# ICFP M2 Advanced Quantum Mechanics: Problem #4: Scattering resonances

D.J. Papoular and F. Chevy

November 27, 2024

## 1 Optical Fabry–Perot interferometer

In this introductory section, we consider the well–known wave optics experiment sketched in Fig. 1: A (classical) electromagnetic field impinges onto a Fabry–Perot cavity consisting of two semi–reflecting mirrors. The field propagates perpendicularly to the mirrors, i.e. along the direction defined by $\boldsymbol{e}_z$. The two mirrors are identical, and they are characterised by their transmission and reflection coefficients $t$ and $r$, which we choose to be real. Hence, the conservation of energy requires $t^2 + r^2 = 1$.

1. Briefly explain why, in each of the three spatial regions $\alpha = A$ (to the left of the mirrors), $B$ (between the mirrors), and $C$ (to the right of the mirrors), the electric field may be sought as $\boldsymbol{E} = [\mathcal{E}_{\alpha+} e^{i(kz-\omega t)} + \mathcal{E}_{\alpha-} e^{i(-kz-\omega t)} + \text{h.c.}]\,\boldsymbol{\epsilon}$, where $\mathcal{E}_{\alpha+}$ and $\mathcal{E}_{\alpha-}$ are two complex amplitudes. Recall the link between the wavevector $k$ and the frequency $\omega$ for propagation in vacuum, and the condition on the polarisation $\boldsymbol{\epsilon}$ with respect to the propagation direction $\boldsymbol{e}_z$.

2. Justify that, for the problem to be fully determined, two conditions must be imposed on the six complex amplitudes.

3. We consider the 1D scattering problem where $\mathcal{E}_{A+} = \mathcal{E}_I$ is the known incident field and $\mathcal{E}_{C-} = 0$. Show that the (complex) transmission coefficient $t_{\text{FP}} = \mathcal{E}_{C+} e^{ikL}/\mathcal{E}_{A+}$ and reflection coefficient $r_{\text{FP}} = \mathcal{E}_{A-}/\mathcal{E}_{A+}$ are given by:

$$t_{\text{FP}} = \frac{t^2\, e^{ikL}}{1 - r^2\, e^{2ikL}} \quad \text{and} \quad r_{\text{FP}} = r\, \frac{1 - e^{2ikL}}{1 - r^2 e^{2ikL}}\ . \tag{1}$$

4. Show that, if $2kL$ is an integer multiple of $2\pi$, then $|t_{\text{FP}}|^2 = 1$ and $r_{\text{FP}} = 0$: all the incident intensity is transmitted. Interpret this as a resonance phenomenon. Explain why the phase of the electromagnetic wave plays a key role; why may this experiment be seen as an example of multiple–wave interference?

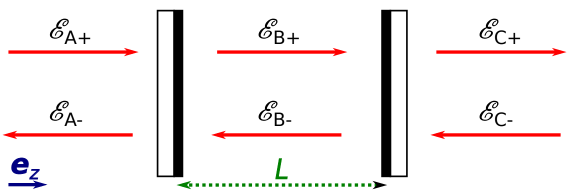


**Figure 1** The six complex amplitudes describing the electric field in the optical Fabry–Perot experiment.

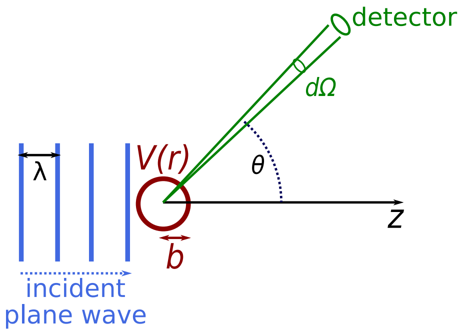


**Figure 2** Scattering of an incident plane wave (blue wavefronts) by the spherically–symmetric potential $V(r)$ of range $b$ (red). The scattered wave is measured in the direction $\theta, \phi$, within the solid angle $d\Omega$ (green).

## 2 Partial waves, $s$–wave phase shift, and scattering length

We turn to the quantum description of the collision between two particles whose masses are $m_1$ and $m_2$, and whose positions are $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Assuming that the interaction between the two particles is described by a potential $V(\boldsymbol{r}_1, \boldsymbol{r}_2) = V(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ which only depends on the relative position $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, the centre–of–mass and relative motions separate. We choose the referential where the centre of mass is fixed, and describe the relative motion in terms of a single fictitious particle of mass $m = m_1 m_2/(m_1 + m_2)$ at the position $\boldsymbol{r}$. We further assume that the scattering potential $V(r)$ is spherically symmetric. Hence, the Hamiltonian reads $H = \boldsymbol{p}^2/(2m) + V(r)$.

