WARM–UP: creation and annihilation operators

- $a^\dagger$ and a are creation/annihilation operators for a given mode

  In the language of second quantisation, a 'mode' is a single–particle state

  Consider the operators $b^\dagger = a$ and $b = a^\dagger$

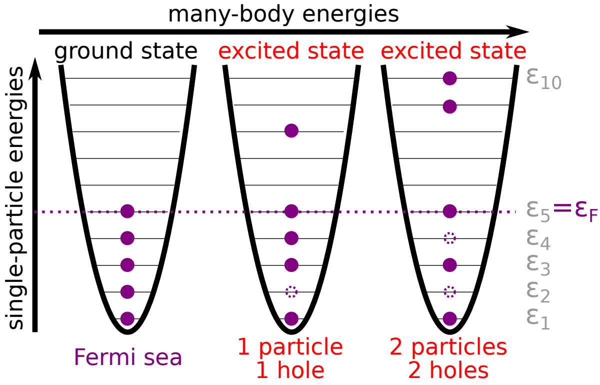
- May $b^\dagger$ and b be understood as creation/annihilation operators ...

  1. for bosons ?
  2. for fermions ?

Particle and hole excitations in a polarised Fermi gas

- Examples of Fermi gases: ultracold atomic gas, electrons in a metal, white–dwarf stars

  'Polarised' means all fermions are in the same spin state $|\uparrow\rangle$



- 1st quantisation: the many–fermion wavefunction is a Slater determinant

  2nd quantisation: anticommuting creation/annihilation operators

- The ground state is a Fermi sea: $|\text{FS}\rangle = a^\dagger_{\varepsilon_5}\, a^\dagger_{\varepsilon_4}\, a^\dagger_{\varepsilon_3}\, a^\dagger_{\varepsilon_2}\, a^\dagger_{\varepsilon_1}\, |\text{vac}\rangle$

  We are interested in the excited states.

Particle and hole excitations in a polarised Fermi gas

- Hamiltonian for the ideal polarised Fermi gas: $H - \varepsilon_F N = \sum_{\mathbf{k}} (\varepsilon_k - \varepsilon_F)\, a^\dagger_{\mathbf{k}}\, a_{\mathbf{k}}$

  Single–particle energies measured from Fermi surface ε_F (grand–canonical Hamiltonian, temp$^{\text{ture}}$ $T = 0$)

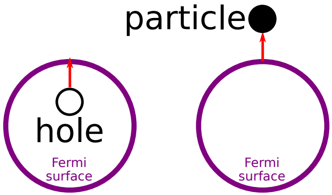
- The ground state is a Fermi sea $|\mathrm{FS}\rangle$, with energy $E_{\mathrm{FS}} > 0$

  All single–particle states with energy $\varepsilon < \varepsilon_F$ are occupied; all those with $\varepsilon > \varepsilon_F$ are free

  The Fermi energy $\varepsilon_F \neq E_{\mathrm{FS}}$: calculate both for a uniform ideal Fermi gas with density n

- There are two types of elementary excitations:

1. Promote 1 particle from inside Fermi sea to Fermi surface: $\varepsilon_{\mathrm{hole}} = \varepsilon_F - \varepsilon_k > 0$
2. Promote 1 particle from Fermi surface to higher–energy state: $\varepsilon_{\mathrm{part}} = \varepsilon_k - \varepsilon_F > 0$



- 'Elementary' because

  all excitations may be written in terms of these

  How to express the excitation of an atom from inside the Fermi sea to an excited single–particle state?

- The vacuum state of the $\{a_{\mathbf{k}}\}$ is $|\mathrm{vac}\rangle$, not $|\mathrm{FS}\rangle$

  For $\varepsilon_k < \varepsilon_F$, $\quad a_{\mathbf{k}}\, |\mathrm{FS}\rangle \neq 0$

From particles to quasi–particles

- KEY IDEA: consider the ground state as a new vacuum,
  introduce new operators $\{b^\dagger_{\mathbf{k}}\}$ creating the *excitations* = *quasiparticles*

  see also (fermionic or bosonic) Bogoliubov theory, BCS theory, ...

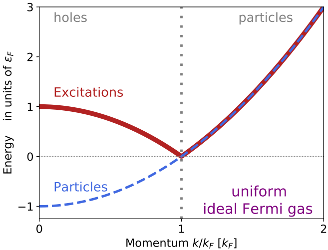
- **Excitations in an ideal Fermi gas:**

  If $\varepsilon_k > \varepsilon_F$: $\quad b^\dagger_{\mathbf{k}} = a^\dagger_{\mathbf{k}} \qquad \varepsilon_{\text{part}} = \varepsilon_k - \varepsilon_F > 0$

  If $\varepsilon_k < \varepsilon_F$: $\quad b^\dagger_{\mathbf{k}} = a_{-\mathbf{k}} \qquad \varepsilon_{\text{hole}} = \varepsilon_F - \varepsilon_k > 0$

  Expelling particle with momentum $\mathbf{k}$ creates hole with momentum $-\mathbf{k}$

  Dispersion relation: $\varepsilon^{\text{excit}}(k) = |\varepsilon_k - \varepsilon_F|$



- $H - N\varepsilon_F = (E_{\text{FS}} - N\varepsilon_F) \quad + \sum_{\mathbf{k}} |\varepsilon_k - \varepsilon_F|\, b^\dagger_{\mathbf{k}}\, b_{\mathbf{k}}$

  Creating an excitation increases the energy!

- If the fermions are charged, what is the charge of the hole?

[J.M. Ziman, *Elements of Advanced Quantum Theory*, CUP (1969), §2.7]