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THE CENTRAL LIMIT THEOREM FOR STATIONARY ASSOCIATED SEQUENCES

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Abstract. We study the problem of convergence in distribution of a suitably normalized sum of stationary associated random variables. We focus on the infinite variance case. New results are announced.

1. Introduction

Let $(X_n)_{n\in\mathbf{N}}$ be a stationary sequence of associated random variables i.e.

$$\mathrm{Cov}\,\big(f(X_1,\dots,X_n),g(X_1,\dots,X_n)\big) \geqq 0,$$

for all coordinatewise non-decreasing functions f, g and all $n\in\mathbf{N}$. We refer to Esary et al. [11] for this notion as well as for its main properties. Association describes the positive dependence structure of several models from reliability theory (cf. Barlow and Proschan [2]), statistical physics (cf. Newman [21]) and percolation theory (cf. Cox and Grimmett [7]). Let $S_n=X_1+\dots+X_n$. The purpose of this paper is to give sufficient conditions ensuring the existence of numerical sequences A_n, B_n for which the quantity

$$\frac{S_n-B_n}{A_n} \tag{1}$$

converges in distribution. We first review the existing results.

Independent observations. For i.i.d. sequences, which are also associated (cf. $(\mathcal{P}_5)$ of Esary et al. [11]), the above mentioned problem is completely solved and the limit distribution of (1) is Gaussian or a non-Gaussian stable law (cf. Feller [12] or Araujo and Giné [1], Ch. 2). The condition

$$H:x\to\mathbf{E}\big(X_1^2\mathbf{1}_{|X_1|\leqq x}\big)\quad\text{is a slowly varying function,} \tag{2}$$

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is necessary and sufficient for the convergence in distribution to a non-degenerate normal law of the quantity in (1) with $B_n = n\mathbf{E}\big(X_1 \mathbf{1}_{|X_1| \leqq t_n}\big)$ and $A_n = \sqrt{nH(t_n)}$, where the truncation sequence (t_n) is defined by

$$(3) \qquad t_n = \sup\left\{x > 0,\ \frac{H(x)}{x^2} \geqq \frac{1}{n}\right\} \quad \text{i.e.} \quad \frac{nH(t_n)}{t_n^2} \to 1 \quad \text{as} \quad n \to +\infty.$$

Now the following two conditions:

$$(4) \qquad H : x \to \mathbf{E}\big(X_1^2 \mathbf{1}_{|X_1| \leqq x}\big) \quad \text{is regularly varying of order} \quad 2 - \alpha,$$

for some $\alpha \in]0, 2[$ and

$$(5) \quad \frac{\mathbf{P}(X_1 > x)}{\mathbf{P}\big(|X_1| > x\big)} \to \frac{c_1}{c_1 + c_2}, \quad \frac{\mathbf{P}(X_1 \leqq -x)}{\mathbf{P}\big(|X_1| > x\big)} \to \frac{c_2}{c_1 + c_2}, \quad \text{as} \quad x \to +\infty$$

where $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 > 0$, are necessary and sufficient for the convergence in distribution to the stable law $c\,\mathrm{Pois}\big(\mu(c_1, c_2, \alpha)\big)$. In that case, the normalizing sequence (A_n) is such that

$$(6) \qquad n\frac{H(A_n)}{A_n^2} \to 1 \quad \text{as} \quad n \to +\infty,$$

while the centering sequence is given by $B_n = n\mathbf{E}\big(X_1 \mathbf{1}_{|X_1| \leqq \tau A_n}\big)$, for some $\tau > 0$.

Dependent observations. For strong mixing sequences with finite variance, convergence to the normal law may hold with the normalization $A_n = \sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$ rather than $\sqrt{\mathrm{Var}\, S_n}$ (Merlevède and Peligrad [20], Dehling et al. [10]). We refer also to Peligrad [24], Berkes and Philipp [4] for analogous problems under ϕ-mixing condition. Bradley [6] proved a central limit theorem for ρ-mixing sequences with infinite variance. Lin [17] proved it under m-dependence.

Associated observations. Stationary associated sequences $(X_n)_{n \in \mathbf{N}}$ with finite variance and fulfilling

$$(7) \qquad L_1(n) := \mathbf{E}(X_1^2) + 2\sum_{r=2}^{n} \mathrm{Cov}\,(X_1, X_r) \to \sigma^2 < +\infty \quad \text{as} \quad n \to +\infty,$$

satisfy

$$(8) \qquad \frac{S_n - B_n}{A_n} \Rightarrow \mathcal{N}(0, 1),$$

where $\Rightarrow$ denotes convergence in distribution, $\mathcal{N}(0,1)$ is the standard Gaussian distribution, $B_n = n\mathbf{E}(X_1)$ and $A_n = \sqrt{n\sigma^2}$ (Newman and Wright [22]). Herrndorf [13] provides a sequence of centered stationary associated random variables with finite variance, such that $L_1(n) \sim \log\, n$ but (8) does not hold for the standardization $A_n = \sqrt{nL_1(n)}$. At the end of his paper, he conjectures that *possibly there exists a different standardization which yields asymptotic normality.* Recently, Lewis [16] gives a necessary and sufficient condition ensuring (8) with the normalization $A_n = \sqrt{nL_1(n)}$ when $L_1(n)$ is a slowly varying function. As far as we know, in the infinite variance setting, there is only one result concerning the convergence of (1) to the Gaussian law (see Matula [19]): the convergence (8) holds with $B_n = 0$ and for some sequence (A_n) if the centered associated sequence (X_n) fulfills $\mathcal{L}(X_1) \in \mathcal{D}(A_n)$ and

$$\lim_{n\to+\infty} \frac{1}{A_n^2} \sum_{1\leqq k<m\leqq n} \operatorname{Cov}(X_k, X_m) = 0, \tag{9}$$

as soon as $\mathbf{E}|X_k X_m| < +\infty$, for $k \neq m$. Here $\mathcal{D}(A_n)$ denotes the domain of attraction of the standard normal law.

Dabrowski and Jakubowski [8] were the first to study the problem of convergence to a non-Gaussian stable limit under association. Let us recall briefly their result: if the stationary associated sequence (X_i) belongs to the domain of strict normal attraction of a jointly strictly α-stable process (Y_i), $\alpha \in\,]0,2[$ (see the definition in Dabrowski and Jakubowski [8], pp. 4–5; this notion depends not only on α, but also on a stable process (Y_i), this is a terminology differing from the usage in the i.i.d. case) and if for some $A > 0$

$$\sum_{k=2}^{\infty} I_\alpha^A(X_1, X_k) < +\infty, \tag{10}$$

where

$$I_\alpha^A(X_i, X_j) = \sup_{a\geqq A} a^{\alpha-2} \int_{-a}^{a}\int_{-a}^{a} \operatorname{Cov}(\mathbf{1}_{X_i\leqq x}, \mathbf{1}_{X_j\leqq y})\, dx\, dy,$$

then $S_n/n^{1/\alpha}$ converges in distribution to a strictly α-stable law. The more general situation i.e. when the limit law is $c\,\mathrm{Pois}\big(\mu(c_1, c_2, \alpha)\big)$ was studied by Jakubowski [15].