The considered scattering experiment is sketched in Fig. 2. We assume that the potential has a finite range $b$, that is, $V(r)$ is negligible for $r \gtrsim b$. A plane wave (blue) with the energy $E = \hbar^2 k^2/(2m)$ (and the wavelength $\lambda = 2\pi/k$, propagating along the $z$ axis, impinges on the scattering potential. The scattered wave is measured outside the range of the potential ($r \gg b$), in the direction defined by the polar angles $\theta, \phi$, within the small solid angle $d\Omega$.

The experiment of Fig. 2 corresponds to a stationary state described by a wavefunction $|\Psi\rangle$ with the well–defined energy $E$ which satisfies the following boundary coundition for large $r$:

$$\Psi(\boldsymbol{r}) \underset{r\to\infty}{=} e^{ikz} + f(\boldsymbol{k}, \boldsymbol{r}) \frac{e^{ikr}}{r} \ . \tag{2}$$

In Eq. (2), the scattering amplitude $f(\boldsymbol{k}, \boldsymbol{r})$ is not known a priori: it should be determined from the solution of the Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$.

5. Recall the meaning of each of the terms on the right–hand side of Eq. (2).
   What is the unit of $f(\boldsymbol{k}, \boldsymbol{r})$, and what is its relation to the scattering cross–section $d\sigma/d\Omega$?

6. Is $|\Psi\rangle$ square–integrable? Which physical property of the state $|\Psi\rangle$ does this reflect?
   What plays the role of the normalisation condition?

7. For $E = 0$, check that the wavefunction defined by Eq. (2) is spherically symmetric. Recall how the scattering length $a$ may be extracted from it.

The object of the remainder of this section is to point out the important link between the scattering length $a$ and the $s$–wave phase shift $\delta_0(k)$ (defined in Question 11 below).

8. For $E > 0$, explain why, despite the spherical symmetry of $H$, the wavefunction $\Psi(\boldsymbol{r})$ only exhibits cylindrical symmetry about the $z$ axis.

9. Justify that $|\Psi\rangle$ may be expanded onto a basis of wavefunctions corresponding to well–defined values of the energy $E = \hbar^2 k^2/2m$, the total angular momentum $\boldsymbol{\ell}^2$, and its projection $\ell_z$. Enforcing the cylindrical symmetry along $z$, conclude that $|\Psi\rangle$ may be written as:

$$\Psi(\boldsymbol{r}) = \sum_{\ell \geq 0} \Psi_\ell(\boldsymbol{r}) = \sum_{\ell \geq 0} \alpha_\ell \, R_\ell(r) P_\ell(\cos\theta) \ , \ \text{where } P_\ell(u) = \text{Legendre polynomial of order } \ell. \tag{3}$$

In Eq. (3), the sum spans all integers $\ell$. The coefficients $\alpha_\ell$ will be determined later.
HINTS: The spherical harmonics $Y_l^m(\theta,\phi)$ satisfy $\boldsymbol{L}^2\,|Y_\ell^m\rangle = \ell(\ell+1)\,|Y_\ell^m\rangle$ and $L_z\,|Y_\ell^m\rangle = m\,|Y_\ell^m\rangle$. The dependence on $\phi$ of $|Y_\ell^m\rangle$ is $e^{im\phi}$. For $m=0$, $Y_\ell^0(\theta) = [(2\ell+1)/(4\pi)]^{1/2} P_\ell(\cos\theta)$.

10. Show that the radial wavefunction $R_l(r)$ satisfies the following Schrödinger equation:

$$\frac{1}{r}\frac{d^2(rR_\ell)}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2mV(r)}{\hbar^2}\right] R_\ell = 0\ . \tag{4}$$

HINT: The Laplacian operator $\Delta$ acting on the function $f(r,\theta,\phi)$ satisfies $\Delta = \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2}(rf) - \dfrac{\boldsymbol{\ell}^2}{r^2} f$ .

