$

$$\langle \chi_{\det(\mathrm{F}_\psi^{(\infty,0')}(\mathcal{F}_{|\mathbb{P}_{(\infty)}}))} \rangle (t')^{-1} = \chi_{\mathrm{cyc}}^{-\mathrm{rk}(\mathcal{F})} \varepsilon_{\overline{k}}(\mathbb{P}_{(\infty)}, \mathcal{F}, dt),$$

which follow respectively from Theorem III.11.8 and from Corollary III.11.10.

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Titre : Facteurs locaux ℓ-adiques

Mots clés : Géométrie algébrique, Cohomologie étale, Facteurs epsilon

Résumé : Cette thèse est composée de deux parties indépendantes. Dans la première, nous donnons une démonstration alternative du théorème d'aplatissement par éclatements de Raynaud-Gruson. Celle-ci repose sur la construction et l'étude de certains espaces valuatifs, et nous permet de dégager la notion de Φ-anneau, qui fournit un substitut algébrique aux anneaux topologiques adiques : la notion correspondante de Φ-schéma est aux schémas ce que les espaces rigides sont aux schémas formels.
Dans une seconde partie, nous nous inspirons de travaux de Laumon et de Deligne pour démontrer l'existence de facteurs epsilon locaux dans un cadre géometrique. Nous démontrons ensuite, en usant de la méthode la phase stationnaire ℓ-adique, une formule du produit pour le déterminant de la cohomologie d'un faisceau ℓ-adique sur une courbe en caractéristique $p \neq \ell$ positive : cela étend des résultats précédemment connus pour un corps de base fini. Parmi les outils utilisées figure la théorie du corps de classes géométrique, dont nous donnons une démonstration s'inspirant de l'approche de Deligne pour le cas non ramifié.

Title : Local factors in ℓ-adic cohomology

Keywords : Algebraic geometry, Étale cohomology, Epsilon factors

Abstract : This thesis is composed of two independent parts. In the first part, we give an alternative demonstration of Raynaud-Gruson's theorem regarding flattening by blow-ups. This is based on the construction and study of certain valuative spaces, and motivates the introduction of the notion of Φ-ring, which provides an algebraic substitute for adic topological rings: the corresponding Φ-schemes are to schemes what rigid spaces are to formal schemes.
In the second part, we draw inspiration from the work of Laumon and Deligne to prove the existence of local ε-factors in a geometric setting. We then prove, using the ℓ-adic stationary phase method, a product formula for the determinant of the cohomology of a ℓ-adic sheaf on a curve of positive characteristic $p \neq \ell$: this extends results previously known for a finite base field. Class field theory is replaced in our approach by its geometric counterpart, of which we give a demonstration based on Deligne's approach to the unramified case.

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