In this paper, we intend to discuss the convergence in distribution of a suitably normalized sum of associated r.v.'s under one of the conditions

$$\mathbf{E}(X_1^2)\begin{cases} <+\infty \quad \text{and} \quad \displaystyle\sum_{n=2}^{+\infty}\mathrm{Cov}\,(X_1,X_n)=+\infty, \\ =+\infty \quad \text{and convergence to the normal law,} \\ =+\infty \quad \text{and} \quad \displaystyle\sum_{k=2}^{\infty} I_\alpha^A(X_1,X_k)=+\infty. \end{cases}$$

The rest of the paper is organized as follows. In Section 2, we make several extensions to the above results. Theorems 1 and 2 below unify the Gaussian and the non-Gaussian cases: the limit distribution depends on the behavior of the marginal law, more generally on the law of a partial sum S_p i.e. if it belongs to the domain of attraction of a Gaussian or a non-Gaussian stable law. In Section 3, we prove the results. With additional technical details, the proofs involve essentially Newman's lemma [cf. Lemma 1]. An appendix is dedicated to the proofs of some intermediate results.

2. Main results

Define, for $x>0$, $y\in\mathbf{R}$,

$$f_x(y)=(x\wedge y)\vee(-x).$$

The function f_x is non-decreasing and will play an important role in the sequel, since its monotonicity preserves association (cf. $(\mathcal{P}_4)$ of Esary et al. [11]). The dependence structure of an associated sequence (X_n) is conveniently described by the truncated covariance function, defined for $x>0$ by

$$G_i(x)=\mathrm{Cov}\left(f_x(X_1),f_x(X_i)\right).$$

The association of the sequence (X_n) implies that the function $x\to G_i(x)$ is positive and non-decreasing with respect to x for every fixed i. Moreover, in the finite variance case

$$G_i(x)\leqq G_i(+\infty)=\mathrm{Cov}\,(X_1,X_i).$$

Instead of the truncated moment H defined in (2), we consider

$$h(x):=\mathbf{E}\big(f_x^2(X_1)\big)=\mathbf{E}\big(x\wedge|X_1|\big)^2=2\int_0^x t\mathbf{P}\big(|X_1|>t\big)\,dt.$$

Clearly

$$h(x) = H(x) + x^2 \mathbf{P}\bigl(|X_1| > x\bigr).$$

The slowly varying property of H is equivalent to that of h and in such a case $h \sim H$ (cf. Lemma 3 in Rosalsky [25]). But from a technical point of view, working with the function h is much easier than with H. We can now state our main results.

THEOREM 1. *Let $(X_n)_{n\in\mathbf{N}}$ be a stationary sequence of associated r.v.'s. Assume that there exist sequences T_n, A_n, $p_n \leqq n$, all tending to infinity with n, and a characteristic function ϕ such that the conditions*

$$\lim_{n\to+\infty} n\mathbf{P}\bigl(|X_1| \geqq \varepsilon T_n\bigr) = 0, \tag{11}$$

$$\lim_{n\to+\infty} \left\{ \mathbf{E}\exp\left(it \frac{S_{p_n} - p_n \mathbf{E}\bigl(f_{\varepsilon T_n}(X_1)\bigr)}{A_n} \right) \right\}^{[n/p_n]} = \phi(t), \tag{12}$$

$$\lim_{n\to+\infty} \frac{1}{A_n^2} \left(\operatorname{Var} \bar{S}_n - \left[\frac{n}{p_n}\right] \operatorname{Var} \bar{S}_{p_n} \right) = 0, \tag{13}$$

hold for some $\varepsilon > 0$, where $\bar{S}_m = \sum_{i=1}^m \bigl[f_{\varepsilon T_n}(X_i) - \mathbf{E}\bigl(f_{\varepsilon T_n}(X_i)\bigr)\bigr]$ and square bracket denotes integer part. Then the characteristic function of $A_n^{-1}\bigl(S_n - n\mathbf{E}\bigl(f_{\varepsilon T_n}(X_1)\bigr)\bigr)$ converges to ϕ.

REMARKS. 1. If all the assumptions of Theorem 1 hold except (12) and (13) which are replaced by

$$\lim_{p\to+\infty} \limsup_{n\to+\infty} \left| \left\{ \mathbf{E}\exp\left(it \frac{S_p - p\mathbf{E}\bigl(f_{\varepsilon T_n}(X_1)\bigr)}{A_n} \right) \right\}^{[n/p]} - \phi(t) \right| = 0,$$

$$\lim_{p\to+\infty} \limsup_{n\to+\infty} \frac{1}{A_n^2} \left(\operatorname{Var} \bar{S}_n - \left[\frac{n}{p}\right] \operatorname{Var} \bar{S}_p \right) = 0,$$

then the conclusion of Theorem 1 still holds and Newman's, Dabrowski and Jakubowski's central limit theorems follow from Theorem 1.

2. Condition (13) is close to Condition B of Jakubowski [15].

If Condition (12) of Theorem 1 holds only for some constant sequence $p_n =: p$, then we obtain

THEOREM 2. *Let $(X_n)_{n\in\mathbf{N}}$ be a stationary sequence of associated r.v.'s. Suppose that there exist a positive integer p, two sequences T_n and A_n both* *tending to infinity with* n*, a characteristic function* ϕ_p *such that, for some* $\varepsilon > 0$*, the limits*

$$\lim_{n\to+\infty}\left\{\mathbf{E}\exp it\left(\frac{S_p - p\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)}{A_n}\right)\right\}^{[n/p]} = \phi_p(t) \tag{14}$$

and

$$\lim_{n\to\infty}\frac{1}{A_n^2}\operatorname{Var}\big(f_{\varepsilon T_n}(X_1)\big) = 0, \qquad \lim_{n\to\infty}\frac{n}{A_n^2}\sum_{i=2}^{n} G_i(\varepsilon T_n) = 0 \tag{15}$$

together with (11) *hold. Then the characteristic function of*

$$A_n^{-1}\big(S_n - n\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)\big)$$

converges to ϕ_p.

Remarks. 1. We suppose that the requirements of Theorem 2 hold, but instead of (11) and (14) we assume

$$\lim_{a\to+\infty}\ \limsup_{n\to+\infty} n\mathbf{P}\big(|X_1| \geqq aT_n\big) = 0, \tag{16}$$

and

$$\lim_{a\to+\infty}\ \limsup_{n\to+\infty}\left|\left\{\mathbf{E}\exp it\left(\frac{S_p - p\mathbf{E}\big(f_{aT_n}(X_1)\big)}{A_n}\right)\right\}^{[n/p]} - \phi_p(t)\right| = 0. \tag{17}$$

Then we deduce, arguing as in the proof of Theorem 2, that

$$\lim_{a\to+\infty}\ \limsup_{n\to+\infty}\left|\left\{\mathbf{E}\exp it\left(\frac{S_n - n\mathbf{E}\big(f_{aT_n}(X_1)\big)}{A_n}\right)\right\} - \phi_p(t)\right| = 0.$$

2. Let us note that, in the case when $p = 1$, it is not necessary to suppose the first limit in (15) (we refer the reader to the proof of Theorem 2).