11. Justify that $R_\ell$ must remain finite for $r \to 0$, so that $\lim_{r\to 0}(rR_\ell) = 0$.
Show that $R_\ell$ may be chosen such that $R_\ell(r) \underset{r\to\infty}{=} \sin(kr - \ell\pi/2 + \delta_\ell)/r$,
and explain how the phase shift $\delta_\ell$ is determined.

12. Combining Eq. (3) with the asymptotic behaviour of $R_\ell$,
show that $\alpha_\ell = (2\ell+1) i^\ell e^{i\delta_\ell}/k$ and that the scattering amplitude reads:

$$f(\boldsymbol{k},\boldsymbol{r}) = f(k,\theta) \quad = \quad \sum_{\ell\geq 0} (2\ell+1)\, P_\ell(\cos\theta)\, \frac{e^{2i\delta_\ell}-1}{2ik}\ . \tag{5}$$

HINT: Asymptotic expansion of a plane wave: $e^{ikz} \underset{r\to\infty}{=} \dfrac{1}{2ikr} \sum_{\ell\geq 0} (2\ell+1)\, P_\ell(\cos\theta)\, \left[(-1)^{\ell+1} e^{-ikr} + e^{ikr}\right]$ .

13. From now on, we focus on the scattering of slow particles: $kb \ll 1$.
   a) For $b \ll r \ll 1/k$, show that $R_\ell = c_1 r^\ell + c_2/r^{\ell+1}$.
   b) For $r \sim 1/k$, express $R_l$ using $c_1$, $c_2$, and the spherical Bessel functions $j_\ell(kr)$, $y_\ell(kr)$:

$$R_l = c_1 \frac{(2\ell+1)!!}{k^\ell}\, j_\ell(kr) - c_2 \frac{k^{\ell+1}}{(2\ell-1)!!}\, y_\ell(kr)\ , \tag{6}$$

   HINTS: $j_\ell(kr)$ and $y_\ell(kr)$ are two independent solutions of Eq. (4) for $V(r) = 0$.
   For small $\rho$, $j_\ell(\rho) \underset{\rho\to 0}{=} \rho^\ell/(2\ell+1)!!$ and $y_\ell(\rho) \underset{\rho\to 0}{=} -(2\ell-1)!!/\rho^{2\ell+1}$.

   c) Assuming that $\delta_\ell$ is small, show that $\delta_\ell \approx \tan\delta_\ell = \dfrac{c_2}{c_1}\dfrac{k^{2\ell+1}}{(2\ell-1)!!(2\ell+1)!!}$ .
   HINT: For large $\rho$, $j_\ell(\rho) \underset{\rho\to\infty}{=} \sin(\rho - \ell\pi/2)/\rho$ and $y_\ell(\rho) \underset{\rho\to\infty}{=} -\cos(\rho-\ell\pi/2)/\rho$.

   d) Conclude that the scattering amplitude $f_\ell = (e^{2i\delta_\ell} - 1)/(2ik)$ in the partial wave $\ell$ is proportional to $k^{2\ell}$. Recall the intuitive explanation for the suppression of $f_\ell$ for $\ell \geq 1$.

14. Show that, for $kb \ll 1$, $\delta_0 = -ka$. Conclude that, outside the range of the potential (i.e. for $r \gg b$), the $s$–wave component $\Psi_{\ell=0}$ of the complete scattering wavefunction $|\Psi\rangle$ satisfies:

$$\Psi_{\ell=0}(r) \underset{r\gg b}{=} \frac{\sin[k(r-a)]}{kr}\ , \tag{7}$$

whereas the higher partial waves $\Psi_{\ell\geq 1}$ are unaffected.
HINT: Use first your answer to Question 7, and then the expansion of Eq. (3), remembering that $P_0(u) = 1$.

Equation (7) confirms that the scattering length determines the rate at which the $s$–wave phase changes with $k$. We shall now see how this phase may undergo resonant phenomena similar to the Fabry–Perot resonance described in Sec. 1, where the interaction potential $V(r)$ plays a role which is analogous to the Fabry–Perot cavity in the optical experiment.

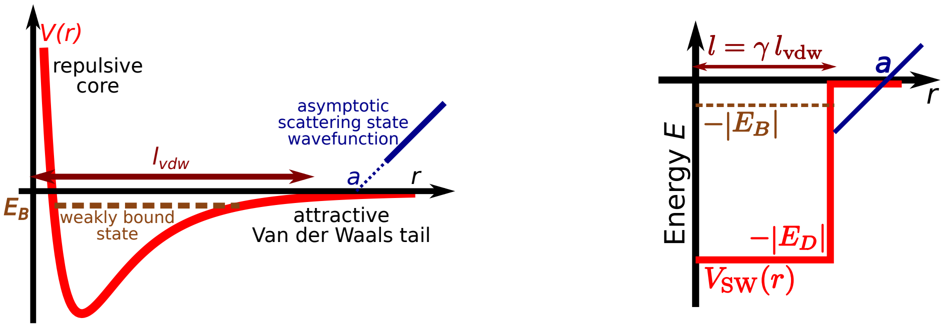


**Figure 3** The potential representing the interaction between two colliding atoms. Left: sketch of the real potential. Right: the spherical square well potential $V_{\rm SW}$ (Eq. (8)) used in Sec. 3.

## 3 One scattering channel: shape resonance

We consider a low–energy collision between two atoms. In this section, the atomic internal states do not change in the course of the collision, so that they do not enter its description. The potential representing the interaction between the two atoms is sketched in Fig. 3 (left). We wish to replace this real potential by a simplified one which allows for analytical calculations. Hence, we consider the spherically–symmetric square well potential $V_{\rm SW}$ shown in Fig. 3 (right) and defined by:

$$V_{\rm SW}(r) = -E_D \text{ for } r \leq l; \quad V_{\rm SW}(r) = 0 \text{ for } r > l. \tag{8}$$

In Eq. (8), the length $l$ is the range of the potential, and the energy $E_D > 0$ sets the well depth.

15. Using Question 7, show that the scattering length $a$ associated with the potential $V_{\rm SW}$ is:

$$a/l = 1 - \frac{\tan(k_D l)}{k_D l} \ , \quad \text{with } k_D l = (E_D/E_l)^{1/2} = (2ml^2 E_D/\hbar^2)^{1/2}. \tag{9}$$

16. Recall why $V_{\rm SW}(r)$ supports bound states which are purely $s$–wave.
Show that their energies $E_B = -\hbar^2\kappa^2/(2m)$ satisfy:

$$\kappa l = -\frac{k_1 l}{\tan(k_1 l)} \quad \text{and} \quad (k_D l)^2 = (k_1 l)^2 + (\kappa l)^2 \ . \tag{10}$$

17. Solve the two coupled Eqs. (10) graphically in the $(k_1 l, \kappa l)$ plane (both $k_1$ and $\kappa$ are positive). Show that, if $(2n_B - 1)\,\pi/2 < k_D l < (2n_B + 1)\,\pi/2$, where $n_B$ is a positive integer, then $V_{\rm SW}$ supports exactly $n_B$ $s$–wave bound states.

18. Comparing Eqs. (9) and (10), check that the potential depths at which a new bound state appears exactly correspond to those for which the scattering length diverges (see Fig. 4).

19. Point out the analogy with the optical Fabry–Perot resonance analysed in Sec. 1. In particular, what is the analog, for the matter wave, of the cavity within which the optical wave undergoes multiple reflections?

The effect analysed in this section is the *scattering resonance* whose theoretical description is the simplest. It involves a single scattering channel (i.e. the atomic internal states do not change). It is achieved by tuning the shape of the interaction potential (we have elected to vary its depth $E_D$). Hence, it is known as a *shape resonance*.

The experimental implementation of shape resonances is challenging (but not impossible: see e.g. Ref. [1]). Experimentalists routinely rely on another type of scattering resonance, the *Feshbach resonance*, which is easier to achieve in the laboratory. Its theoretical description is more involved: we investigate it in Sec. 4.

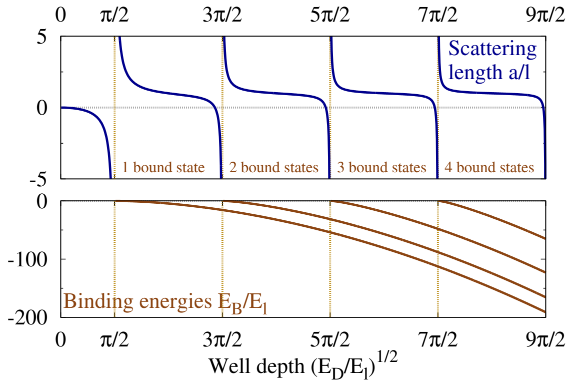


**Figure 4** Scattering length $a$ (top) and bound–state energies $E_B$ for the spherical square well potential of Eq. (8). Lengths are in units of the potential range $l$ and energies in units of $E_l = \hbar^2/(2ml^2)$.