3. As noticed by Jakubowski [15], there exists a 1-dependent associated sequence strictly stationary for which (14) holds with $p = 1$ but not with $p = 2$.

4. *Rates of convergence in the weak law of large numbers.* Suppose here that $\mathbf{E}X_1 = 0$ and that $\mathbf{E}|X_1|^q < +\infty$ for some $q \in [1, 2[$. Then it is not hard to deduce that $\lim_{a\to+\infty}\ \lim_{n\to+\infty} n^{1-1/q}\mathbf{E}\big(f_{an^{1/q}}(X_1)\big) = 0$, that Condition

(16) and the first limit in (15) hold with $T_n = A_n = n^{1/q}$. We conclude, using the Marcinkiewicz–Zygmund strong laws for independent sequences (cf. Baum and Katz [3]) and some standard estimations, that (17) holds with $p = 1$, $\phi_1(t) = 1$ and $A_n = n^{1/q}$. In that case, the second limit in Condition (15) can be written as

$$\lim_{n\to\infty} \frac{1}{n^{2/q-1}} \sum_{i=2}^{n} G_i(\varepsilon n^{1/q}) = 0, \tag{18}$$

and leads to rates of convergence in the weak law of large numbers for the stationary associated sequence (X_n) (recall that

$$\lim_{a\to+\infty} \lim_{n\to+\infty} n^{1-1/q}\mathbf{E}\big(f_{an^{1/q}}(X_1)\big) = 0).$$

5. *Association and* m*-dependence.* If the associated sequence is m-dependent, then the conditions of Theorem 2 (with $p = 1$) are close to that of Lin [17].

In the sequel, we discuss special cases of Theorems 1 and 2.

2.1. $\mathbf{E}X_1^2 < +\infty$ **and** $\sum_{j=2}^{+\infty} \mathrm{Cov}\,(X_1, X_j) = +\infty$**.** In this situation the normalization A_n may or not be $\sqrt{\mathrm{Var}\,(S_n)}$.

Proposition 1. *Let* $(X_n)_{n\in\mathbf{N}}$ *be a stationary sequence of associated and centered r.v.'s fulfilling* $\mathbf{E}(X_1^2) < \infty$ *and* $\mathrm{Var}\, S_n = nL(n)$ *where* L *is a slowly varying function. Let* (A_n) *and* (p_n) *be two sequences tending to infinity with* n *such that*

$$\liminf_{n\to+\infty} \frac{A_n^2}{\mathrm{Var}\,(S_n)} > 0, \tag{19}$$

$$\lim_{n\to+\infty} \frac{n}{p_n} = +\infty, \qquad \lim_{n\to+\infty} \frac{L(p_n)}{L(n)} = 1, \tag{20}$$

$$\lim_{n\to+\infty} \left\{\mathbf{E}\left(\exp it\frac{S_{p_n}}{A_n}\right)\right\}^{[n/p_n]} = \exp\left(-\frac{t^2}{2}\right), \quad \textit{for any} \quad t \in \mathbf{R}. \tag{21}$$

Then (8) *holds for this sequence* (A_n) *and for* $B_n = 0$.

Remarks. 1. Let us explain how to deduce Proposition 1 from Theorem 1. We deduce from $\mathbf{E}(X_1) = 0$, $\mathbf{E}(X_1^2) < +\infty$ and the Chebyshev inequality that

$$\frac{n}{A_n}\big|\,\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)\big| \leqq \frac{n}{\varepsilon T_n A_n}\mathbf{E}X_1^2. \tag{22}$$

Choosing T_n to grow very rapidly, Condition (11) together with

$$\lim_{n\to+\infty}\frac{n}{A_n}\left|\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)\right| = 0 \tag{23}$$

will clearly hold and the order of magnitude of $\operatorname{Var}\bar{S}_n$ and $\operatorname{Var}\bar{S}_{p_n}$ will be the same as that of $\operatorname{Var}S_n$ and $\operatorname{Var}S_{p_n}$ and thus (13) will follow from (19) and (20). Condition (12) with $\phi(t)=e^{-t^2/2}$ follows from (23) and (21). Those facts together with (23) prove Proposition 1 from Theorem 1.

2. The situation when $A_n^2=\operatorname{Var}(S_n)$ was studied by Lewis [16]: if the conditions in the first sentence of Proposition 1 hold, then the following two statements are equivalent:

$$\left(\frac{S_n^2}{\operatorname{Var}(S_n)}\right)_{n\geqq 1} \text{ is uniformly integrable} \Leftrightarrow \frac{S_n}{\sqrt{\operatorname{Var}(S_n)}} \Rightarrow \mathcal{N}(0,1).$$

The first part of the equivalence is also deduced from Proposition 1 via the Lindeberg's theorem which guarantees Condition (21) (cf. Theorem 7.2 of Billingsley [5]).

3. Let us note that Proposition 1 remains true if Conditions (21) and (19) are fulfilled for n belonging to an infinite set of integers Q. In that case the convergence in distribution of $\frac{S_n}{A_n}$ holds when $n\in Q$ and $n\to+\infty$.

We now give sufficient conditions for the normalizing sequence to be equal to $\sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$. Note that $\sqrt{\frac{2}{\pi}}=\mathbf{E}|Z|$, where Z is a r.v.'s distributed as the standard Gaussian law.

Corollary 1. *Let $(X_n)_{n\in\mathbf{N}}$ be a stationary sequence of associated and centered r.v.'s. Suppose that $\mathbf{E}(X_1^2)<\infty$, that $\operatorname{Var}S_n=nL(n)$ where L is a slowly varying function and that* (19) *holds with $A_n=\sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$. If moreover*

$$\mathbf{E}|S_n| = \sqrt{n}\tilde{L}(n), \tag{24}$$

where $\tilde{L}$ is a slowly varying function, then (8) *holds with $B_n=0$ and $A_n=\sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$.*

We prove Corollary 1 in Section 3. Let us note that it follows from Proposition 1 and its proof by adapting Dehling et al. [10] methods.

2.2. $\mathbf{E}X_1^2=+\infty$ and convergence to the normal law. If Condition (2) holds, then Conditions (11), (14) and the first limit in (15) are fulfilled with $T_n=A_n=t_n$, $p=1$, $\varepsilon=1$, where t_n is defined by (3) (cf. Lemma 3.1 of Bradley [6] for some limits properties of this truncate sequence (t_n)). Theorem 2 yields then:

PROPOSITION 2. *Let* $(X_n)_{n\in\mathbf{N}}$ *be a stationary sequence of associated r.v.'s. Suppose that* $\mathbf{E}(X_1^2)=+\infty$ *and that*

$$h: x\to \mathbf{E}\big(x\wedge |X_1|\big)^2 \quad \textit{is a slowly varying function.} \tag{25}$$

If moreover

$$\lim_{n\to\infty}\frac{1}{h(t_n)}\sum_{i=2}^{n}G_i(t_n)=0, \tag{26}$$

then

$$\frac{S_n-n\mathbf{E}\big(X_1\mathbf{1}_{|X_1|\leqq t_n}\big)}{\sqrt{nh(t_n)}}\Rightarrow \mathcal{N}(0,1),$$

where t_n *is defined as in* (3).