# 4 Two coupled scattering channels: Feshbach resonance

In this section, we still consider a low–energy collision between two atoms. However, we now allow for a change in the internal atomic states as a function of $r$. We make three assumptions:

- For large interatomic distances, the two–atom internal state is $|\text{op}\rangle$. There, the atoms interact via the potential $V_{\text{op}}(r)$. The atoms come in with an energy $E$ which is greater than the scattering threshold for $V_{\text{op}}(r)$, (see Fig. 5), so that the state $|\text{op}\rangle$ defines an *open channel*.
- The two–atom Hamiltonian supports a bound state corresponding to a *different* two–atom internal state $|\text{cl}\rangle$. The interaction potential $V_{\text{cl}}(r)$ for the two atoms in the state $|\text{cl}\rangle$ is different from the one for the two atoms in the state $|\text{op}\rangle$. In particular, we assume that $E$ is below the scattering threshold for $V_{\text{cl}}(r)$, so that the state $|\text{cl}\rangle$ defines a *closed channel*.
- The states $|\text{op}\rangle$ and $|\text{cl}\rangle$ are *coupled*, so that the internal state may change as a function of $r$: it is $|\text{op}\rangle$ for large values of $r$, but it may be a combination of $|\text{op}\rangle$ and $|\text{cl}\rangle$ for smaller $r$.

We shall show that if the closed–channel bound state energy is close to the open–channel threshold energy, a scattering resonance occurs. This two–channel resonance, called the *Feshbach resonance*, is routinely used in cold–atom experiments to manipulate the nature (attractive or repulsive) and the strength of the interaction between these atoms.

We describe the collision using a two–channel Hamiltonian:

$$H = \begin{pmatrix} H_{\text{op}} & W \\ W & H_{\text{cl}} \end{pmatrix} \ . \tag{11}$$

In Eq. (11), $H_{\text{op}} = \boldsymbol{p}^2/(2m) + V_{\text{op}}(r)$ is the open–channel Hamiltonian, $H_{\text{cl}} = \boldsymbol{p}^2/(2m) + V_{\text{cl}}(r)$ is the closed–channel Hamiltonian, and $W(r)$ is the coupling operator (which we assume to be Hermitian). We seek an eigenstate $|\Psi\rangle = \Psi_{\text{op}}(r)\,|\text{op}\rangle + \Psi_{\text{cl}}(r)\,|\text{cl}\rangle$ of $H$ with the energy $E$, which satisfies the scattering boundary condition of Eq. (2), *all atoms being in* $|\text{op}\rangle$ *for large* $r$:

$$\Psi(\boldsymbol{r}) \underset{r\to\infty}{=} \left[ e^{ikz} + f(\boldsymbol{k}, \boldsymbol{r}) \frac{e^{ikr}}{r} \right] |\text{op}\rangle \ . \tag{12}$$

Thus, the closed–channel component $\Psi_{\text{cl}}(r)$ vanishes for large $r$, but *it is non–zero for finite* $r$.

20. Write the eigenvalue equation for the matrix Hamiltonian $H$ in the following form:

$$\begin{cases} (E - H_{\text{op}})\,|\Psi_{\text{op}}\rangle &= W\,|\Psi_{\text{cl}}\rangle \\ (E - H_{\text{cl}})\,|\Psi_{\text{cl}}\rangle &= W\,|\Psi_{\text{op}}\rangle \end{cases} \tag{13}$$

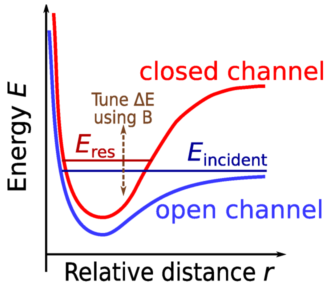


**Figure 5** The two coupled scattering channels giving rise to a Feshbach resonance. The blue and red curve show the interaction potential in the open and closed channels, respectively.