REMARKS. 1. Since $G_i(t_n)\leqq \operatorname{Cov}(X_1,X_i)$, Matula's result is then deduced from the previous proposition.

2. An analogue of Proposition 2 can also be deduced under Conditions (4), (5), (15) with $T_n=A_n$. The normalized and centered sequences are defined as in (6). The limit law is in that case $c\,\mathrm{Pois}\,\big(\mu(c_1,c_2,\alpha)\big)$.

3. Let $(Z_i)_{i\in\mathbf{N}}$ be a sequence of independent and identically distributed (i.i.d.) random variables, such that H_{Z_1} is a slowly varying function (for a random variable X_1, H_{X_1} denotes the truncate moment defined by (2)). The requirements of Proposition 2 are fulfilled by the stationary 2-dependent sequence $(X_i):=(Z_iZ_{i+1})_i$. In fact it follows from Theorem 1 of Maller [18] that $H_{Z_1Z_2}$ is a slowly varying function as soon as H_{Z_1} fulfills this property. Condition (25) is thus satisfied by $X_1=Z_1Z_2$. Since $G_i(x)=0$, for $i\geqq 3$, Condition (26) is thus reduced to

$$\lim_{n\to+\infty}\frac{n}{\beta_n^2}G_2(\beta_n)=0, \tag{27}$$

where the normalizing coefficients β_n fulfill

$$\lim_{n\to+\infty}n\beta_n^{-2}\mathbf{E}\Big(Z_1^2Z_2^2\mathbf{1}_{|Z_1Z_2|\leqq\beta_n}\Big)=1,\quad \lim_{n\to+\infty}n\mathbf{P}\Big(|Z_1Z_2|\geqq\beta_n\Big)=0. \tag{28}$$

Now some standard estimations based on (28) prove that Condition (27) is equivalent to

$$\lim_{n\to\infty}\frac{n}{\beta_n^2}\mathbf{E}\Big(Z_1Z_2\mathbf{1}_{|Z_1Z_2|\leqq\beta_n}Z_2Z_3\mathbf{1}_{|Z_2Z_3|\leqq\beta_n}\Big)=0. \tag{29}$$

The last limit is proved by Davis and Resnick [9] (cf. the proof of their Condition (2.5)).

Now, the association property of the sequence (X_n) can be deduced if one supposes that the i.i.d. sequence (Z_i) is of positive random variables. In fact in this situation X_i can be written as a non-decreasing function of the associated vector (Z_i, Z_{i+1}) and the conclusion follows from properties $(\mathcal{P}_4)$ and $(\mathcal{P}_5)$ of Esary et al. [11].

2.3. $\mathbf{E}X_1^2 = +\infty$ **and** $\sum_{j=2}^{+\infty} I_\alpha^A(X_1, X_j) = +\infty$**.** In this section, we apply Theorem 2 (cf. also Remark 1 below Theorem 2) with $p = 1$. In this situation, convergence in distribution to a non-degenerate stable law may hold:

COROLLARY 2. *Let* $(X_n)_{n\in\mathbf{N}}$ *be a stationary sequence of centered associated r.v.'s. Suppose that* $\mathcal{L}(X_1) \in \tilde{\mathcal{D}}(n^{1/\alpha})$*, where* $\tilde{\mathcal{D}}(n^{1/\alpha})$ *denotes the domain of attraction of a non-degenerate* α*-stable law* μ_α *(*$\alpha \in]1,2[$*). If, for some* $\varepsilon > 0$

$$\lim_{n\to+\infty} n^{1-2/\alpha} \sum_{i=1}^{n} G_i(\varepsilon n^{1/\alpha}) = 0,$$

then

$$\frac{S_n}{n^{1/\alpha}} \Longrightarrow \mu_\alpha.$$

REMARKS. 1. Let us give more details about the proof of Corollary 2. We apply Theorem 2 (cf. also Remark 1 below Theorem 2) with $p = 1$, $A_n = T_n = n^{1/\alpha}$. Since $\mathcal{L}(X_1)$ is in the domain of attraction of a non-degenerate α-stable law, there is a constant C such that $\mathbf{P}\big(|X_1| > x\big) \leqq Cx^{-\alpha}$. Hence Condition (16) and the first limit in (15) are satisfied with $T_n = A_n = n^{1/\alpha}$. It is not hard to check that the requirements of Theorem 2 are satisfied from the assumptions of Corollary 2. Let us just precise the limiting behavior of $B_n/A_n = n^{1-1/\alpha}\mathbf{E}f_{aT_n}(X_1)$. Some standard estimations based on $\mathbf{E}(X_1) = 0$ and $\mathbf{P}\big(|X_1| > x\big) \leqq Cx^{-\alpha}$ prove that $\lim_{a\to+\infty} \lim_{n\to+\infty} B_n/A_n = 0$.

2. Suppose that there exists a positive function $L : \lim_{x\to+\infty} L(x) = 0$ and a sequence $(\psi_i)_i$ for which $G_i(x) \leqq \psi_i x^{2-\alpha} L(x)$. If

$$\lim_{n\to+\infty} L(n^{1/\alpha}) \sum_{i=1}^{n} \psi_i = 0,$$

then a stable limit theorem holds, while it is possible to have

$$\sum_{j=2}^{+\infty} I_\alpha^A(X_1, X_j) = +\infty.$$

This situation can occur if one supposes that $\mathbf{P}\big(|X_1| \geqq x\big) \sim x^{-\alpha}L(x)$ where L is a slowly varying function, non-increasing, locally bounded in $[0,+\infty[$, fulfilling $\lim_{x\to+\infty} L(x) = 0$. In such a case there exists a positive constant c_α depending only on α such that

$$G_i(v) \leqq c_\alpha L(v) v^{2-\alpha}.$$

Let us justify the last bound. Clearly $G_i(v) \leqq \operatorname{Var}\big(f_v(X_1)\big) = G_1(v)$. Now write as in Proposition 2.10 of Dabrowski and Jakubowski [8]:

$$G_1(v) = \int_{-v}^{v}\int_{-v}^{v} \mathbf{P}(X_1 \geqq x \vee y)\mathbf{P}(X_1 < x \wedge y)\,dx\,dy$$

$$\leqq c\int_{-v}^{v}\int_{-v}^{v} \min\Big(L\big(|x\wedge y|\big)\,|x\wedge y|^{-\alpha}, L\big(|x\vee y|\big)\,|x\vee y|^{-\alpha}\Big)\,dx\,dy$$

$$\leqq c\int_{0}^{v}\int_{0}^{v} L^{1/2}(x)|x|^{-\alpha/2}L^{1/2}(y)|y|^{-\alpha/2}\,dx\,dy \leqq c_\alpha L(v)v^{2-\alpha}.$$

Let $\psi_i = \sup_{v>0} \frac{v^{\alpha-2}}{L(v)} G_i(v)$. Clearly $I_\alpha^A(X_1, X_i) \leqq \text{const.} \times \psi_i$ and under a suitable rate of divergence of $\sum_{i=1}^n \psi_i$, convergence in distribution of $\frac{S_n}{n^{1/\alpha}}$ holds while $\sum_{i=1}^n I_\alpha^A(X_1, X_i)$ can diverge.