21. We first consider the second line of Eq. (11). Justify that the operator $E - H_{\rm cl}$ is invertible, and that its inverse $G_{\rm cl} \approx |\phi_{\rm res}\rangle \langle\phi_{\rm res}| / (E - E_{\rm res})$. Here, $|\phi_{\rm res}\rangle$ is the resonant bound state in the closed channel with the energy $E_{\rm res}$. Conclude that $|\Psi_{\rm cl}\rangle$ satisfies:

$$|\Psi_{\rm cl}\rangle = |\phi_{\rm res}\rangle \, \frac{\langle\phi_{\rm res}|W|\Psi_{\rm op}\rangle}{E - E_{\rm res}} \, . \tag{14}$$

22. We now turn to the first line of Eq. (13). Justify that the operator $E - H_{\rm op}$ is non–invertible. Hence, we introduce the Green's function $G^+_{\rm op} = (E - H_{\rm op} + i0^+)^{-1}$. Justify that $|\Psi_{\rm op}\rangle$ satisfies:

$$|\Psi_{\rm op}\rangle = |\psi^+_{\boldsymbol{k}}\rangle + G^+_{\rm op} W \, |\phi_{\rm res}\rangle \, \frac{\langle\phi_{\rm res}|W|\Psi_{\rm op}\rangle}{E - E_{\rm res}} \, . \tag{15}$$

In Eq. (15), $|\psi^+_{\boldsymbol{k}}\rangle$ satisfies Eq. (2) for the Hamiltonian $H_{\rm op}$ with $E = \hbar^2 k^2/(2m)$.
Hints: First, think about the states for which $E - H_{\rm op}$ is not invertible. Then, use Eq. (14).

23. Express $\langle\phi_{\rm res}|W|\Psi_{\rm op}\rangle$ in terms of the 'bare' states $|\phi_{\rm res}\rangle$, $|\psi^+_{\boldsymbol{k}}\rangle$ and their energies $E_{\rm res}$, $E$:

$$\frac{\langle\phi_{\rm res}|W|\Psi_{\rm op}\rangle}{E - E_{\rm res}} = \frac{\langle\phi_{\rm res}|W|\psi^+_{\boldsymbol{k}}\rangle}{E - E_{\rm res} - \langle\phi_{\rm res}|W G^+_{\rm op} W|\phi_{\rm res}\rangle} \, . \tag{16}$$

The final step involves the asymptotic behaviour of $G^+_{\rm op}(\boldsymbol{r})$. It can be shown (see the complementary Sec. 4.1) that, for any square–integrable function $u(\boldsymbol{r})$:

$$\langle\boldsymbol{r}|G^+_{\rm op}|u\rangle \underset{r\to\infty}{=} -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\,\langle\psi^-_{\boldsymbol{k}}|u\rangle \;\;,\;\text{with } \boldsymbol{k} = k\boldsymbol{r}/r. \tag{17}$$

In Eq. (17), $|\psi^-_{\boldsymbol{k}}\rangle$ is the scattering state with ingoing–wave boundary conditions:
$\psi^-_{\boldsymbol{k}}(\boldsymbol{r}) = [\psi^+_{-\boldsymbol{k}}(\boldsymbol{r})]^* \underset{r\to\infty}{=} e^{ikz} + f^-(\boldsymbol{k},\boldsymbol{r})\, e^{-ikr}/r$.

24. Combining Eqs. (14), (15), and (17), show that $|\Psi_{\rm op}\rangle$ has the following asymptotic behaviour:

$$\Psi_{\rm op}(\boldsymbol{r}) \underset{r\to\infty}{=} e^{ikz} + f_{\rm op}\frac{e^{ikr}}{r} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\frac{\langle\psi^-_{\boldsymbol{k}}|W|\phi_{\rm res}\rangle\,\langle\phi_{\rm res}|W|\psi^+_{\boldsymbol{k}}\rangle}{E - E_{\rm res} - \langle\phi_{\rm res}|W G^+_{\rm op} W|\phi_{\rm res}\rangle} \, . \tag{18}$$

25. Finally, take the zero–energy limit of Eq. (18) to show that the scattering length has a resonant behaviour:

$$a = a_{\rm op} + \frac{m}{2\pi\hbar^2}\frac{|\langle\phi_{\rm res}|W|\psi_{\boldsymbol{0}}\rangle|^2}{E - E_{\rm res} - \langle\phi_{\rm res}|W G^+_{\rm op} W|\phi_{\rm res}\rangle} \, , \tag{19}$$

where $a_{\rm op}$ is the scattering length of $H_{\rm op}$, namely, the value of the scattering length if the bound state is detuned far from the resonance ($E - E_{\rm res}$ large). The state $|\psi_{\boldsymbol{0}}\rangle = |\psi^+_{\boldsymbol{0}}\rangle = |\psi^-_{\boldsymbol{0}}\rangle$ is the scattering state for $H_{\rm op}$ with $E = 0$.