3. Proofs

We first recall Newman's Lemma which is the fundamental tool in all the subsequent derivations.

LEMMA 1 (Newman [21]). *If (X_n) is a sequence of associated r.v.'s with finite variance, then for any $t \in \mathbf{R}$ there holds:*

$$\left|\mathbf{E}\big(\exp(itS_n)\big) - \prod_{j=1}^n \mathbf{E}\big(\exp(itX_j)\big)\right| \leqq \frac{t^2}{2}\left[\operatorname{Var} S_n - \sum_{i=1}^n \mathbf{E}X_i^2\right].$$

The proofs of our results will follow from the following lemma.

LEMMA 2. *Let $(X_n)_{n\in\mathbf{N}}$ be a stationary sequence of associated random variables. Let $S_n = X_1 + \ldots + X_n$. Suppose that $n = rp$, with $r, p \in \mathbf{N}$. Let $M \in \bar{\mathbf{R}}_+$ be fixed. Finally let $\bar{S}_n = \sum_{i=1}^n \big[f_M(X_i) - \mathbf{E}\big(f_M(X_i)\big)\big]$ with $f_M(x) = (x \wedge M) \vee (-M)$. Then the following estimations hold, for any $t \in \mathbf{R}$, $\tau > 0$, $b \in \mathbf{R}$:*

$$\left|\mathbf{E}\left(\exp it\left(\frac{S_n - rb}{\tau}\right)\right) - \phi(t)\right| \leqq 4n\mathbf{P}\big(|X_1| \geqq M\big) \tag{30}$$

$$+\frac{t^2}{\tau^2}\big(\operatorname{Var}(\bar S_n)-r\operatorname{Var}(\bar S_p)\big)+\left|\left\{\mathbf{E}\left(\exp it\left(\frac{S_p-b}{\tau}\right)\right)\right\}^r-\phi(t)\right|.$$

If we suppose moreover that the stationary sequence (X_n) *is of centered random variables, then for any* $t\in\mathbf{R}$, $x>0$, $\tau>0$,

$$\left|\left\{\mathbf{E}\left(\exp it\left(\frac{S_p}{\tau}\right)\right)\right\}^r-\exp\left(-\frac{t^2}{2}\right)\right|\leqq 2\frac{t^2\vee t^4}{r}+2r\mathbf{P}\big(|S_p|\geqq x\big) \tag{31}$$

$$+\frac{rt}{\tau}\left|\mathbf{E}\big(S_p\mathbf{1}_{|S_p|\geqq x}\big)\right|+\frac{t^2}{2}\left|1-\frac{r}{\tau^2}\mathbf{E}\big(S_p^2\mathbf{1}_{|S_p|<x}\big)\right|$$

$$+\big(t^2\vee|t|^3\big)r\mathbf{E}\left(\left[\frac{S_p^2}{\tau^2}\wedge\frac{|S_p|^3}{\tau^3}\right]\mathbf{1}_{|S_p|<x}\right).$$

PROOF. The proof of Lemma 2 is very classical. An analogue approximation lemma under mixing assumptions can be found in Dehling et al. [10] or in Berkes et al. [4]. For the sake of clarity, we give its proof in detail. Clearly

$$\frac{S_n-rb}{\tau}=\frac{1}{\tau}\left(\sum_{i=1}^n f_M(X_i)-rb\right)+\frac{1}{\tau}\sum_{i=1}^n\big(X_i-f_M(X_i)\big)=:\frac{S_{1,n}-rb}{\tau}+S_{2,n}.$$

We deduce, from $\mathbf{P}(S_{2,n}\neq 0)\leqq n\mathbf{P}\big(|X_1|\geqq M\big)$, that

$$\left|\mathbf{E}\Big(e^{it\frac{S_n-rb}{\tau}}\Big)-\mathbf{E}\Big(e^{it\frac{S_{1,n}-rb}{\tau}}\Big)\right|\leqq 2n\mathbf{P}\big(|X_1|\geqq M\big). \tag{32}$$

We also obtain using the trivial fact

$$|x_1\dots x_m-y_1\dots y_m|\leqq\sum_{i=1}^m|x_i-y_i|,\quad\text{for}\quad x_i,y_i\in\mathbf{C},\quad |x_i|,|y_i|\leqq 1, \tag{33}$$

$$\left|\Big\{\mathbf{E}(e^{it\frac{S_{1,p}-b}{\tau}})\Big\}^r-\Big\{\mathbf{E}(e^{it\frac{S_p-b}{\tau}})\Big\}^r\right|\leqq 2n\mathbf{P}\big(|X_1|\geqq M\big). \tag{34}$$

The random variable $S_{1,n}$ is the sum of r associated r.v.'s distributed as $S_{1,p}$ (recall that $n=rp$), thus Newman's inequality yields:

$$\left|\mathbf{E}\Big(e^{it\frac{S_{1,n}-rb}{\tau}}\Big)-\Big\{\mathbf{E}(e^{it\frac{S_{1,p}-b}{\tau}})\Big\}^r\right|\leqq\frac{t^2}{\tau^2}\big(\operatorname{Var}(\bar S_n)-r\operatorname{Var}(\bar S_p)\big). \tag{35}$$

The first part of Lemma 2 is proved collecting inequalities (32), (34) and (35). We now prove the second part. We suppose that $t^2 \leqq r$, otherwise (31) is trivial. Again (33) yields

$$\left|\left\{\mathbf{E}(e^{it\frac{S_p}{\tau}})\right\}^r - \left\{1 - \frac{t^2}{2r}\right\}^r\right| \leqq r\left|\mathbf{E}\left(e^{it\frac{S_p}{\tau}}\right) - \left(1 - \frac{t^2}{2r}\right)\right| \tag{36}$$

$$\leqq r\left|\mathbf{E}\left(e^{it\frac{S_p}{\tau}}\mathbf{1}_{|S_p|\geqq x}\right)\right| + r\left|\mathbf{E}\left(e^{it\frac{S_p}{\tau}}\mathbf{1}_{|S_p|<x}\right) - \left(1 - \frac{t^2}{2r}\right)\right|$$

$$\leqq 2r\mathbf{P}\big(|S_p| \geqq x\big) + \frac{rt}{\tau}\left|\mathbf{E}\big(S_p\mathbf{1}_{|S_p|\geqq x}\big)\right| + \frac{t^2}{2}\left|1 - \frac{r}{\tau^2}\mathbf{E}\big(S_p^2\mathbf{1}_{|S_p|<x}\big)\right|$$

$$+ r\left|\mathbf{E}\left[\left(e^{it\frac{S_p}{\tau}} - 1 - i\frac{t}{\tau}S_p + \frac{t^2}{2\tau^2}S_p^2\right)\mathbf{1}_{|S_p|<x}\right]\right|$$

Now we use the inequalities

$$\left|e^{iy} - 1 - iy + \frac{y^2}{2}\right| \leqq y^2 \wedge |y|^3, \quad \text{for any} \quad y \in \mathbf{R}$$

and

$$|e^x - 1 - x| \leqq x^2, \quad \text{for any} \quad x : |x| \leqq \frac{1}{2}$$

to deduce that

$$r\left|\mathbf{E}\left[\left(e^{it\frac{S_p}{\tau}} - 1 - i\frac{t}{\tau}S_p + \frac{t^2}{2\tau^2}S_p^2\right)\mathbf{1}_{|S_p|<x}\right]\right| \tag{37}$$

$$\leqq (t^2 \vee |t|^3)r\mathbf{E}\left(\left[\frac{S_p^2}{\tau^2} \wedge \frac{|S_p|^3}{\tau^3}\right]\mathbf{1}_{|S_p|<x}\right),$$

and

$$\left|e^{\frac{-t^2}{2}} - \left\{1 - \frac{t^2}{2r}\right\}^r\right| \leqq \frac{t^4}{4r}. \tag{38}$$

The second part of Lemma 2 is proved by collecting inequalities (36), (37) and (38).

Proof of Theorem 1. We obtain, taking $M = \varepsilon T_n$, $p = p_n$, $r = [n/p_n]$, $b = p_n\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)$ and $\tau = A_n$ in (30) and using (11), (12) and (13) (together with the inequality $\operatorname{Var}\bar{S}_{p_n[n/p_n]} \leqq \operatorname{Var}\bar{S}_n$ which follows from association)

$$\lim_{n\to+\infty} \mathbf{E}\left(\exp\left(it\frac{S_{p_n[n/p_n]} - p_n[n/p_n]\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)}{A_n}\right)\right) = \phi(t). \tag{39}$$

We deduce from (39) and (11) (using estimations as in (32))

$$\lim_{n\to+\infty} \mathbf{E}\left(\exp\left(it\frac{\bar{S}_{p_n[n/p_n]}}{A_n}\right)\right) = \phi(t). \tag{40}$$

Recall that, for a positive integer m, $\bar{S}_m = \sum_{i=1}^m \big[f_M(X_i) - \mathbf{E}\big(f_M(X_i)\big)\big]$ and $M = \varepsilon T_n$. Now the association property leads to

$$\begin{aligned}\frac{1}{A_n^2}\operatorname{Var}\big(\bar{S}_n - \bar{S}_{p_n[n/p_n]}\big) &\leqq \frac{1}{A_n^2}\big(\operatorname{Var}\bar{S}_n - \operatorname{Var}\bar{S}_{p_n[n/p_n]}\big)\\ &\leqq \frac{1}{A_n^2}\left(\operatorname{Var}\bar{S}_n - \left[\frac{n}{p_n}\right]\operatorname{Var}\bar{S}_{p_n}\right)\end{aligned}$$

The last bound together with (13) yields

$$\lim_{n\to+\infty} \frac{1}{A_n^2}\operatorname{Var}\big(\bar{S}_n - \bar{S}_{p_n[n/p_n]}\big) = 0. \tag{41}$$

The limit in (41) together with (40) and some standard estimations yields

$$\lim_{n\to+\infty} \mathbf{E}\left(\exp\left(it\frac{\bar{S}_n}{A_n}\right)\right) = \phi(t). \tag{42}$$

We deduce from (11) (arguing as in (32))

$$\lim_{n\to+\infty} \left|\mathbf{E}\left(\exp\left(it\frac{S_n - n\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)}{A_n}\right)\right) - \mathbf{E}\left(\exp\left(it\frac{\bar{S}_n}{A_n}\right)\right)\right| = 0. \tag{43}$$

The proof of Theorem 1 is complete, by (43) and (42).

Proof of Theorem 2. In order to prove this theorem, it suffices to take $M = \varepsilon T_n$, $r = \left[\frac{n}{p}\right]$, $b = p\mathbf{E}\big(f_{\varepsilon T_n}(X_1)\big)$ and $\tau = A_n$ in (30), to note that

$$\frac{1}{A_n^2}\left(\operatorname{Var}\left(\bar{S}_n\right) - \left[\frac{n}{p}\right]\operatorname{Var}(\bar{S}_p)\right) \tag{44}$$

$$\leqq \frac{2n}{A_n^2}\sum_{i=2}^{n} G_i(M) + \frac{p}{A_n^2}\operatorname{Var}\left(f_M(X_1)\right),$$

and to argue exactly as in the proof of Theorem 1. Let us note that for $p=1$, the left hand side of (44) is bounded only by the first term on the right hand side.

PROOF OF COROLLARY 1. We shall check the assumptions of Proposition 1 for n belonging to a set of infinite integers $Q \subset \mathbf{N}$ (according to Remark 3, following Proposition 1). To this end, fix $p \in \mathbf{N}$. According to (31) and the Chebyshev inequality, the requirements of Proposition 1 for $n \in Q$ (to be defined later) are satisfied if there exist two sequences $r = r_p \to +\infty$, $x = x_p \to +\infty$ as $p \to +\infty$ such that

($\mathcal{C}_1$) $\lim_{p\to+\infty} x^{-2} r \operatorname{Var} S_p = 0$,

($\mathcal{C}_2$) $\lim_{p\to+\infty} \frac{L(rp)}{L(p)} = 1$.

($\mathcal{C}_3$) $\limsup_{p\to+\infty} \frac{r \operatorname{Var} S_p}{\tau^2} \leqq$ const., with $\tau^2 = r\mathbf{E}\left(S_p^2 \mathbf{1}_{|S_p|<x}\right)$,

($\mathcal{C}_4$) $\lim_{p\to+\infty} r\mathbf{E}\left(\frac{|S_p|^3}{\tau^3}\mathbf{1}_{|S_p|<x}\right) = 0$, $\lim_{p\to+\infty} \frac{r \operatorname{Var} S_p}{x\tau} = 0$.

The existence of such sequences x and r is guaranteed by the following lemma that we discuss in the appendix (let us note that this lemma does not require any dependence assumptions).

LEMMA 3. *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of stationary and centered r.v.'s having finite variance. Suppose that* $\operatorname{Var} S_n = nL(n)$ *with L a slowly varying function and that* (19) *holds with $A_n = \sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$. Then there exist two sequences $r = r_p \to +\infty$, $x = x_p \to +\infty$ as $p \to +\infty$ fulfilling conditions* ($\mathcal{C}_1$), ($\mathcal{C}_2$), ($\mathcal{C}_3$) *and* ($\mathcal{C}_4$).