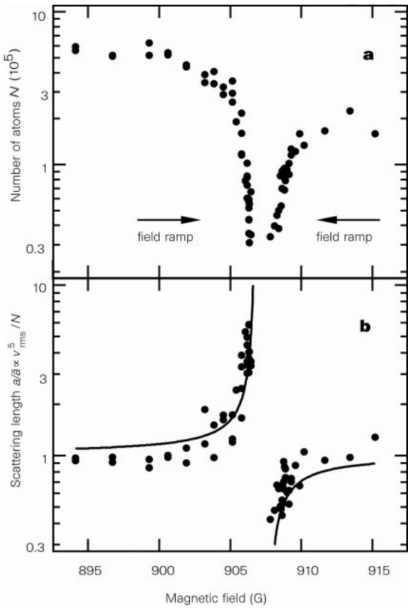


**Figure 6** First reported observation of a Feshbach resonance in a cold atomic gas. The experiment was performed on a gas of Bose–condensed $^{23}$Na atoms. *(a)* Near the resonance, inelastic collisions are enhanced, so that the atoms are lost from the trap. *(b)* For a given static magnetic field $B$, the scattering length is measured by releasing the atoms from the trap and measuring their kinetic energy after a time–of–flight expansion. The measured curve $a(B)$ (normalised to its off–resonant value $a_{\text{op}}$) matches the hyperbolic prediction of Eq. (19).
(Figure reproduced from Ref. [2]).

26. Sketch $a$ as a function of $E$. What is the resonant energy, and why is it slightly different from $E_{\text{res}}$? Which parameter sets the strength of the resonance (also called 'resonance width')?

27. Why may a static magnetic field may be used to tune $E - E_{\text{res}}$ and, hence, vary $a$?
**Difficult:** What is the coupling $W$ due to?
Hint: For two 2 atoms, list the 4 relevant angular momenta. How do they couple for large $r$? for small $r$?

## 4.1 Complement: asymptotic behaviour of the Green's function $G^+(r)$

The object of the last few questions is to prove the asymptotic behaviour for large $r$ of the single–channel Green's function $G^+_{\text{op}}(\boldsymbol{r}, \boldsymbol{r}')$ given by Eq. (17) above.

We recall the exact expression for the Green's function $G_0^+$ of the (single channel) free Hamiltonian $H_0 = \boldsymbol{p}^2/(2m)$:

$$G_0^+ = \frac{1}{E - H_0 + i0^+} \quad \text{and} \quad G_0^+(\boldsymbol{r}, \boldsymbol{r}') = \langle \boldsymbol{r} | G_0^+ | \boldsymbol{r}' \rangle \quad = -\frac{m}{2\pi\hbar^2} \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r} - \boldsymbol{r}'|} \ . \tag{20}$$

We alse define $G_0^- = (E - H_0 - i0^+)^{-1} = [G_0^+]^\dagger$ which is an ingoing wave.

Similarly, we shall use two Green's functions for $H_{\text{op}}$. The first one, $G^+_{\text{op}} = (E - H_{\text{op}} + i0^+)^{-1}$, has already been introduced in Question 22. The second one is $G^-_{\text{op}} = (E - H_{\text{op}} - i0^+)^{-1} = [G^+_{\text{op}}]^\dagger$.

28. We call $|\phi_{\boldsymbol{k}}\rangle$ the plane wave with the wavevector $\boldsymbol{k}$: $\phi_{\boldsymbol{k}}(\boldsymbol{r}) = \langle \boldsymbol{r} | \phi_{\boldsymbol{k}} \rangle = \exp(i\boldsymbol{k} \cdot \boldsymbol{r})$.
Starting from Eq. (20), show that Eq. (17) holds for the free problem:

$$\text{For any square–integrable function } u(\boldsymbol{r}), \quad \langle \boldsymbol{r} | G_0^+ | \boldsymbol{u} \rangle \underset{r\to\infty}{=} -\frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \langle \phi_{\boldsymbol{k}} | u \rangle \ . \tag{21}$$

29. Show that the ingoing–wave scattering state $|\psi^-_{\boldsymbol{k}}\rangle$ (in the open channel) satisfies:

$$|\psi^-_{\boldsymbol{k}}\rangle = (1 + G^-_{\text{op}} V) \, |\phi_{\boldsymbol{k}}\rangle \ . \tag{22}$$

Hint: Prove $|\phi_{\boldsymbol{k}}\rangle = (1 - G_0^- V) \, |\psi^-_{\boldsymbol{k}}\rangle$. Then, apply $(A^{-1} - B^{-1}) = A^{-1}(B - A)B^{-1}$ to $G^-_{\text{op}}$ and $G_0^-$.

30. Show that $G^+_{\text{op}} = G_0^+(1 + V G^+_{\text{op}})$.
Apply Eq. (21) to $|v\rangle = (1 + V G^+_{\text{op}}) \, |u\rangle$, and conclude using Eq. (22).
Hint: For the first step, apply $(A^{-1} - B^{-1}) = B^{-1}(B - A)A^{-1}$ to $G^+_{\text{op}}$ and $G_0^+$.

## Further reading

INTRODUCTORY

- A detailed discussion of Fabry–Perot interference and, more generally, multiple–beam interference, may be found in Ref. [3, Sec. 9.6].
- Chapter 8 of Ref. [4] provides an introduction to quantum scattering, including a discussion of the partial waves expansion and the properties of the free spherical waves. The spherical harmonics are reviewed in Ref. [5, chap. VI, complement A].
- The application of quantum scattering theory to ultracold gases is vividly presented in Ref. [6]. This reference contains useful ideas concerning the optical theorem and the key role of the scattering length, as well as an experimentally–informed discussion of elastic and inelastic processes.

EXPERIMENTAL REFERENCES ON SCATTERING RESONANCES

- The $s$–wave shape resonances of Sec. 3 in this problem are challenging to exploit experimentally. However, shape resonances involving non–$s$–wave bound states were observed even before Feshbach resonances [1].
- The first observation of a Feshbach resonance in an ultracold gas is due to W. Ketterle [2]. In this early experiment, the change in the interaction energy (and, hence, in the scattering length) was measured by time–of–flight techniques. The key figure of this paper is reproduced here as Fig. 6.
- There is another important type of scattering resonance, called *confinement–induced resonance*, which may occur when the system is trapped in highly anisotropic geometries [7].

MORE ADVANCED

- A mathematically–oriented presentation of spherical harmonics, which stresses their link with group representation theory, may be found in Ref. [8, chap. 7].
- Another famous example where waves are expanded onto a basis of spherical harmonics is the multipole expansion of the electromagnetic field [9, chap. 4].
- The review article by Chin et al [10] provides references to the theoretical and experimental literature on Feshbach resonances in cold gases. Their useful Table IV lists many of the known Feshbach resonances.
- Messiah gives a remarkably thorough and concise presentation of quantum scattering theory [11, chap. XIX].

## References


[1] H. M. J. M. Boesten, C. C. Tsai, J. R. Gardner, D. J. Heinzen, B. J. Verhaar, Phys. Rev. A **55**, 636 (1997).

[2] S. Inouye, M. R. Andrews, J. Stenger, H. Miesner, D. M. Stamper-Kurn, W. Ketterle, Nature **392**, 151 (1998).

[3] E. Hecht, *Optics*, Addison-Wesley, 4th ed. (2002).

[4] C. Cohen-Tannoudji, B. Diu, F. Laloë, *Quantum Mechanics, volume 1*, Wiley (1991).

[5] C. Cohen-Tannoudji, B. Diu, F. Laloë, *Quantum Mechanics, volume 2*, Wiley (1991).

[6] J. Dalibard, in *Proceedings of the International School of Physics Enrico Fermi, Course CXL: Bose-Einstein condensation in gases*, Società Italiana di Fisica (1998).

[7] E. Haller, M. Gustavsson, M. J. Mark, J. G. Danzl, R. Hart, G. Pupillo, H. C. Nägerl, Science **325**, 1224 (2009).

[8] Y. Kosmann-Schwarzbach, *Groups and symmetries: from finite groups to Lie groups*, Springer (2010).

[9] J. D. Jackson, *Classical Electrodynamics*, Wiley, 3rd ed. (1999).

[10] C. Chin, R. Grimm, P. Julienne, E. Tiesinga, Rev. Mod. Phys. **82**, 1225 (2010).

[11] A. Messiah, *Quantum Mechanics, volume II*, North Holland (1962).