We define for the sequences (r_p) and (x_p) (i.e. those for which ($\mathcal{C}_1$), ($\mathcal{C}_2$), ($\mathcal{C}_3$) and ($\mathcal{C}_4$) are satisfied)

$$Q := \{n_p := pr_p,\ p \in \mathbf{N}\} \quad \text{and} \quad \tau_{n_p}^2 := r_p\mathbf{E}\left(S_p^2\mathbf{1}_{|S_p|<x_p}\right). \tag{45}$$

Lemma 2, together with Lemma 3, yields then

$$\lim_{n\to+\infty,\ n\in Q}\left\{\mathbf{E}\exp it\left(\frac{S_p}{\tau_n}\right)\right\}^{n/p} = \exp\left(-\frac{t^2}{2}\right).$$

We deduce, from ($\mathcal{C}_2$), ($\mathcal{C}_3$) and $\operatorname{Var} S_n = nL(n)$, that

$$\limsup_{p\to+\infty}\mathbf{E}\left(\frac{S_{n_p}}{\tau_{n_p}}\right)^2 \leqq \text{const.} \tag{46}$$

Condition (19) of Proposition 1 is thus satisfied for $A_n = \tau_n$ and for $n \in Q$, $n \to +\infty$. We then deduce from Proposition 1 (for more clarity combine Inequalities (30), (31) with $M = +\infty$, $b = 0$, $\phi(t) = \exp\left(-\frac{t^2}{2}\right)$, $n := pr_p$, $r := r_p$, $x := x_p$ and $\tau := \tau_{n_p}$ (r_p, x_p are as above) and apply $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$.)

$$\lim_{n\to+\infty,\ n\in Q} \mathbf{E} \exp it \left(\frac{S_n}{\tau_n}\right) = \exp\left(-\frac{t^2}{2}\right). \tag{47}$$

Let us now identify the normalizing sequence $(\tau_n)_{n\in Q}$. We deduce from (46) that the sequence $\left(\frac{S_n}{\tau_n}\right)_{n\in Q}$ is uniformly integrable. Thus by (47) and using Theorem 5.4 in Billingsley [5]

$$\lim_{n\to+\infty,\ n\in Q} \frac{\mathbf{E}\,|S_n|}{\tau_n} = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} |x| \exp\left(-\frac{1}{2}x^2\right)\, dx = \sqrt{\frac{2}{\pi}}. \tag{48}$$

In order to finish the proof of Corollary 1, we need the following lemma that we discuss in the Appendix. In the sequel, we denote $\sigma_n^2 = \operatorname{Var} S_n$ and $\rho_n = \sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$.

Lemma 4. *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of stationary and centered r.v.'s. Suppose that Condition* (24) *holds, with a slowly varying function $\tilde{L}$. Let r_p be such that*

$$\lim_{p\to+\infty} \frac{\tilde{L}(pr_p)}{\tilde{L}(p)} = 1. \tag{49}$$

Suppose that there exists a sequence (g_p) tending to infinity with p for which

$$\lim_{p\to+\infty} \frac{r_p \mathbf{E}\left(S_p^2 \mathbf{1}_{|S_p|\leqq g_p\sigma_p}\right)}{\rho_{pr_p}^2} = 1 \tag{50}$$

then

$$\lim_{p\to+\infty} \frac{1}{\rho_p^2}\mathbf{E}\left(S_p^2 \mathbf{1}_{|S_p|\leqq g_p\sigma_p}\right) = 1. \tag{51}$$

End of the Proof of Corollary 1. We follow exactly the lines of the proofs of Theorems 2 and 3 of [10]. An outline is the following. Without loss of generality the sequence $r = r_p$ (whose existence is guaranteed by

Lemma 3) can moreover satisfy Condition (49) (we refer the reader to the proof of Lemma 4 for a justification). We obtain combining (45) and (48)

$$\lim_{p\to+\infty}\frac{r_p\mathbf{E}\left(S_p^2\mathbf{1}_{|S_p|\leqq x_p}\right)}{\rho^2_{pr_p}}=1 \tag{52}$$

(recall that $\rho_n=\sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$). Now we note that $x_p=g_p\sigma_p$ and that the sequence (g_p) tends to infinity with p (for this cf. (62) and (63) below). This fact together with (52) proves that Condition (50) is satisfied. Since the sequence (r_p) satisfies Condition (49), we deduce then according to Lemma 4, that (51) holds.

Now we argue as in the proof of (4.10) in [10]: we construct, using the limit in (51), two sequences $h(n_p)$ and $j(n_p)$ tending to infinity with p, fulfilling $h(n_p)\leqq j(n_p)\leqq g_{n_p}$ and

$$w^2(n_p):=\frac{1}{\sigma^2_{n_p}}\mathbf{E}\left(S^2_{n_p}\mathbf{1}_{\sqrt{h(n_p)}\sigma_{n_p}\leqq|S_{n_p}|\leqq j(n_p)\sigma_{n_p}}\right)\to 0.$$

Define the sequence $\left(d(n)\right)_{n\in Q}$ by (4.11) of [10]:

$$\lim_{n\in Q,\ n\to+\infty}d(n)=0,\quad d(n)\geqq\max\left(2h^{-1/2}(n),w^2(n)\right),\quad n\in Q. \tag{53}$$

Now suppose that the elements of Q are arranged in an increasing order, say $Q=\{n_k,\,k\geqq 1\}$ (the sequence (n_k) is increasing) and let

$$J_k=\left[n_kh^2(n_k)d(n_k),n_kj^2(n_k)d(n_k)\right].$$

The sequences $h(n_k)$ and $j(n_k)$ are chosen in such a way that for k sufficiently large, say $k\geqq k_0$, one has

$$J_k\cap J_{k+1}\neq\emptyset \tag{54}$$

(we refer the reader to (4.12) of [10] for more details).

We deduce from (53) and the choice of the sequences $h(n_k)$ and $j(n_k)$ that the left endpoint of J_k tends to infinity with k. This remark together with (54) yields, for m sufficiently large, the existence of a positive integer $k\geqq k_0$ such that $m\in J_k$. Thus, we have for some $g\in\left[h(n_k),j(n_k)\right]$ and some $\theta\in[0,1]$ (cf. (4.13) of [10])

$$m=n_kg^2d(n_k)=n_k\left[g^2d(n_k)\right]+\theta n_k=:M_k+\theta n_k. \tag{55}$$

Let us first prove that the sequence (S_{M_k}) suitably normalized converges in distribution to the standard Gaussian law. For this we note that M_k belongs to the set Q defined by (45). In fact let

$$p := n_k, \quad r := r(n_k) := \left[g^2 d(n_k)\right], \quad x = x(n_k) = g\sigma_{n_k}. \tag{56}$$

We have to check that those sequences fulfill conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ (we shall check this property in the Appendix). If so, $M_k = n_k\left[g^2 d(n_k)\right] = pr$ (with p and r are as defined by (56)) is an element of the set Q. Hence we conclude using (46) and (47)

$$\limsup_{k\to+\infty} \mathbf{E}\left(\frac{S_{M_k}}{\tau_{M_k}}\right)^2 \leqq \text{const.}, \qquad \lim_{k\to+\infty} \mathbf{E}\exp it\left(\frac{S_{M_k}}{\tau_{M_k}}\right) = \exp\left(-\frac{t^2}{2}\right). \tag{57}$$

Next we prove (cf. Appendix) that, for m and M_k as in (55)

$$\lim_{k\to+\infty} \tau_{M_k}^{-2}\mathbf{E}(S_m - S_{M_k})^2 = 0. \tag{58}$$

We set $\tau_m := \tau_{M_k}$ for m and M_k as in (55). Then by (57), (58) and (55)

$$\lim_{m\to+\infty} \mathbf{E}\exp it\left(\frac{S_m}{\tau_m}\right) = \exp\left(-\frac{t^2}{2}\right). \tag{59}$$

Our task now is to identify the normalizing constants τ_m. We deduce from the first bound in (57) and from (58) that the sequence $\left(\frac{S_m}{\tau_m}\right)_{m\in\mathbf{N}}$ is uniformly integrable and we conclude using (59) (as for (48)) that

$$\lim_{m\to+\infty} \frac{\mathbf{E}|S_m|}{\tau_m} = \sqrt{\frac{2}{\pi}}. \tag{60}$$

The proof of Corollary 1 is now complete by (59) and (60).

4. Appendix

PROOF OF LEMMA 3. The proof of this lemma follows along the lines of the proof of Theorem 1 of Dehling et al. [10]. In the sequel we give an outline of this proof in order to emphasize its validity without any mixing conditions.

As it is noticed by (3.14) of [10], the slowly varying property of L yields the existence of a sequence $(\mathcal{X}_p)$ such that

$$\lim_{p\to+\infty} \mathcal{X}_p = +\infty \quad \text{and} \quad \lim_{p\to+\infty} \sup_{1\leqq t\leqq \mathcal{X}_p} \left|\frac{L(pt)}{L(p)} - 1\right| = 0. \tag{61}$$

Let $(z_p)_p$ be a sequence of real numbers fulfilling $z_p \to +\infty$ as $p \to +\infty$ and $z_p \leqq \sqrt{\mathcal{X}_p}$.

Define the sequence $(i_p)_p$ as (3.8) of [10]:

$$\lim_{p\to+\infty} i_p = +\infty, \quad \lim_{p\to+\infty} 2^{-i_p} \log z_p = +\infty.$$

Let $k = k_p$ be a positive integer less than i_p and fulfilling (3.9) of [10]:

$$\frac{1}{\mathrm{Var}\, S_p}\mathbf{E}\Big(S_p^2 \mathbf{1}_{\frac{|S_p|}{\sqrt{\mathrm{Var}\,(S_p)}}\in I_{k_p}}\Big) \leqq \frac{1}{i_p}, \quad \text{with} \quad I_i(p) := \Big]z_p^{2^{-i-1}}, z_p^{2^{-i}}\Big].$$

Define the sequence (g_p) and (v_p) as in (3.10) and (3.5) of [10]:

$$g_p := z_p^{2^{-k_p}}, \quad v_p^2 := \frac{1}{\mathrm{Var}\, S_p}\mathbf{E}\left(S_p^2 \mathbf{1}_{\sqrt{g_p}<\frac{|S_p|}{\sqrt{\mathrm{Var}\, S_p}}\leqq g_p}\right).$$

Finally let

(62)
$$x := x_p = g_p\sqrt{\mathrm{Var}\, S_p}, \quad r = r_p = g_p^2 c_p \quad \text{where} \quad c_p := \max\big(2g_p^{-1/2}, v_p^2\big).$$

Note that

(63) $$\lim_{p\to+\infty} g_p = +\infty, \quad \lim_{p\to+\infty} c_p = 0, \quad \lim_{p\to+\infty} x_p = +\infty, \quad \lim_{p\to+\infty} r_p = +\infty.$$

Let us now check conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ for those sequences (x_p) and (r_p).

1. The expressions of x_p and r_p yield $(\mathcal{C}_1)$.

2. The choices of g_p and r_p yield $r_p \leqq \mathcal{X}_p$. The definition of $\mathcal{X}_p$ (cf. (61)) implies $(\mathcal{C}_2)$.

3. Condition (19) with $A_n = \sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$, yields $(\mathcal{C}_3)$ (cf. (3.11) of [10]).

4. Condition $(\mathcal{C}_3)$, together with the expressions of the sequences r_p, g_p, v_p yields $(\mathcal{C}_4)$ (argue as (2.16) and (2.17) of [10]).

Proof of Lemma 4. Since $\tilde{L}$ is a slowly varying function, then as in (61) (cf. (3.14) of [10] for a proof) there exists a sequence $(\mathcal{X}'_p)_p$ with

(64) $$\lim_{p\to+\infty} \mathcal{X}'_p = +\infty \quad \text{and} \quad \lim_{p\to+\infty} \sup_{1\leqq t\leqq \mathcal{X}'_p} \left|\frac{\tilde{L}(pt)}{\tilde{L}(p)} - 1\right| = 0.$$

Hence there exists a sequence (r_p) fulfilling (49). The proof of Lemma 4 is complete, noting that Condition (24) together with the choice of r_p leads to $\rho^2_{pr_p} \sim r_p \rho^2_p$, as p tends to infinity.

Let us note that without loss of generality, we can assume that (64) is satisfied by the sequence $(\mathcal{X}_p)_p$ of (61) (instead of $(\mathcal{X}'_p)_p$). If so, we can choose a sequence (r_p) tending to infinity with p fulfilling both $(\mathcal{C}_2)$ and (49).

More details. Let us check that the sequences defined by (56) fulfill conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$. Noting that $x^{-2} r \operatorname{Var} S_p \leqq d(n_k)$ and using (53), we deduce that $(\mathcal{C}_1)$ is satisfied. Choosing the sequence $j(n_p)$ to satisfy moreover $j(n_p) \leqq \frac{1}{2}\sqrt{\mathcal{X}_p}$, we deduce that $r = \left[g^2 d(n_k)\right] \leqq j^2(n_k) d(n_k) < \frac{1}{2}\mathcal{X}(n_k)$, we conclude using (61) that $(\mathcal{C}_2)$ is satisfied. Condition (19) with $A_n = \rho_n = \sqrt{\frac{\pi}{2}}\mathbf{E}|S_n|$ leads $(\mathcal{C}_3)$ (arguing as (3.11) of [10]). The first limit in $(\mathcal{C}_4)$ is satisfied by (2.16) and (2.17) of [10]. Finally the second limit in $(\mathcal{C}_4)$ is also satisfied, in fact, we obtain using $(\mathcal{C}_3)$ and standard estimations, that there exists a constant C such that, $x^{-1}\tau^{-1} r \operatorname{Var} S_p \leqq C\sqrt{d(n_k)}$. Since $d(n_k)$ goes to 0 as n_k tends to infinity, we deduce that the last bound of $(\mathcal{C}_4)$ is satisfied.

We now check (58) for m and M_k as in (55). We deduce from (55) and the association property (i.e. the sequence $(\sigma^2_p)_p$ is non-decreasing) that

$$\mathbf{E}(S_m - S_{M_k})^2 = \mathbf{E}S^2_{\theta n_k} \leqq \sigma^2_{n_k}. \tag{65}$$

We deduce from $r = \left[g^2 d(n_k)\right] < \mathcal{X}(n_k)$, the limit in (61) and the expressions of M_k and r (cf. (55) and (56)), that for k sufficiently large (cf. also [10], p. 1368)

$$\frac{\operatorname{Var} S_{M_k}}{r \operatorname{Var} S_{n_k}} \geqq \frac{1}{2}. \tag{66}$$

Since $r = r(n_k)$ tends to infinity with k, we deduce then combining (65) and (66) that

$$\lim_{k\to+\infty} \sigma^{-2}_{M_k}\mathbf{E}(S_m - S_{M_k})^2 = 0.$$

The last limit together with the first bound in (57) proves (58).

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